DETERMINATION OF SOME ALGEBRAIC PROPERTIES OF BASARAB LOOPS

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EFFIONG, GIDEON OKON (B.Sc., UYO, M.Sc., IFE)

(20174078398)

A DISSERTATION SUBMITTED TO

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FEDERAL UNIVERSITY OF TECHNOLOGY, OWERRI

IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE AWARD OF THE DEGREE OF DOCTOR OF PHILOSOPHY (Ph.D) IN MATHEMATICS

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Certification

This is to certify that this work **"Determination of Some Algebraic Properties of Basarab Loops"** was carried out by Effiong, Gideon Okon (20174078398) in partial fulfilment for the award of the degree of Doctor of Philosophy in the Department of Mathematics, Federal University of Technology, Owerri, Imo State, Nigeria.

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Dedication

This thesis is dedicated to God Almighty for the gift of life, and for His abundant grace and wisdom which guided me throughout my studies, and in the writing of this research.

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I am grateful to God, the Father of our Lord Jesus Christ whose love and mercy has brought me up to this level in my pursuit of knowledge. God has been with me through the ups and downs of my academic life. Looking back the memory lane in my studies, indeed I would say, I have crossed many big rivers to be here and God has been faithful.

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Abstract

Basarab loops are non-associative generalizations of groups and are classified as loops of non Bol-Moufang type. They are G-loops with deep algebraic and structural properties. Not much were known about the form of isotopes, holomorphs, associators, center, and subloops of Basarab loops. This work was to determine some algebraic properties of Basarab loops. The objectives of the study were to construct a Basarab loop, investigate the relationship between Basarab loop and other loops like conjugacy closed loop, abelian inner mapping loop, and Osborn loop, examine the isotopes of a Basarab loop, investigate the holomorphs and associators of a Basarab loop, and characterize some subloops of a Basarab loop. Basarab loop identities were considered and some algebraic properties of loops were investigated. Loop notions such as the use of parentheses, multiplication group, isotopy theory, and holomorphy theory, total multiplication group were examined on a Basarab loop through the governing laws of Basarab loop. Some constructions of Basarab loops were given and some algebraic properties of Basarab loops were determined. The results obtained have shown that the centrum of a Basarab loop is a subloop and it is equal to the center of a Basarab loop, and that a Basarab loop with the left (right) inverse property, or inverse property is an extra loop. Necessary and sufficient conditions for isotopes and principal isotopes of a Basarab loop were determined. It was proved that every principal isotope of a Basarab loop is a Basarab loop. It was proved that any Osborn loop is a Basarab loop if and only if it is a left (right) Basarab loop. Also, the holomorphs of a Basarab loop were investigated by considering a group $A(Q)$ of automorphisms of a loop. Some necessary and sufficient conditions for an $A(Q)$-holomorph of a loop $(Q,\cdot)$ to be left (right) Basarab loop, and Basarab loop were established. Some left (right) translation mapping of the holomorph of a left (right) Basarab loop was shown to be left (right) regular. It was shown that an $A(Q)$-holomorph of a loop $(Q,\cdot)$ which satisfies the inverse property is a Basarab loop if and only if $(Q,\cdot)$ is a Basarab loop and every automorphism of Q is nuclear. Some subloops of a Basarab loop which are characterized by permutations were obtained. It was proved that a Basarab loop is a centrum-abelian inner mapping loop. Relationship between associators and inner mappings of a Basarab loop was defined. It was shown that the associator of any three elements of a Basarab loop is contained in the center and centrum of a Basarab loop. This study has presented additional properties of Basarab loops which are now available for applications. Therefore, it is recommended that researchers and cryptographers should use the properties of Basarab loops determined by this study for further research and applications.

Keywords: Basarab loop, isotopy, holomorphy, associator, inner mapping

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List of Symbols

$a \in S$	a is an element of set S.
$A \cap B$	The intersection of the sets A and B.
$\{a \in S \mid a \text{ satisfies } P\}$	The subset of elements of S satisfying P.
$A \implies B$	A implies B.
$A \iff B$	Set A if and only if set B.
$A = B$	Sets A and B are equal.
$\mathbb{N}$	Set of Natural numbers.
$\mathbb{Z}$	Set of integers.
$\mathbb{Z}_n$	Set of integers mod n.
$\|G\|$	Order of the loop G.
G/H	Quotient of the loop G by a subloop H.
$R_x : G \to G$	Right translation of the loop G.
$L_x : G \to G$	Left translation of the loop G.
$J : G \to G$	Inverse translation of the loop G.
$I : G \to G$	Identity map of the loop G.
$N(G, \cdot)$	Nucleus of the loop G.
$SYM(G)$	Symmetric group of the loop G.
$AUT(G, \cdot)$	Autotopism group of the loop G.
$AUM(G, \cdot)$	Automorphism group of the loop G.

CHAPTER ONE

INTRODUCTION

1.1 Background of the Study

Plain subtraction of natural numbers is the oldest known non-associative operation used by man. In the year 1845, Arthur Cayley constructed the first example of an abstract non-associative system. Cayley numbers is the name given to the example of an abstract non-associative system constructed by Arthur Cayley. Cayley numbers were later generalized by Dickson, and are known as Cayley-Dickson algebras, (Pfulgfelder, 2000). Dickson (1906) published the paper titled *Linear algebras in which division is always uniquely possible*. In his paper, 'non-associative division rings' were investigated. As at that time, 'non-associative division rings' had less attention. It was seen that 'non-associative division rings' had a loop as the multiplicative structure, but this time, without weak associativity. Dickson also constructed examples of loop which are not Moufang. Also, Goodaire, Jespers & Miles (1996) examined alternative rings from loop point of view.

Loop theory emerged from pure Mathematics areas in a simultaneous manner. These areas include algebra, geometry and topology. In Algebra, the researchers Max Zorn and Hans

Zassenhaus who were the students of Artin in Hamburg researched on the Algebraic aspect of non-associative systems through alternative algebras. Researcher like Suschkewitsch investigated non-associative systems through binary systems. Thus, Suschkewitsch approach to non-associative systems was different from the approach taken by Max Zorn and Hans Zassenhaus. Researcher Artin had proved theorems which were used by researchers to advancing results in pure Mathematics, (Pflugfelder, 2000). Worthy of mention is Ruth Moufang who used a theorem proved by Artin in her well-cited paper titled 'On Quasigroups '(Moufang, 1935).

In Geometry, Wilhelm Blaschke introduced another branch of Mathematics and it is known as Web Geometry. The Three major areas from which loop theory came to be are brought together by Web Geometry. The research followed the Hilbert's principle which states that geometric axioms of the planes correspond to the algebraic properties of the coordinating system. The two prominent papers written by Moufang (1935) and Bol (1937) marked the official beginning of loop theory. Moufang was a geometer, and one of the first three women who became famous in Mathematics, the other two women are Sofia Kovalevskaya and Emmy Noether, (Pflugfelder, 2000). In her paper, she began with an alternative field, and she proved Artin's theorem by the use of only the multiplicative system. She defines a structure, which she calls a Quasigroup Q^*, satisfying the following postulates :

(i) Closure;

(ii) Existence of an identity element and unique inverse;

(iii) $a(a'b) = (aa')b$ and $(ba')a = b(a'a)$;

(iv) $[a(ca)]b = a[c(ab)]$;

She also defines a system Q^{**}, believing it to be different from Q^*. Thus, Q^{**} satisfies an additional identity :

(v) $(ab)(ca) = a[(bc)a]$;

Bol showed that (iv) implies (v), and Bruck later proved that they are both equivalent to two other identities :

(vi) $[(ab)c]b = a[b(cb)]$; and

(vii) $(ab)(ca) = [a(bc)]a$.

The system Q^* is known as a Moufang loop, and it can be defined using any one of the identities (iv) through (vii) which are called the Moufang identities. According to Moufang, the system Q^* is diassociative and obeys the theorem called Moufang's theorem and it echoes Artin's theorem.

Moufang's Theorem states that every Moufang loop is di-associative, that is, any two elements generate a subgroup. Kiechle (2002) noted that Moufang in her paper presented a definition which is now known as a 'Bol loop with the right inverse property'. Kunen (1996) proves that each of the Moufang identities in a quasigroup implies that the quasigroup is a loop. Kunen (1996) noted that a loop identity need not always imply its mirror.

Bol's used Web-geometry to establish his results. He began his research by developing three new configurations U_1, U_2, U_3. He then asked a question, whether the closure of these three figures implies associativity. Bol proved that the three U figures together imply only the law $a[b(cb)] = [(ab)c]b$. This law is one of the Moufang identities. So, Bol was able to answer the question he asked in the negative. Details of this result are found in Nagy & Vojtěchovský (2003), Blaschke & Bol (1938), and Pflugfelder (1970). Bol made clear the algebraic meaning of each of the U figures. He showed that U_1 and U_2 correspond to the loop identities known as the left Bol and right Bol identities. These identities are : $(b(cb))a = b(c(ba))$ and $a((bc)b) = ((ab)c)b$. Bol justified the following properties implied by the U figures :

U_1 the right inverse property : $(dc)c' = d$

U_2 the left inverse property : $a'(ab) = b$

U_3 the anti-automorphic inverse law : $(ab)' = b'a'$

Bol proves that U_1 and U_2 together imply U_3, and when all three are closed, a Moufang's quasigroup Q^* is obtained. According to Bol's results, Moufang's Quasigroup Q^* obeys the law : $b(cb) = (bc)b$. This law is called the elastic law. Bol was able to show that a loop is Moufang if only if it is both right and left Bol. By this, Bol split the Moufang identities into two. He presented an example of Bol loops which are not Moufang, and credited this example to Zassenhaus. Bol also gave an example of a commutative Moufang loop, which is not a group, again, Bol credited this result to Zassenhaus. In the 1960s, Robinson investigated Bol loops from separate point of view. Details of this result are found in Ajimal (1978), Robinson (1964), Chein & Goodaire (2005), Foguel, Kinyon & Phillips (2006), Solarin (1986), and Solarin & Sharma (1984). Kiechle (2002) emphasized that the class of Bol loops is not self-dual, that the Bol identity $a(b \cdot ac) = (a \cdot bac)$ does not imply the right Bol identity. At the usage of the left Bol identity, the left inverse property, the left alternative law, etc. fit nicely with the mappings applied. For Bol loops, it is known that every loop isotope of a Bol loop is a Bol loop, (Pflugfelder, 1990).

Some renowned mathematicians like Max Zorn and Emil Artin who were previously exposed to the subject of quasigroups arrived at the United States and continued their research in this field. Besides this reason, a robust interest in non-associative structures previously existed in the United states, remarkably at the University of Chicago. Leonard Dickson, whose name we know from Cayley-Dickson algebra, was teaching at Chicago. Thus, Chicago became a new center of quasigroups research in the 1940s, just like Hamburg had become in the previous decade. In addition to research in alternative algebra, there were already American publications on quasigroups and they used the term 'quasigroup' in a broader sense, the way it is used now.

The terminology of quasigroup theory undertook a historic change at this time. It became apparent that it was necessary to distinguish between two classes of quasigroups : those with and those without an identity element. A new name was needed to designate the system with identity. This occurred around 1942, among people of the Albert's circle in Chicago, who coined the word 'loop' after the Chicago Loop. For Chicago locals, the term 'Loop' refers to the main business area and the elevated train that literally made a loop around this part of the city. The name loop, was an excellent choice in numerous ways. First, the word 'loop' rhymes with 'group'. Second, it expresses a sense of closure. Third, it is short and simple so that it could be adopted in other languages. Today, it is used in many languages, with slight variations : for example, DIE LOOP in German (first used by Pickert) and LUPA in Russian, (Jaiyeola, 2005).

The first publications introducing the term 'loop' were the two very important papers by Albert (1943, 1944) titled *Quasigroups I* and *Quasigroups II*. In addition to the introduction of the new term 'loop', a highly significant aspect of the *Quasigroups I* paper was the introduction of the concept of isotopy for quasigroups. Albert's papers were followed by two very important publications by Richard Hubert Bruck : *Some Results in the Theory of Quasigroups* (1944) and *Contributions to the Theory of Loops* (1946). American period of loop theory, extending from the 1940s through 1960s, has most important role credited to Albert and Bruck and their schools. The volume of the research done during this period by Albert and Bruck and the followers are enormous. Among the dominant topics of their research were the following:

(i) Isotopy theory; properties of isotopic quasigroups and loops; isotopic invariants, autotopisms, pseudo-automorphisms, isotopy-isomorphy properties;

(ii) Loops with different inverse properties: left, right, weak, cross, automorphic and anti-automorphic inverse properties;

(iii) Basic concepts of subquasigroups, cosets, ‘characteristic’ property π and nilpotency with respect to π;

(iv) Homomorphy theory;

(v) Groups of permutations on loops: multiplication groups, inner mappings and the notion of A-loops, semi-automorphism;

(vi) Moufang loops: commutative Moufang loops;

(vii) Bol loops and their subvariety of Bruck loops; and

(viii) Different classes of quasigroups: totally symmetric, distributive, abelian and through them, different geometric and combinatorial systems.

Bruck (1958) became the most reference book on loops. After some years, Jaiyeola (2009a) became another interesting book on loop theory as it extends the notion of Smarandache on loops.

The subject of Latin squares is, of course, much older than loop theory. Mutually orthogonal Latin squares were already studied by Euler in the XVIII century from a combinatorial point of view. However, as loop theory developed, there appeared connections between the combinatorial and several quasigroup- theoretical aspects of Latin squares. For example, combinatorial structures such as bloc designs or Steiner triple systems can be associated with algebraic varieties of Steiner quasigroups and totally symmetric loops, Denes & Keedwell (1974), Pelling & Rogers (1979). As discussed in Effiong (2017), some scientists in England during the 1930s, showed that every 6×6 Latin square belongs to a set of six so-called ‘adjugates’, which we now know as ‘conjugates’ or ‘inverse’ quasigroup operations, or ‘parastrophes’. The concept of parastrophes in general was introduced by Sade in France and subsequently, Rafael Artzy introduced isostrophes as product of parastrophes and isotopes, (Kannappan, 2009).

Loop theory is applied prominently in the science of protection of a set of information given out from the sender to the receiver. The information transformed is protected from an unlawful user, and error which may occur during the process. This science of protecting information is divided into parts, and the most common parts are cryptology and coding theory. Cryptology is further divided into two parts, namely cryptography and cryptanalysis. The knowledge of cryptography and cryptanalysis are much needed by a good cryptographer. In fact, a good cryptanalyst is a good cryptographer. This is because, cryptography is a science on procedures of transformation of information having an aim of securing the information transformed from an unauthorized beneficiary, while cryptanalysis is a science on procedures and strategies of attacking or penetrating an already secured information. Obviously, a cryptographer works to protecting a given information, while a cryptanalyst thrives to break down that particular information. Thus, the work in cryptography is to create new ciphers (that is, method of transformation of information) while the work in cryptanalysis is to search for means of breaking down a secured information (Shcherbacov, 2003; Jaiyeola & Adeniran, 2010; Effiong, 2017).

Denes & Denes (2001), noted that cryptology is based generally on fields which are commutative and associative. Shcherbacov (2003) supported this view, according to him, many development of error detecting and error correcting codes, cryptographic algorithms and enciphering systems have made use of associative algebraic structures, in which groups and fields are not exemption. Magliveras, Stinson & Trung (2002) presented a new approach to designing public key cryptosystems using one-way function and trap doors via associative algebraic structure. It is interesting to note that non-associative structures such as quasigroups and loops work wonderfully well, and to some extent perform better than associative structures. Jaiyeola (2008d) examined a double cryptography using the Smarandache Keedwell cross inverse quasigroup, while Jaiyeola (2012a) studied the application of Keedwell cross inverse quasigroup to cryptography. Jaiyeola & Smarandache (2018) worked on the inverse properties in Neutrosophic Triplet loop and their application to cryptography. Jaiyeola (2011a) studied middle universal m-inverse Quasigroups and their application to cryptography. Accordingly, Denes & Keedwell (2002) and Koscienlny (1997) noted that ciphers constructed using non-associative systems give excellent possibilities than ciphers constructed using associative systems. Denes & Keedwell (1992) studied equipment of hardware encryption and theoretical construction of cryptosystems using quasigroups. Denes & Keedwell (1974) and Shcherbacov (2003) noted that quasigroups are 'generalized permutations, 'of some kind and the number of quasigroups of order n is larger than $n! \cdot (n-1)! \cdot \; ... \; \cdot 2! \cdot 1!$. These results in general have prompted the effectiveness of applications of quasigroups and loops in cryptography and cryptanalysis.

Coding Theory is a science of protecting a given set of information from an unexpected error during transmission. Coding theorist ensures that errors occurred during transmission of information are corrected. According to Verhoeff (1969) and Beckley (1967), errors during transmission of data are mostly made by human operators. And the outstanding errors made by human are called single errors. These are errors made by interchanging adjacent digits. For instance, instead of $...ab...$ one rather obtained $...ba...$. Other types of single errors are insertion and deletion errors. Coding theory makes use of a loop transversal, a linear code and a channel. Smith (2000) explained that an abelian group channel is the set of errors corrected by the code. He considered the length 3 binary repetition code $C = \{000, \ 111\}$, and set $\mathbb{Z}_2^3$ to be the channel and $T = \{000, \ 001, \ 010, \ 100\}$ to be the set of errors corrected by the linear code C, and used it to define an abelian group homomorphism. He emphasized that an effective way of defining a code in a good channel is to specify the loop structure (an abelian group) on the set of errors.

Shcherbacov (2003) define a loop transversal as follows:

Let $(Q, \cdot)$ be a group, $(H, \cdot)$ be its subgroup, e be the identity element of this group. A complete system T of representatives of the left cosets as, $a \in Q$ is called a left transversal in group $(Q, \cdot)$ by subgroup $(H, \cdot)$. That is, from any coset $a_i \cdot H$ we take only one element, for example, element a_i. Thus, $T = \{1,\ a_1,\ a_2,\ ...,\ a_n,\ ...\}$ is a left transversal. Define on the set T an operation $*$ in the following way: $a * b = a \cdot b(mod\ H)$. Shcherbacov claimed that $(T,\ *)$ is a right quasigroup with identity element e. This means that, the equation $a * x = b$ has unique solution for any $a, b \in Q$ and $e * s = s * e$ for all $s \in T$. He concluded that if $(T,\ *)$ is a loop, then $(T,\ *)$ is called a loop transversal. According to Koscienlny (1997) and Denes & Keedwell (2002), the use of codes based on non-associative structures like quasigroups and loops in coding theory are of great advantage and research have shown that they present better possibilities than codes based on associative systems. Apart from cryptology, quasigroup theory are also applied in music theory. Effiong (2021) discussed some applications of quasigroups in music theory.

Research on loop theory which began over seven decades ago, has spread into various aspect of Mathematics and beyond. In Algebra, loop theory is extended into theories like Ring and Field, while in Functional Analysis, Topological loops have been investigated to some extent. In Physics, loop theory has been examined under near-field structures (Bulut, 2005; Figula, 2009). Further studies on loop theory showed that loop could be classified depending on the identities or conditions such a loop constitute. It is in this regard that loop theorists boast of Bol loop, Moufang loop, Buchsteiner loop, and other well-known loop structures. These loops over the years, have been much helpful in the science of securing information. Basarab loop is another type of a loop, and could serve as a variety and substitute, when loops are needed for a better security system in dissemination of information (Jaiyeola & Adeniran, 2010; Effiong, 2017).

1.2 Statement of the Problem

A Basarab loop is a type of loop which was introduced by Basarab (1992). He investigated the nucleus of a Basarab loop and also examined the relationship between Basarab loop and other loops like VD-loop, Generalized Moufang loop and Osborn loop. Some properties of a Basarab loop and its relationship with other loops were known which include Bol-Moufang loops and some inverse property loops, (Effiong 2017; Jaiyeola & Effiong 2018). The form of isotopes, holomorphs and associators of a Basarab loop are not yet known. Hence, there is need to investigate the isotopes, holomorphs and associators of a Basarab loop which form the main problem of this study.

1.3 Objectives of the Study

The main objective of this study is to establish some properties of isotopes, holomorphs and associators of a Basarab loop, using the middle inner mapping as well as the left and right translations of a loop.

The specific objectives of the study are :

(a) to construct a Basarab loop, and investigate the relationship between Basarab loop and some loops like (left, right) CC-loop, CC-loop, and Osborn loop;

(b) to examine the isotopes of a Basarab loop, and show that the centrum of a Basarab loop is a subloop and it is equal to the center of a Basarab loop;

(c) to investigate the holomorphic structure of a Basarab loop, and obtain some subloops of a Basarab loop that are characterized by permutations and then find the relationship among them; and

(d) to examine the algebraic properties of associators of a Basarab loop, and show that a Basarab loop is a centrum-Abelian inner mappings loop.

1.4 Justification of the Study

Loop theory has been an area of interest to many researchers considering its applications in coding theory and cryptography. Depending on the technical requirements for the dissemination pattern of a given information, the loop identities that will be used for the programming must be rightly chosen. Basarab loop identities are known to be pertinent tools for cryptographers, yet sufficient theoretical and practical investigations have not been carried out on the isotopes, holomorphs and associators of a Basarab loop. For instance, Jaiyeola (2011b, 2013a) worked on the application of some quasigroups and also on the discovery of cryptographic identities. The results have not been studied for a Basarab loop. The need of isotopism as a means of covering the message under transmission is that any unauthorized beneficiary at the receiving end, who has an access to the message, will not see the real message correctly in its isotopic form. The isotopic properties of a Basarab loop which could be used for this cryptic message, and holomorphs of a Basarab loop are yet to be investigated. This study hopes to provide some isotopic properties of a Basarab loop, some properties of holomorphs of a Basarab loop, and examine some algebraic properties of the associators of a Basarab loop. These results shall be useful to cryptographers, cryptanalysts and musicians.

1.5 Scope of the Study

This investigation is limited to the centrum, isotopes, holomorphs and associators of a Basarab loop, construction of a Basarab loop, the relationship between a Basarab loop and centrum-Abelian inner mappings loop, and to obtain some subloops of a Basarab loop that are characterized by permutations and the relationship among them.

CHAPTER TWO

LITERATURE REVIEW

2.1 Related Literature

Basarab loops (also called K-loops) are non-associative generalizations of groups. Basarab (1992) studied the nuclues of a Basarab loop . It was established that the nucleus of a Basarab loop is nontrivial, and it is a normal subloop. If $(Q, \cdot)$ is a Basarab loop and $N(Q, \cdot)$ be the nucleus, it was proved that the quotient loop $Q/N(Q, \cdot)$ is an abelian. A Basarab loop $(Q, \cdot)$ is solvable, if the nucleus $N(Q, \cdot)$ of a Basarab loop $(Q, \cdot)$ has an odd order. Basarab (1996) studied the Nucleus of a Basarab loop and it was proved that any Basarab loop (any VD-loop) is a G-loop, and any Basarab loop (VD-loop) is an Osborn loop. A generalized Moufang loop $(Q, \cdot)$ was shown to be a VD-loop, if $x^4 \in N$ whenever $x \in Q$, and N being the nucleus of the generalized Moufang loop$(Q, \cdot)$. It was proved that the three nuclei of each VD-loop $(Q, \cdot)$ coincide, that is, $N_r = N_m = N_l = N$. Moreover, N was proved to be a normal subloop in $(Q, \cdot)$. It was established that a Basarab loop $(Q, \cdot)$ is a VD-loop if $x^2 \in N$ for any $x \in Q$, where N is the nucleus of $(Q, \cdot)$; and a VD-loop $(Q, \cdot)$ is a Basarab loop if $x^2 \in N$ for any $x \in Q$, where N is the nucleus of $(Q, \cdot)$. Basarab (1997) studied the Nucleus of a Basarab Loop and it was established that if a loop $(Q, \cdot)$ has a nontrivial nucleus N, which is a normal subloop of $(Q,\cdot)$ and (x,y,z) is the associator of elements $x,y,z \in Q$, then $(x,y,z)n = n(x,y,z)$, where $n \in N$. It was shown that if a (non-group) loop $(Q,\cdot)$ has a nontrivial nucleus N which is a normal subloop of $(Q,\cdot)$ and the associator of any three elements of $(Q,\cdot)$ belongs to N, then N has a nontrivial centre $Z(N)$. The nucleus $N(Q,\cdot)$ of a Basarab loop $(Q,\cdot)$ which is not a group, was proved to have a nontrivial centre. The nucleus $N(Q,\cdot)$ of a Basarab loop $(Q,\cdot)$ was proved to contain the associator of any three elements of Q. The centre $Z(N)$ of the nucleus $N(Q,\cdot)$ of a Basarab loop $(Q,\cdot)$ was shown to be a normal subloop of $(Q,\cdot)$. If a Basarab loop $(Q,\cdot)$ is not a group, it was proved that the quotient loop $Q/Z(N)$ is a group. A Basarab loop generated by one element was proved to be solvable, and it was established that every subloop of a Basarab loop generated by one element is solvable. It was proved that the centre and the nucleus of a Basarab loop with an automorphic inverse property coincide, and that every Basarab loop with an automorphic inverse property is nilpotent.

Effiong (2017) studied a Basarab loop. The results showed that the left, right and middle nuclei of the Basarab loop coincide, and the nucleus of the Basarab loop was obtained to be the set of elements x of the Basarab loop permitting their middle inner mapping T_x to be in the automorphism group of the Basarab loop. The following results were also established: a Basarab loop is flexible if and only if it has an alternative property; a Basarab loop has a cross inverse property if and only if it is an abelian group; a Basarab loop has a left inverse property if and only if it is flexible; a Basarab loop has a right inverse property if and only if it is flexible; a Basarab loop has an inverse property if and only it is flexible; a Basarab loop is a Moufang loop if and only if it is an extra loop; a Basarab loop is a right central loop if and only if it is a right Bol loop; a Basarab loop is a left central loop if and only if it is a left Bol loop; a Basarab loop is a central loop if and only if it is a Moufang loop; a Basarab loop is an extra loop if and only if it is a Moufang loop; a Basarab loop is a group if and only if the middle inner mapping of the Basarab loop is an automorphism; a Basarab loop is a right Bol loop if and only if it is a right central loop; a Basarab loop is a left Bol loop if and only if it is a left central loop; a Basarab loop is a Bol loop if and only if it is a flexible central loop; and a Basarab loop is a Buchsteiner loop if and only if it is a Moufang loop.

Jaiyéọlá $\&$ Effiong (2018) investigated Basarab loop and its variance with inverse properties. It was shown that a Basarab loop $(Q, \cdot)$ has the cross inverse property if and only if $(Q, \cdot)$ is an abelian group or all left (right) translations of $(Q, \cdot)$ are right (left) regular. In a Basarab loop, the following properties are equivalent: flexibility property, right inverse property, left inverse property, inverse property, right alternative property, left alternative property and alternative property. The following were proved: a Basarab loop is a weak inverse property loop if it is flexible such that the middle inner mapping is contained in a permutation group; a Basarab loop is an automorphic inverse property loop if a semi-commutative law is obeyed such that the middle inner mapping is contained in a permutation group; a Basarab loop is an anti-automorphic inverse property loop if every element has a two-sided inverse such that the middle inner mapping is contained in a permutation group; a Basarab loop is a semi-automorphic inverse property loop if the Basarab loop is flexible, the middle inner mapping is contained in a permutation group such that a semi-cross inverse property holds; and a Basarab loop with the m-inverse property such that a permutation condition is true is a cross inverse property loop if it is flexible. Jaiyéọlá $\&$ Effiong (2021) studied the generators of the total multiplication group of Basarab loop and it was shown that a Basarab loop is a totally automorphic loop if and only if it is a commutative and flexible loop.

2.2 Specific Literature

Chiboka (1990) constructed a certain finite order G-loops. In Chiboka $\&$ Solarin (1991), the holomorphs of conjugacy closed loops were studied. The following results were established:

if a group of automorphisms of a loop $(Q, \cdot)$ is given which satisfies the inverse property, then the holomorph of $(Q, \cdot)$ is conjugacy closed if and only if $(Q, \cdot)$ is conjugacy closed and every automorphism of Q is nuclear; and if $(Q, \cdot)$ is an inverse property conjugacy closed loop, then each right inner mapping is a nuclear automorphism. Jaiyeola (2006a) studied the holomorphs in the Smarandache concept in loops and the following were established: if two loops are isomorphic, then their holomorphs are also isomorphic, conversely, if their holomorphs are isomorphic, then the loops are isotopic; a loop is a Smarandache loop if and only if its holomorph is a Smarandache loop; a loop is an inverse property Smarandache loop if and only if its holomorph is an inverse property Smarandache loop; and a loop is a weak inverse property Smarandache loop if and only if its holomorph is a weak inverse property Smarandache loop. Jaiyeola (2008c) studied the holomorphic structure of automorphic inverse property quasigroup [AIPQ and (AIPL)] and cross inverse property quasigroups and loops [CIPQ and (CIPL)]. It was proved that the holomorph of a loop is a Smarandache; AIPL, CIPL, K-loop, Bruck-loop or Kikkawa-loop if and only if its Smarandache automorphism group is trivial and the loop is itself a Smarandache; AIPL, CIPL, K-loop, Bruck-loop or Kikkawa-loop.

Adéníran, Jaiyéolá, & Idowu (2014) studied the holomorph of generalized Bol loops. The notion of the holomorph of a generalized Bol loop and generalized flexible-Bol loop were characterized. With the aid of two self-mappings on th holomorph of a loop, the following results were proved: the loop is a generalized Bol loop if and only if its holomorph is a generalized Bol loop; the loop is a generalized flexible-Bol loop if and only if its holomorph is a generalized flexible-Bol loop. Jaiyeola (2015) characterized afresh the notion of the holomorph of a generalized Bol looop (GBL). The following results were established: the holomorph of a right inverse property loop (RIPL) is a GBL if and only if the loop is a GBL and some bijections of the loop are (middle) regular; and the holomorph of a RIPL is a GBL if and only if the loop is a GBL and some elements of the loop are right (middle) nuclear. Isere, Adéníran & Jaiyéolá

(2015) deduced some commutative diagrams for holomorphy of Osborn loops by considering isomorphisms among the various groups of regular bijections and the nucleus of the loop. Jaiyéolá, David, Ilojide & Oyebo (2017) explored the structure of the holomorph of a middle Bol loop. The following results were established: for some automorphisms, the holomorph of a commutative loop is a commutative middle Bol loop if and only if the loop is a middle Bol loop and its automorphism group is abelian and a subgroup of both the group of middle regular mappings and the right multiplication group; commutativity (flexibility) is a necessary and sufficient condition for holomorphic invariance under the existing isostrophy between middle Bol loops; and the right combined holomorph of a middle Bol loop is equal to the holomorph of the middle Bol loop if and only if the automorphism group is abelian and a subgroup of the multiplication group of the middle Bol loop. Ogunrinade, Ajala, Olaleru, & Jaiyeola (2019) considered the holomorph of self-distributive quasigroup with key laws. It was shown that given a quasigroup $(Q, \cdot)$, its holomorph obeys the key laws if and only if there exists a subsemigroup which obeys the key laws and the generalized key laws in $(Q, \cdot)$. Ilojide, Jaiyeola & Olatinwo (2019) investigated the holomorphy of Fenyves BCI-Algebras. It was proved that if a loop and its holomorph are BCI-algebras, then the former is a BCI-algebra if and only if the latter has a BCK-subalgebra.

Jaiyeola (2005) carried out an isotopic study of properties of central loops. The following results were established: constuction of a known C-loop and its isotope; the automorphism group of a loop is a group of exponent 2 which characterizes the isotopism of loops with the same carrier set by inner mapping and automorphism; the left central and right central identities are isotopic invariant properties, and the same is true for the central identity if the loops are alternative central square; there exists a system of isotopic central loops that obeys a kind of generalized distributive law; and a C-loop of exponent 4 and a central square C-loop exist if some autotopisms exist. Jaiyeola & Adeniran (2006) studied the derivatives of central loops.

The right (left) derivative isotopes of C-loop were proved to be C-loops. It was shown that C-loops are isotopic to some finite indecomposable groups of the classes $D_i, i = 1,2,3,4,5$ and that the center of such C-loops have a rank of $1,2$ or 3. Jaiyeola (2009c) examined the universality of central loops. The following results were established: an LC (RC)-loop is a left (right) universal loop; an LC (RC)-loop is a universal loop if and only if it is a right (left) universal loop; if a RC-loop (LC-loop, C-loop) is universal, then it is a right Bol loop (left Bol, Moufang loop) respectively; if a loop and its right or left isotope are commutative then the loop is a C-loop if and only if its right or left isotope is a C-loop; and if a C-loop is central square and its right or left isotope is an alternative central square loop, then the latter is a C-loop. Jaiyeola $\&$ Adeniran (2009a) investigated the representation sets of central loops and the obtained results were used to construct a finite C-loop. It was proved that for certain types of isotopisms, the central identities are isotopic invariant. Jaiyeola (2008c) studied the holomorphic structure of automorphic inverse property quasigroups and loops [AIPQ and (AIPL)] and cross inverse property quasigroups and loops [CIPQ and (CIPL)], it is established that the holomorph of a loop is a Smarandache; AIPL, CIPL, K-loop, Bruck-loop or Kikkawa-loop if and only if its Smarandache autophism group is trivial and the loop is trivial and the loop is itself a Smarandache; AIPL, CIPL, K-loop, Bruck-loop or Kikkawa-loop.

Jaiyeola (2008b) presented a special type of isotopism under which m-inverse quasigroups are isotopic invariant. It was shown that if two distinct quasigroups are isotopic under $\mathcal{T}$- condition and any one of them is an m-inverse quasigroup and has a trivial set of m-weak inverse permutatios, then the two quasigroups are both m-inverse quasigroups that are isomorphic. Jaiyeola $\&$ Adeniran (2009c) proved two distinct isotopy-isomorphy conditions for a weak inverse property loop. It was established that if $(G,\dot{})$ and $(H,\circ)$ are two distinct loops that are isotopic under the triple (A,B,C). In addition, if the pair of $(G,\dot{})$ and $(H,\circ)$ obey the $\mathcal{T}$-condition then $(G,\dot{})$ is a WIPL if and only if $(H,\circ)$ is a WIPL. Jaiyeola (2011c) examined Smarandache isotopy of second Smarandache isotope of a special loop is Smarandache isomorphic to a Smarandache principal isotope of the special loop. It was shown that every special loop that is Smarandache isotopic to a second Smarandache Bol loop is itself a second Smarandache Bol loop. Still on Smarandache concept, Jaiyeola (2006b) studied the universality of some Smarandache loops of Bol-Moufang type.

Over the years, researchers have delved into the properties of Osborn loops. Jaiyeola $\&$ Adeniran (2009b) solved Michael Kinyon's 2005 open problem. This problem was focused on the universality of Osborn loops, and these authors proved that it is not every Osborn loop that is universal. Jaiyeola (2008e) worked on the universality of Osborn loops. The following were proved: an Osborn loop is universal if and only if any of its f, g -principal isotopes is isomorphic to a left principal isotopes of the loop and right universal if and only if any of its f, e-right principal isotopes is ismorphic to some principal isotopes of the loop.

Jaiyeola (2012b) re-affirmed that an Osborn loop is universal if and only if some of its principal isotopes are isomorphic to some other principal isotopes of the loop. Jaiyeola (2013b) established some new identities for universal Osborn loop and left (right) universal Osborn loop. It was shown that these identities are true for Moufang loops, extra loops, CC-loops, VD-loops and universal WIP loops. Jaiyeola (2014) examined universal Osborn loops. The following results were established: a loop is a universal Osborn loop if and only if it has a particular simplicial complex; a loop is a universal Osborn loop and obeys two new identities if and only if it has another simplicial complex; a universal Osborn loop pyramid and four of its isotopes form a rectangular pyramid in a 3- dimentional space. A simplicial complex is a pair (V, S) where V is a set of points called vertices and S is a given family of finite subsets, called simplexes, so that all points of V are simplexes, and non-empty subset of a simplex is a simplex.

Two new identities that characterize universal (left and right universal) Osborn loops were established in Jaiyeola $\&$ Adeniran (2009d). It was further proved that a conjugacy closed loop is said to be diassociative if and only if it is power associative and has a weak form of diassociativity. Jaiyeola & Adeniran (2011) established eight non equivalent sufficient conditions under which an arbitrary loop is a G- Osborn loop. Jaiyéola, Adéníran & Sòlárìn (2011a) expressed the 2005 open problem, ”Does there exist a proper Osborn loop with a trivial nucleus?” for finite Osborn loops in terms of the orders of the nucleus, 2^{nd} Bryant-Schneider and automorphism groups of the loop. Some sufficient conditions for the non-existence of a universal (left,right universal) Osborn loop with trivial nucleus were also deduced.

Isere, Adéníran & Jaiyéọlá (2021) studied some properties of Latin quandles that are applicable in cryptography. They established some necessary and sufficient conditions for four distinct cores of Osborn loop and used these results to build cipher algorithms. In Jaiyéola, Adéníran & Sòlárìn (2011b), for universal (left and right universal) Osborn loops, it was shown that every CC-quasigroup is isotopic to an Osborn loop if and only if every CC-quasigroup obeys any of the two identities that characterize universal (left and right universal) Osborn loops. It was further shown that an Osborn loop is universal if and only if any of its f, g- principal isotopes is isomorphic to some principal isotopes of the loop. Jaiyeola, Ilojide & Popoola (2013) studied some polynomials that generate quasigroups over the ring $\mathbb{Z}_n$ with regards to their isotopy structure. Jaiyeola, Ilojide, Saka & Ilori (2020) studied some varieties of Fenyves Quasi Neutrosophic Triplet loop and their isotopy. The results obtained were applied to health sciences. Necessary and sufficient conditions for a groupoid istope of a BCI-algebra to be a BCI-algebra were established. BCI-algebras are quasi neutrosophic loops, while Fenyves BCI are BCI-algebras that satisfy the 60 Bol-Moufang identites. It was shown that for BCI-algebras, associativity is isotopic invariant. These authors applied their results to the initial (old, sick or healthy) state of a person, and the final (new, healthy or sick) state of the person as a result of the prescribed medical treatment. Interestingly, these authors proved an isotopism in health sciences to be a measure of the change from the old state of a body condition to the new state.

Adeniran, Oyebo & Mohammed (2011) investigated certain isotopic maps of a C-loop. It was shown that the holomorph of a C-loop is a C-loop if each element of the automorphism group of the loop is left nuclear. Condition under which an element of the Bryant-Schneider group of a C-loop forms an automorphism was established. It was proved that elements of the Bryant-Schneider group of C-loop can be expressed as a product of pseudo-automorphisms and right translations of elements of the nucleus of the loop. These authors were able to established that the Bryant-Schneider group of a C-loop is a kind of generalized holomorph of the loop. Necessary and sufficient conditions for the holomorph of a loop to be a C-loop were established. Kinyon & Kunen (2004) studied the structure of extra loops. It was shown that every finite nonassociative extra loop has a nontrivial center. Also, an infinite nonassociative extra loop with a trivial center was constructed, and necessary and sufficient conditions for a loop to be an extra loop were given. The authors described those groups which could form the nuclei of nonassociative extra loops. They proved that if there is a nonassociative extra loop with the nucleus being a group, then the center of the nucleus contains an element of order 2. Also, an extra loop is solvable if and only if its nucleus is solvable.

Kinyon, Kunen & Phillips (2004) studied the diassociativity in conjugacy closed loops (CC-loop). It was proved that the center of the nucleus of a CC-loop contains all the associators of elements of the loop. If in addition, the CC-loop is diassociative (that is, an extra loop), then all the associators of elements of the CC-loop have order 2. It was shown that a CC-loop with the AAIP is an extra loop. In any CC-loop, the nuclei of the loop coincide and the nucleus is a normal subloop of the loop. The authors further established that every commutative CC-looop is a group. These authors also investigated the associators and inner mappings of a CC-loop. It was established that the nucleus of a nonassociative CC-loop has a nontrival center which contains the subgroup generated by the associators.

Kinyon & Kunen (2006) studied CC-loops and power-associative CC-loops (PACC-loops). It was proved that if Q is a PACC-loop with nucleus N, the quotient Q/N is an abelian group of exponent 12. The authors studied some sets such as the set of Moufang elements, the set of weak inverse property (WIP) elements, and the set of extra elements. It was proved that in a CC-loop, the set WIP and the sets of extra elements are normal subloops. The authors also investigated the associator subloops of a loop, and they established some properties for some associator subloops of the given loop to be subgroups of the nucleus. Equivalent forms for identities of a CC-loop were obtained and used in establishing more properties of a CC-loop. Interestingly, examples of a CC-loop and PACC-loop were constructed respectively.

Nagy & Strambach (1994) investigated left CC-loops (LCC-loops). It was proved that a loop with the inverse property is left conjugacy closed if and only if it is an extra loop. It was further emphasised that such a loop is conjugacy closed. The authors constructed examples of LCC-loops and CC-loops. One of such constructions was used to establish that there exist LCC-loops which are not conjugacy closed but are G-loops. Goodaire & Robinson (1982), defined the center of a loop to be the set of elements of the loop that are commutative and are in the nucleus of the loop. It was shown that in any CC-loop, the right and left inner mappings are automorphisms. Goodaire & Robinson (1990) established that a CC-loop with the inverse property is an extra loop.

Drapal (2004) studied CC-loop and their multiplication groups. The author stated clearly that groups are those loops in which the right translation mappings R_x (and also the left translation mappings L_x) are closed under composition, that is, R_xR_y is always equal to some R_z). The author identified and proved some basic properties of a CC-loop. It was established that if a loop is conjugacy closed, then its left and right multiplication groups are normal subgroups of its multiplication group. It was shown that in a CC-loop, the left and right inner mapping groups are equal. The inner mapping group of a CC-loop Q was shown to be generated by the set of its middle inner mapping T_x, for every x in Q, and the right and left inner mapping group of a CC-loop Q is a normal subgroup of the inner mapping group of Q. The nucleus of a CC-loop was proved to be a normal subloop. It was also proved that every CC-loop is a G-loop. Drápal & Kinyon (2020) studied normality and nuclear squares of some loops. They showed that every Osborn loop has a normal nucleus and this nucleus coincides with the left, right and middle nuclei.

Syrbu & Drapal (2019) worked on total multiplication groups of loops. It was established that the multiplication group of an inverse property loop Q is a normal subgroup of index two of the total multiplication group of Q. The authors gave a set of mappings which generates a total multiplication group. The authors investigated the centre of the total multiplication group, and they established that if two loops are isostrophic then the center of their total multiplication group are isomorphic. It was proved that for a middle Bol loop, its multiplication group is a normal subgroup of its total multiplication group, and that its inner mapping group is a normal subgroup of its total inner mapping group.

Phillips & Stanovský (2012) studied Bruck loops with abelian inner mapping groups and it was established that Bruck loops with abelian inner mapping groups are centrally nilpotent of class at most 2. The condition that inner mapping of a loop is abelian was expressed in a set of equations. Using these equations, the authors were able to described the definition of centrally nilpotent of class 2 in a form of equation. Kinyon, Veroff & Vojtěchovský (2013) investigated loops with abelian inner mapping group and it was proved that the quotient $G/N(G)$ is an abelian group where G is an abelian inner mappings left central loop and $N(G)$ is the nucleus of G.

All the above work did not look into the form of isotopes, holomorphs and associators of a Basarab loop. This research work will be focused on these properties.

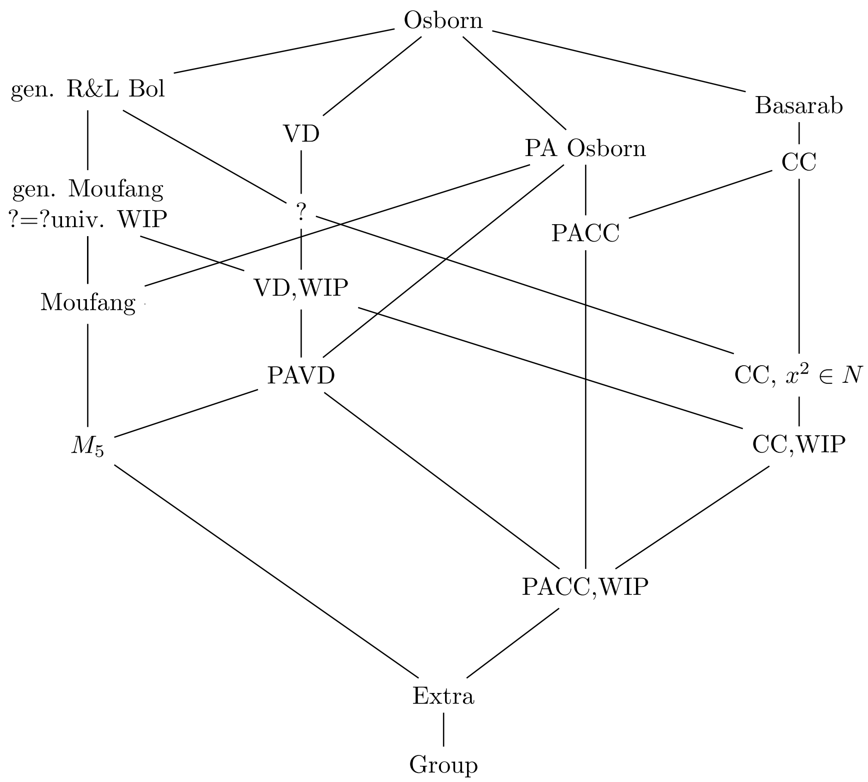

Figure 2.1: Hasse Diagram for Basarab loops

CHAPTER THREE

METHODOLOGY

3.1 Introduction

Let Q be a non-empty set, and let $\cdot$ be a binary operation defined on Q. If, for every $x,\ y \in Q,\ \ x \cdot y\ \in Q$, then $(Q, \cdot)$ is called a groupoid. Bruck (1966) used the name groupoid to refer to a magma. Let $(Q, \cdot)$ be a groupoid, and let x be a fixed element in $(Q, \cdot)$. Then the left and right translation maps of $Q,\ \ L_x,\ R_x : Q \longrightarrow Q$ can be defined respectively by

$$yL_x = x \cdot y \text{ and } yR_x = y \cdot x$$

Let $(Q, \cdot)$ be a groupoid. If for every $x,\ y \in Q,$ the equations $a \cdot x = b$ and $y \cdot a = b$ have unique solutions for x and y respectively, then $(Q, \cdot)$ is said to be a quasigroup.

Equivalently, a groupoid $(Q, \cdot)$ is a quasigroup if its left and right translation mappings are bijections or permutations. Then for quasigroups, the inverse mappings L_x^{-1} and R_x^{-1} exist, and $x\backslash y = yL_x^{-1}$ and $x/y = xR_y^{-1}$, for every $x, y \in Q$. Here,

$$x\backslash y = z \iff x \cdot z = y \text{ and } x/y = z \iff z \cdot y = x.$$

Thus, $(Q, \backslash)$ and $(Q, /)$ are also quasigroups.

A quasigroup $(Q, \cdot, /, \backslash)$ is a set Q equipped with three binary operations of multiplication $(\cdot)$, right division $(/)$, and left division $(\backslash)$ such that for all $x, y \in Q$:

(i) $x \cdot (x \backslash y) = y$, $(y/x) \cdot x = y$; and

(ii) $x \backslash (x \cdot y) = y$, $(y \cdot x)/x = y$.

Here, $x \cdot y/z$ and $x \cdot y \backslash z$ stand for $x(y/z)$ and $x(y \backslash z)$ respectively. Suppose that $(Q.\cdot)$ be a quasigroup with a unique element $e \in Q$ such that for every $x \in Q$, $x \cdot e = e \cdot x = x$, then $(Q, \cdot)$ is said to be a loop.

There exists an element $x^\lambda \in Q$ for every $x \in Q$ such that $x^\lambda \cdot x = e$ and the element x^λ is called the left inverse element of $x \in Q$. Also, there exists an element $x^\rho \in Q$, for every $x \in Q$ such that $x \cdot x^\rho = e$ and the element x^ρ is called the right inverse element of $x \in Q$. Suppose that e^λ is an element of a quasigroup $(Q, \cdot)$ and for every $x \in Q$, $e^\lambda x = x$, then e^λ is called a left identity element. Similarly, if e^ρ is an element of a quasigroup $(Q, \cdot)$ and for every $x \in Q$, $xe^\rho = x$, then e^ρ is called a right identity element. The element $x^\rho = xJ_\rho \in Q$ is called right inverse element if $xx^\rho = e^\rho$ for all $x \in Q$. The element $x^\lambda = xJ_\lambda \in Q$ is called left inverse element if $x^\lambda x = e^\lambda$ for all $x \in Q$, (Jaiyeola, 2009b). A loop $(Q, \cdot)$ is called a Basarab loop (or K-loop), if the identities:

$$(x \cdot y(xJ_\rho)) \cdot (xz) = x \cdot yz, \quad (y \cdot x) \cdot ((xJ_\lambda \cdot z) \cdot x) = yz \cdot x$$

$$(xJ_\rho = x^{-1},\ xJ_\lambda = {}^{-1}x,\ xJ_\lambda \cdot z = {}^{-1}x \cdot z)$$

hold for all $x, y, z \in Q$. A loop $(Q, \cdot)$ is called an automorphic inverse property Basarab loop (or IK-loop) if $(Q, \cdot)$ is a Basarab loop and the mapping J of $(Q, \cdot)$ is an automorphism of the loop $(Q, \cdot)$, (Basarab, 1992).

Theorem 3.1.1. (Basarab, 1997)

Let $(Q, \cdot)$ be a Basarab loop.

1. $N(Q, \cdot)$ contains the associator of any three elements of Q.

2. The quotient loop $Q/Z(Q, \cdot)$ is a group.

3. If $(Q, \cdot)$ is generated by one element, then it is solvable.

4. If $(Q, \cdot)$ has the automorphic inverse property, then it is nilpotent.

Theorem 3.1.2. (Basarab, 1992)

Let $(Q, \cdot)$ be a Basarab loop.

1. $N(Q, \cdot)$ is a nontrivial normal subloop.

2. The quotient loop $Q/N(Q, \cdot)$ is an abelian.

3. If $N(Q, \cdot)$ has an odd order, then $(Q, \cdot)$ is solvable.

Theorem 3.1.3. (Basarab, 1996)

1. Any Basarab loop (any VD-loop) is a G-loop.

2. Any Basarab loop (VD-loop) is an Osborn loop.

3. A Basarab loop $(Q, \cdot)$ is a VD-loop if $x^2 \in N(Q, \cdot)$ for any $x \in Q$.

4. A VD-loop $(Q, \cdot)$ is a Basarab loop if $x^2 \in N(Q, \cdot)$ for any $x \in Q$.

3.2 The Use of Parentheses

Let Q be a groupoid. If $yx = e$ for all $x, y \in Q$, y is called a left inverse of x. If y is the unique left and right inverse of x, we write $y = x^{-1}$. Let $x \in Q$ and $m \in \mathbb{N}$, then we can speak of $x^0 = 1$, $x(x^{m-1}) = x^m$. Also, if x^{-1} is the unique inverse then $(x^{-1})^m = x^{-m}$. Applying this to left translation mapping we have $eL_x^m = x^m$, where e is the identity element. The use of parentheses is applied in the study of quasigroup and loops. Hence, the dot-convention are sometimes applied for parentheses. For instance, let $x, y, z \in Q, ; x(yz) = x{\cdot}yz,\ (xy)z = xy{\cdot}z$, and $x^n y = (x^n)y,\ zy^n = z(y^n)$ for $n \in \mathbb{Z}$. Let Q be a groupoid, an element $x \in Q$ is called left (right) alternative if $x^2 y = x \cdot xy$ $(yx^2 = yx \cdot x)$ for all $y \in Q$. Also, an element $x \in Q$ is a left power alternative if for all $n \in \mathbb{Z},\ x$ and L_x each have an inverse such that $L_{x^n} = L_x^n$.

Lemma 3.2.1. (Kiechle, 2000)

Let Q be a groupoid, then the following are true:

(i) if every element in Q has a unique inverse, then $(x^{-1})^{-1} = x$, for $x \in Q$;

(ii) an element $x \in Q$ is left alternative if and only if $L_{x^2} = L_x^2$;

(iii) Q is a left loop if and only if L_y is bijective for every $y \in Q$

Also, an element $x \in Q$ is a right power alternative if for all $n \in \mathbb{Z},\ x$ and R_x each have an inverse such that $R_{x^n} = R_x^n$.

Lemma 3.2.2. (Kiechle, 2000)

Let Q be a groupoid, then the following are true:

(i) if every element in Q has a unique inverse, then $(x^{-1})^{-1} = x$, for $x \in Q$;

(ii) an element $x \in Q$ is right alternative if and only if $R_{x^2} = R_x^2$;

(iii) Q is a right loop if and only if R_y is bijective for every $y \in Q$

3.3 Construction of a loop

Lemma 3.3.1. (Nagy & Strambach (1994))

Let $\mathcal{F}$ be a field with 3 elements and denote by $\mathcal{F}^*$ the multiplicative group of $\mathcal{F}$. Define on

$Q = \mathcal{F} \times \mathcal{F}^*$ the multiplication $(a\,,\alpha)(b\,,\beta) = (ab\,,(a-1)(b-1) + \alpha b + \beta)$. Then a loop $Q = \mathcal{F} \times \mathcal{F}^*$, with $e = (1\,,0)$, $g = (1\,,1)$ and $h = (-1\,,0)$ is obtained.

Theorem 3.3.1. (Nagy & Strambach (1994))

Let G and H be abelian groups and $g : G \times G \longrightarrow H$ be a mapping with $g(a\,,0) = g(0\,,a) = 0$ for any $a \in G$. Then the multiplication $\circ$ on $G \times G$ given by

$$(a\,,\alpha) \circ (b\,,\beta) = (a+b\,,\alpha+\beta+g(a\,,b))$$

defines a loop $Q(g)$ with the identity $(0\,,0)$. If $(a\,,\alpha)(b\,,\beta) = (c\,,\gamma)$ then

$$(a\,,\ \alpha) = (c\,,\gamma)/(b\,,\beta) = (c-b\,,\gamma-\beta-g(c-b\,,b)),$$

$$(b\,,\beta) = (a\,,\alpha)\backslash(c\,,\gamma) = (c-a\,,\gamma-\alpha-g(a\,,c-a)).$$

Corollary 3.3.1. (Nagy & Strambach (1994))

The loop $Q(g)$ is a left conjugacy closed loop if and only if

$$(a\,,\alpha) \circ [(b\,,\beta) \circ (c\,,\gamma)] = \{[(a\,,\alpha) \circ (b\,,\beta)]/(a\,,\alpha)\} \circ [(a\,,\alpha) \circ (c\,,\gamma)],$$

for all $a, b, c \in G$ and $\alpha, \beta, \gamma \in H$. This identity is equivalent to the relation

$$g(b\,,c) + g(a\,,b+c) = g(a\,,b) - g(b\,,a) + g(a\,,c) + g(b\,,a+c)$$

for all $a, b, c \in G$.

Corollary 3.3.2. (Nagy & Strambach (1994))

Let G and H be abelian groups and $g : G \times G \longrightarrow H$ be a mapping with $g(a\,,0) = g(0\,,a) = 0$ for any $a \in G$. If $g(a\,,\cdot) : G \longrightarrow H$ is a homomorphism of abelian groups for any fixed $a \in G$ then $Q(g)$ is a left conjugacy closed loop.

3.4 Multiplication Group

A loop $(Q, \cdot)$ is a set Q with a binary operation such that the following conditions are satisfied:

(i) for each $x \in Q$, the left translation $L_x : Q \to Q;\ y \mapsto xy$ is a bijection;

(ii) for each $x \in Q$, the right translation $R_x : Q \to Q;\ y \mapsto yx$ is a bijection;

(iii) there exist $e \in Q$ such that $e \cdot x = x \cdot e = x$ for all $x \in Q$.

Given a loop $(Q, \cdot)$, the set of all left translations $L_x : Q \to Q;\ y \mapsto xy$ generates a permutation group on Q which is known as the left multiplication group of Q, and denoted by $\mathcal{M}_\lambda(Q, \cdot)$. The set of all right translation $R_x : Q \to Q;\ y \mapsto yx$ generates a permutation group on Q, called the right multiplication group of Q, and denoted by $\mathcal{M}_\rho(Q, \cdot)$. Then the multiplication group of Q is the set $\mathcal{M}(L, \cdot) = \Big\langle \{R_x, R_x^{-1}, L_x, L_x^{-1}\ :\ x \in L\} \Big\rangle$.

Lemma 3.4.1. (Figula, 2009)

Let L be a loop with multiplication group $\mathcal{M}(L, \cdot)$ and e its identity element.

(i) Let α be a homomorphism of the loop L onto the loop αL with kernel N. Then N is a normal subloop of L and α induces a homomorphism of the group $\mathcal{M}(L, \cdot)$ onto group $\mathcal{M}(\alpha(L, \cdot))$

(ii) For every normal subgroup N of $\mathcal{M}(L, \cdot)$ the orbit $N(e)$ is a normal subloop of L. Moreover, $N \leq \mathcal{M}(N(e))$.

Theorem 3.4.1. (Smith, 2007)

Let G be a group. Then there are group isomorphisms $R : G \longrightarrow \mathcal{M}_\rho(G, \cdot); g \longmapsto R_g$ and $L : G \longrightarrow \mathcal{M}_\lambda(G, \cdot); g \longmapsto L_{g^{-1}}$. Moreover, there is an exact sequence

$$e \longrightarrow Z(G) \xrightarrow{\triangle} G \times G \xrightarrow{T} \mathcal{M}(G, \cdot) \longrightarrow e$$

of groups with $\triangle : z \longmapsto (z\ , z)$ and $T : (a\ , b) \longmapsto R_b L_a^{-1}$.

Theorem 3.4.2. (Drapal, 2004)

Let G be a loop, and let $\mathcal{M}_\lambda(G, \cdot)$ and $\mathcal{M}_\rho(G, \cdot)$ be its left and right multiplication groups, respectively. Then $\mathcal{M}_{\lambda_1}(G, \cdot) = \langle L_y L_x L_{xy}^{-1}; x, y \in G \rangle, \mathcal{M}_{\rho_1}(G, \cdot) = \langle R_y R_x R_{yx}^{-1}; x, y \in G \rangle$, and $Inn(G, \cdot)$ is generated by $\mathcal{M}_{\lambda_1}(G, \cdot) \bigcup \mathcal{M}_{\rho_1}(G, \cdot) \bigcup \{T_x : x \in G\}$.

Lemma 3.4.2. (Drapal, 2004)

Let $(G, \cdot)$ be a loop and let $\mathcal{M}_\lambda(G, \cdot)$ and $\mathcal{M}_\rho(G, \cdot)$ be its left and right multiplication groups, respectively. Let $C(G)$ be the centrum of $(G, \cdot)$ and put $H = \mathcal{M}_\lambda(G, \cdot)$. Then $C_H(\mathcal{M}_\rho(G, \cdot)) = \{L_x : x \in N_\lambda\}$ and $C_H(\mathcal{M}_\lambda(G, \cdot)) = \{R_x : x \in N_\rho\}$.

Theorem 3.4.3. (Drapal, 2004)

Let $(G, \cdot)$ be a loop and let $\mathcal{M}_\lambda(G, \cdot)$ and $\mathcal{M}_\rho(G, \cdot)$ be its left and right multiplication groups, respectively. Then $Z(\mathcal{M}_\rho(G, \cdot)) = \{L_x : x \in N_\rho\} \bigcap \mathcal{M}_\rho(G, \cdot)$and $Z(\mathcal{M}_\lambda(G, \cdot)) = \{R_x : x \in N_\rho\} \bigcap \mathcal{M}_\lambda(G, \cdot)$.

3.5 Total Multiplication Group

Let $(G,\ \cdot)$ be a quasigroup and $x \in G$. The left, right and middle translations by x, denoted by L_x, R_x, M_x, respectively are mappings from the symmetric group S_G, defined as follows: $yL_x = x \cdot y,\ yR_x = y \cdot x,\ yM_x = y \backslash x$, for all $x, y \in G$. The groups $\mathcal{M}(G, \cdot) = \langle L_x,\ R_x \mid x \in G \rangle$ and $\mathcal{T}_\mathcal{M}(L, \cdot) = \langle L_x,\ R_x,\ M_x \mid x \in G \rangle$ are called the multiplication group and the total multiplication group of $(G,\ \cdot)$, respectively. If $(G,\ \cdot)$ is a loop, then its inner mapping group is denoted by $Inn(G, \cdot)$ and the total inner mapping group of $(G, \cdot)$ is denoted by $\mathcal{T}_{Inn}(G, \cdot)$, (Stanovský & Vojtěchovský, 2014).

Theorem 3.5.1. (Syrbu & Drapal (2019))

If $(G,\ \cdot)$ is an IP–loop then $\mathcal{M}(G, \cdot)$ is a normal subgroup of index two of the group $\mathcal{T}_\mathcal{M}(L, \cdot)$

Theorem 3.5.2. (Syrbu & Drapal (2019))

If $(G,\ \cdot)$ is an RIP–loop or an LIP–loop then $\mathcal{T}_{\mathcal{M}}(L,\cdot) = \big\langle \mathcal{M}(G,\cdot),\ I \big\rangle$.

Lemma 3.5.1. (Syrbu & Drapal (2019))

If $(G,\cdot)$ is loop with unit e, then the generators of $Inn_e(G,\cdot) = Inn(G,\cdot)$ are: $L_{x,y} = L_yL_xL_{xy}^{-1},\ R_{x,y} = R_xR_yR_{xy}^{-1},\ T_x = R_xL_x^{-1}$.

Theorem 3.5.3. (Syrbu & Drapal (2019))

If $(G,\cdot)$ is a loop, then $\mathcal{T}_{Inn}(G,\cdot) = \big\langle L_{x,y};\ R_{x,y};\ T_x;\ P_{x,y};\ U_x \mid x,y \in G \big\rangle$, where $P_{x,y} = M_xM_yL_xR_y^{-1},\ U_x = R_xM_x$.

Corollary 3.5.1. (Syrbu & Drapal (2019))

If $(G,\cdot)$ is a power associative loop, then

$\mathcal{T}_{Inn}(G,\cdot) = \big\langle L_{x,y};\ R_{x,y}\ P_{x,y};\ U_x \mid x,y \in G \big\rangle$, where $P_{x,y} = M_xM_yL_xR_y^{-1},\ U_x = R_xM_x$.

Corollary 3.5.2. (Syrbu & Drapal (2019))

If $(G,\cdot)$ is a middle Bol loop, then

$\mathcal{T}_{Inn}(G,\cdot) = \big\langle R_{x,y}\ P_{x,y};\ U_x \mid x,y \in G \big\rangle$, where $P_{x,y} = M_xM_yL_xR_y^{-1},\ U_x = R_xM_x$.

Theorem 3.5.4. (Syrbu & Drapal (2019))

If $(G,\cdot)$ is a middle Bol loop, then $\mathcal{M}(G,\cdot) \trianglelefteq \mathcal{T}_{\mathcal{M}}(L,\cdot)$ and $Inn(G,\cdot) \trianglelefteq \mathcal{T}_{Inn}(G,\cdot)$

Lemma 3.5.2. (Syrbu & Drapal (2019))

A loop $(G,\cdot)$ is called a middle Bol loop if the corresponding e-loop $(G,\ \cdot\ /,;\backslash)$ satisfies the identity $x(yz\backslash x) = (x/z)(y\backslash x)$.

3.6 Isotopy Theory

The triple (U,V,W) such that $U,V,W \in SYM(Q)$ is called an autotopism of Q, if and only if $aU \cdot bV = (a\cdot b)W$, for all $a,b \in Q$. If $(Q,\cdot)$ and $(G,\circ)$ are any two loops. Then the triple $(U, V, W) : (Q, \cdot) \to (G, \circ)$ such that $U, V, W : Q \to G$ are bijections is called a loop isotopism if and only if $aU \circ bV = (a \cdot b)W$, for all $a, b \in Q$. By component-wise composition, the set of all autotopisms of Q forms a group called the autotopism group of Q.

Definition 3.6.1. (Pflugfelder, 1990)

Let $(G, \cdot)$ be a quasigroup. Then

1. a bijection U is called autotopic if there exists $(U, V, W) \in AUT(G, \cdot)$; the set of all such mappings forms a group $\Sigma(G, \cdot)$.

2. a bijection U is called ρ-regular if there exists $(I, U, U) \in AUT(G, \cdot)$; the set of all such mappings forms a group $\mathcal{P}(G, \cdot)$.

3. a bijection U is called λ-regular if there exists $(U, I, U) \in AUT(G, \cdot)$; the set of all such mappings forms a group $\Lambda(G, \cdot) \leq \Sigma(G, \cdot)$.

4. a bijection U is called μ-regular if there exists a bijection U' such that $(U, U'^{-1}, I) \in AUT(G, \cdot)$. U' is called the adjoint of U. The set of all μ-regular mappings forms a group $\Phi(G, \cdot) \leq \Sigma(G, \cdot)$. The set of all adjoint mapping forms a group $\Psi(G, \cdot)$.

Theorem 3.6.1. (Bruck, 1966)

(i) Every isotope of a quasigroup is a quasigroup.

(ii) If the loop $(Q, \circ)$ is isotopic to the loop $(Q, \cdot)$, the left, right and two-sided multiplication groups of $(Q, \circ)$ are respectively isomorphic.

(iii) If the loop $(Q, \circ)$ is isotopic to the loop $(Q, \cdot)$, the left, middle, and right nuclei of $(Q, \cdot)$ are respectively isomorphic to those of $(Q, \circ)$ and the center of $(Q, \cdot)$ is isomorphic to that of $(Q, \circ)$.

(iv) If the loop $(Q, \circ)$ is isotopic to the loop $(Q, \cdot)$ and if ϕ is a homomorphism of $(Q, \cdot)$ upon a loop $(G, \cdot)$, then ϕ induces a homomorphism of $(Q, \circ)$ upon a loop $(G, \circ)$.

(v) Every loop isotopic to a Moufang loop is Moufang.

Let the triple (U, V, W) be an isotopism from a loop $(Q, \cdot)$ onto another loop $(G, \circ)$, then Q and G are called loop isotopes. If $Q = G$ and $W = I$ (identity mapping) then (U, V, I) is called a principal loop isotopism. Then $(G, \circ)$ is called a principal loop isotope of $(G, \cdot)$. If $U = V = W$, then U is called an isomorphism. This means, an isotopism with three equal components forms an isomorphism.

Theorem 3.6.2. (Smith, 2007)

If a loop is isotopic to a group, then it is isomorphic to the group. In particular, isotopic groups are isomorphic.

Proof. Let $(G, \cdot)$ be a loop. It suffices to consider a principal isotopy

$$(g\ , h\ , e_G) : (G\ , \circ\ , e) \longrightarrow (G\ , \circ\ , /\ , \backslash)$$

between a loop structure $(G\ , \circ\ , e)$ and a group structure $(G\ , \circ\ , /\ , \backslash)$ on a set G, so that $a^g \cdot b^h = a \circ b$ for $a, b \in G$. Since, $e^g \cdot b^h = e \circ b = b$, it follows that $b^h = e^g \backslash b$. Also, $a^g = a/e^h$. Then $(e^g\ \cdot a\ \cdot e^h) \circ (e^g\ \cdot b\ \cdot e^h) = e^g \cdot a \cdot b \cdot e^h$, which make $(G, \cdot) \longrightarrow (G, \cdot); f \longmapsto e^g \cdot f \cdot e^h$ as the required isomorphism. □

Theorem 3.6.3. (Nagy & Strambach (1994))

Let Q be a loop and let L be the group generated by the left translations $L_h : Q \longrightarrow, h \in Q$. The following conditions are equivalent for the loop Q:

(i) the set $\{L_h,\ h \in Q\}$ is invariant under the inner automorphisms of L;

(ii) $a\backslash(bc) = a\backslash(ba) \cdot a\backslash c$ for all $a, b, c \in Q$;

(iii) $a \cdot bc = (ab)/a \cdot ac$ for all $a, b, c \in Q$;

(iv) the triples $(R_a L_a^{-1},\ L_a^{-1},\ L_a^{-1})$ are autotopisms of Q for all $a \in Q$;

(v) the triples $(L_aR_a^{-1},\ L_a,\ L_a)$ are autotopisms of Q for all $a \in Q$.

Proof. Putting $(ab)/a = x$ and $ac = y$ then the equivalence of the conditions (ii) and (iii) is obtained. The conditions (ii) and (iv), respectively (iii) and (v) are equivalent by the definition of an autotopism of Q. If the condition (i) holds, then for any $a, b \in Q$ there exixts an element $c \in Q$ such that $L_aL_bL_a = L_c$. This implies, $aL_a^{-1}L_bL_a = ab = ca$ and $c = (ab)/a$ or $xL_a^{-1}L_bL_a = xL_{ab/a}$. And with $c = xL_a^{-1}$ the last relation gives $a \cdot bc = (ab)/a \cdot ac$. From (iii) above, (i) is obtained immediately. □

Lemma 3.6.1. (Nagy & Strambach (1994))

Let $(Q, \cdot)$ be a loop and let $(Q^*, *)$ be a principal isotope of Q with the multiplication $(a,\ b) \longmapsto a * b$ given by $a * b = a/x \cdot y\backslash b$, where x, y are fixed elements in Q.

The loops with multiplication $a * b = a/x \cdot b$ or $a * b = a \cdot y\backslash b$ are said to be left or right isotopic to Q, respectively.

Corollary 3.6.1. (Nagy & Strambach (1994))

Every principal isotopism is a composition of a left and a right isotopism.

Corollary 3.6.2. (Nagy & Strambach (1994))

The class of left conjugacy closed loops is closed with respect to left isotopisms. Moreover, the left translations are isomorphisms between left isotopic left conjugacy closed loops.

Proof. Let Q be a loop and let $\bar{Q}$ be a loop which is left isotopic to Q. Then the left translations $\bar{L_a}$ of $\bar{Q}$ are given by $\bar{L_a} = L_{a/x}$ for some fixed $x \in Q$. Thus, the set $\{\bar{L_a},\ a \in \bar{Q}\}$ of left translations of $\bar{Q}\}$ coincides with the set $\{L_a,\ a \in Q\}$ of left translations of Q. □

Nagy & Strambach (1994) noted that the multiplication $b \circ c = b/a \cdot c$ which defines a loop left isotopic to Q is related to the multiplication of Q by the isomorphism $b \longmapsto ab$ because of the identity in Theorem 3.6.3 (iii).

Lemma 3.6.2. (Drapal, 2004)

Let G be a loop and let U and V be permutations of G such that (U, V, W) or (V, U, U) is an autotopism. If $U(e) = e$, then $U = V$ and V is an automorphism.

Lemma 3.6.3. (Isere, Adéníran & Jaiyéolá 2015)

Let $(G, \cdot)$ be a loop. Let

$$\omega : \mathcal{P}(G, \cdot) \to N_\rho(G, \cdot) \uparrow \psi(U) = eU, \vartheta : \Lambda(G, \cdot) \to N_\lambda(G, \cdot) \uparrow \vartheta(U) = eU,\ \varphi : \Phi(G, \cdot) \to \Psi(G, \cdot)$$

$\uparrow \varphi(U) = U'$, $\sigma : \Phi(G, \cdot) \to N_\mu(G, \cdot) \uparrow \sigma(U) = eU$ and $\beta : \Psi(G, \cdot) \to N_\mu(G, \cdot) \uparrow \beta(U') = eU'$

Then $\mathcal{P}(G, \cdot) \overset{\omega}{\cong} N_\rho(G, \cdot)$, $\Lambda(G, \cdot) \overset{\vartheta}{\cong} N_\lambda(G, \cdot)$, $\Phi(G, \cdot) \overset{\varphi}{\cong} \Psi(G, \cdot)$, $\Phi(G, \cdot) \overset{\sigma}{\cong} N_\mu(G, \cdot)$, $\Psi(G, \cdot) \overset{\beta}{\cong} N_\mu(G, \cdot)$.

3.7 Holomorphy of a Loop

Let $(Q, \cdot)$ be a loop, $A(Q)$ is a group of automorphisms of loop $(Q, \cdot)$ and let $G = A(Q) \times Q$ and define $(U,\ a) \circ (V,\ b) = (UV,\ aV \cdot b)$ for all $(U, a),\ (V,\ b) \in G$. Then the loop $(G, \circ)$ is called $A(Q)-$holomorph of $(Q, \cdot)$ or simply holomorphy of $(Q, \cdot)$, (Chiboka & Solarin, 1991).

Theorem 3.7.1. (Adeniran et al. 2011)

Let $(Q, \cdot)$ be an LC– loop and $A(Q)$ be a group of automorphism of $(Q, \cdot)$. Then the $A(Q)-$holomorph $(G, \circ)$ of $(Q, \cdot)$ is an LC– loop if and only if $(aU \cdot ab)c = aU(a \cdot bc)$ for all $a, b, c \in Q$ and for all $U \in A(Q)$.

Proof. Suppose, $A(Q)-$holomorph $(G, \circ)$ of $(Q, \cdot)$ is an LC– loop then

$$\{(U,\ a) \circ [(U,\ a) \circ (V,\ b)]\} \circ (W,\ c) = (U,\ a) \circ \{(U,\ a) \circ [(V,\ b) \circ (W,\ c)]\}$$

for all $a, b, c \in Q$ and for all $U, V, W \in A(Q)$. Thus,

$$\{(U,\ a) \circ (UV,\ aV \cdot b)\} \circ (W,\ c) = (U,\ a) \circ \{(U,\ a) \circ (VW,\ bW \cdot c)\}$$

$$\{U \cdot UV,\ aUV \cdot (aV \cdot b)\} \circ (W,\ c) = (U,\ a) \circ \{(U \cdot VW,\ aVW \cdot (bW \cdot c))\}$$

$$\{(U \cdot UV)W,\ [aUV \cdot (aV \cdot b)]W \cdot c\} = \{U(U \cdot VW),\ (aU \cdot VW) \cdot (a \cdot VW)(bW \cdot c)\}$$

for all $a, b, c \in Q$ and for all $U, V, W \in A(Q)$. Therefore,

$$(aUV \cdot (aV \cdot b))W \cdot c = a(U \cdot VW) \cdot ((a \cdot VW) \cdot (bW \cdot c))$$

for all $a, b, c \in Q$ and for all $U, V, W \in A(Q)$. Therefore,

$$(aUVW \cdot (aVW \cdot bW)) \cdot c = a(U \cdot VW) \cdot ((a \cdot VW) \cdot (bW \cdot c))$$

for all $a, b, c \in Q$ and for all $U, V, W \in A(Q)$. Putting $X = VW$ gives

$$(aUX \cdot (aX \cdot bW)) \cdot c = a(U \cdot X) \cdot ((a \cdot X) \cdot (bW \cdot c))$$

Hence, $(aU \cdot (a \cdot bWX^{-1})) \cdot cX^{-1} = (aU \cdot a(bWX^{-1} \cdot cX^{-1}))$ for all $a, b, c \in Q$ and for all $U, V, W \in A(Q)$. Setting $\bar{b} = bWX^{-1}$ and $\bar{c} = cX^{-1}$, then $(aU \cdot a\bar{b})\bar{c} = aU \cdot (a \cdot \bar{b}\bar{c})$. Thus, $(aU \cdot ab)c = aU(a \cdot bc)$ is obtained by replacing $\bar{b}$ and $\bar{c}$ with b and c respectively, for all $a, b, c \in Q$ and for all $U, V, W \in A(Q)$. By reserving the process, the converse is obtained. □

Corollary 3.7.1. (Adeniran et al. 2011)

Let $(Q, \cdot)$ be a loop, and $A(Q)$ be the group of all automorphism of Q, then Q is LC– loop if and only if $B = \left\langle L_a L_{aU},\ I,\ L_a L_{aU} \right\rangle$ is an autotopism of Q, for all $a \in Q$ and for all $U \in A(Q)$.

Theorem 3.7.2. (Adeniran et al. 2011)

Let $(Q, \cdot)$ be a loop and $A(Q)$ be a group of automorphism of $(Q, \cdot)$. Then the

$A(Q)-$holomorph $(G,\circ)$ of $(Q,\cdot)$ is an RC–loop if and only if $b((c\cdot aU)a) = (bc\cdot aU)a$ for all $a,b,c\in Q$ and for all $U\in A(Q)$.

Corollary 3.7.2. (Adeniran et al. 2011)

Let $(Q,\cdot)$ be a loop, and $A(Q)$ be the group of all automorphism of Q, then Q is RC– loop if and only if $B=\langle I,\ R_{aU}R_a,\ R_{aU}R_a\rangle$ is an autotopism of Q, for all $a\in Q$ and for all $U\in A(Q)$.

Theorem 3.7.3. (Adeniran et al. 2011)

Let $(Q,\cdot)$ be a loop and $A(Q)$ be a group of automorphism of $(Q,\cdot)$. Then the $A(Q)-$holomorph $(G,\circ)$ of $(Q,\cdot)$ is an C–loop if and only if $(b\cdot bU)a\cdot c = b(aU\cdot ac)$ for all $a,b,c\in Q$ and for all $U\in A(Q)$.

Corollary 3.7.3. (Adeniran et al. 2011)

Let $(Q,\cdot)$ be a loop, and $A(Q)$ be the group of all automorphism of Q, then Q is C– loop if and only if $B=\langle R_{aU}R_a,\ L_{aU}^{-1}L_{x^{-1}},\ I\rangle$ is an autotopism of Q, for all $a\in Q$ and for all $U\in A(Q)$.

3.8 AIM Loops

A loop G is said to be an AIM loop (for Abelian Inner Mappings) if $Inn(G)$ is an abelian group (Kinyon et al. 2013).

Lemma 3.8.1. (Kinyon et al. 2013)

A nonempty subset L of a loop G is a subloop ($L\leq G$) if it is closed under the three operations $\cdot$, $\backslash$, and $/$.

Lemma 3.8.2. (Kinyon et al. 2013)

A subloop L of G is normal $(L\trianglelefteq G)$ if $L\varphi=\{\ l\varphi\mid l\in L\}$ is equal to L for every $\varphi\in Inn(G)$.

Let G be a loop. Then the left, right, and middle nuclei of Q are given by:

$$N_\lambda(G) = \{\ a \in G \mid ax \cdot y = a \cdot xy,\ \forall\ x, y \in G\ \}$$

$$N_\rho(G) = \{\ a \in G \mid xy \cdot a = x \cdot ya,\ \forall\ x, y \in G\ \}$$

$$N_\mu(G) = \{\ a \in G \mid xa \cdot y = x \cdot ay,\ \forall\ x, y \in G\ \}$$

and the nucleus is $N(G) = N_\lambda(G) \cap N_\rho(G) \cap N_\mu(G)$.

Thus the nucleus $N(G)$ consists of all elements $a \in G$ that associate with all $x, y \in G$, the commutant $C(G)$ consists of all elements $a \in G$ that commute with all $x \in G$, and the center $Z(G)$ consists of all elements $a \in G$ that commute and associate with all $x, y \in G$.

Lemma 3.8.3. (Kinyon et al. 2013)

Let G be a loop, then the inclusions $N_\lambda(G) \leq N(G), N_\rho(G) \leq N(G)$, $N_\mu(G) \leq N(G), N(G) \leq Z(G)$ and $C(G) \leq Z(G)$ hold, but not necessarily the equalities.

Lemma 3.8.4. (Kinyon et al. 2013)

A loop G is an AIM loop if and only if the following identities hold:

$$T_xT_y = T_yT_x,\ L_{x,y}L_{z,w} = L_{z,w}L_{x,y},$$

$$R_{x,y}R_{z,w} = R_{z,w}R_{x,y},\ L_{x,y}T_z = T_zL_{x,y},$$

$$R_{x,y}T_z = T_zR_{x,y},\ L_{x,y}R_{z,w} = R_{z,w}L_{x,y}$$

for all $x, y, z, w \in G$.

Thus, let $(Q, \cdot)$ be a loop with a subloop $H \neq \emptyset$, then $(Q, \cdot)$ is called an H- AIM loop or H-Abelian inner mappings loop if $Inn(Q, \cdot)|_H$ is an Abelian group. Now, given a normal subloop L of a loop G, G/L denotes the factor loop G modulo L whose elements are the subsets (left cosets) $aL = \{al \mid l \in G\}$ for $a \in G$, and $aL \cdot bL = (a \cdot b)L$. The associator, $(a,\ b,\ c)$ of a, b and c in G is defined as $ab \cdot c = (a \cdot bc)(a,\ b,\ c)$ and the commutator, $[a,\ b]$ of a and b in G, is defined as $ab = ba \cdot [a,\ b]$. The mirror of the associator and the commutator respectively is given as $[a,\ b,\ c] = (a \cdot bc)/(ab \cdot c)$ and $[a,\ b] = (ba)/(ab)$.

Lemma 3.8.5. (Kinyon et al. 2013)

If $N(G) \trianglelefteq G$ then $G/N(G)$ is an abelian group if and only if the following identities hold:

$$[[a,\ b,\ c], x\ y] = [x,\ [a,\ b,\ c],\ y] = [x,\ y, [a,\ b,\ c]] = e$$

$$[[a,\ b], c,\ x] = [c,\ [a,\ b],\ x] = [c,\ x, [a,\ b]] = e, \text{ for all } a, b, c, x, y \in G.$$

Proof. Let $L \trianglelefteq G$. The following conditions are equivalent: $(aL \cdot bL) \cdot cL = aL \cdot (bL \cdot cL)$, $(ab \cdot c)L = (a \cdot bc)L$, $((a \cdot bc)\backslash(ab \cdot c))L = L$, $[a,\ b,\ c]L = L, [a,\ b,\ c] \in L$. Thus Q/L is a group if and only if $[a,\ b,] \in L$ for all $a, b, \in G$. Similarly, G/L is commutative if and only if $[a, b] \in G$ for all $a, b \in G$.

The condition $[[a,\ b,\ c],\ x;\ y] = [x,\ [a,\ b,\ c],\ y] = [x,\ y,\ [a,\ b,\ c]] = e$ implies $[a,\ b,\ c] \in N(G)$ for all $a, b, c \in G$, and the condition $[[a,\ b],\ c, x] = [c, [a,\ b],\ x] = [c,\ x,\ [a,\ b]] = e$ implies $[a,\ b] \in N(G)$ for all $a, b \in G$. $\square$

It is noted that an element $x \in G$ is in the left nucleus $N_\lambda(G)$ if and only if $[x,\ b,\ c] = e$ for all $b, c \in G$. The other nuclei are similarly characterized.

Theorem 3.8.1. (Kinyon et al. 2013)

Let G be an AIM LC loop. Then $G/N(G)$ is an abelian group and $G/Z(G)$ is a group.

LC loops were introduced by Fenyves (1969) as one of the loops of Bol-Moufang type (the element occuring twice on both sides has no other element separating it from itself). If G is an LC loop then for all $x, y, z \in G,\ x(x(yz)) = (x(xy))z$.

CHAPTER FOUR

MAIN RESULTS

4.1 Introduction

In this chapter, some algebraic properties of Basarab loops are determined using some loop notions and some basic features of Basarab loops. The study concentrates on the centrum, isotopes, holomorphs and associators of a Basarab loop, construction of a Basarab loop, the relationship between a Basarab loop and centrum abelian inner mappings loop, and to obtain some subloops of a Basarab loop that are characterized by permutations. It is proved that an Obsorn loop is a Basarab loop if and only if it is both left and right Basarab loop. Investigation is carried out on the isotopes of a Basarab loop and it is proved that every principal isotope of a Basarab loop is a Basarab loop. It is shown that the centrum of a Basarab loop is a subloop and it is equal to the center of a Basarab loop. Some necessary and sufficient conditions for the holomorphs of a loop to be a Basarab loop are determined. Also, some subloops of a basarab loop are obtained and characterized. The algebraic properties of associators of a Basarab loop are examined and it is found that the associator of any three elements of a Basarab loop is contained in the center and centrum of a Basarab loop. It is also proved that a Basarab loop is a centrum abelian inner mapping loop.

4.2 Basarab loop Constructions

In this section, some constructions are carried out for a Basarab loop. Two abelian groups are considered. A mapping f which takes a cross product of one of the abelian groups into the other is considered with a condition. A multiplication $\circ$ satisfying some laws with respect to the elements of these abelian groups and the mapping f is defined, on the cross product of these abelian groups with an identity $(0, 0)$. This algebraic structure with multiplication $\circ$ is shown to be a loop. Specifically, the necessary and sufficient conditions for such an algebraic structure with multiplication $\circ$ to be a left (right) Basarab loop, and Basarab loop are established, respectively. Equivalent forms of these necessary and sufficient conditions are expressed using mapping f only.

Theorem 4.2.1. Let $(A, +)$ and $(B, +)$ be abelian groups, and $f : A \times A \longrightarrow B$ be a mapping with $f(a, 0) = f(0, a) = 0$ for any $a \in A$. Let $\circ$ be defined on $A \times B$ by

$$(a, x) \circ (b, y) = (a + b, x + y + f(a, b))$$

. Then, the pair $(A \times B, \circ)$ is a left Basarab loop with the identity $(0, 0)$ if and only if

$$[(a, x) \circ ((b, y) \circ (a, x)^{\rho})] \circ [(a, x) \circ (c, z)] = (a, x) \circ [(b, y) \circ (c, z)]$$

for all $a, b, c \in A$ and $x, y, z \in B$, and $(a, x)^{\rho} = (-a,\ -x - f(a, -a))$.

Proof. This result is the consequence of Theorem 3.3.1 and Corollary 3.3.1 in the case of left Basarab law. Next, it is shown that the permutation $(a, x)^{\rho}$ on the left Basarab loop exists. The permutation $(a, x)^{\rho}$ is obtained by solving

$$(a, x) \circ (b', y') = (a + b',\ x + y' + f(a, b')) = (0, 0)$$

$$a+b'=0 \implies a=-b' \text{ or } -b'=-a$$

$$x+y'+f(a,b')=0 \implies x+y'+f(a,-a)=0 \implies y'=-f(a,-a)-x$$

$$\implies (b',y')=(-a,-f(a,-a)-x)$$

$$\text{Therefore, } (a,x)\circ(a,x)^{\rho}=(a,x)\circ(-a,-x-f(a,-a))$$

$$=(a-a,x-x-f(a,-a)+f(a,-a)=(0,0)$$

So, $(a,x)^{\rho}=(-a,-x-f(a,-a))$. □

Theorem 4.2.2. Let $(A,+)$ and $(B,+)$ be abelian groups, and $f:A\times A\longrightarrow B$ be a mapping with $f(a,0)=f(0,a)=0$ for any $a\in A$. Let $\circ$ be defined on $A\times B$ by

$$(a,x)\circ(b,y)=(a+b,x+y+f(a,b))$$

. Then, the pair $(A\times B,\circ)$ is a left Basarab loop with the identity $(0,0)$ if and only if

$$f(b,c)+f(a,b+c)=-f(a,-a)+f(a,b-a)+f(b,-a)+f(a,c)+f(b,a+c)$$

for all $a,b,c\in A$.

Proof. From Theorem 4.2.1,

$$[(a,x)\circ((b,y)\circ(a,x)^{\rho})]\circ[(a,x)\circ(c,z)]=(a,x)\circ[(b,y)\circ(b,y)\circ(c,z)]$$

$$\text{and } (a,x)^{\rho}=(-a,-x-f(a,-a)).$$

$$(b,y)\circ(a,x)^{\rho}=(b,y)\circ(-a,-x-f(a,-a))$$

$$=(b-a,y-x-f(a,-a)+f(b,-a))$$

.

Also, $(a, x) \circ ((b, y) \circ (a, x)^{\rho}) = (a, x) \circ (b - a, y - x - f(a, -a) + f(b, -a)$

$= (a + b - a, x + y - x - f(a, -a) + f(b, -a) + f(a, b - a)$

$= (b, y - f(a, -a) + f(b, -a) + f(a, b - a).$

Then, $(a, x) \circ (c, z) = (a + c, x + z + f(a, c))$, so that

$[(a, x) \circ ((b, y) \circ (a, x)^{\rho}] \circ [(a, x) \circ (c, z)] =$

$(b, y - f(a, -a) + f(b, -a) + f(a, b - a)) \circ (a + c, x + z + f(a, c)) =$

$(b + a + c, y - f(a, -a) + f(b, -a) + f(a, b - a) + x + z + f(a, c) + f(b, a + c)) =$

$$(a + b + c, x + y + z - f(a, -a) + f(a, b - a) + f(b, -a) + f(a, c) + f(b, a + c)) \tag{4.1}$$

On the other hand,

$$(a, x) \circ ((b, y) \circ (c, z)) = (a, x) \circ (b + c, y + z + f(b, c)) =$$

$$(a + b + c, x + y + z + f(b, c) + f(a, b + c)) \tag{4.2}$$

Joining Equations 4.1 and 4.2 together, and comparing both sides:

$$x + y + z - f(a, -a) + f(a, b - a) + f(b, -a) + f(a, c) + f(b, a + c)$$

$$= x + y + z + f(b, c) + f(a, b + c),$$

it follows that,

$$f(b, c) + f(a, b + c) = -f(a, -a) + f(a, b - a) + f(b, -a) + f(a, c) + f(b, a + c)$$

for all $a, b, c \in A$. $\square$

Corollary 4.2.1. Let A and B be abelian groups and $f : A \times A \longrightarrow B$ be a mapping with $f(a\ ,0) = f(0\ ,a) = 0$ for any $a \in A$. If $f(a\ ,\cdot) : A \longrightarrow B$ is a homomorphism of abelian groups for any fixed $a \in A$ then $(A \times B, \circ)$ is a left Basarab loop.

Proof. This result follows from Theorem 4.2.2. $\square$

Theorem 4.2.3. Let $(A, +)$ and $(B, +)$ be abelian groups, and $f : A \times A \longrightarrow B$ be a mapping with $f(a, 0) = f(0, a) = 0$ for any $a \in A$. Let $\circ$ be defined on $A \times B$ by

$$(a, x) \circ (b, y) = (a + b, x + y + f(a, b))$$

. Then, the pair $(A \times B, \circ)$ is a right Basarab loop with the identity $(0, 0)$ if and only if

$$((b, y) \circ (a, x))) \circ [((a, x)^{\lambda} \circ (c, z)) \circ (a, x)] = ((b, y) \circ (c, z)) \circ (a, x)$$

for all $a, b, c \in A$ and $x, y, z \in B$, and $(a, x)^{\lambda} = (-a,\ -x - f(-a, a))$.

Proof. This result is the consequence of Theorem 3.3.1 and Corollary 3.3.1 in the case of right Basarab law. Next, it is shown that the permutation $(a, x)^{\lambda}$ on the right Basarab loop exists. The permutation $(a, x)^{\lambda}$ is obtained by solving

$$(b', y') \circ (a, x) = (b' + a, y' + x + f(b', a)) = (0, 0) \implies b' + a = 0 \implies a = -b' \text{ or } b' = -a$$

$$y' + x + f(b', a) = 0 \implies y' + x + f(-a, a) = 0 \implies y' = -x - f(-a, a)$$

$$\implies (b', y') = (-a, -x - f(-a, a)).$$

Clearly, $(a, x)^{\lambda}(a, x) = (-a, -x - f(-a, a))(a, x)$

$$= (-a + a, -x - f(-a, a) + x + f(-a, a)) = (0, 0).$$

□

Theorem 4.2.4. Let $(A,+)$ and $(B,+)$ be abelian groups, and $f : A \times A \longrightarrow B$ be a mapping with $f(a,0) = f(0,a) = 0$ for any $a \in A$. Let $\circ$ be defined on $A \times B$ by

$$(a,x) \circ (b,y) = (a+b, x+y+f(a,b))$$

. Then, the pair $(A \times B, \circ)$ is a right Basarab loop with the identity $(0,0)$ if and only if

$$f(b,c) + f(b+c,a) = f(b,a) - f(-a,a) + f(-a,c) + f(-a+c,a) + f(b+a,c)$$

for all $a,b,c \in A$.

Proof. From Theorem 4.2.3,

$$((b,y) \circ (a,x)) \circ [((a,x)^{\lambda} \circ (c,z)) \circ (a,x)] = ((b,y) \circ (c,z)) \circ (a,x)$$

and $(a,x)^{\lambda} = (-a, -x - f(-a,a))$.

$$\text{Thus } ((b,y) \circ (a,x)) \circ [((-a, -x - f(-a,a)) \circ (c,z)) \circ (a,x)]$$

$$= ((b,y) \circ (a,x)) \circ [(-a+c, -x - f(-a,a) + z + f(-a,c)) \circ (a,x)]$$

$$= ((b,y) \circ (a,x)) \circ [-a+c+a, -x - f(-a,a) + z + f(-a,c) + x + f(-a+c,a)]$$

$$= ((b,y) \circ (a,x)) \circ [c, -f(-a,a) + z + f(-a,c) + f(-a+c,a)]$$

$$= (b+a, y+x+f(b,a)) \circ [c, -f(-a,a) + z + f(-a,c) + f(-a+c,a)]$$

$$= (b+a+c, y+x+f(b,a) - f(-a,a) + z + f(-a,c) + f(-a+c,a) + f(b+a,c)) =$$

$$(a+b+c, x+y+z+f(b,a) - f(-a,a) + f(-a,c) + f(-a+c,a) + f(b+a,c)) \quad (4.3)$$

On the other hand,

$$((b,y)\circ(c,z))\cdot(a,x) = (b+c, y+z+f(b,c))\circ(a,x)$$

$$= (b+c+a, y+z+x+f(b,c)+f(b+c,a))$$

$$= (a+b+c, x+y+z+f(b,c)+f(b+c,a) \tag{4.4}$$

Joining Equations 4.3 and 4.4 together, and comparing both sides:

$$x+y+z+f(b,a)-f(-a,a)+f(-a,c)+f(-a+c,a)+f(b+a,c)$$

$$= x+y+z+f(b,c)+f(b+c,a)$$

It follows that, for all $a,b,c \in A$,

$$f(b,c)+f(b+c,a) = f(b,a)-f(-a,a)+f(-a,c)+f(-a+c,a)+f(b+a,c).$$

□

Corollary 4.2.2. Let A and B be abelian groups and $f : A\times A \longrightarrow B$ be a mapping with $f(a\ ,0) = f(0\ ,a) = 0$ for any $a \in A$. If $f(\cdot\ ,\ a) : A \longrightarrow B$ is a homomorphism of abelian groups for any fixed $a \in A$ then $(A\times B, \circ)$ is a right Basarab loop.

Proof. This result follows from Theorem 4.2.4. □

Corollary 4.2.3. Let $(A,+)$ and $(B,+)$ be abelian groups, and $f : A\times A \longrightarrow B$ be a mapping with $f(a,0) = f(0,a) = 0$ for any $a \in A$. Let $\circ$ be defined on $A\times B$ by

$$(a,x)\circ(b,y) = (a+b, x+y+f(a,b))$$

. Then, the pair $(A \times B, \circ)$ is a Basarab loop with the identity $(0,0)$ if and only if

$$[(a,x) \circ ((b,y) \circ (a,x)^{\rho})] \circ [(a,x) \circ (c,z)] = (a,x) \circ [(b,y) \circ (c,z)]$$

and $((b,y) \circ (a,x))) \circ [((a,x)^{\lambda} \circ (c,z)) \circ (a,x)] = ((b,y) \circ (c,z)) \circ (a,x)$

for all $a,b,c \in A$ and $x,y,z \in B$, $(a,x)^{\lambda} = (-a,\ -x - f(-a,a))$

and $(a,x)^{\rho} = (-a,\ -x - f(a,-a))$.

Proof. The proof follows Theorems 4.2.1 and 4.2.3 by definition of a Basarab loop. □

Corollary 4.2.4. Let $(A,+)$ and $(B,+)$ be abelian groups, and $f : A \times A \longrightarrow B$ be a mapping with $f(a,0) = f(0,a) = 0$ for any $a \in A$. Let $\circ$ be defined on $A \times B$ by

$$(a,x) \circ (b,y) = (a+b, x+y+f(a,b))$$

. Then, the pair $(A \times B, \circ)$ is a Basarab loop with the identity $(0,0)$ if and only if

$$f(b,c) + f(a,b+c) = -f(a,-a) + f(a,b-a) + f(b,-a) + f(a,c) + (b,a+c)$$

and $f(b,c) + f(b+c,a) = f(b,a) - f(-a,a) + f(-a,c) + f(-a+c,a) + f(b+a,c)$

for all $a,b,c \in A$.

Proof. The proof follows from Theorems 4.2.2 and 4.2.4 and by the definition of a Basarab loop. □

Corollary 4.2.5. Let A and B be abelian groups and $f : A \times A \longrightarrow B$ be a mapping with $f(a\ ,0) = f(0\ ,a) = 0$ for any $a \in A$. If $f(a\ ,\ \cdot),\ f(\cdot\ ,\ a) : A \longrightarrow B$ are homomorphisms of abelian groups for any fixed $a \in A$ then $(A \times B, \circ)$ is a Basarab loop.

Proof. This result follows from Corollary 4.2.4. □

4.3 Basarab loops with some other loops

This section examines a Basarab loop as a type of an Osborn loop, and proves that Basarab loop and CC-loop are Osborn loops. It is proved that any Osborn loop is a Basarab loop if and only if it is a left (right) Basarab loop; and a Basarab loop is flexible if and only if it is an extra loop. Also, an extra loop is proved to be both a Buchsteiner loop and a Basarab loop. Necessary and sufficient conditions for a Basarab loop to be a CC-loop, an RCC-loop, and an LCC-loop are given. It is established that the center and centrum of a Basarab loop coincide, and both are contained in the Nucleus of the Basarab loop. Necessary and sufficient conditions for a Basarab loop to be power alternative are given. The characteristics of automorphism and some inverse properties like Anti-Automorphic Inverse Property (AAIP) and Inverse Property (IP) in a Basarab loop are considered, and the following results are proved: the left (right) inner mapping of a Basarab loop is an automorphism; a Basarab loop with the RIP or LIP is an extra loop; a Basarab loop with the IP (AAIP) is an extra loop; a left (right) Basarab loop with the RIP (LIP) is an extra loop; and the middle inner mapping generates the inner mapping group of a Basarab loop. It was also proved that, the left and right inner mappings of a Basarab loop are nuclear.

4.3.1 Basarab loops and Osborn loops

Definition 4.3.1. A loop $(Q,\cdot)$ is called an Osborn loop if it obeys any of the three identities:

$$x(yz\cdot x)=(xJ_\lambda\backslash y)\cdot zx \tag{4.5}$$

$$x(yz\cdot x)=x(y(xJ_\lambda)\cdot x)\cdot zx \tag{4.6}$$

$$x(yz\cdot x)=x(yx\cdot xJ_\rho)\cdot zx \tag{4.7}$$

Lemma 4.3.1. Every Basarab loop is an Osborn loop.

Proof. Let $(Q,\cdot)$ be a Basarab loop, then $T_x^{-1} = L_xR_x^{-1} = R_{xJ_\rho}L_x$ and $T_x = R_xL_x^{-1} = L_{xJ_\lambda}R_x$, for all $x \in Q$. The Basarab loop autotopisms give:

$$(R_{xJ_\rho}L_x,\ L_x,\ L_x)(R_x,\ L_{xJ_\lambda}R_x,\ R_x) \in AUT(Q,\cdot)$$

$$\implies (R_{xJ_\rho}L_xR_x,\ L_xL_{xJ_\lambda}R_x,\ L_xR_x) \in AUT(Q,\cdot).$$

Now, for all $x,y,z \in Q,$ it follows that

$$yR_{xJ_\rho}L_xR_x \cdot zL_xL_{xJ_\lambda}R_x = (yz)L_xR_x.$$

Setting $R_{xJ_\rho}L_x = L_xR_x^{-1}$ and $L_{xJ_\lambda}R_x = R_xL_x^{-1}$, we have

$$yL_xR_x^{-1}R_x \cdot zL_xR_xL_x^{-1} = (yz)L_xR_x \implies yL_x \cdot zL_xR_xL_x^{-1} = (yz)L_xR_x$$

$$\implies yL_x \cdot (zL_xR_xL_x^{-1}R_x^{-1} \cdot x) = (yz)L_xR_x \implies$$

$xy\cdot(((x\backslash(xz\cdot x))/x)\cdot x) = (x\cdot yz)x \implies xy\cdot(\Theta_x z\cdot x) = (x\cdot yz)x;$ where $\Theta_x = L_xR_xL_x^{-1}R_x^{-1}$.

This implies Osborn loop. □

Lemma 4.3.2. Let $(Q,\cdot)$ be a Basarab loop, then for all $x,y \in Q$,

$$xy = (x \cdot yx^\rho)x,\ yx = x(x^\lambda y \cdot x).$$

Proof. These are the immediate consequences of the middle inner map of a Basarab loop. □

Theorem 4.3.1. An Osborn loop satisfies the left Basarab identity if and only if the right Basarab identity holds.

Proof. Let $(Q,\cdot)$ be an Osborn loop. In Equation 4.5, setting $z=e$ implies $x(yx)=(xJ_\lambda\backslash y)\cdot x \implies yR_xL_x = yL^{-1}_{xJ_\lambda}R_x \implies L_{xJ_\lambda}R_xL_x = R_x \implies L_{xJ_\lambda}R_x = R_xL_x^{-1} = T_x$. So, for all $x,y,z\in Q$;

$$xy\cdot(((x\backslash(xz\cdot x))/x)\cdot x) = (x\cdot yz)x \implies (L_x,\ L_xR_xL_x^{-1},\ L_xR_x)\in AUT(Q,\cdot).$$

Then, $x\in Q$ satisfies the left Basarab identity if and only if

$$(L_xR_x^{-1},\ L_x,\ L_x)\in AUT(Q,\cdot) \iff (R_xL_x^{-1},\ L_x^{-1},\ L_x^{-1})\in AUT(Q,\cdot)$$

$$\iff (R_xL_x^{-1},\ L_x^{-1},\ L_x^{-1})(L_x,\ L_xR_xL_x^{-1},\ L_xR_x)\in AUT(Q,\cdot)$$

$$\iff (R_x,\ R_xL_x^{-1},\ R_x)\in AUT(Q,\cdot)$$

$$\iff (R_x,\ L_{xJ_\lambda}R_x,\ R_x)\in AUT(Q,\cdot).$$

This implies right Basarab identity holds. □

Theorem 4.3.2. An Osborn loop satisfies the right Basarab identity if and only if the left Basarab identity holds.

Proof. Let $(Q,\cdot)$ be an Osborn loop. In Equation 4.7, setting $z=e$ implies

$$x(yx) = x(yx\cdot xJ_\rho)\cdot x \implies yR_xL_x = yR_xR_{xJ_\rho}L_xR_x \implies R_xL_x = R_xR_{xJ_\rho}L_xR_x$$

$$\implies L_x = R_{xJ_\rho}L_xR_x \implies L_xR_x^{-1} = R_{xJ_\rho}L_x = T_x^{-1}.$$

Then every $x\in Q$ satisfies the right Basarab identity if and only if

$$(R_x,\ R_xL_x^{-1},\ R_x)\in AUT(Q,\cdot) \iff (R_x^{-1},\ L_xR_x^{-1},\ R_x^{-1})\in AUT(Q,\cdot)$$

$$\iff (L_x,\ L_xR_xL_x^{-1},\ L_xR_x)(R_x^{-1},\ L_xR_x^{-1},\ R_x^{-1}) \in AUT(Q,\cdot)$$

$$\iff (L_xR_x^{-1},\ L_xR_xL_x^{-1}L_xR_x^{-1},\ L_xR_xR_x^{-1}) \in AUT(Q,\cdot)$$

$$(L_xR_x^{-1},\ L_x,\ L_x) \in AUT(Q,\cdot) \iff (R_{xJ_\rho}L_x,\ L_x,\ L_x) \in AUT(Q,\cdot)$$

This implies left Basarab identity holds. □

Corollary 4.3.1. Any Osborn loop $(Q,\cdot)$ is a Basarab loop if and only if $(Q,\cdot)$ is a right (left) Basarab loop.

Proof. This is true by the proof of Theorem 4.3.2. □

4.3.2 Basarab loop, extra loop and Buchsteiner loop

Theorem 4.3.3. A Basarab loop is flexible if and only if it is an extra loop.

Proof. Let $(Q,\cdot)$ be a Basarab loop, then

$$(T_x^{-1},\ L_x,\ L_x),\ (R_x,\ T_x,\ R_x) \in AUT(Q,\cdot).$$

Now, if $(Q,\cdot)$ is flexible then $(R_x^{-1}L_x,\ L_x,\ L_x) \in AUT(Q,\cdot)$ and $(R_x,\ L_x^{-1}R_x,\ R_x) \in AUT(Q,\cdot)$. Thus,

$$(R_x^{-1}L_x,\ L_x,\ L_x) \in AUT(Q,\cdot) \implies yR_x^{-1}L_x \cdot zL_x = (yz)L_x,\ \forall\ x,y,z \in Q$$

$$\implies xy \cdot xz = x(yx \cdot z), \forall\ x,y,z \in Q.$$

Also,

$$(R_x,\ L_x^{-1}R_x,\ R_x) \in AUT(Q,\cdot) \implies yR_x \cdot zL_x^{-1}R_x = (yz)R_x,\ \forall\ x,y,z \in Q$$

$$\implies yx \cdot zx = (y \cdot xz)x,\ \forall\ x,y,z \in Q.$$

□

Remark 4.3.1. A flexible Basarab loop is an extra loop.

Corollary 4.3.2. A left (right) Basarab loop is flexible if and only if it is an extra loop.

Proof. This is true, using Theorem 4.3.3. □

Theorem 4.3.4. An extra loop is both a Buchsteiner loop and a Basarab loop.

Proof. Let $(Q,\cdot)$ be an extra loop, then for all $x, y, z \in Q$; $(xy \cdot z)x = x(y \cdot zx)$ has the autotopism: $(L_x,\ R_x^{-1},\ L_xR_x^{-1}) \in AUT(Q,\cdot)$. So, $(Q,\cdot)$ satisfies flexibility property implies

$$(L_x,\ R_x^{-1},\ L_xR_x^{-1}) = (L_x,\ R_x^{-1},\ R_x^{-1}L_x) \in AUT(Q,\cdot)$$

Thus,

$$(L_x,\ R_x^{-1},\ R_x^{-1}L_x) \in AUT(Q,\cdot) \implies yL_x \cdot z'R_x^{-1} = (y \cdot z')R_x^{-1},\ x, y, z' \in Q$$

$$\implies (yL_x \cdot z)L_x^{-1} = (y \cdot zR_x)R_x^{-1},\ \forall\ x, y, z \in Q \implies x\backslash(xy \cdot z) = (y \cdot zx)/x,\ \forall\ x, y, z \in Q.$$

This implies, an extra loop is a Buchsteiner loop. With Theorem 4.3.3, in an extra loop $(Q,\cdot)$;

$$(L_x,\ R_x^{-1},\ R_x^{-1}L_x),\ (T_x^{-1},\ L_x,\ L_x),\ (R_x,\ T_x,\ R_x) \in AUT(Q,\cdot).$$

□

4.3.3 Basarab loops and CC-loops

Theorem 4.3.5. A loop $(Q,\cdot)$ is a Basarab loop if and only if it is a CC-loop and satisfies $xy = (x \cdot yx^{\rho})x,\ yx = x(x^{\lambda}y \cdot x)$, for all $x, y \in Q$.

Proof. In a Basarab loop $(Q,\cdot)$, $L_xR_x^{-1} = R_{xJ_\rho}L_x$ and $R_xL_x^{-1} = L_{xJ_\lambda}R_x$, for all $x \in Q$. From the autotopisms of $(Q,\cdot)$,

$$(T_x^{-1},\ L_x,\ L_x) \in AUT(Q,\cdot) \iff (R_{xJ_\rho}L_x,\ L_x,\ L_x) \in AUT(Q,\cdot)$$

and

$$(R_x,\ T_x,\ R_x) \in AUT(Q,\cdot) \iff (R_x,\ L_{xJ_\lambda}R_x,\ R_x) \in AUT(Q,\cdot).$$

Thus,

$$(T_x^{-1},\ L_x,\ L_x) \in AUT(Q,\cdot) \implies yL_xR_x^{-1} \cdot z'L_x = (y \cdot z'L_x),\ \forall\ x, y, z' \in Q.$$

This means,

$$yL_xR_x^{-1} \cdot z = (y \cdot zL_x^{-1})L_x,\ \forall\ x, y, z \in Q$$

$$\implies (xy)R_x^{-1} \cdot z = (y(x\backslash z))L_x \implies ((xy)/x)z = x \cdot y(x\backslash z);\ \forall\ x, y, z \in Q.$$

Also,

$$(R_x,\ T_x,\ R_x) \in AUT(Q,\cdot) \implies yR_x \cdot z'R_xL_x^{-1} = (yz')R_x,\ \forall\ x, y, z' \in Q.$$

Which means,

$$y \cdot zR_xL_x^{-1} = (yR_x^{-1} \cdot z)R_x \implies y((zx)L_x^{-1}) = (yR_x^{-1} \cdot z)R_x$$

$$\implies y(x\backslash(zx)) = (y/x)z \cdot x;\ \forall\ x, y, z \in Q.$$

□

Corollary 4.3.3. A loop $(Q,\cdot)$ is a left Basarab loop if and only if it is an LCC-loop and satisfies

$xy = (x \cdot yx^{\rho})x$, for all $x, y \in Q$.

Proof. This is obtained from the proof of Theorem 4.3.5. □

Corollary 4.3.4. A loop $(Q, \cdot)$ is a right Basarab loop if and only if it is an RCC-loop and satisfies $yx = x(x^{\lambda}y \cdot x)$, for all $x, y \in Q$.

Proof. This is obtained from the proof of Theorem 4.3.5. □

Corollary 4.3.5. An extra loop is a Buchsteiner, a Basarab and a CC-loop.

Proof. This is true by Theorems 4.3.4 and 4.3.5. □

Corollary 4.3.6. Every CC-loop is an Osborn loop.

Proof. By Lemma 4.3.1 and the proof of Theorem 4.3.5, the result follows. □

4.3.4 Center of a Basarab loop

Theorem 4.3.6. Let $(Q, \cdot)$ be a left Basarab loop. Let $a \in N_{\rho}(Q, \cdot)$ or $(Q, \cdot)$ be an LIP loop. Then, $a \in C(Q, \cdot)$ if and only if $L_a \in \rho(Q, \cdot)$.

Proof. Consider the left Basarab identity, $(x \cdot y(xJ_{\rho})) \cdot xz = x \cdot yz$. If $a \in C(Q, \cdot) \cap N_{\rho}(Q, \cdot)$, and for $y = a$, it follows that

$$(x \cdot a(xJ_{\rho})) \cdot xz = x \cdot az \implies (x \cdot xJ_{\rho}a) \cdot xz = x \cdot az, \ \forall\, a, x, z \in Q.$$

Also,

$$a \in C(Q, \cdot) \cap N_{\rho}(Q, \cdot) \implies (x \cdot (xJ_{\rho})) \cdot xz = x \cdot az, \ a, x, z \in Q$$

$$\implies (x(xJ_{\rho}) \cdot a) \cdot xz = x \cdot az, \ \forall\, a, x, z \in Q.$$

By the definition of xJ_{ρ}, it follows that

$$a \cdot xz = x \cdot az, \ \forall\, a, x, z \in Q \implies (I, \ L_a, \ L_a) \in AUT(Q, \cdot).$$

This means, $a \in C(Q,\cdot) \cap N_\rho(Q,\cdot) \implies \forall\, a \in Q,\ L_a \in \rho(Q,\cdot)$. □

Theorem 4.3.7. Let $(Q,\cdot)$ be a right Basarab loop. Let $a \in N_\lambda(Q,\cdot)$ or $(Q,\cdot)$ be an RIP loop. Then, $a \in C(Q,\cdot)$ if and only if $R_a \in \lambda(Q,\cdot)$.

Proof. Consider the right Basarab identity

$$(y \cdot x) \cdot ((xJ_\lambda z) \cdot x) = yz \cdot x.$$

If $a \in C(Q,\cdot) \cap N_\lambda(Q,\cdot)$, and for $z = a$ in the Basarab law, then

$$(y \cdot x) \cdot ((xJ_\lambda)a \cdot x) = ya \cdot x,\ \forall\, a, x, y \in Q \implies (y \cdot x) \cdot (a(xJ_\lambda) \cdot x) = ya \cdot x,\ \forall\, x, z \in Q;$$

Since

$$a \in C(Q,\cdot) \cap N_\lambda(Q,\cdot),\ (y{\cdot}x){\cdot}((a{\cdot}xJ_\lambda){\cdot}x) = ya{\cdot}x \implies (y{\cdot}x){\cdot}(a{\cdot}(xJ_\lambda{\cdot}x)) = ya{\cdot}x,\ \forall\, a, x, y \in Q.$$

It follows from the definition of xJ_λ that $yx \cdot a = ya \cdot x,\ \forall\, a, x, y \in Q \implies (R_a,\ I,\ R_a) \in AUT(Q,\cdot)$. Thus,

$$a \in C(Q,\cdot) \cap N_\lambda(Q,\cdot) \implies \forall\, a \in Q,\ R_a \in \lambda(Q,\cdot)$$

□

Corollary 4.3.7. Let $(Q,\cdot)$ be a Basarab loop. Let $a \in N(Q,\cdot)$ or $(Q,\cdot)$ be an inverse property loop. Then, $a \in C(Q,\cdot)$ if and only if $L_a \in \rho(Q,\cdot)$ or $R_a \in \lambda(Q,\cdot)$.

Proof. This result follows from Theorems 4.3.6 and 4.3.7, and the fact that an inverse property loop has both left and right inverse properties. □

Corollary 4.3.8. Let $(Q, \cdot)$ be a Basarab loop and let $Z(Q, \cdot)$ be the center of $(Q, \cdot)$. Then,

$$Z(Q, \cdot) = \big\{a \in Q : a \cdot xy = x \cdot ay,\ yx \cdot a = ya \cdot x,\ \forall\, x, y \in Q\big\}.$$

Proof. This result follows from Theorems 4.3.6 and 4.3.7. And

$$a \in N,\ C \iff a \in Z \implies a \in W.$$

If $a \in W$, then $a \cdot xy = x \cdot ay$. With $y = e,\ ax = x \cdot a \implies a \in C$. Thus,

$$a \cdot xy = x \cdot ay \text{ and } xy \cdot a = xa \cdot y$$

implies $a \in N_\mu = N$. Then $a \in N \cap C \implies a \in Z$. Hence, $Z = W$. □

Lemma 4.3.3. Let $(Q, \cdot)$ be a Basarab loop and let

$$\mathcal{P}'(Q, \cdot) = \big\{L_a \in \mathcal{P}(Q, \cdot) |\ a \in C(Q, \cdot)\big\} \text{ and } \Lambda'(Q, \cdot) = \big\{R_a \in \Lambda(Q, \cdot) |\ a \in C(Q, \cdot)\big\}.$$

(a) $Z(Q, \cdot)$ is anti-monomorphic and monomorphic to $\mathcal{P}(Q, \cdot)$ and $\Lambda(Q, \cdot)$.

(b) $Z(Q, \cdot)$ is embeddable in $\mathcal{P}'(Q, \cdot)$ and $\Lambda'(Q, \cdot)$ i.e. $Z(Q, \cdot) \cong \mathcal{P}'(Q, \cdot)$ and $Z(Q, \cdot) \cong \Lambda'(Q, \cdot)$.

(c) If $(Q, \cdot)$ is finite, then, $|Z(Q, \cdot)| \leq |\mathcal{P}(Q, \cdot)|,\ |Z(Q, \cdot)| \leq |\Lambda(Q, \cdot)|,\ |Z(Q, \cdot)| = |\mathcal{P}'(Q, \cdot)|$ and $|Z(Q, \cdot)| = |\Lambda'(Q, \cdot)|$.

Proof. **(a)** Let $\alpha : Z(Q, \cdot) \longrightarrow \mathcal{P}(Q, \cdot) \uparrow \alpha(x) = L_x$. Then, for any $a, b \in Z(Q, \cdot) :\ \alpha(ab) = L_{ab} = L_b L_a = \alpha(b)\alpha(a)$ and $\alpha(ba) = L_{ba} = L_a L_b = \alpha(a)\alpha(b)$. So, α is both an homomorphism and an anti-homomorphism. Also, for any $x_1, x_2 \in Z(Q, \cdot),\ \alpha(x_1) = \alpha(x_2) \implies L_{x_1} = L_{x_2} \implies x_1 = x_2$. So, α is both an homomorphism and an anti-homomorphism.

The proof is similar for when $\alpha : Z(Q,\cdot) \longrightarrow \Lambda(Q,\cdot) \uparrow \alpha(x) = R_x$.

(b) This is based on (a) and Theorems 4.3.6 and 4.3.7.

(c) These are consequences of (a) and (b) for finite $(Q,\cdot)$.

□

Lemma 4.3.4. If a left Basarab loop $(Q,\cdot)$ has a left power alternativity, then:

(a) $T_x^{-n} = (R_{xJ_\rho}L_x)^n = (L_xR_x^{-1})^n = R_x^{-n}L_x^n$;

(b) $|T_x| = n$ if and only if $x^n \in C(Q,\cdot)$.

Proof. Let $(Q,\cdot)$ be a left Basarab loop, then $(T_x^{-1},\ L_x,\ L_x) \in AUT(Q,\cdot)$. If $(Q,\cdot)$ is a right power alternative loop then there exist $(T_x^{-n},\ L_x^n,\ L_x^n)$, so that for all $x, y, z \in Q$;

$$yT_x^{-n} \cdot zL_x^n = (yz)L_x^n \implies yT_x^{-n} \cdot x^n z = x^n(yz).$$

Put $z = x^{-n}$ to get $yT_x^{-n} = x^n(yx^{-n})$. This implies,

$$yT_x^{-n} = yR_{x^{-n}}L_{x^n} = yR_x^{-n}L_x^n \implies T_x^{-n} = R_x^{-n}L_x^n.$$

Thus, $T_x^{-n} = [(R_xL_x^{-1})^{-1}]^n = (L_xR_x^{-1})^n \implies T_x^{-n} = (L_xR_x^{-1})^n = R_x^{-n}L_x^n = (R_{xJ_\rho}L_x)^n$. For (b),

$$|T_x| = n \iff L_x^n = R_x^n \iff L_{x^n} = R_{x^n} \iff x^ny = yx^n \iff x^n \in C(Q,\cdot).$$

□

Lemma 4.3.5. If a right Basarab loop $(Q,\cdot)$ has a right power alternativity, then:

(a) $T_x^n = (L_{xJ_\lambda} R_x)^n = (R_x L_x^{-1})^n = L_x^{-n} R_x^n$;

(b) $|T_x| = n$ if and only if $x^n \in C(Q, \cdot)$.

Proof. Let $(Q, \cdot)$ be a right Basarab loop, then $(R_x,\ T_x,\ R_x) \in AUT(Q, \cdot)$. If $(Q, \cdot)$ is a left power alternative loop then there exist $(R_x^n,\ T_x^n,\ R_x^n)$. This means, for all $x, y, z \in Q$;

$$yR_x^n \cdot zT_x^n = (yz)R_x^n \implies yx^n \cdot zT_x^n = (yz)x^n.$$

Put $y = x^{-n}$ to get $zT_x^n = x^{-n}z \cdot x^n$. This implies,

$$T_x^n = L_x R_x^n \implies T_x^n = (R_x L_x^{-1})^n = L_x^{-n} R_x^n = (L_{xJ_\lambda} R_x)^n.$$

To obtain (b),

$$|T_x| = n \iff L_x^n = R_x^n \iff L_{x^n} = R_{x^n} \iff x^n y = yx^n \iff x^n \in C(Q, \cdot).$$

□

Lemma 4.3.6. Let $(Q, \cdot)$ be a Basarab loop, then the following are true for all $x, y, z \in Q$:

(a) $L_x R_{xz} = R_{xJ_\rho}^{-1} R_z L_x$

(b) $R_x L_{yx} = L_{xJ_\lambda}^{-1} L_y R_x$

Proof. Consider the Basarab law $(x \cdot y(xJ_\rho)) \cdot xz = x \cdot yz,\ \forall\ x, y, z \in Q$. This means,

$$yR_{xJ_\rho} L_x R_{xz} = yR_z L_x \implies R_{xJ_\rho} L_x R_{xz} = R_z L_x \implies L_x R_{xz} = R_{xJ_\rho}^{-1} R_z L_x.$$

To obtain (b), consider the Basarab law $(y \cdot x) \cdot ((xJ_\lambda)z \cdot x) = yz \cdot x,\ \forall\ x, y, z \in Q$. Then

$$zL_{xJ_\lambda} R_x L_{yx} = zL_y R_x \implies L_{xJ_\lambda} R_x L_{yx} = L_y R_x \implies R_x L_{yx} = L_{xJ_\lambda}^{-1} L_y R_x.$$

□

Lemma 4.3.7. Let $(Q,\cdot)$ be a Basarab loop, then the following are true for all $x,y,z\in Q$:

(a) $R_{xz}L_x^{-1}=L_{xJ_\lambda}R_xR_z$

(b) $L_{yx}R_x^{-1}=R_{xJ_\rho}L_xR_y$.

Proof. From Lemma 4.4.2 (a), $L_xR_{xz}=R_{xJ_\rho}^{-1}R_zL_x$. In a Basarab loop,

$$R_{xJ_\rho}L_x=L_xR_x^{-1}\implies L_x=R_{xJ_\rho}^{-1}L_xR_x^{-1}\implies L_xR_xL_x^{-1}=R_{xJ_\rho}^{-1}.$$

$$\text{So, } L_xR_{xz}=R_{xJ_\rho}^{-1}R_zL_x\implies L_xR_{xz}=L_xR_xL_x^{-1}R_zL_x\implies L_xR_{xz}L_x^{-1}=L_xR_xL_x^{-1}R_z$$

$$\implies R_{xz}L_x^{-1}=R_xL_x^{-1}R_z\implies R_{xz}L_x^{-1}=L_{xJ_\lambda}R_z$$

. From Lemma 4.4.2 (b), $R_xL_{yx}=L_{xJ_\lambda}^{-1}L_yR_x$. In a Basarab loop,

$$L_{xJ_\lambda}R_x=R_xL_x^{-1}\implies R_x=L_{xJ_\lambda}^{-1}R_xL_x^{-1}\implies R_xL_xR_x^{-1}=L_{xJ_\lambda}^{-1}.$$

$$\text{So, } R_xL_{yx}=L_{xJ_\lambda}^{-1}L_yR_x\implies R_xL_{yx}=R_xL_xR_x^{-1}L_yR_x\implies L_{yx}=L_xR_x^{-1}L_yR_x$$

$$\implies L_{yx}R_x^{-1}=L_xR_x^{-1}L_y\implies L_{yx}R_x^{-1}=R_{xJ_\rho}L_xL_y.$$

□

Corollary 4.3.9. Let $(Q,\cdot)$ be a Basarab loop. Then the following are true for all $x,y,z\in Q$:

(a) $L_{yx}=R_{xJ_\rho}L_xR_yR_x$

(b) $R_{xz}=L_{xJ_\lambda}R_xR_zL_x$.

Proof. These are true by Lemma 4.4.3. □

Theorem 4.3.8. In a Basarab loop, $Z(Q,\cdot)=C(Q,\cdot)$.

Proof. Let $(Q,\cdot)$ be a Basarab loop. From corollary 4.3.9 (a),

$$L_{yx} = R_{xJ_\rho}L_xR_yR_x,\ \forall\ x,y \in Q \implies L_{yx} = L_xR_x^{-1}L_yR_x.$$

Since $R_{xJ_\rho}L_x = L_xR_x^{-1}$ in a Basarab loop. An element $a \in C(Q,\cdot) \implies L_a = R_a \implies L_a^{-1} = R_a^{-1}$. Let $x = a$ in $L_{yx} = L_xR_x^{-1}L_yR_x$ then $L_{ya} = L_aL_a^{-1}L_yR_a \implies L_{ya} = L_yR_a \implies L_{ya} = L_yL_a$. So, for every $x \in Q$,

$$xL_{ya} = xL_yL_a \implies ya\cdot x = a\cdot yx \implies ya\cdot x = yx\cdot a,\ \forall\ y \in Q \text{ and } a \in C(Q,\cdot).$$

Also, from corollary 4.3.9 (b), $R_{xz} = L_{xJ_\lambda}R_xR_zL_x,\ \forall\ x,z \in Q$. This means $R_{xz} = R_xL_x^{-1}R_zL_x,\ \forall\ x,z \in Q$ since, $L_{xJ_\lambda}R_x = R_xL_x^{-1}$. For every $a \in C(Q,\cdot)$ and set $x = a$ we have,

$$R_{az} = R_aL_a^{-1}R_zL_a \implies R_{az} = R_aR_a^{-1}R_zL_a \implies R_{az} = R_zL_a \implies R_{az} = R_zR_a$$

$$\implies \forall\ y \in Q,\ y\cdot az = yz\cdot a \implies y\cdot az = a\cdot yz,\ \forall\ y,z \in Q \text{ and } a \in C(Q,\cdot)$$

which means $x\cdot ay = a\cdot xy,\ \forall\ x,y \in Q$. Therefore,

$$C(Q,\cdot) = \big\{a \in Q : ya\cdot x = yx\cdot a,\ x\cdot ay = a\cdot xy,\ \forall\ x,y \in Q\big\}.$$

Hence, in a Basarab loop $(Q,\cdot)$, $Z(Q,\cdot) = C(Q,\cdot)$. □

Theorem 4.3.9. Let $(Q,\cdot)$ be a Basarab loop, then $Z(Q,\cdot) \le N(Q,\cdot)$.

Proof. Let $(Q,\cdot)$ be a loop then $Z(Q,\cdot) = C(Q,\cdot) \cap N(Q,\cdot)$. In Basarab loop,

$$Z(Q,\cdot) = C(Q,\cdot) \implies Z(Q,\cdot) = Z(Q,\cdot) \cap N(Q,\cdot) \implies Z(Q,\cdot) \le N(Q,\cdot)$$

□

Corollary 4.3.10. Let $(Q,\cdot)$ be a Basarab loop, then $C(Q,\cdot) \leq N(Q,\cdot)$.

Proof. In Basarab loop $(Q,\cdot)$, $C(Q,\cdot) = Z(Q,\cdot)$.

Thus, $Z(Q,\cdot) \leq N(Q,\cdot) \implies C(Q,\cdot) \leq N(Q,\cdot)$. □

Lemma 4.3.8. Let $(Q,\cdot)$ be a Basarab loop with the inverse property, then the following are true for all $x, y, z \in Q$:

(a) $L_x R_{xz} = R_x R_z L_x$;

(b) $R_x L_{yx} = L_x L_y R_x$.

Proof. From Lemma 4.4.2 (a), $L_x R_{xz} = R^{-1}_{xJ_\rho} R_z L_x \implies L_x R_{xz} = R_x R_z L_x$.

From Lemma 4.4.2 (b), $R_x L_{yx} = L^{-1}_{xJ_\lambda} L_y R_x \implies R_x L_{yx} = L_x L_y R_x$. □

Theorem 4.3.10. Let $(Q,\cdot)$ be a Basarab loop with the inverse property then $L(x,y) = [R_x, L_{yx}]$ and $R(x,z) = [L_x, R_{xz}]$ for all $x, y, z \in Q$.

Proof. From Lemma 4.4.4 (a),

$$L_x R_{xz} = R_x R_z L_x \implies L_x R_{xz} L_x^{-1} = R_x R_z$$

$$\implies L_x R_{xz} L_x^{-1} R_{xz}^{-1} = R_x R_z R_{xz}^{-1} \implies [L_x, R_{xz}] = R_x R_z R_{xz}^{-1} = R(x,z)$$

From Lemma 4.4.4 (b),

$$R_x L_{yx} = L_x L_y R_x \implies R_x L_{yx} R_x^{-1} = L_x L_y$$

$$\implies R_x L_{yx} R_x^{-1} L_{yx}^{-1} = L_x L_y L_{yx}^{-1} \implies [R_x, L_{yx}] = L_x L_y L_{yx}^{-1} = L(x,y).$$

□

Corollary 4.3.11. Let $(Q,\cdot)$ be a Basarab loop with the inverse property, then $L(x,y) = [R_x, L_{yx}]$ and $R(x,y) = [L_x, R_{xy}]$ for all $x, y \in Q$.

Proof. This is true by setting $z = y$ in Theorem 4.3.10. □

Corollary 4.3.12. Let $(Q,\cdot)$ be a Basarab loop then $(R_x R_{xJ_\rho})^{-1} = L_x L_{xJ_\lambda}$ for all $x \in Q$.

Proof. In a Basarab loop $(Q,\cdot)$, $T_x = R_x L_x^{-1} = L_{xJ_\lambda} R_x$ and

$$T_x^{-1} = L_x R_x^{-1} = R_{xJ_\rho} L_x.$$

$$\text{Thus, } T_x = R_x L_x^{-1} \implies T_x^{-1} = L_x R_x^{-1} = (L_{xJ_\lambda} R_x)^{-1} = R_x^{-1} L_{xJ_\lambda}^{-1}.$$

$$\text{And } T_x^{-1} = R_{xJ_\rho} L_x \implies T_x^{-1} = R_x^{-1} L_{xJ_\lambda}^{-1} = R_{xJ_\rho} L_x$$

$$\implies R_x^{-1} = R_{xJ_\rho} L_x L_{xJ_\lambda} \implies R_{xJ_\rho}^{-1} R_x^{-1} = L_x L_{xJ_\lambda} \implies (R_x R_{xJ_\rho})^{-1} = L_x L_{xJ_\lambda}.$$

□

Corollary 4.3.13. A Basarab loop has the RIP if and only if it has the LIP.

Proof. Let $(Q,\cdot)$ be a Basarab loop, from corollary 4.3.12,

$$R_{xJ_\rho}^{-1} R_x^{-1} = L_x L_{xJ_\lambda} \implies R_x R_x^{-1} = L_x L_{xJ_\lambda} \implies I = L_x L_{xJ_\lambda} \implies L_{xJ_\lambda} = L_x^{-1} \implies LIP.$$

□

Corollary 4.3.14. A Basarab loop has the LIP if and only if it has the RIP.

Proof. The proof is similar to Corollary 4.3.13. □

Corollary 4.3.15. A Basarab loop with the RIP or LIP is an extra loop.

Proof. By Corollaries 4.3.13 and 4.3.14, this is true since a Basarab loop with an inverse property is an extra loop. □

4.3.5 Automorphism and Inverse Property

Lemma 4.3.9. Let $(Q,\cdot)$ be a Basarab loop, then for all $x,y,z\in Q;\ R_{xJ_\rho}L_x=R_zL_xR_{xz}^{-1}$ and $L_{xJ_\lambda}R_x=L_yR_xL_{yx}^{-1}$.

Proof. Consider the Basarab law $(x\cdot y(xJ_\rho))\cdot xz=x\cdot yz$. Then, for all $x,y,z\in Q$;

$$yR_{xJ_\rho}L_xR_{xz}=yR_zL_x \implies R_{xJ_\rho}L_xR_{xz}=R_zL_x \implies R_{xJ_\rho}L_x=R_zL_xR_{xz}^{-1},\ \forall\ x,y\in Q.$$

Also, for all $x,y,z\in Q$, the Basarab law

$$(y\cdot x)(xJ_\lambda z\cdot x)=yz\cdot x \implies zL_{xJ_\lambda}R_xL_{yx}=zL_yR_x \implies L_{xJ_\lambda}R_x=L_yR_xL_{yx}^{-1}.$$

□

Corollary 4.3.16. In a Basarab loop $(Q,\cdot),\ R_{xJ_\rho}L_xL_{xJ_\lambda}R_x=L_{xJ_\lambda}R_xR_{xJ_\rho}L_x=I$

Proof. This is the result obtained when T_x and T_x^{-1} of a Basarab loop are multiplied. □

Corollary 4.3.17. In a left Basarab loop $(Q,\cdot),\ R_{xJ_\rho}L_x=R_yL_xR_{xy}^{-1}$ for all $x,y\in Q$.

Proof. This is obtained by a left Basarab law translation. □

Corollary 4.3.18. In a right Basarab loop $(Q,\cdot),\ L_{xJ_\lambda}R_x=L_yR_xL_{yx}^{-1}$ for all $x,y\in Q$.

Proof. This is obtained by a right Basarab law translation. □

Corollary 4.3.19. In a left Basarab loop $(Q,\cdot),\ R_{xy}^{-1}R_{xz}=L_x^{-1}R_y^{-1}R_zL_x$ for all $x,y,z\in Q$.

Proof. From Lemma 4.4.1 and Corollary 4.3.17,

$$R_{xJ_\rho}L_x=R_zL_xR_{xz}^{-1} \text{ and } R_{xJ_\rho}L_x=R_yL_xR_{xy}^{-1}$$

respectively, are true for left Basarab law, which implies $R_{xJ_\rho}L_x = R_zL_xR_{xz}^{-1} = R_yL_xR_{xy}^{-1}$. This means, $R_zL_xR_{xz}^{-1} = R_yL_xR_{xy}^{-1} \implies L_x^{-1}R_y^{-1}R_zL_x = R_{xy}^{-1}R_{xz}$. □

Theorem 4.3.11. Let $(Q,\cdot)$ be a Basarab loop, then $R(x,y)$ is an automorphism.

Proof. Let $(Q,\cdot)$ be a Basarab loop, then from the right Basarab law, $(R_x,\ L_{xJ_\lambda}R_x, R_x) \in AUT(Q,\cdot)$. For all $x,y \in Q, (R_x,\ L_{xJ_\lambda}R_x, R_x), (R_y,\ L_{yJ_\lambda}R_y,\ R_y) \in AUT(Q,\cdot)$. This implies that

$$(R_x,\ L_{xJ_\lambda}R_x,\ R_x)(R_y,\ L_{yJ_\lambda}R_y,\ R_y) \in AUT(Q,\cdot)$$

$$\implies (R_xR_y,\ L_{xJ_\lambda}R_xL_{yJ_\lambda}R_y,\ R_xR_y) \in AUT(Q,\cdot).$$

Also, let $x = xy$ in $(R_x,\ L_{xJ_\lambda}R_x,\ R_x) \in AUT(Q,\cdot)$ then

$$(R_{xy},\ L_{(xy)J_\lambda}R_{xy},\ R_{xy}) \in AUT(Q,\cdot) \iff (R_{xy}^{-1},\ [L_{(xy)J_\lambda}R_{xy}]^{-1},\ R_{xy}^{-1}) \in AUT(Q,\cdot)$$

For all $x,y \in Q$,

$$(R_xR_y,\ L_{xJ_\lambda}R_xL_{yJ_\lambda}R_y,\ R_xR_y)(R_{xy}^{-1},\ [L_{(xy)J_\lambda}R_{xy}]^{-1},\ R_{xy}^{-1}) \in AUT(Q,\cdot)$$

$$(R_xR_yR_{xy}^{-1},\ L_{xJ_\lambda}R_xL_{yJ_\lambda}R_y[L_{(xy)J_\lambda}R_{xy}]^{-1},\ R_xR_yR_{xy}^{-1}) \in AUT(Q,\cdot)$$

For all $v,w \in Q$,

$$vR_xR_yR_{xy}^{-1} \cdot wL_{xJ_\lambda}R_xL_{yJ_\lambda}R_y[L_{(xy)J_\lambda}R_{xy}]^{-1} = (vw)R_xR_yR_{xy}^{-1}.$$

Set $v = e$,

$$eR_xR_yR_{xy}^{-1} \cdot wL_{xJ_\lambda}R_xL_{yJ_\lambda}R_y[L_{(xy)J_\lambda}R_{xy}]^{-1} = wR_xR_yR_{xy}^{-1}.$$

But $eR_xR_yR_{xy}^{-1} = e,$ this means

$$wL_{xJ_\lambda}R_xL_{yJ_\lambda}R_y[L_{(xy)J_\lambda}R_{xy}]^{-1} = wR_xR_yR_{xy}^{-1}.$$

Therefore,

$$R(x,y) = R_xR_yR_{xy}^{-1} = L_{xJ_\lambda}R_xL_{yJ_\lambda}R_y[L_{(xy)J_\lambda}R_{xy}]^{-1},$$

which implies $(R_xR_yR_{xy}^{-1},\ R_xR_yR_{xy}^{-1},\ R_xR_yR_{xy}^{-1}) \in AUT(Q,\cdot)$. Hence, $R(x,y)$ is an automorphism. □

Theorem 4.3.12. Let $(Q,\cdot)$ be a Basarab loop, then $L(x,y)$ is an automorphism.

Proof. Let $(Q,\cdot)$ be a Basarab loop, then from the left Basarab law, $(R_{xJ_\rho}L_x,\ L_x,\ L_x) \in AUT(Q,\cdot)$. For all $x,y \in Q$,

$$(R_{xJ_\rho}L_x,\ L_x,\ L_x), (R_{yJ_\rho}L_y,\ L_y,\ L_y) \in AUT(Q,\cdot).$$

This implies that

$$(R_{xJ_\rho}L_x,\ L_x,\ L_x)(R_{yJ_\rho}L_y,\ L_y,\ L_y) \in AUT(Q,\cdot)$$

$$\implies (R_{xJ_\rho}L_xR_{yJ_\rho}L_y,\ L_xL_y,\ L_xL_y) \in AUT(Q,\cdot).$$

Let $x = yx$ in $(R_{xJ_\rho}L_x,\ L_x,\ L_x) \in AUT(Q,\cdot)$ then

$$(R_{(yx)J_\rho}L_{yx},\ L_{yx},\ L_{yx}) \in AUT(Q,\cdot) \iff ([R_{(yx)J_\rho}L_{yx}]^{-1},\ L_{yx}^{-1},\ L_{yx}^{-1}) \in AUT(Q,\cdot)$$

This means that for all $x,y \in Q$,

$$(R_{xJ_\rho}L_xR_{yJ_\rho}L_y,\ L_xL_y,\ L_xL_y)([R_{(yx)J_\rho}L_{yx}]^{-1},\ L_{yx}^{-1},\ L_{yx}^{-1}) \in AUT(Q,\cdot)$$

$$\implies (R_{xJ_\rho}L_xR_{yJ_\rho}L_y[R_{(yx)J_\rho}L_{yx}]^{-1},\ L_xL_yL_{yx}^{-1},\ L_xL_yL_{yx}^{-1}) \in AUT(Q,\cdot).$$

For all $v, w \in Q$,

$$vR_{xJ_\rho}L_xR_{yJ_\rho}L_y[R_{(yx)J_\rho}L_{yx}]^{-1} \cdot wL_xL_yL_{yx}^{-1} = (vw)L_xL_yL_{yx}^{-1}$$

Set $w = e$,

$$vR_{xJ_\rho}L_xR_{yJ_\rho}L_y[R_{(yx)J_\rho}L_{yx}]^{-1} \cdot eL_xL_yL_{yx}^{-1} = vL_xL_yL_{yx}^{-1}.$$

Since, $eL_xL_yL_{yx}^{-1} = e$, then

$$vR_{xJ_\rho}L_xR_{yJ_\rho}L_y[R_{(yx)J_\rho}L_{yx}]^{-1} = vL_xL_yL_{yx}^{-1}.$$

Therefore,

$$L(x,y) = L_xL_yL_{yx}^{-1} = R_{xJ_\rho}L_xR_{yJ_\rho}L_y[R_{(yx)J_\rho}L_{yx}]^{-1},$$

which means

$$(L_xL_yL_{yx}^{-1},\ L_xL_yL_{yx}^{-1},\ L_xL_yL_{yx}^{-1}) \in AUT(Q,\cdot).$$

Thus, $L(x,y)$ is an automorphism. $\square$

Theorem 4.3.13. In a Basarab loop $(Q,\cdot)$, the following are true: (i) $T_y = L_xR_yL_{xy}^{-1}$ (ii) $T_y^{-1} = R_xL_yR_{yx}^{-1}$

Proof. Let $(Q,\cdot)$ be a Basarab loop. From Theorem 4.3.11, $R(x,y) \in AUM(Q,\cdot)$ means

$$L_{xJ_\lambda}R_xL_{yJ_\lambda}R_y[L_{(xy)J_\lambda}R_{xy}]^{-1} = R_xR_yR_{xy}^{-1}.$$

This implies,

$$L_{xJ_\lambda}R_xL_{yJ_\lambda}R_yT_{xy}^{-1} = R_xR_yR_{xy}^{-1} \implies T_xT_yT_{xy}^{-1} = R_xR_yR_{xy}^{-1} \implies T_xT_yL_{xy}R_{xy}^{-1} = R_xR_yR_{xy}^{-1}$$

$$T_xT_yL_{xy} = R_xR_y \implies R_xL_{xy} = L_xR_y \implies T_y = L_xR_yL_{xy}^{-1}.$$

Also, $L(x,y)\in AUM(Q,\cdot)$ from Theorem 4.3.12 means

$$R_{xJ_\rho}L_xR_{yJ_\rho}L_y[R_{(yx)J_\rho}L_{yx}]^{-1}=L_xL_yL_{yx}^{-1}$$

This implies

$$R_{xJ_\rho}L_xR_{yJ_\rho}L_yT_{yx}=L_xL_yL_{yx}^{-1}\implies R_{xJ_\rho}L_xR_{yJ_\rho}L_yR_{yx}L_{yx}^{-1}=L_xL_yL_{yx}^{-1}$$

$$\implies R_{xJ_\rho}L_xR_{yJ_\rho}L_yR_{yx}=L_xL_y\implies T_x^{-1}T_y^{-1}R_{yx}=L_xL_y\implies L_xR_x^{-1}T_y^{-1}R_{yx}=L_xL_y$$

$$\implies R_x^{-1}T_y^{-1}R_{yx}=L_y\implies T_y^{-1}=R_xL_yR_{yx}^{-1}.$$

□

Corollary 4.3.20. In a Basarab loop $(Q,\cdot)$,

$$R(x,y)=R_xR_yR_{xy}^{-1}=L_{xJ_\lambda}R_xL_{yJ_\lambda}R_y[L_{(xy)J_\lambda}R_{xy}]^{-1}$$

and

$$L(x,y)=L_xL_yL_{yx}^{-1}=R_{xJ_\rho}L_xR_{yJ_\rho}L_y[R_{(yx)J_\rho}L_{yx}]^{-1}.$$

Proof. This result follows from Theorems 4.3.11 and 4.3.12. □

Corollary 4.3.21. In a right Basarab loop $(Q,\cdot)$, $T_y=L_xR_yL_{xy}^{-1}$ for all $x,y\in Q$.

Proof. This result follows from the proof of Theorem 4.3.13. □

Corollary 4.3.22. In a left Basarab loop $(Q,\cdot)$, $T_y^{-1}=R_xL_yR_{yx}^{-1}$ for all $x,y\in Q$.

Proof. This result follows from the proof of Theorem 4.3.13. □

Theorem 4.3.14. A Basarab loop with the IP is an extra loop.

Proof. Let $(Q,\cdot)$ be a Basarab loop, then

$$(R_{xJ_\rho}L_x,\ L_x,\ L_x),(R_x,\ L_{xJ_\lambda}R_x,\ R_x)\in AUT(Q,\cdot).$$

If $(Q,\cdot)$ is with the IP then, $R_{xJ_\rho}=R_x^{-1}$ and $L_{xJ_\lambda}=L_x^{-1}$. This implies

$$(R_{xJ_\rho}L_x,\ L_x,\ L_x)\in AUT(Q,\cdot)\iff(R_x^{-1}L_x,\ L_x,\ L_x)\in AUT(Q,\cdot)$$

This means for all $x,y,z\in Q,\ yR_x^{-1}L_x\cdot zL_x=(yz)L_x\implies xy\cdot xz=x(yx\cdot z)$; and

$$(R_x,\ L_{xJ_\lambda}R_x,\ R_x)\in AUT(Q,\cdot)\iff(R_x,\ L_x^{-1}R_x,\ R_x)\in AUT(Q,\cdot)$$

$$\implies yR_x\cdot zL_x^{-1}R_x=(yz)R_x,\ \forall\ x,y,z\in Q\implies yx\cdot zx=(y\cdot xz)x,\forall\ x,y,z\in Q.$$ □

Corollary 4.3.23. A left Basarab loop with the RIP is an extra loop.

Proof. This is true since in a left Basarab loop, RIP, LIP, and IP are equivalent. □

Corollary 4.3.24. A right Basarab loop with the LIP is an extra.

Proof. This is true since in a right Basarab loop, RIP, LIP, and IP are equivalent. □

Lemma 4.3.10. Let $(Q,\cdot)$ be a Basarab loop, then for all $x,y,z\in Q$, the following hold:
(i) $(x\cdot yx^\rho)\cdot xy^\rho=x$ (ii) $(y^\lambda\cdot x)\cdot((x^\lambda\cdot y)\cdot x)=x$.

Proof. Consider the left Basarab law $(x\cdot yx^\rho)\cdot xz=x\cdot yz$ and set $z=y^\rho$, it follows that

$$(x\cdot yx^\rho)\cdot xy^\rho=x\cdot yy^\rho\implies(x\cdot yx^\rho)\cdot xy^\rho=x.$$

Also, consider the right Basarab law $(y\cdot x)\cdot(x^\lambda z\cdot x)=yz\cdot x$ and set $z=y^\rho$. This yields,

$$(y\cdot x)\cdot(x^\lambda y^\rho\cdot x)=yy^\rho\cdot x\implies(y\cdot x)\cdot(x^\lambda y^\rho\cdot x)=x.$$

Then setting $y = y^\lambda$ gives $(y^\lambda \cdot x) \cdot (x^\lambda y \cdot x) = x, \forall\, x, y \in Q$. □

Theorem 4.3.15. A Basarab loop $(Q, \cdot)$ with the AAIP is an extra loop.

Proof. If a loop $(Q, \cdot)$ has the AAIP then $(xy)^\rho = y^\rho x^\rho$ and $(xy)^\lambda = y^\lambda x^\lambda$ are true for all $x, y \in Q$. This means, $\lambda = \rho$. Let $(Q, \cdot)$ be a Basarab loop and set $yx^\rho = z$ in Lemma 4.3.5(i), then for AAIP, $z = yx^\rho \implies z^\rho = (yx^\rho)^\rho = xy^\rho$.

Also, set $z = y^\lambda x$ in Lemma 4.3.5(ii), then in AAIP, $z = y^\lambda x \implies z^\lambda = (y^\lambda x)^\lambda = x^\lambda y$. From Lemma 4.3.5, we obtain respectively that $xz \cdot z^{-1} = x$ and $z \cdot z^{-1}x = x$. These imply that the inverse property is satisfied. As proved in Theorem 4.3.14, a Basarab loop with the inverse property is an extra loop. Thus, a Basarab loop $(Q, \cdot)$ with the AAIP is an extra loop. □

Corollary 4.3.25. A left(right) Basarab loop with the AAIP is an extra loop.

Proof. This follows from the proof of Theorem 4.3.15. □

Theorem 4.3.16. In a Basarab loop, the middle inner mapping generates the inner mapping group.

Proof. From Corollary 4.3.20, in a Basarab loop,

$$R(x, y) = T_x T_y T_{xy}^{-1} \text{ and } L(x, y) = T_x^{-1} T_y^{-1} T_{yx}.$$

$$Inn(Q, \cdot) = \Big\langle \{R(x, y),\ L(x, y),\ T(x) : x, y \in Q\} \Big\rangle$$

$$= \Big\langle \{T_x T_y T_{xy}^{-1},\ T_x^{-1} T_y^{-1} T_{yx},\ T(x) : x, y \in Q\} \Big\rangle$$

$$= \Big\langle \{T(x) : x \in Q\} \Big\rangle.$$

□

Corollary 4.3.26. Let $(Q, \cdot)$ be a Basarab loop, then $y \cdot zx = y \cdot (x \cdot zT_x)$ and $xy \cdot z = (yT_x^{-1} \cdot x)z$ hold for all $x, y, z \in Q$.

Proof. Let $(Q,\cdot)$ be a Basarab loop, then

$$T_x = L_{xJ_\lambda}R_x = R_xL_x^{-1} \implies R_x = L_{xJ_\lambda}R_xL_x,\ \forall\, w \in Q,\ wR_x = wL_{xJ_\lambda}R_xL_x,$$

also for $u \in Q,\ u \cdot wR_x = u \cdot wL_{xJ_\lambda}R_xL_x \implies u \cdot wx = u \cdot (x \cdot wL_{xJ_\lambda}R_x)$

$\implies u \cdot wx = u \cdot (x \cdot wT_x)$. So, for all $x, y, z \in Q,\ y \cdot zx = y(x \cdot zT_x)$.

Also, $T_x^{-1} = L_xR_x^{-1} \implies \forall\, y \in Q,\ yL_xR_x^{-1} = yT_x^{-1} \implies yL_x = yT_x^{-1}R_x$

$\implies xy = yT_x^{-1} \cdot x,$ for every $z \in Q,\ xy \cdot z = (yT_x^{-1} \cdot x)z.$

□

Theorem 4.3.17. Let $(Q,\cdot)$ be a loop, then:

(a) a right Basarab law is equivalent to $(y, z, x) = (y, x, zT_x)$ and $yx = x(x^\lambda y \cdot x)$;

(b) a left Basarab law is equivalent to $[x, y, z] = [yT_x^{-1}, x, z]$ and $xy = (x \cdot yx^\rho)x$.

Proof. **(a)** Consider the right Basarab law $(y \cdot x) \cdot ((xJ_\lambda)z \cdot x) = yz \cdot x$ for a loop $(Q,\cdot)$. Then for all $x, y, z \in Q$,

$$(y \cdot x) \cdot zL_{xJ_\lambda}R_x = yz \cdot x \implies yx \cdot zT_x = yz \cdot x.$$

So that, $yx \cdot zT_x = (y \cdot (x \cdot zT_x)) \cdot (y, x, zT_x)$ and $y \cdot zx = y \cdot (x \cdot zT_x)$ by Corollary 4.3.26. Combining these, we have

$$yx \cdot zT_x = (y \cdot (x \cdot zT_x)) \cdot (y, x, zT_x) = (y \cdot zx)(y, x, zT_x)$$

$$\implies yx \cdot zT_x = (y \cdot zx)(y, x, zT_x) \implies yz \cdot x = (y \cdot zx)(y, x, zT_x)$$

. Since $(yz \cdot x) = (y \cdot zx)(y, z, x)$ by definition, then

$$(y \cdot zx)(y, z, x) = (y \cdot zx)(y, x, zT_x) \implies (y, z, x) = (y, x, zT_x).$$

(b) Consider the left Basarab law $(x \cdot y(xJ_\rho)) \cdot xz = x \cdot yz$ for all $x, y, z \in Q$. Then

$$yR_{xJ_\rho}L_x \cdot xz = x \cdot yz \implies yT_x^{-1} \cdot xz = x \cdot yz.$$

This means, $yT_x^{-1} \cdot xz = [yT_x^{-1}, x, z]((yT_x^{-1} \cdot x) \cdot z)$ and $xy \cdot z = (yT_x^{-1} \cdot x)z$ by Corollary 4.3.26. So that,

$$yT_x^{-1} \cdot xz = [yT_x^{-1}, x, z]((yT_x^{-1} \cdot x) \cdot z) = [yT_x^{-1}, x, z](xy \cdot z)$$

$$\implies yT_x^{-1} \cdot xz = yT_x^{-1}, x, z(xy \cdot z) \implies x \cdot yz = [yT_x^{-1}, x, z](xy \cdot z)$$

. Also, $(x \cdot yz) = [x, y, z](xy \cdot z)$ by definition, this means

$$[yT_x^{-1}, x, z](xy \cdot z) = [x, y, z](xy \cdot z) \implies [yT_x^{-1}, x, z] = [x, y, z].$$

□

4.3.6 Associators, Moufang and extra elements

Definition 4.3.2. Let $(Q, \cdot)$ be a loop, then the associators $(a, b, c),\ [a, b, c]$ for all $a, b, c \in Q$; are defined respectively by $(ab \cdot c) = (a \cdot bc)(a, b, c)$ and $(a \cdot bc) = [a, b, c](ab \cdot c)$.

Definition 4.3.3. An element a of a loop $(Q, \cdot)$ is a Moufang element if and only if

$$\begin{cases} \forall\ x, y \in Q,\ a(xy \cdot a) = ax \cdot ya\ , \\ \forall\ x, y \in Q,\ (a \cdot xy)a = ax \cdot ya\ . \end{cases}$$

Definition 4.3.4. An element a of a loop $(Q, \cdot)$ is an extra element if and only if for all $x, y \in Q,\ a(x \cdot ya) = (ax \cdot y)a$.

Definition 4.3.5. Let $(Q, \cdot)$ be a Basarab loop with a nucleus N. An automorphism α of $(Q, \cdot)$ is nuclear if and only if $x\alpha \in xN$ for each $x \in Q$.

Lemma 4.3.11. Let $(Q, \cdot)$ be a Basarab loop, then for all $x, y, z \in Q$:

(a) $zL(x, y) = z[y, x, z]$

(b) $zR(x, y) = (z, x, y)z$.

Proof. (a) Let $(Q, \cdot)$ be a Basarab loop, then $L(x, y) = L_x L_y L_{yx}^{-1}$ for all $x, y \in Q$.

$$\text{For } z \in Q,\ zL(x, y) = zL_x L_y L_{yx}^{-1} \implies zL(x, y) = (yx)\backslash(y \cdot xz)$$

$$\implies (yx)\backslash[(yx \cdot z)[y, x, z]] = zL(x, y) \implies [(yx)\backslash(yx \cdot z)][y, x, z] = zL(x, y)$$

Since associator of any three elements of a Basarab loop $(Q, \cdot)$ contains in the nucleus of $(Q, \cdot)$ then $[(yx)\backslash(yx \cdot z)][y, x, z] = zL(x, y)$

$$\implies [(yx)(yx\backslash z)][y, x, z] = zL(x, y) \implies z[y, x, z] = zL(x, y).$$

(b) Also, $R(x, y) = R_x R_y R_{xy}^{-1}$ for all $x, y \in Q$.

$$\text{For } z \in Q,\ zR(x, y) = zR_x R_y R_{xy}^{-1} \implies zR(x, y) = zR_x R_y/(xy)$$

$$\implies zR(x, y) = (zx \cdot y)/(xy) \implies zR(x, y) = (z \cdot xy)(z, x, y)/(xy)$$

$$\implies (z, x, y)[(z \cdot xy)/(xy)] = zR(x, y) \implies zR(x, y) = (z, x, y)z = z(z, x, y)$$

$$\implies zR(x, y) = (z, x, y)z.$$

□

Corollary 4.3.27. Let $(Q,\cdot)$ be a Basarab loop, then $R(x,y)$ and $L(x,y)$ are nuclear.

Proof. Let $(Q,\cdot)$ be a Basarab loop. The associator of any three elements of a Basarab loop contains in the nucleus of $(Q,\cdot)$ implies $zL(x,y) = z[y,x,z] \implies zL(x,y) = zN$ and $zR(x,y) = (z,x,y)z \implies R(x,y) = Nz$. Since N of a Basarab loop is normal, $zR(x,y) = zN$. □

Theorem 4.3.18. Let $(Q,\cdot)$ be a Basarab loop and $Z(N)$ center of $N(Q,\cdot)$ then $zR(x,y),\ zL(x,y) \in Z(N)$ for every $x,y \in Q$ and $z \in Z(N)$.

Proof. Let $(Q,\cdot)$ be a Basarab loop, then for all $x,y \in Q$ and $c \in N$;

$$c \in N \implies (cx)/x \implies [c\cdot(xy/y)]/x \implies [(c\cdot xy)/y]/x \implies cR_{xy}R_y^{-1}R_x^{-1} \in N.$$

If $z \in Z(N)$ then

$$z\cdot cR_{xy}R_y^{-1}R_x^{-1} = cR_{xy}R_y^{-1}R_x^{-1}\cdot z \tag{4.8}$$

The autotopism $T = (R_x,\ L_{xJ_\lambda}R_x,\ R_x)$ of $(Q,\cdot)$ implies that

$$(R_xR_yR_{xy}^{-1},\ R_xR_yR_{xy}^{-1},\ R_xR_yR_{xy}^{-1}) \in AUT(Q,\cdot) \tag{4.9}$$

Applying Equation 4.9 to Equation 4.8,

$$zR_xR_yR_{xy}^{-1}\cdot cR_{xy}R_y^{-1}R_x^{-1}R_xR_yR_{xy}^{-1} = (cR_{xy}R_y^{-1}R_x^{-1}\cdot z)R_xR_yR_{xy}^{-1}$$

$$
\begin{aligned}
zR_xR_yR_{xy}^{-1}\cdot c &= (cR_{xy}R_y^{-1}R_x^{-1}\cdot z)R_xR_yR_{xy}^{-1}\\
zR(x,y)\cdot c &= ((c\cdot xy)R_y^{-1}R_x^{-1}\cdot z)R(x,y)\\
&= ((cx\cdot y)R_y^{-1}R_x^{-1}\cdot z)R(x,y)\\
&= ((cx)R_yR_y^{-1}R_x^{-1}\cdot z)R(x,y)\\
&= (cR_xR_x^{-1}\cdot z)R(x,y) = (cz)R(x,y)\\
&= zL_cR(x,y) = zR(x,y)L_c
\end{aligned}
$$

This implies $zR(x,y)\cdot c = c\cdot zR(x,y)$, then $zR(x,y)\in Z(N)$.

Also, for every $x,y\in Q$ and $c\in N$;

$$
c\in N \implies x\backslash(xc) \implies x\backslash[(y\backslash yx)\cdot c] \implies x\backslash[y\backslash(yx\cdot c)] \implies cL_{yx}L_y^{-1}L_x^{-1}\in N.
$$

If $z\in Z(N)$, then

$$
z\cdot cL_{yx}L_y^{-1}L_x^{-1} = cL_{yx}L_y^{-1}L_x^{-1}\cdot z \tag{4.10}
$$

The autotopism $S=(R_{xJ_\rho}L_x,\ L_x,\ L_x)$ of $(Q,\cdot)$ implies that

$$
(L_xL_yL_{yx}^{-1},\ L_xL_yL_{yx}^{-1},\ L_xL_yL_{yx}^{-1})\in AUT(Q,\cdot). \tag{4.11}
$$

Applying Equation 4.15 to Equation 4.14,

$$
zL_xL_yL_{yx}^{-1}\cdot cL_{yx}L_y^{-1}L_x^{-1}L_xL_yL_{yx}^{-1} = (cL_{yx}L_y^{-1}L_x^{-1}\cdot z)L_xL_yL_{yx}^{-1}
$$

$$zL_xL_yL_{yx}^{-1} \cdot c = (cL_{yx}L_y^{-1}L_x^{-1} \cdot z)L_xL_yL_{yx}^{-1}$$

$$zL(x,y) \cdot c = ((yx \cdot c)L_y^{-1}L_x^{-1} \cdot z)L(x,y)$$

$$= ((y \cdot xc)L_y^{-1}L_x^{-1} \cdot z)L(x,y)$$

$$= ((xc)L_yL_y^{-1}L_x^{-1} \cdot z)L(x,y)$$

$$= (cL_xL_x^{-1} \cdot z)L(x,y) = (cz)L(x,y)$$

$$= zL_cL(x,y) = zL(x,y)L_c.$$

This means $zL(x,y) \cdot c = c \cdot zL(x,y)$. Thus, $zL(x,y) \in Z(N)$. □

Lemma 4.3.12. An element a of a Basarab loop $(Q, \cdot)$ is a Moufang element if and only if it is flexible.

Proof. Let $(Q, \cdot)$ be a Basarab loop, then for $a \in Q$,

$$(T_a^{-1},\ L_a,\ L_a),\ (R_a,\ T_a,\ R_a) \in AUT(Q, \cdot)$$

$$\implies (T_a^{-1},\ L_a,\ L_a)(R_a,\ T_a,\ R_a) \in AUT(Q, \cdot)$$

$$\implies (L_aR_a^{-1}R_a,\ L_aR_aL_a^{-1},\ L_aR_a) \in AUT(Q, \cdot)$$

$$\implies (L_a,\ L_aR_aL_a^{-1},\ L_aR_a) \in AUT(Q, \cdot)$$

$\implies (L_a,\ R_a,\ L_aR_a) \in AUT(Q, \cdot)$ if $(Q, \cdot)$ is flexible. This gives

$$ax \cdot ya = (a \cdot xy)a,\ \forall\ x, y \in Q.$$

Also, $(R_a,\ T_a,\ R_a)(T_a^{-1},\ L_a,\ L_a) \in AUT(Q, \cdot) \implies (R_aT_a^{-1},\ T_aL_a,\ R_aL_a) \in AUT(Q, \cdot) \implies (R_aL_aR_a^{-1},\ R_a,\ R_aL_a) \in AUT(Q, \cdot) \implies (L_a,\ R_a,\ R_aL_a) \in AUT(Q, \cdot)$ if $(Q, \cdot)$ is flexible. This gives $ax \cdot ya = a(xy \cdot a),\ \forall\ x, y \in Q.$ □

Lemma 4.3.13. Let $(Q, \cdot)$ be a Basarab loop. For every $a, b \in Q$, the following are equivalent:

1. $a \cdot ba = ab \cdot a$
2. $ab \cdot ax = a(ba \cdot x),\ \forall\, x \in Q$
3. $xa \cdot ba = (x \cdot ab)a,\ \forall\, x \in Q$
4. $ba \cdot a^{\rho} = b$
5. $a^{\lambda} \cdot ab = b$
6. $(x, a, a^{\rho}) = e,\ \forall\, x \in Q$
7. $(a^{\lambda}, a, x) = e,\ \forall\, x \in Q$.

Proof. Let $(Q, \cdot)$ be a Basarab loop.

1. Let property 1 holds.

2. Consider the Basarab law $(x \cdot y(xJ_{\rho})) \cdot xz = x \cdot yz$. Fix $a, b \in Q$ such that $x = a$ and $y = ba$. Then

$$(a \cdot (ba \cdot (aJ_{\rho}))) \cdot az = a \cdot (ba \cdot z) \implies bR_aR_{xJ_{\rho}}La \cdot az = a(ba \cdot z)$$

$$\implies bR_aL_aR_a^{-1} \cdot az = a(ba \cdot z).$$

If $(Q, \cdot)$ is flexible then $bLa \cdot az = a(ba \cdot z)$ holds and $ab \cdot az = a(ba \cdot z)$ follows. This implies extra property for $a, b \in Q$.

3. Consider the Basarab law $(y \cdot x) \cdot ((xJ_{\lambda})z \cdot x) = yz \cdot x$. Fix $a, b \in Q$ such that $x = a$ and $z = ab$. Then

$$(y \cdot a) \cdot ((aJ_{\lambda} \cdot ab) \cdot a) = (y \cdot ab) \cdot a \implies (y \cdot a) \cdot bL_aL_{aJ_{\lambda}}R_a = (y \cdot ab) \cdot a$$

$$\implies (y \cdot a) \cdot bL_aR_aL_a^{-1} = (y \cdot ab) \cdot a.$$

By flexibility, $(y \cdot a) \cdot bR_aL_aL_a^{-1} = (y \cdot ab) \cdot a \implies ya \cdot ba = (y \cdot ab)a$. Another extra property is satisfied.

4. From the proof of 1, $(a \cdot (ba \cdot (aJ_\rho))) \cdot az = a \cdot (ba \cdot z) \implies bR_aR_{xJ_\rho}La \cdot az = a(ba \cdot z)$. If $R_aR_{a^\rho} = I$ holds then $bR_aR_{xJ_\rho}La \cdot az = a(ba \cdot z) \implies bL_a \cdot az = a(ba \cdot z)$ satisfies 2.

5. From the proof of 2, $(y \cdot a) \cdot ((aJ_\lambda \cdot ab) \cdot a) = (y \cdot ab) \cdot a \implies (y \cdot a) \cdot bL_aL_{aJ_\lambda}R_a = (y \cdot ab) \cdot a$. If $L_aL_{aJ_\lambda} = I$ holds then $(y \cdot a) \cdot bR_a = (y \cdot ab)a$ satisfies 3.

6. Use 4.

7. Use 5.

□

Theorem 4.3.19. An element of a Basarab loop $(Q, \cdot)$ is contained in the set of Moufang elements of $(Q, \cdot)$ if and only if $(x, a, a^\rho) = e$ for all $x \in Q$.

Proof. This true by using Lemma 4.4.5. □

Lemma 4.3.14. (Jaiyeola & Effiong (2018))

Let $(Q, \cdot)$ be a Basarab loop. The following are equivalent:

1. Flexibility
2. Right inverse property
3. Left inverse property
4. Inverse property
5. Right alternative property
6. Left alternative property
7. Alternative property.

Corollary 4.3.28. Let $(Q, \cdot)$ be a Basarab loop. An element $a \in Q$ is an extra element if the following are equivalent:

1. a is flexible
2. a has RIP

3. a has LIP
4. a has IP
5. a has RAP
6. a has LAP
7. a has AP

Proof. This is true by Theorem 4.3.19 and Lemma 4.3.14. □

Corollary 4.3.29. Let $(Q,\cdot)$ be a Basarab loop, an element $a \in Q$ is a Moufang element if and only if it is an extra element and $a^2 \in N(Q,\cdot)$.

Proof. Let $(Q,\cdot)$ be a Basarab loop, then a Moufang element of $(Q,\cdot)$ satisfies Corollary 4.3.17. Thus a is Moufang if and only if $(L_a,\ R_a^{-1}, L_aR_a^{-1})(L_a,\ R_a,\ L_aR_a) \in AUT(Q,\cdot)$

$$\iff (L_a^2,\ I,\ L_aR_a^{-1}L_aR_a) \in AUT(Q,\cdot) \iff (L_a^2,\ I,\ L_a^2) \in AUT(Q,\cdot)$$

$$\iff (L_{a^2},\ I,\ L_{a^2}) \in AUT(Q,\cdot) \iff a^2 \in N_\lambda(Q,\cdot).$$

Thus, $a^2 \in N(Q,\cdot)$, since the nuclei of a Basarab loop coincide. □

4.4 Isotopes of Basarab loop

In this section, the isotopes of a Basarab loop is investigated. It is shown that every right isotope of a right Basarab loop is a right conjugacy closed loop and every left isotope of a left Basarab loop is a left conjugacy closed loop. The left(right) isotope of a right(left) Basarab loop is shown to be a right(left) Basarab loop. It is established that every principal isotope of a left Basarab loop is a left Basarab loop and every principal isotope of a right Basarab loop is a right Basarab loop. Hence, every principal isotope of a Basarab loop is a Basarab loop. Necessary and sufficient conditions for isotopes and principal isotopes of a Basarab loop are determined.

Definition 4.4.1. Let $(L,\cdot)$ be a loop. The loop with the multiplication $x \star y = x/a \cdot y$ is said to be left isotopic to a loop $(L, \cdot)$ for all $a, x, y \in Q$. The loop with the multiplication $x \star y = x \cdot b \backslash y$ is said to be right isotopic to a loop $(L, \cdot)$ for all $b, x, y \in Q$.

Theorem 4.4.1. Every right isotope of a right Basarab loop is an RCC-loop.

Proof. Consider the right Basarab loop identity

$$(y \star x) \star ((J_{\lambda'} x) z \star x) = (y \star z) \star x$$

and apply $x \star y = x \cdot b \backslash y$ to obtain

$$(y \cdot b \backslash x) \cdot b \backslash (((J_{\lambda'} x) \cdot b \backslash z) \cdot b \backslash x) = (y \cdot b \backslash z) \cdot (b \backslash x) \tag{4.12}$$

Set $y = e$ in Equation 4.12, then

$$(b \backslash x) \cdot b \backslash (((J_{\lambda'} x) \cdot b \backslash z) \cdot b \backslash x) = (b \backslash z) \cdot (b \backslash x).$$

This implies,

$$b \backslash (((J_{\lambda'} x) \cdot b \backslash z) \cdot b \backslash x) = (b \backslash x) \backslash [(b \backslash z) \cdot (b \backslash x)]. \tag{4.13}$$

Put Equation 4.13 into Equation 4.12 then

$$(y \cdot b \backslash x) \cdot [(b \backslash x) \backslash [(b \backslash z) \cdot (b \backslash x)]] = (y \cdot b \backslash z) \cdot (b \backslash x).$$

Replacing z by bz and gives

$$(y \cdot b \backslash x) \cdot [(b \backslash x) \backslash [z \cdot (b \backslash x)]] = (y \cdot z) \cdot (b \backslash x).$$

which means, $yR_{(b\backslash x)} \cdot zR_{(b\backslash x)}L^{-1}_{(b\backslash x)} = (yz)R_{(b\backslash x)}$. Therefore, setting $b\backslash x = w$ implies that

$$(R_w,\ R_wL_w^{-1},\ R_w) \in AUT(Q,\cdot) \implies y \cdot zR_wL_w^{-1} = (yR_w^{-1} \cdot z)R_w$$

$$\implies y((zw)L_w^{-1}) = (yR_w^{-1} \cdot z)R_w \implies y(w\backslash(zw)) = (y/w)z \cdot w;\ \forall\ w, y, z \in Q.$$

□

Lemma 4.4.1. Every right isotope of a right Basarab loop $(Q,\cdot)$ satisfies any of the following for all $b, x, y, z \in Q$:

1. $(y \cdot b\backslash x) \cdot b\backslash(((J_{\lambda'}x) \cdot z) \cdot b\backslash x) = (y \cdot z) \cdot (b\backslash x)$
2. $b\backslash((J_{\lambda'}x) \cdot b\backslash x) = e$
3. $(y \cdot b\backslash x) \cdot b\backslash(b\backslash x) = (y \cdot x) \cdot (b\backslash x)$
4. $(y \cdot b\backslash x) \cdot b\backslash(((J_{\lambda'}x) \cdot (J_{\rho'}y)) \cdot b\backslash x) = b\backslash x$
5. $(y \cdot b\backslash x) \cdot b\backslash(((J_{\lambda'}x) \cdot y) \cdot b\backslash x) = y^2 \cdot (b\backslash x)$
6. $(b\backslash x) \cdot b\backslash(((J_{\lambda'}x) \cdot z) \cdot b\backslash x) = z \cdot (b\backslash x)$
7. $((J_{\lambda'}z) \cdot b\backslash x) \cdot b\backslash(((J_{\lambda'}x) \cdot z) \cdot b\backslash x) = (b\backslash x)$
8. $(y \cdot x) \cdot b\backslash(((J_{\lambda'}(bx)) \cdot b\backslash z) \cdot x) = (y \cdot b\backslash z) \cdot x.$

Proof. Let $(Q,\cdot)$ be a right Basarab loop. Replace z by bz in Equation 4.12 to obtain 1. Then 2 to 7 are obtained either by identifying z and y, also by equating each of z and y accordingly in Equation 4.12, while 8 is obtained by replacing x with bx in Equation 4.12. □

Lemma 4.4.2. Every right isotope of a right Basarab loop $(Q,\cdot)$ satisfies any of the following for all $b, x \in Q$:

1. $(R_{(b\backslash x)},\ L_{J_{\lambda'}x}R_{(b\backslash x)}L_b^{-1},\ R_{(b\backslash x)}) \in AUT(Q,\cdot)$

2. $(R_x,\ L_{J_{\lambda'}(bx)}R_xL_b^{-1},\ R_x) \in AUT(Q,\cdot)$

3. $(R_x,\ R_xL_{J_{\lambda'}x}^{-1},\ R_x) \in AUT(Q,\cdot)$

4. $(R_x,\ R_xM_x,\ R_x) \in AUT(Q,\cdot)$

5. $(R_{(b\backslash x)},\ R_{(b\backslash x)}L_{(b\backslash x)}^{-1},\ R_{(b\backslash x)}) \in AUT(Q,\cdot)$

Proof. From (1) of Lemma 4.4.1,

$$yR_{(b\backslash x)} \cdot zL_{J_{\lambda'}x}R_{(b\backslash x)}L_b^{-1} = (yz)R_{(b\backslash x)} \implies (R_{(b\backslash x)},\ L_{J_{\lambda'}x}R_{(b\backslash x)}L_b^{-1},\ R_{(b\backslash x)}) \in AUT(Q,\cdot).$$

Setting $x = bx$ gives $(R_x,\ L_{J_{\lambda'}(bx)}R_{(b\backslash x)}L_b^{-1},\ R_x) \in AUT(Q,\cdot)$. If $b = J_{\lambda'}x$, we obtain $(R_x,\ L_{J_{\lambda'}e'}R_xL_{J_{\lambda'}x}^{-1},\ R_x) \in AUT(Q,\cdot)$. Thus,

$$(R_x,\ L_{J_{\lambda'}e'}R_xL_{J_{\lambda'}x}^{-1},\ R_x) \in AUT(Q,\cdot) \implies (R_x,\ R_xL_{J_{\lambda'}x}^{-1},\ R_x) \in AUT(Q,\cdot).$$

From Lemma 4.4.1(2), $b\backslash((J_{\lambda'}x) \cdot b\backslash x) = e \implies J_{\lambda'}x \cdot b\backslash x = b$. Using the permutation M_x defined for $x \in Q$ as $yM_x = y\backslash x$ we write $b\backslash x$ as bM_x. So that

$$J_{\lambda'}x \cdot b\backslash x = b \implies J_{\lambda'}x \cdot bM_x = b \implies bM_xL_{J_{\lambda'}x} = b \implies M_xL_{J_{\lambda'}x} = I \implies M_x = L_{J_{\lambda'}x}^{-1}$$

for all $x \in Q$. Putting $M_x = L_{J_{\lambda'}x}^{-1}$ in (3) of Lemma 4.4.2, gives $(R_x,\ R_xM_x,\ R_x) \in AUT(Q,\cdot)$. From (6) of Lemma 4.4.1,

$$(b\backslash x) \cdot b\backslash(((J_{\lambda'}x) \cdot z) \cdot b\backslash x) = z \cdot (b\backslash x) \implies L_{J_{\lambda'}}R_{(b\backslash x)L_b^{-1}}L_{(b\backslash x))} = R_{(b\backslash x)}$$

$$\implies L_{J_{\lambda'}x}R_{(b\backslash x)}L_b^{-1}L_{(b\backslash x)} = R_{(b\backslash x)} \implies L_{J_{\lambda'}x}R_{(b\backslash x)}L_b^{-1} = R_{(b\backslash x)}L_{(b\backslash x)}^{-1}.$$

Putting this in (1) of Lemma 4.4.2 gives $(R_{(b\backslash x)},\ R_{(b\backslash x)}L_{(b\backslash x)}^{-1},\ R_{(b\backslash x)}) \in AUT(Q,\cdot)$.

□

Theorem 4.4.2. Every left isotope of a left Basarab loop is an LCC-loop.

Proof. Consider the left Basarab identity

$$(x \star (y \star J_{\rho'} x)) \star (x \star z) = x \star (y \star z)$$

and apply $x \star y = x/a \cdot y$ then

$$(x/a \cdot (y/a \cdot J_{\rho'} x))/a \cdot (x/a \cdot z) = (x/a) \cdot (y/a \cdot z). \tag{4.14}$$

Replace y by ya to get

$$(x/a \cdot (y \cdot J_{\rho'} x))/a \cdot (x/a \cdot z) = (x/a) \cdot (y \cdot z).$$

Setting $z = e$ in Equation 4.14 gives

$$(x/a \cdot (y/a \cdot J_{\rho'} x))/a \cdot (x/a) = (x/a) \cdot (y/a).$$

This implies

$$(x/a \cdot (y/a \cdot J_{\rho'} x))/a = [(x/a) \cdot (y/a)]/(x/a). \tag{4.15}$$

Putting Equation 4.15 into Equation 4.14 gives

$$[[(x/a) \cdot (y/a)]/(x/a)] \cdot (x/a \cdot z) = (x/a) \cdot (y/a \cdot z).$$

Replacing y by ya implies

$$[[(x/a) \cdot y]/(x/a)] \cdot (x/a \cdot z) = (x/a) \cdot (y \cdot z).$$

Therefore, setting $x/a = v$ implies $[(v \cdot y)/v] \cdot (v \cdot z) = v \cdot (y \cdot z)$, which means

$$yL_vR_v^{-1} \cdot zL_v = (yzL_v) \implies (L_vR_v^{-1},\ L_v,\ L_v) \in AUT(Q, \cdot).$$

This gives $yL_vR_v^{-1} \cdot z = (y \cdot zL_v^{-1})L_v,\ \forall\ v, y, z \in Q$

$$\implies (vy)R_v^{-1} \cdot z = (y(v\backslash z))L_v \implies ((vy)/v)z = v \cdot y(v\backslash z);\ \forall\ v, y, z \in Q.$$

□

Lemma 4.4.3. Every left isotope of a left Basarab loop $(Q, \cdot)$ satisfies any of the following for all $a, x, y, z \in Q$:

1. $(x/a \cdot (y \cdot J_{\rho'}x))/a \cdot (x/a \cdot z) = (x/a) \cdot (y \cdot z)$
2. $(x/a \cdot J_{\rho'}x)/a = e$
3. $(x/a)/a \cdot (x/a \cdot z) = (x/a) \cdot (x.z)$
4. $(x/a \cdot ((J_{\lambda'}z) \cdot J_{\rho'}x))/a \cdot (x/a \cdot z) = (x/a)$
5. $(x/a \cdot (z \cdot J_{\rho'}x))/a \cdot (x/a \cdot z) = (x/a) \cdot z^2$
6. $(x/a \cdot (y \cdot J_{\rho'}x))/a \cdot (x/a) = (x/a) \cdot y$
7. $(x/a \cdot (y \cdot J_{\rho'}x))/a \cdot (x/a \cdot J_{\rho'}y) = (x/a)$
8. $(x \cdot (y/a \cdot (J_{\rho'}(xa))))/a \cdot xz = x \cdot (y/a \cdot z)$.

Proof. Let $(Q, \cdot)$ be a left Basarab loop. Replace y by ya in Equation 4.14 to obtain 1. Then 2 to 7 are obtained either by identifying z and y, also by equating each of z and y accordingly in Equation 4.14, and 8 is obtained by replacing x with xa in Equation 4.14. □

Lemma 4.4.4. Every left isotope of a left Basarab loop $(Q,\cdot)$ satisfies any of the following for all $a,x \in Q$:

1. $(R_{J_{\rho'}x}L_{(x/a)}R_a^{-1},\ L_{(x/a)},\ L_{(x/a)}) \in AUT(Q,\cdot)$
2. $(R_{J_{\rho'}(xa)}L_xR_a^{-1},\ L_x,\ L_x) \in AUT(Q,\cdot)$
3. $(L_xR^{-1}_{J_{\rho'}x},\ L_x,\ L_x) \in AUT(Q,\cdot)$
4. $(L_xM_x^{-1},\ L_x,\ L_x) \in AUT(Q,\cdot)$
5. $(L_{(x/a)}R^{-1}_{(x/a)},\ L_{(x/a)},\ L_{(x/a)}) \in AUT(Q,\cdot)$.

Proof. From (1) of Lemma 4.4.3,

$$yR_{J_{\rho'}x}L_{(x/a)}R_a^{-1}\cdot zL_{(x/a)} = (yz)L_{(x/a)} \implies (R_{J_{\rho'}x}L_{(x/a)}R_a^{-1},\ L_{(x/a)},\ L_{(x/a)}) \in AUT(Q,\cdot).$$

Letting $x = xa$ gives $(R_{J_{\rho'}(xa)}L_xR_a^{-1},\ L_x,\ L_x) \in AUT(Q,\cdot)$. If $a = J_{\rho'}x$, then $(R_{J_{\rho'}(xJ_{\rho'}x)}L_xR_a^{-1},\ L_x,\ L_x) \in AUT(Q,\cdot)$. This implies, $(L_xR^{-1}_{J_{\rho'}x},\ L_x,\ L_x) \in AUT(Q,\cdot)$. From (2) of Lemma 4.4.3, $(x/a\cdot J_{\rho'}x)/a = e \implies x/a\cdot J_{\rho'}x = a$. Applying the permutation M_x^{-1} defined for $x\in Q$ as $yM_x^{-1} = x/y$ to obtain $aM_x^{-1} = x/a$. This means,

$$x/a\cdot J_{\rho'}x = a \implies aM_x^{-1}R_{J_{\rho'}x} = a \implies M_x^{-1}R_{J_{\rho'}x} = I$$

$$\implies M_x^{-1} = R^{-1}_{J_{\rho'}x}.$$

Then (3) of Lemma 4.4.3 means $(L_xM_x^{-1},\ L_x,\ L_x) \in AUT(Q,\cdot)$. From (6) of Lemma 4.4.3,

$$(x/a\cdot(y\cdot J_{\rho'}x))/a\cdot(x/a) = (x/a)\cdot y$$

$$\implies R_{J_{\rho'}x}L_{(x/a)}R_a^{-1}R_{(x/a)} = L_{(x/a)} \implies R_{J_{\rho'}x}L_{(x/a)}R_a^{-1} = L_{(x/a)}R^{-1}_{(x/a)}.$$

Putting this in (1) of Lemma 4.4.3 above, gives $(L_{(x/a)}R^{-1}_{(x/a)},\ L_{(x/a)},\ L_{(x/a)}) \in AUT(Q,\cdot)$.

$\square$

Theorem 4.4.3. Every left isotope of a right Basarab loop is a right Basarab loop.

Proof. Consider the right Basarab loop identity

$$(y \star x) \star ((J_{\lambda'}x)z \star x) = (y \star z) \star x$$

and apply $x \star y = x/a \cdot y$ to obtain

$$(y/a \cdot x)/a \cdot ((J_{\lambda'}x/a \cdot z)/a \cdot x) = (y/a \cdot z)/a \cdot x.$$

Replace y by ya to get

$$(y \cdot x)/a \cdot ((J_{\lambda'}x/a \cdot z)/a \cdot x) = (y \cdot z)/a \cdot x. \tag{4.16}$$

Setting $z = e$ in Equation 4.16 gives

$$(y \cdot x)/a \cdot ((J_{\lambda'}x/a)/a \cdot x) = y/a \cdot x.$$

Also, setting $y = e$ in Equation 4.16 yields

$$(x/a) \cdot ((J_{\lambda'}x/a \cdot z)/a \cdot x) = (z/a) \cdot x.$$

Which means

$$((J_{\lambda'}x/a \cdot z)/a \cdot x) = (x/a)\backslash((z/a) \cdot x). \tag{4.17}$$

Putting Equation 4.17 in Equation 4.16 gives

$$(y \cdot x)/a \cdot [(x/a)\backslash((z/a) \cdot x)] = (y \cdot z)/a \cdot x.$$

Then $yR_xR_a^{-1} \cdot zR_a^{-1}R_xL_{(x/a)}^{-1} = (yz)R_a^{-1}R_x \implies (R_xR_a^{-1},\ R_a^{-1}R_xL_{(x/a)},\ R_a^{-1}R_x) \in AUT(Q, \cdot)$. Setting $x = xa$ implies $(R_{xa}R_a^{-1},\ R_a^{-1}R_{xa}L_x^{-1},\ R_a^{-1}R_{xa}) \in AUT(Q, \cdot)$. Next, set $x = J_\lambda a$, then $(R_a^{-1},\ R_a^{-1}L_{J_\lambda a},\ R_a^{-1}) \in AUT(Q, \cdot) \implies (R_a,\ L_{J_\lambda a}R_a,\ R_a) \in AUT(Q, \cdot) \implies \forall\, a, b, c \in Q, (b \cdot a) \cdot ((J_\lambda a)c \cdot a) = bc \cdot a.$ □

Theorem 4.4.4. Every right isotope of a left Basarab loop is a left Basarab loop.

Proof. Consider the left Basarab identity

$$(x \star (y \star J_{\rho'}x)) \star (x \star z) = x \star (y \star z)$$

and apply $x \star y = x \cdot b\backslash y$ then

$$(x \cdot b\backslash(y \cdot b\backslash J_{\rho'}x)) \cdot b\backslash(x \cdot b\backslash z) = x \cdot b\backslash(y \cdot b\backslash z).$$

Replace z by bz to get

$$(x \cdot b\backslash(y \cdot b\backslash J_{\rho'}x)) \cdot b\backslash(x \cdot z) = x \cdot b\backslash(y \cdot z). \tag{4.18}$$

Set $z = e$ in Equation 4.18 and obtain

$$(x \cdot b\backslash(y \cdot b\backslash J_{\rho'}x)) \cdot (b\backslash x) = x \cdot b\backslash y.$$

Which means

$$(x \cdot b\backslash(y \cdot b\backslash J_{\rho'}x)) = (x \cdot b\backslash y)/(b\backslash x). \tag{4.19}$$

Putting Equation 4.19 in Equation 4.18 then

$$(x \cdot b\backslash y)/(b\backslash x) \cdot b\backslash(x \cdot z) = x \cdot b\backslash(y \cdot z). \tag{4.20}$$

Then,

$$yL_b^{-1}L_{bx}R_x^{-1} \cdot zL_{bx}L_b^{-1} = (yz)L_b^{-1}L_{bx} \implies (L_b^{-1}L_{bx}R_x^{-1},\ L_{bx}L_b^{-1},\ L_b^{-1}L_{bx}) \in AUT(Q, \cdot).$$

Setting $x = J_{\rho'}b$ then

$$(L_b^{-1}R_{J_{\rho'}x}^{-1},\ L_b^{-1},\ L_b^{-1}) \in AUT(Q, \cdot) \implies (R_{J_{\rho'}b}L_b,\ L_b,\ L_b) \in AUT(Q, \cdot)$$

$$\implies \forall\ b, c, d \in Q,\ (b \cdot cJ_{\rho'}b) \cdot bd = b \cdot (c \cdot d).$$

□

4.4.1 Principal Isotopes of Basarab loop

Definition 4.4.2. Let $(L, \cdot)$ be a loop. Every principal isotopism is a composition of a left and a right isotopism.

Theorem 4.4.5. Every principal isotope of a right Basarab loop is a right Basarab loop.

Proof. Consider the right Basarab loop identity

$$(y \star x) \star ((J_{\lambda'}x)z \star x) = (y \star z) \star x$$

and apply $x \star y = x \circ b\backslash y$ to obtain

$$(y \circ b\backslash x) \circ b\backslash((J_{\lambda'}x \circ b\backslash z) \circ b\backslash x) = (y \circ b\backslash z) \circ (b\backslash x).$$

Set $b\backslash x = u,\ b\backslash z = w$ then

$$(y \circ u) \circ b\backslash((J_{\lambda'}(bu) \circ w) \circ u) = (y \circ w) \circ u.$$

Apply $x \circ y = x/a \cdot y$ to get

$$(y/a \cdot u) \circ b\backslash((J_{\lambda'}(bu)/a \cdot w) \circ u) = (y/a \cdot w)/a \cdot u.$$

This implies

$$(y/a \cdot u)/a \cdot b\backslash((J_{\lambda'}(bu)/a \cdot w)/a \cdot u) = (y/a \cdot w)/a \cdot u.$$

Set $y/a = s$, to obtain

$$(s \cdot u)/a \cdot b\backslash((J_{\lambda'}(bu)/a \cdot w)/a \cdot u) = (s \cdot w)/a \cdot u. \tag{4.21}$$

Next, set $s = e$ in Equation 4.21 and get

$$(u/a) \cdot b\backslash((J_{\lambda'}(bu)/a \cdot w)/a \cdot u) = (w/a) \cdot u.$$

This implies

$$b\backslash((J_{\lambda'}(bu)/a \cdot w)/a \cdot u) = (u/a)\backslash((w/a) \cdot u). \tag{4.22}$$

Putting Equation 4.22 in Equation 4.21

$$(s \cdot u)/a \cdot [(u/a)\backslash((w/a) \cdot u)] = (s \cdot w)/a \cdot u.$$

This means $sR_uR_a^{-1} \cdot wR_aR_uL_{(u/a)}^{-1} = (sw)R_a^{-1}R_u$

$$\implies (R_uR_a^{-1},\ R_aR_uL_{(u/a)}^{-1},\ R_a^{-1}R_u) \in AUT(Q, \cdot).$$

Setting $u = ua$ gives

$$(R_{ua}R_a^{-1},\ R_aR_{ua}L_u^{-1},\ R_a^{-1}R_{ua}) \in AUT(Q,\cdot).$$

Next, setting $u = J_\lambda a$ gives

$$(R_eR_a^{-1},\ R_a^{-1}R_eL_{J_\lambda a}^{-1},\ R_a^{-1}R_e) \in AUT(Q,\cdot) \implies (R_a^{-1},\ R_a^{-1}L_{J_\lambda a}^{-1},\ R_a^{-1}) \in AUT(Q,\cdot)$$

$$\implies (R_a,\ L_{J_\lambda a}R_a,\ R_a) \in AUT(Q,\cdot) \implies \forall\, a,b,c \in Q, (b\cdot a)\cdot((J_\lambda a)c\cdot a) = (b\cdot c)\cdot a.$$

□

Theorem 4.4.6. Every principal isotope of a left Basarab loop is a left Basarab loop.

Proof. Consider the left Basarab identity

$$(x \star (y \star J_{\rho'}x)) \star (x \star z) = x \star (y \star z)$$

and apply $x \star y = x/a \circ y$ to get

$$(x/a \circ (y/a \circ J_{\rho'}x))/a \circ (x/a \circ z) = (x/a) \circ (y/a \circ z).$$

Set $x/a = g$, and $y/a = h$ to obtain

$$(g \circ (h \circ J_{\rho'}(ga)))/a \circ (g \circ z) = g \circ (h \circ z).$$

Apply $x \circ y = x \cdot b\backslash y$ to get

$$(g \cdot b\backslash(h \circ J_{\rho'}(ga)))/a \cdot b\backslash(g \circ z) = g \cdot b\backslash(h \circ z)$$

This implies

$$(g \cdot b\backslash(h \cdot b\backslash J_{\rho'}(ga)))/a \cdot b\backslash(g \cdot b\backslash z) = g \cdot b\backslash(h \cdot b\backslash z).$$

Set $q = b\backslash z$ to obtain

$$(g \cdot b\backslash(h \cdot b\backslash J_{\rho'}(ga)))/a \cdot b\backslash(g \cdot q) = g \cdot b\backslash(h \cdot q) \tag{4.23}$$

Set $q = e$ to obtain

$$(g \cdot b\backslash(h \cdot b\backslash J_{\rho'}(ga)))/a \cdot b\backslash g = g \cdot b\backslash h.$$

This implies

$$(g \cdot b\backslash(h \cdot b\backslash J_{\rho'}(ga)))/a = (g \cdot b\backslash h)/(b\backslash g) \tag{4.24}$$

Putting Equation 4.24 into Equation 4.23 to get

$$[(g \cdot b\backslash h)/(b\backslash g)] \cdot b\backslash(g \cdot q) = g \cdot b\backslash(h \cdot q).$$

This means, $hL_b^{-1}L_gR_{(b\backslash g)}^{-1} \cdot qL_gL_b^{-1} = (h \cdot q)L_b^{-1}L_g$

$$\implies (L_b^{-1}L_gR_{(b\backslash g)}^{-1},\ L_gL_b^{-1},\ L_b^{-1}L_g) \in AUT(Q, \cdot).$$

Putting $bg = g$ gives $(L_b^{-1}L_{bg}R_g^{-1},\ L_{bg}L_b^{-1},\ L_b^{-1}L_{bg}) \in AUT(Q, \cdot)$. Next, setting $g = J_{\rho'}b$ gives

$$(L_b^{-1}L_eR_{J_{\rho'}b}^{-1},\ L_eL_b^{-1},\ L_b^{-1}L_e) \in AUT(Q, \cdot)$$

$$\implies (L_b^{-1}R_{J_{\rho'}b}^{-1},\ L_b^{-1},\ L_b^{-1}) \in AUT(Q, \cdot) \implies (R_{J_{\rho'}b}L_b,\ L_b,\ L_b) \in AUT(Q, \cdot)$$

$$\implies \forall\, b, c, d \in Q,\ (b \cdot cJ_{\rho'}b) \cdot (b \cdot d) = b \cdot (c \cdot d).$$

□

Corollary 4.4.1. Every principal isotope of a Basarab loop is a Basarab loop.

Proof. This result follows from Theorems 4.4.5 and 4.4.6. □

Lemma 4.4.5. Let $(Q,\cdot)$ be a right Basarab loop. Then the principal isotope of $(Q,\cdot)$ satisfies any of the following for all $a,b,s,u,w \in Q$:

1. $(su)/a \cdot b\backslash((J_{\lambda'}(bu)/a)/a \cdot u) = (s/a)\cdot u$
2. $(s\cdot u)/a \cdot b\backslash((J_{\lambda'}(bu)/a \cdot J_\rho s)/a \cdot u) = /a \cdot u$
3. $(s\cdot u)/a \cdot b\backslash((J_{\lambda'}(bu)/a \cdot s)/a \cdot u) = s^2/a \cdot u$
4. $(s\cdot u)/a \cdot b\backslash((J_{\lambda'}(bu)/a \cdot J_\rho[J_{\lambda'}(bu)/a])/a \cdot u) = (s \cdot J_\rho[J_{\lambda'}(bu)/a])/a \cdot u$
5. $u^2/a \cdot b\backslash((J_{\lambda'}(bu)/a \cdot w)/a \cdot u) = (uw)/a \cdot u$
6. $u/a \cdot b\backslash((J_{\lambda'}(bu)/a \cdot w)/a \cdot u) = (w/a)\cdot u$
7. $/a \cdot b\backslash((J_{\lambda'}(bu)/a \cdot w)/a \cdot u) = (J_\lambda u \cdot w)/a \cdot u$
8. $(J_\lambda w \cdot u)/a \cdot b\backslash((J_{\lambda'}(bu)/a \cdot w)/a \cdot u) = /a \cdot u$

Proof. The lemma is obtained by identifying s and w and also by equating s and w accordingly using Equation 4.21. □

Lemma 4.4.6. Let $(Q,\cdot)$ be a left Basarab loop. Then the principal isotope of $(Q,\cdot)$ satisfies any of the following for all $a,b,g,h,q \in Q$:

1. $(g \cdot b\backslash(b\backslash J_{\rho'}(ga)))/a \cdot b\backslash(g\cdot q) = g \cdot b\backslash q$
2. $(g \cdot b\backslash(J_\lambda q \cdot b\backslash J_{\rho'}(ga)))/a \cdot b\backslash(g\cdot q) = g \cdot b\backslash$
3. $(g \cdot b\backslash(q \cdot b\backslash J_{\rho'}(ga)))/a \cdot b\backslash(g\cdot q) = g \cdot b\backslash q^2$
4. $(g \cdot b\backslash)/a \cdot b\backslash(g\cdot q) = g \cdot b\backslash(J_\lambda[b\backslash J_{\rho'}(ga)]\cdot q)$

5. $(g \cdot b\backslash(h \cdot b\backslash J_{\rho'}(ga)))/a \cdot (b\backslash g) = g \cdot (b\backslash h)$

6. $(g \cdot b\backslash(h \cdot b\backslash J_{\rho'}(ga)))/a \cdot b\backslash(g \cdot J_\rho h) = g \cdot b\backslash$

7. $(g \cdot b\backslash(h \cdot b\backslash J_{\rho'}(ga)))/a \cdot b\backslash = g \cdot b\backslash(h \cdot J_\rho g)$

8. $(g \cdot b\backslash(h \cdot b\backslash J_{\rho'}(ga)))/a \cdot b\backslash g^2 = g \cdot b\backslash(h \cdot g)$

Proof. The lemma is obtained by identifying h and q and also by equating h and q accordingly using Equation 4.23. □

Lemma 4.4.7. Every principal isotope of a right Basarab loop $(Q, \cdot)$ satisfies any of the following for all $a, b, u \in Q$:

1. $(R_uR_a^{-1},\ L_{J_{\lambda'}(bu)/a}R_a^{-1}R_uL_b^{-1},\ R_a^{-1}R_u) \in AUT(Q, \cdot)$

2. $L_{J_{\lambda'}(bu)/a}R_a^{-1}R_uL_b^{-1}L_{(u^2/a)} = L_uR_a^{-1}R_u$

3. $(R_uR_a^{-1},\ L_uR_a^{-1}R_uL_{(u^2/a)},\ R_a^{-1}R_u) \in AUT(Q, \cdot)$

4. $L_{J_{\lambda'}(bu)/a}R_a^{-1}R_uL_b^{-1} = R_a^{-1}R_uL_{(u/a)}^{-1}$

5. $(R_uR_a^{-1},\ R_a^{-1}R_uL_{(u/a)}^{-1},\ R_a^{-1}R_u) \in AUT(Q, \cdot)$

6. $L_{J_{\lambda'}(bu)/a}R_a^{-1}R_uL_b^{-1} = L_{J_\lambda u}R_a^{-1}R_uL_{/a}^{-1}.$

Proof. From (3) of Lemma 4.4.5, $sR_uR_a^{-1} \cdot sL_{J_{\lambda'}(bu)/a}R_a^{-1}R_uL_b^{-1} = s^2R_a^{-1}R_u$

$$\Longrightarrow (R_uR_a^{-1},\ L_{J_{\lambda'}(bu)/a}R_a^{-1}R_uL_b^{-1},\ R_a^{-1}R_u) \in AUT(Q, \cdot).$$

From (5) of Lemma 4.4.5, (2) is obtained. Next, using (2) in (1), (3) is achieved. Also, (4) follows from (5) of Lemma 4.4.5, (5) is obtained by considering (4) in (1), and (6) is obtained using (7) of Lemma 4.4.5. □

Lemma 4.4.8. Every principal isotope of a left Basarab loop $(Q,\cdot)$ satisfies any of the following for all $a,b,g\in Q$:

1. $(R_{b\backslash J_{\rho'}(ga)}L_b^{-1}L_gR_a^{-1},\ L_gL_b^{-1},\ L_b^{-1}L_g)\in AUT(Q,\cdot)$

2. $R_{b\backslash J_{\rho'}(ga)}L_b^{-1}L_gR_a^{-1}=L_b^{-1}L_gR_{(b\backslash g)}^{-1}$

3. $(L_b^{-1}L_gR_{(b\backslash g)}^{-1},\ L_gL_b^{-1},\ L_b^{-1}L_g)\in AUT(Q,\cdot)$

4. $R_{b\backslash J_{\rho'}(ga)}L_b^{-1}L_gR_a^{-1}=R_{J_\rho g}L_b^{-1}L_gR_{b\backslash}$

5. $R_{b\backslash J_{\rho'}(ga)}L_b^{-1}L_gR_a^{-1}=R_gL_b^{-1}L_gR_{(b\backslash g^2)}$

6. $(R_gL_b^{-1}L_gR_{(b\backslash g^2)},\ L_gL_b^{-1},\ L_b^{-1}L_g)\in AUT(Q,\cdot)$

Proof. From (3) of Lemma 4.4.6, (1) is obtained. From (5) of Lemma 4.4.6, (2) is obtained. Next, using (2) in (1), (3) is achieved. Also, (4) follows from (7) of Lemma 4.4.6, (5) is obtained by using (8) of Lemma (4.4.6), and (6) is obtained by considering (5) in (1). □

4.5 Holomorphy of a Basarab loop

In this section, the holomorphy of a Basarab loop is investigated by considering a group $A(Q)$ of automorphisms of a loop. Some necessary and sufficient conditions for an $A(Q)$-holomorph of a loop $(Q,\cdot)$ to be left (right) Basarab loop, and Basarab loop are established, respectively. These necessary and sufficient conditions are also expressed in terms of autotopisms of a loop. Some left (right) translation mapping of the holomorph of a left (right) Basarab loop is shown to be left (right) regular. It is proved that $A(Q)$ -holomorph of a loop $(Q,\cdot)$ which satisfies the inverse property is a Basarab loop if and only if $(Q,\cdot)$ is a Basarab loop and every automorphism of Q is nuclear.

4.5.1 Holomorphy of left Basarab loop

Theorem 4.5.1. *Let $(Q,\cdot)$ be a loop with $A(Q)$-holomorph $(H,\circ)$. Then, $(H,\circ)$ is a left Basarab loop if and only if*

$$(x\alpha\cdot yx^{\rho})\cdot xz = x\alpha\cdot yz$$

for all $x,y,z\in Q$ and for all $\alpha\in A(Q)$.

Proof. Let $(Q,\cdot)$ be a loop and $A(Q)\le AUM(Q,\cdot)$. Let $H=A(Q)\times Q$ such that $(\alpha,x)\circ(\beta,y)=(\alpha\beta,x\beta\cdot y)$ for all $(\alpha,x),(\beta,y)\in H$. Clearly,

$$(\alpha,x)^{\rho}=(\alpha^{-1},x^{\rho}\alpha^{-1})=(\alpha^{-1},(x\alpha^{-1})^{\rho}).\quad \text{Therefore,}$$

$$\{(\alpha,x)\circ((\beta,y)\circ(\alpha,x)^{\rho}\}\circ((\alpha,x)\circ(\gamma,z))=(\alpha,x)\circ((\beta,y)\circ(\gamma,z))$$

$$\iff \{(\alpha,x)\circ((\beta,y)\circ(\alpha^{-1},x^{\rho}\alpha^{-1})\}\circ((\alpha,x)\circ(\gamma,z))=(\alpha,x)\circ((\beta,y)\circ(\gamma,z))$$

$$\iff \{(\alpha,x)\circ(\beta\alpha^{-1},y\alpha^{-1}\cdot x^{\rho}\alpha^{-1})\}\circ(\alpha\gamma,x\gamma\cdot z)=(\alpha,x)\circ(\beta\gamma,y\gamma\cdot z)$$

$$\iff \big(\alpha(\beta\alpha^{-1}),x\beta\alpha^{-1}\cdot(y\alpha^{-1}\cdot x^{\rho}\alpha^{-1})\big)\circ(\alpha\gamma,x\gamma\cdot z)=(\alpha(\beta\gamma),x\beta\gamma\cdot(y\gamma\cdot z))$$

$$\iff \big((\alpha(\beta\alpha^{-1}))\alpha\gamma,\big(x\beta\alpha^{-1}\cdot(y\alpha^{-1}\cdot x^{\rho}\alpha^{-1})\big)\,\alpha\gamma\cdot(x\alpha\cdot z)\big)=(\alpha\beta\gamma,x\beta\gamma\cdot(y\gamma\cdot z))$$

$$\iff \big(\alpha\beta\gamma,(x\beta\alpha^{-1}\cdot(y\alpha^{-1}\cdot x^{\rho}\alpha^{-1}))\alpha\gamma\cdot(x\alpha\cdot z)\big)=(\alpha\beta\gamma,x\beta\gamma\cdot(y\gamma\cdot z))$$

$$\iff \big((x\beta\alpha^{-1}\cdot(y\alpha^{-1}\cdot x^{\rho}\alpha^{-1}))\alpha\gamma\big)\cdot(x\alpha\cdot z)=x\beta\gamma\cdot(y\gamma\cdot z)$$

$$\iff \big((x\beta\alpha^{-1}\alpha\gamma)\cdot(y\alpha^{-1}\cdot x^{\rho}\alpha^{-1})\alpha\gamma\big)\cdot(x\gamma\cdot z)=x\beta\gamma\cdot(y\gamma\cdot z)$$

$$\iff \big((x\beta\alpha^{-1})\cdot(y\alpha^{-1}\cdot x^{\rho}\alpha^{-1})\alpha\gamma\big)\cdot(x\alpha)\cdot z)=x\beta\gamma\cdot(y\gamma\cdot z)$$

$$\iff \big((x\beta\gamma)\cdot(y\alpha^{-1}\alpha\gamma\cdot x^{\rho}\alpha^{-1}\alpha\gamma)\big)\cdot(x\gamma\cdot z)=x\beta\gamma\cdot(y\gamma\cdot z)$$

$$\iff ((x\beta\gamma)\cdot(y\gamma\cdot x^{\rho}\gamma))\cdot(x\gamma\cdot z) = x\beta\gamma\cdot(y\gamma\cdot z)$$

$$\iff ((x\beta\gamma)\cdot(y\gamma\cdot x^{\rho}\gamma))\,\gamma^{-1}\cdot(x\gamma\cdot z)\gamma^{-1} = (x\beta\gamma\cdot(y\gamma\cdot z))\gamma^{-1} \iff$$

$$(x\beta\cdot(y\cdot x^{\rho}))\cdot(x\cdot z\gamma^{-1}) = x\beta\cdot(y\cdot z\gamma^{-1}) \tag{4.25}$$

Setting $\bar{z} = z\gamma^{-1}$ in Equation (4.25) gives

$$(x\beta\cdot(y\cdot x^{\rho}))\cdot(x\cdot\bar{z}) = x\beta\cdot(y\cdot\bar{z}) \tag{4.26}$$

In Equation (4.26), setting $\bar{z} = z$ and $\beta = \alpha$ gives

$$(x\alpha\cdot yx^{\rho})\cdot xz = x\alpha\cdot yz$$

for all $x, y, z \in Q$, and $\alpha \in A(Q)$. □

Corollary 4.5.1. *Let $(Q,\cdot)$ be a loop with $A(Q)$-holomorph $(H,\circ)$. Then, $(H,\circ)$ is a left Basarab loop if and only if $(R_{x^{\rho}}L_{x\alpha}, L_x, L_{x\alpha}) \in AUT(Q,\cdot)$ for all $x \in Q$ and for all $\alpha \in A(Q)$.*

Proof. This follows from Theorem 4.5.1. □

Lemma 4.5.1. *Let $(Q,\cdot)$ be a left Basarab loop with $A(Q)$-holomorph $(H,\circ)$. Then, $(H,\circ)$ is a left Basarab loop if and only if any of the following is true:*

1. *$(x\alpha\cdot(x\backslash y))\cdot z = x\alpha\cdot(x\backslash(yz))$ for all $x, y, z \in Q$ and for all $\alpha \in A(Q)$.*
2. *$(L_x^{-1}L_{x\alpha}, I, L_x^{-1}L_{x\alpha}) \in AUT(Q,\cdot)$ for all $x \in Q$ and for all $\alpha \in A(Q)$.*
3. *$L_x^{-1}L_{x\alpha}$ is λ-regular for all $x \in Q$ and for all $\alpha \in A(Q)$.*
4. *$L_x^{-1}L_{x\alpha} \in \Lambda(Q,\cdot)$ for all $x \in Q$ and for all $\alpha \in A(Q)$.*

Proof. Let $(Q,\cdot)$ be a left Basarab loop. Then the autotopism $T = (R_{x^{\rho}}L_x, L_x, L_x)$ holds.

$(H,\circ)$ is a left Basarab loop if and only if $S=(R_{x^\rho}L_{x\alpha},L_x,L_{x\alpha})\in AUT(Q,\cdot)$.

$$T=(R_{x^\rho}L_x,L_x,L_x)\in AUT(Q,\cdot)\implies T^{-1}=(L_x^{-1}R_{x^\rho}^{-1},L_x^{-1},L_x^{-1})\in AUT(Q,\cdot).$$

$(H,\circ)$ is a left Basarab loop if and only if

$$\begin{aligned}T^{-1}S&=(L_x^{-1}R_{x^\rho}^{-1},L_x^{-1},L_x^{-1})(R_{x^\rho}L_{x\alpha},L_x,L_{x\alpha})\\&=(L_x^{-1}R_{x^\rho}^{-1}R_{x^\rho}L_{x\alpha},L_x^{-1}L_x,L_x^{-1}L_{x\alpha})=(L_x^{-1}L_{x\alpha},I,L_x^{-1}L_{x\alpha})\in AUT(Q,\cdot).\end{aligned}$$

Also, for all $x,y,z\in Q$ and for all $\alpha\in A(Q)$,

$$(L_x^{-1}L_{x\alpha},I,L_x^{-1}L_{x\alpha})\in AUT(Q,\cdot)\iff yL_x^{-1}L_{x\alpha}\cdot z=(yz)L_x^{-1}L_{x\alpha}$$

$$\iff (x\alpha\cdot(x\backslash y))\cdot z=x\alpha\cdot(x\backslash(yz)).$$

The rest follows by Definition 3.6.1(3). □

Corollary 4.5.2. *Let $(Q,\cdot)$ be a loop with $A(Q)$-holomorph $(H,\circ)$. Then, $(H,\circ)$ is a left Basarab loop if and only if $(Q,\cdot)$ is a left Basarab loop and any of the following equivalent conditions holds:*

1. $L_x^{-1}L_{x\alpha}\in\Lambda(Q,\cdot)$ *for all $x\in Q$ and for all $\alpha\in A(Q)$.*
2. $x\alpha\cdot x^\rho\in N_\lambda(Q,\cdot)$ *for all $x\in Q$ and for all $\alpha\in A(Q)$.*
3. $\alpha=J_\rho M_n^{-1}$ *for all $\alpha\in A(Q)$ and some $n\in N_\lambda(Q,\cdot)$.*
4. $A(Q)\le\Lambda(Q,\cdot)$.

Proof. Let $(H,\circ)$ be a left Basarab loop, then $\{I\}\times Q\le(H,\circ)$ and $\{I\}\times Q\cong(Q,\cdot)$. Hence, $(Q,\cdot)$ is a left Basarab loop. Thus, by Lemma 4.5.1, $L_x^{-1}L_{x\alpha}\in\Lambda(Q,\cdot)$ for all $x\in Q$ and for all $\alpha\in A(Q)$.

Conversely, if $(Q,\cdot)$ is a left Basarab loop and 1. holds, then by Lemma 4.5.1, $(H,\circ)$ is a left Basarab loop.

1 $\iff$ 2 By Theorem 3.1.1, $L_x^{-1}L_{x\alpha}\overset{\vartheta}{\cong}eL_x^{-1}L_{x\alpha}\iff eL_x^{-1}L_{x\alpha}\in N_\lambda(Q,\cdot)\iff x\alpha\cdot x^\rho\in N_\lambda(Q,\cdot)$ for all $x\in Q$ and for all $\alpha\in A(Q)$.

2 $\iff$ 3 $x\alpha\cdot x^\rho\in N_\lambda(Q,\cdot)\iff x\alpha\cdot x^\rho=n\iff x\alpha=nR_{x^\rho}^{-1}=x^\rho M_n^{-1}\iff\alpha=J_\rho M_n^{-1}$ for all $\alpha\in A(Q)$ and some $n\in N_\lambda(Q,\cdot)$.

1 $\iff$ 4 For all $x\in Q$ and for all $\alpha\in A(Q)$, $L_x^{-1}L_{x\alpha}\in\Lambda(Q,\cdot)\Leftrightarrow L_{x\alpha}\in L_x\Lambda(Q,\cdot)\iff L_{x\alpha}=L_x\lambda$ for some $\lambda\in\Lambda(Q,\cdot)\iff(\alpha,I,\lambda)\in AUT(Q,\cdot)\iff\alpha\in\Lambda(Q,\cdot)\iff A(Q)\le\Lambda(Q,\cdot)$.

□

Remark 4.5.1. *From Corollary 4.5.2, it can be deduced that the $\Lambda(Q,\cdot)$-holomorph $H(Q,\cdot)$ of a left Basarab loop $(Q,\cdot)$ is precisely a left Basarab loop.*

Corollary 4.5.3. *Let $(Q,\cdot)$ be a loop with $A(Q)$-holomorph $(H,\circ)$. Then, $(H,\circ)$ is an LIP left Basarab loop if and only if $(Q,\cdot)$ is an LIP left Basarab loop and any of the following equivalent conditions holds:*

1. *$JL_x^{-1}L_{x\alpha}J$ is μ-regular with an adjoint $(JL_x^{-1}L_{x\alpha}J)'=L_{x\alpha}^{-1}L_x$ for all $x\in Q$ and for all $\alpha\in A(Q)$.*

2. *$JL_x^{-1}L_{x\alpha}J\in\Phi(Q,\cdot)$ and $(JL_x^{-1}L_{x\alpha}J)'=L_{x\alpha}^{-1}L_x\in\Psi(Q,\cdot)$ for all $x\in Q$ and for all $\alpha\in A(Q)$.*

3. *$x\alpha\cdot x^\rho,\ x\cdot x^\rho\alpha\in N_\mu(Q,\cdot)$ for all $x\in Q$ and for all $\alpha\in A(Q)$.*

4. *$\alpha=J_\rho M_{n_1}^{-1}=\alpha=M_{n_2}J_\lambda$ for all $\alpha\in A(Q)$ and some $n_1,n_2\in N_\mu(Q,\cdot)$.*

5. *$A(Q)\subseteq J\Phi(Q,\cdot)J$ and $(JA(Q)J)'\subseteq J\Phi(Q,\cdot)J\subseteq\Psi(Q,\cdot)$.*

Proof. It should first be noted that $(H,\circ)$ is an LIP loop if and only if $(Q,\cdot)$ is an LIP. Following Corollary 4.5.3, $(H,\circ)$ is a left Basarab loop if and only if $(Q,\cdot)$ is a left Basarab loop and $L_x^{-1}L_{x\alpha}\in\Lambda(Q,\cdot)$ for all $x\in Q$ and for all $\alpha\in A(Q)$.

In an LIPL, $(U,V,W)\in AUT(Q,\cdot)\implies (JUJ,W,U)\in AUT(Q,\cdot)$. So, for all $x\in Q$ and for all $\alpha\in A(Q)$:

$$L_x^{-1}L_{x\alpha}\in\Lambda(Q,\cdot)\Leftrightarrow(L_x^{-1}L_{x\alpha},I,L_x^{-1}L_{x\alpha})\in AUT(Q,\cdot)\iff\big(JL_x^{-1}L_{x\alpha}J,L_x^{-1}L_{x\alpha},I\big)\in AUT(Q,\cdot)$$

$\iff JL_x^{-1}L_{x\alpha}J$ is μ-regular with an adjoint $(JL_x^{-1}L_{x\alpha}J)'=L_{x\alpha}^{-1}L_x$.

1 $\iff$ 2 This is straightforward.

2 $\iff$ 3 By Theorem 3.1.1, $JL_x^{-1}L_{x\alpha}J\overset{\sigma}{\approxeq}eJL_x^{-1}L_{x\alpha}J\iff eJL_x^{-1}L_{x\alpha}J\in N_\mu(Q,\cdot)\iff$ $x\alpha\cdot x^\rho\in N_\mu(Q,\cdot)$ for all $x\in Q$ and for all $\alpha\in A(Q)$.

Furthermore, $L_{x\alpha}^{-1}L_x\overset{\beta}{\approxeq}eL_{x\alpha}^{-1}L_x\iff eL_{x\alpha}^{-1}L_x\in N_\mu(Q,\cdot)\iff x\cdot x^\rho\alpha\in N_\mu(Q,\cdot)$ for all $x\in Q$ and for all $\alpha\in A(Q)$.

3 $\iff$ 4 $x\alpha\cdot x^\rho\in N_\mu(Q,\cdot)\iff x\alpha\cdot x^\rho=n_1\iff x\alpha=n_1R_{x^\rho}^{-1}=x^\rho M_{n_1}^{-1}\iff\alpha=$ $J_\rho M_{n_1}^{-1}$ for all $\alpha\in A(Q)$ and some $n_1\in N_\mu(Q,\cdot)$.

Furthermore, $x\cdot x^\rho\alpha\in N_\mu(Q,\cdot)\iff x\cdot x^\rho\alpha=n_2\Leftrightarrow x^\rho\alpha=n_2L_x^{-1}=xM_{n_2}\Leftrightarrow\alpha=$ $M_{n_2}J_\lambda$ for all $\alpha\in A(Q)$ and some $n_2\in N_\mu(Q,\cdot)$.

1 $\iff$ 5 For all $x\in Q$ and for all $\alpha\in A(Q)$, $JL_x^{-1}L_{x\alpha}J\in\Phi(Q,\cdot)\iff JL_x^{-1}L_{x\alpha}J=\phi$ for some $\phi\in\Phi(Q,\cdot)\iff L_x^{-1}L_{x\alpha}=J\phi J\iff L_{x\alpha}=L_xJ\phi J\iff$ $(\alpha,I,J\phi J)\in AUT(Q,\cdot)\iff(J\alpha J,J\phi J,I)\in AUT(Q,\cdot)\iff J\alpha J\in\Phi(Q,\cdot)$ and $(J\alpha J)'=(J\psi J)=J\psi^{-1}J\in\Psi(Q,\cdot)$ for some $\psi\in\Psi(Q,\cdot)\Leftrightarrow A(Q)\subseteq J\Phi(Q,\cdot)J$ and $(JA(Q)J)'\subseteq J\Phi(Q,\cdot)J\subseteq\Psi(Q,\cdot)$.

Furthermore, for all $x\in Q$ and for all $\alpha\in A(Q)$, $(JL_x^{-1}L_{x\alpha}J)'=L_{x\alpha}^{-1}L_x\in$ $\Psi(Q,\cdot)\iff L_x\in L_{x\alpha}\Psi(Q,\cdot)\iff L_x\in L_{x\alpha}\psi$ for some $\psi\in\Psi(Q,\cdot)\iff$

$(\alpha, I, \psi^{-1}) \in AUT(Q,\cdot) \iff (J\alpha J, I, \psi^{-1}) \in AUT(Q,\cdot) \iff J\alpha J \in \Phi(Q,\cdot)$ and $(J\alpha J)' = \psi \in \Psi(Q,\cdot) \iff A(Q) \subseteq J\Phi(Q,\cdot)J$ and $(JA(Q)J)' \subseteq \Psi(Q,\cdot)$.

$\square$

Corollary 4.5.4. *Let $(Q,\cdot)$ be a loop with $A(Q)$-holomorph $(H,\circ)$. If $(H,\circ)$ is an LIP left Basarab loop, then*

$$\begin{array}{ccc} & & x\alpha\cdot x^{\rho},\ x\cdot x^{\rho}\alpha \\ & \overset{\sigma}{\nearrow} & \uparrow\beta \\ JL_x^{-1}L_{x\alpha}J & \xrightarrow[\text{isomorphism}]{\varphi} & L_{x\alpha}^{-1}L_x \end{array} \quad \in \quad \begin{array}{ccc} & & N_{\mu}(Q,\cdot) \\ & \overset{\sigma}{\nearrow} & \uparrow\beta\ \text{isomorphism} \\ \Phi(Q,\cdot) & \xrightarrow[\text{isomorphism}]{\varphi} & \Psi(Q,\cdot) \end{array} \tag{4.27}$$

for all $x \in Q,\ \alpha \in A(Q)$. i.e. $\sigma = \varphi\beta$.

Proof. This follows from Corollary 4.5.3. $\square$

Lemma 4.5.2. *Let $A(Q)$ be an automorphism group of a left Basarab loop $(Q,\cdot)$. The holomorph $(H,\circ)$ of $(Q,\cdot)$ is a left Basarab loop if and only if for all $x, y, z \in Q$ and $\alpha \in A(Q)$ any of the following is satisfied:*

1. $(R_{x^{\rho}}L_xR_{x^{\rho}}L_{x\alpha}, L_x^2, L_xL_{x\alpha}) \in AUT(Q,\cdot)$
2. $(R_{x^{\rho}}L_{x\alpha}R_{x^{\rho}}L_x, L_x^2, L_{x\alpha}L_x) \in AUT(Q,\cdot)$
3. $(R_{x^{\rho}}L_{x\alpha}L_x^{-1}R_{x^{\rho}}^{-1}, I, L_{x\alpha}L_x^{-1}) \in AUT(Q,\cdot)$

Proof. Let $(Q,\cdot)$ be a left Basarab loop. By applying the autotopisms of $(Q,\cdot)$ and its holomorph, and also using Corollary 4.5.1 and Lemma 4.5.2, the result follows. $\square$

Lemma 4.5.3. *Let $(Q,\cdot)$ be a loop with $A(Q)$-holomorph $(H,\circ)$. If $(H,\circ)$ is a left Basarab loop then the following are true for all $x, z \in Q$ and $\alpha \in A(Q)$:*

1. $(x\alpha\cdot x^{\rho})\cdot z = x\alpha\cdot(x\backslash z)$; 2. $x\alpha\cdot x^{\rho} \in N_{\lambda}(Q,\cdot)$; 3. $(x\alpha\cdot x^{\rho})\cdot xz = x\alpha\cdot z$;

4. $(x\alpha \cdot x^\rho)x = x\alpha$;

5. $(x\alpha \cdot x^\rho, x, z) = e$;

6. $L_{(x\alpha \cdot x^\rho)} = L_x^{-1}L_{x\alpha}$;

7. $L_{(x\alpha \cdot x^\rho)} \in \Lambda(Q, \cdot)$.

Proof. 1. By Lemma 4.5.1, $(H, \circ)$ is a left Basarab loop implies

$$(L_x^{-1}L_{x\alpha}, I, L_x^{-1}L_{x\alpha}) \in AUT(Q, \cdot) \implies yL_x^{-1}L_{x\alpha} \cdot z = (yz)L_x^{-1}L_{x\alpha} \forall\, y, z \in Q.$$

By Corollary 4.5.2, $(H, \circ)$ is a left Basarab loop implies that $(Q, \cdot)$ a left Basarab loop. So, $R_{x^\rho} = L_x R_x^{-1} L_x^{-1}$ which implies $L_x^{-1} = R_x L_x^{-1} R_{x^\rho}$. Thus, for all $y, z \in Q$, $yR_x L_x^{-1} R_{x^\rho} L_{x\alpha} \cdot zI = (yz)L_x^{-1}L_{x\alpha}$ if and only if

$$yT_x R_{x^\rho} L_{x\alpha} \cdot zI = (yz)L_x^{-1}L_{x\alpha}, \forall\, x, y, z \in Q. \tag{4.28}$$

Setting $y = e$ in Equation (4.28), then $eT_x R_{x^\rho} L_{x\alpha} \cdot zI = (ez)L_x^{-1}L_{x\alpha} \implies eR_{x^\rho} L_{x\alpha} \cdot zI = zL_x^{-1}L_{x\alpha} \implies (x\alpha \cdot x^\rho) \cdot z = x\alpha \cdot (x\backslash z)$.

2. From 1, $(x\alpha \cdot x^\rho) \cdot z = x\alpha \cdot (x\backslash z) \iff zL_{(x\alpha \cdot x^\rho)} = zL_x^{-1}L_{x\alpha} \iff L_{(x\alpha \cdot x^\rho)} = L_x^{-1}L_{x\alpha}$. So,

$$\big(L_x^{-1}L_{x\alpha}, I, L_x^{-1}L_{x\alpha}\big) = (L_{(x\alpha \cdot x^\rho)}, I, L_{(x\alpha \cdot x^\rho)}) \in AUT(Q, \cdot) \implies x\alpha \cdot x^\rho \in N_\lambda(Q, \cdot).$$

3. In 1 above, $(x\alpha \cdot x^\rho) \cdot z = x\alpha \cdot (x\backslash z)$, setting $z = xz$ gives $(x\alpha \cdot x^\rho) \cdot xz = x\alpha \cdot z$.

4. If $z = x$ in 3 above, then $(x\alpha \cdot x^\rho)x = x\alpha$.

5. From 3 and 4 above , $(x\alpha \cdot x^\rho) \cdot xz = x\alpha \cdot z$ and $(x\alpha \cdot x^\rho)x \cdot z = x\alpha \cdot z$ respectively. Using these, we get $(x\alpha \cdot x^\rho) \cdot xz = x\alpha \cdot z = (x\alpha \cdot x^\rho)x \cdot z \Rightarrow (x\alpha \cdot x^\rho, x, z) = e$.

6. This is gotten from the proof of 2.

7. This is gotten from the proof of 2.

$\square$

4.5.2 Holomorphy of right Basarab loop

Theorem 4.5.2. *Let $(Q,\cdot)$ be a loop with $A(Q)$-holomorph $(H,\circ)$. Then, $(H,\circ)$ is a right Basarab loop if and only if for all $x,y,z\in Q$ and $\delta\in A(Q)$,*

$$(y\cdot x\delta)\cdot((x^{\lambda}\delta\cdot z)\cdot x)=yz\cdot x \quad \textit{or} \quad (y\cdot x)\cdot((x^{\lambda}\cdot z)\cdot x\delta^{-1})=(y\cdot z)\cdot x\delta^{-1}$$

Proof. Let $(Q,\cdot)$ be a loop and $A(Q)\le AUM(Q,\cdot)$. Let $H=A(Q)\times Q$ such that $(\alpha,x)\circ(\beta,y)=(\alpha\beta,x\beta\cdot y)$, for all $(\alpha,x),(\beta,y)\in H$. Clearly,

$$(\beta,y)^{\lambda}=(\beta^{-1},y^{\lambda}\beta^{-1})=(\beta^{-1},(y\beta^{-1})^{\lambda}).\quad \text{Then,}$$

$$((\beta,y)\circ(\alpha,x))\cdot\{((\alpha,x)^{\lambda}\circ(\gamma,z))\circ(\alpha,x)\}=((\beta,y)\circ(\gamma,z))\circ(\alpha,x)$$

$$\iff ((\beta,y)\circ(\alpha,x))\circ\{((\alpha^{-1},x^{\lambda}\alpha^{-1})\circ(\gamma,z))\circ(\alpha,x)\}=((\beta,y)\circ(\gamma,z))\circ(\alpha,x)$$

$$\iff (\beta\alpha,y\alpha\cdot x)\circ\{(\alpha^{-1}\gamma,x^{\lambda}\alpha^{-1}\gamma\cdot z)\circ(\alpha,x)\}=(\beta\gamma,y\gamma\cdot z)\circ(\alpha,x)$$

$$\iff (\beta\alpha,y\alpha\cdot x)\circ\{\alpha^{-1}\gamma\alpha,(x^{\lambda}\alpha^{-1}\gamma\cdot z)\alpha\cdot x\}=(\beta\gamma\alpha,(y\gamma\cdot z)\alpha\cdot x)$$

$$\iff (\beta\alpha\alpha^{-1}\gamma\alpha,(y\alpha\cdot x)\alpha^{-1}\gamma\alpha\cdot((x^{\lambda}\alpha^{-1}\gamma\cdot z)\alpha\cdot x))=(\beta\gamma\alpha,(y\gamma\cdot z)\alpha\cdot x)$$

$$\iff (y\alpha\cdot x)\alpha^{-1}\gamma\alpha\cdot((x^{\lambda}\alpha^{-1}\gamma\cdot z)\alpha\cdot x)=(y\gamma\cdot z)\alpha\cdot x$$

$$\iff (y\alpha\alpha^{-1}\gamma\alpha\cdot x\alpha^{-1}\gamma\alpha)\cdot((x^{\lambda}\alpha^{-1}\gamma\alpha\cdot z\alpha)\cdot x)=(y\gamma\alpha\cdot z\alpha)\cdot x$$

$$\iff (y\gamma\alpha\cdot x\alpha^{-1}\gamma\alpha)\cdot((x^{\lambda}\alpha^{-1}\gamma\alpha\cdot z\alpha)\cdot x)=(y\gamma\alpha\cdot z\alpha)\cdot x.$$

Replace: $y\gamma\alpha$ with y; $z\alpha$ with z and $\alpha^{-1}\gamma\alpha$ with δ, then it follows that

$$(y \cdot x\delta) \cdot ((x^{\lambda}\delta \cdot z) \cdot x) = (y \cdot z) \cdot x$$

□

Corollary 4.5.5. *Let $(Q,\cdot)$ be a loop with $A(Q)$-holomorph $(H,\circ)$. Then, $(H,\circ)$ is a right Basarab loop if and only if for all $x \in Q$ and $\delta \in A(Q)$,*

$$(R_{x\delta},\ L_{x^{\lambda}\delta}R_x,\ R_x) \in AUT(Q,\cdot) \quad \textit{or} \quad (R_x,\ L_{x^{\lambda}}R_{x\delta^{-1}}, R_{x\delta^{-1}}) \in AUT(Q,\cdot)$$

Proof. The proof follows from Theorem 4.5.2. □

Lemma 4.5.4. *Let $(Q,\cdot)$ be a right Basarab loop with $A(Q)$-holomorph $(H,\circ)$. Then, $(H,\circ)$ is a right Basarab loop if and only if any of the following is true:*

1. $y \cdot ((z/x) \cdot x\delta^{-1}) = ((yz)/x) \cdot x\delta^{-1}$ *holds for all $x \in Q$ and for all $\delta^{-1} \in A(Q)$.*
2. $(I,\ R_x^{-1}R_{x\delta^{-1}}, R_x^{-1}R_{x\delta^{-1}}) \in AUT(Q,\cdot)$ *for all $x \in Q$ and for all $\delta^{-1} \in A(Q)$.*
3. $R_x^{-1}R_{x\delta^{-1}}$ *is ρ-regular for all $x \in Q$ and for all $\delta^{-1} \in A(Q)$.*
4. $R_x^{-1}R_{x\delta^{-1}} \in \mathcal{P}(Q,\cdot)$ *for all $x \in Q$ and for all $\delta^{-1} \in A(Q)$.*

Proof. Let $(Q,\cdot)$ be a right Basarab loop. Then the autotopism $T = (R_x, L_{x^{\lambda}}R_x, R_x)$ holds. By Corollary 4.5.5, the holomorph $(H,\circ)$ of $(Q,\cdot)$ is a right Basarab loop if and only if $S = (R_x, L_{x^{\lambda}}R_{x\delta^{-1}}, R_{x\delta^{-1}})$ is the autotopism of Q.

The autotopism $T = (R_x, L_{x^{\lambda}}R_x, R_x) \in AUT(Q)$ implies $T^{-1} = (R_x^{-1}, R_x^{-1}L_{x^{\lambda}}^{-1}, R_x^{-1}) \in$

$AUT(Q)$. The holomorph $(H,\circ)$ of $(Q,\cdot)$ is a right Basarab loop if and only if

$$\begin{aligned} T^{-1}S &= (R_x^{-1}, R_x^{-1}L_{x^\lambda}^{-1}, R_x^{-1})(R_x, L_{x^\lambda}R_{x\delta^{-1}}, R_{x\delta^{-1}}) \\ &= (R_x^{-1}R_x, R_x^{-1}L_{x^\lambda}^{-1}L_{x^\lambda}R_{x\delta^{-1}}, R_x^{-1}R_{x\delta^{-1}}) \\ &= (I,\ R_x^{-1}R_{x\delta^{-1}}, R_x^{-1}R_{x\delta^{-1}}) \in AUT(Q,\cdot) \end{aligned}$$

Also, for all $x, y, z \in Q$ and for all $\delta \in A(Q)$,

$$\big(I,\ R_x^{-1}R_{x\delta^{-1}}, R_x^{-1}R_{x\delta^{-1}}\big) \in AUT(Q,\cdot) \iff y\cdot zR_x^{-1}R_{x\delta^{-1}} = (yz)R_x^{-1}R_{x\delta^{-1}} \iff$$

$$y\cdot((z/x)\cdot x\delta^{-1}) = ((yz)/x)\cdot x\delta^{-1}.$$

The rest follows by Definition 3.6.1(2). □

Corollary 4.5.6. *Let $(Q,\cdot)$ be a loop with $A(Q)$-holomorph $(H,\circ)$. Then, $(H,\circ)$ is a right Basarab loop if and only if $(Q,\cdot)$ is a right Basarab loop and any of the following equivalent conditions holds:*

1. $R_x^{-1}R_{x\delta} \in \mathcal{P}(Q,\cdot)$ *for all* $x \in Q$ *and for all* $\delta \in A(Q)$.
2. $x^\lambda \cdot x\delta \in N_\rho(Q,\cdot)$ *for all* $x \in Q$ *and for all* $\delta \in A(Q)$.
3. $\delta = J_\lambda M_n$ *for all* $\delta \in A(Q)$ *and some* $n \in N_\rho(Q,\cdot)$.
4. $A(Q) \le \mathcal{P}(Q,\cdot)$.

Proof. Let $(H,\circ)$ be a right Basarab loop, then $\{I\}\times Q \le (H,\circ)$ and $\{I\}\times Q \cong (Q,\cdot)$. Hence, $(Q,\cdot)$ is a right Basarab loop. Thus, by Lemma 4.5.4, $R_x^{-1}R_{x\delta} \in \mathcal{P}(Q,\cdot)$ for all $x \in Q$ and for all $\delta \in A(Q)$.

Conversely, if $(Q,\cdot)$ is a right Basarab loop and 1. holds, then by Lemma 4.5.4, $(H,\circ)$ is a right Basarab loop.

1 $\iff$ 2 By Theorem 3.1.1, $R_x^{-1}R_{x\delta} \overset{\psi}{\cong} eR_x^{-1}R_{x\delta} \iff eR_x^{-1}R_{x\delta} \in N_\rho(Q,\cdot) \iff x^\lambda \cdot x\delta \in N_\rho(Q,\cdot)$ for all $x \in Q$ and for all $\delta \in A(Q)$.

2 $\iff$ 3 $x^\lambda \cdot x\delta \in N_\rho(Q,\cdot) \iff x^\lambda \cdot x\delta = n \iff x\delta = nL_{x^\lambda}^{-1} = x^\lambda M_n \iff \delta = J_\lambda M_n$ for all $\delta \in A(Q)$ and some $n \in N_\rho(Q,\cdot)$.

1 $\iff$ 4 For all $x \in Q$ and for all $\delta \in A(Q)$, $R_x^{-1}R_{x\delta} \in \mathcal{P}(Q,\cdot) \iff R_{x\delta} \in R_x\mathcal{P}(Q,\cdot) \iff R_{x\delta} = R_x\rho$ for some $\rho \in \mathcal{P}(Q,\cdot) \iff (I,\delta,\rho) \in AUT(Q,\cdot)$ $\iff \delta \in \mathcal{P}(Q,\cdot) \iff A(Q) \leq \mathcal{P}(Q,\cdot)$.

□

Remark 4.5.2. *From Corollary 4.5.6, it can be deduced that the $\mathcal{P}(Q,\cdot)$-holomorph $H(Q,\cdot)$ of a right Basarab loop $(Q,\cdot)$ is precisely a right Basarab loop.*

Corollary 4.5.7. *Let $(Q,\cdot)$ be a loop with $A(Q)$-holomorph $(H,\circ)$. Then, $(H,\circ)$ is an RIP right Basarab loop if and only if $(Q,\cdot)$ is an RIP right Basarab loop and any of the following equivalent conditions holds:*

1. *$R_x^{-1}R_{x\delta}$ is μ-regular with an adjoint $(R_x^{-1}R_{x\delta})' = JR_{x\delta}^{-1}R_xJ$ for all $x \in Q$ and for all $\delta \in A(Q)$.*
2. *$R_x^{-1}R_{x\delta} \in \Phi(Q,\cdot)$ and $(R_x^{-1}R_{x\delta})' = JR_{x\delta}^{-1}R_xJ \in \Psi(Q,\cdot)$ for all $x \in Q$ and for all $\delta \in A(Q)$.*
3. *$x^\lambda \cdot x\delta,\ x^\lambda\delta \cdot x \in N_\mu(Q,\cdot)$ for all $x \in Q$ and for all $\delta \in A(Q)$.*
4. *$\delta = J_\lambda M_{n_1} = \delta = M_{n_2}^{-1}J_\rho$ for all $\delta \in A(Q)$ and some $n_1, n_2 \in N_\mu(Q,\cdot)$.*
5. *$A(Q) \subseteq J\Psi(Q,\cdot)J$, $J\Psi(Q,\cdot)J \subseteq \Phi(Q,\cdot)$ and $(J\Psi(Q,\cdot)J)' \subseteq JA(Q)J$.*

Proof. It should first be noted that $(H,\circ)$ is an RIP loop if and only if $(Q,\cdot)$ is an RIP. Following Corollary 4.5.7, $(H,\circ)$ is a right Basarab loop if and only if $(Q,\cdot)$ is a right Basarab loop and $R_x^{-1}R_{x\delta} \in \mathcal{P}(Q,\cdot)$ for all $x \in Q$ and for all $\delta \in A(Q)$.

In an RIPL, $(U,V,W)\in AUT(Q,\cdot) \implies (W,JVJ,U)\in AUT(Q,\cdot)$. So, for all $x\in Q$ and for all $\alpha\in A(Q)$:

$$R_x^{-1}R_{x\delta}\in\mathcal{P}(Q,\cdot) \iff (I,R_x^{-1}R_{x\delta},R_x^{-1}R_{x\delta})\in AUT(Q,\cdot)$$

$$\iff \big(R_x^{-1}R_{x\delta},JR_x^{-1}R_{x\delta}J,I\big)\in AUT(Q,\cdot)$$

$\iff R_x^{-1}R_{x\delta}$ is μ-regular with an adjoint $(R_x^{-1}R_{x\delta})'=JR_{x\delta}^{-1}R_xJ$.

1 $\iff$ 2 This is straightforward.

2 $\iff$ 3 By Theorem 3.1.1, $R_x^{-1}R_{x\delta}\overset{\sigma}{\cong}eR_x^{-1}R_{x\delta} \iff eR_x^{-1}R_{x\delta}\in N_\mu(Q,\cdot) \iff x^\lambda\cdot x\delta\in N_\mu(Q,\cdot)$ for all $x\in Q$ and for all $\delta\in A(Q)$.

Furthermore, $JR_{x\delta}^{-1}R_xJ\overset{\beta}{\cong}eJR_{x\delta}^{-1}R_xJ \iff eJR_{x\delta}^{-1}R_xJ\in N_\mu(Q,\cdot) \iff x^\lambda\delta\cdot x\in N_\mu(Q,\cdot)$ for all $x\in Q$ and for all $\delta\in A(Q)$.

3 $\iff$ 4 $x^\lambda\cdot x\delta\in N_\mu(Q,\cdot) \iff x^\lambda\cdot x\delta=n_1 \iff x\delta=n_1L_{x^\lambda}^{-1}=x^\lambda M_{n_1} \iff \delta=J_\lambda M_{n_1}$ for all $\delta\in A(Q)$ and some $n_1\in N_\mu(Q,\cdot)$.

Furthermore, $x^\lambda\delta\cdot x\in N_\mu(Q,\cdot) \iff x^\lambda\delta\cdot x=n_2 \iff x^\lambda\delta=n_2R_x^{-1}=xM_{n_2}^{-1} \iff \delta=M_{n_2}^{-1}J_\rho$ for all $\delta\in A(Q)$ and some $n_2\in N_\mu(Q,\cdot)$.

1 $\iff$ 5 For all $x\in Q$ and for all $\delta\in A(Q)$, $R_x^{-1}R_{x\delta}\in\Phi(Q,\cdot) \iff R_x^{-1}R_{x\delta}=\phi$ for some $\phi\in\Phi(Q,\cdot) \iff R_{x\delta}=R_x\phi \iff (I,\delta,\phi)\in AUT(Q,\cdot) \iff (\phi,J\delta J,I)\in AUT(Q,\cdot) \iff \phi\in\Phi(Q,\cdot)$ and $\phi'=J\delta^{-1}J\in\Psi(Q,\cdot)$ for some $\phi\in\Phi(Q,\cdot) \iff A(Q)\subseteq J\Psi(Q,\cdot)J$.

Furthermore, for all $x\in Q$ and for all $\delta\in A(Q)$, $(R_x^{-1}R_{x\delta})'=JR_{x\delta}^{-1}R_xJ\in\Psi(Q,\cdot) \iff JR_{x\delta}^{-1}R_xJ=\psi$ for some $\psi\in\Psi(Q,\cdot) \iff R_{x\delta}^{-1}R_x=J\psi J$ for some $\psi\in\Psi(Q,\cdot) \iff R_x=R_{x\delta}^{-1}J\psi J$ for some $\psi\in\Psi(Q,\cdot) \iff (I,\delta,J\psi^{-1}J)\in AUT(Q,\cdot) \iff (J\psi^{-1}J,J\delta J,I)\in AUT(Q,\cdot) \iff J\psi^{-1}J\in\Phi(Q,\cdot)$ and

$(J\psi^{-1}J)' = J\delta J \in \Psi(Q,\cdot) \iff J\Phi(Q,\cdot)J \subseteq \Psi(Q,\cdot)$ and $(J\Psi(Q,\cdot)J)' \subseteq JA(Q)J \subseteq \Psi(Q,\cdot)$.

$\square$

Corollary 4.5.8. *Let $(Q,\cdot)$ be a loop with $A(Q)$-holomorph $(H,\circ)$. If $(H,\circ)$ is an RIP right Basarab loop, then*

$$\begin{array}{ccc} & & x^\lambda\cdot x\delta,\ x^\lambda\delta\cdot x \\ & \overset{\sigma}{\nearrow} & \uparrow \beta \\ R_x^{-1}R_{x\delta} & \xmapsto[\text{isomorphism}]{\varphi} & JR_{x\delta}^{-1}R_xJ \end{array} \quad \in \quad \begin{array}{ccc} & & N_\mu(Q,\cdot) \\ & \overset{\sigma}{\nearrow} & \uparrow \beta\ \text{isomorphism} \\ \Phi(Q,\cdot) & \xrightarrow[\text{isomorphism}]{\varphi} & \Psi(Q,\cdot) \end{array} \tag{4.29}$$

for all $x\in Q,\ \delta\in A(Q)$. i.e. $\sigma=\varphi\beta$.

Proof. This follows from Corollary 4.5.7. $\square$

Lemma 4.5.5. *Let $(Q,\cdot)$ be a loop with $A(Q)$-holomorph $(H,\circ)$. If $(H,\circ)$ is a right Basarab loop then for all $x,y\in Q$ and $\delta\in A(Q)$, the following are true:*

1. $y\cdot(x^\lambda\cdot x\delta^{-1})=(y/x)\cdot x\delta^{-1}$.
2. $x^\lambda\cdot x\delta^{-1}\in N_\rho(Q,\cdot)$.
3. $yx\cdot(x^\lambda\cdot x\delta^{-1})=y\cdot x\delta^{-1}$.
4. $x\cdot(x^\lambda\cdot x\delta^{-1})=x\delta^{-1}$.
5. $\big(y,x,x^\lambda\cdot x\gamma\big)=e$.
6. $R_{x^\lambda\cdot x\delta^{-1}}=R_x^{-1}R_{x\delta^{-1}}$.
7. $R_{x^\lambda\cdot x\delta^{-1}}\in\mathcal{P}(Q,\cdot)$.

Proof. $(H,\circ)$ is a right Basarab loop implies

$$(R_x, L_{x^\lambda}R_{x\delta^{-1}}, R_{x\delta^{-1}})\in AUT(Q) \implies \forall\, y,z\in Q,\ yR_x\cdot zL_{x^\lambda}R_{x\delta^{-1}}=(yz)R_{x\delta^{-1}}.$$

By Corollary 4.5.6, $(Q,\cdot)$ is a right Basarab loop, hence, $L_{x^\lambda}=R_xL_x^{-1}R_x^{-1}$ which implies

$R_x^{-1} = L_x R_x^{-1} L_{x^\lambda}$. Again by Lemma 4.5.4, $(H, \circ)$ is a right Basarab loop if and only if

$$(I,\ R_x^{-1} R_{x\delta^{-1}}, R_x^{-1} R_{x\delta^{-1}}) \in AUT(Q, \cdot).$$

1. So, for all $x, y, z \in Q,\ yI \cdot zR_x^{-1} R_{x\delta^{-1}} = (yz)R_x^{-1} R_{x\delta^{-1}}$ implies for all $x, y, z \in Q$.

   $$yI \cdot zL_x R_x^{-1} L_{x^\lambda} R_{x\delta^{-1}} = (yz)R_x^{-1} R_{x\delta^{-1}} \implies yI \cdot zT_x^{-1} L_{x^\lambda} R_{x\delta^{-1}} = (yz)R_x^{-1} R_{x\delta^{-1}},$$

   Thus, with $z = e$, we have

   $$yI \cdot eT_x^{-1} L_{x^\lambda} R_{x\delta^{-1}} = (ye)R_x^{-1} R_{x\delta^{-1}} \implies yI \cdot eL_{x^\lambda} R_{x\delta^{-1}} = yR_x^{-1} R_{x\delta^{-1}}$$
   $$\implies y \cdot (x^\lambda \cdot x\delta^{-1}) = (y/x) \cdot x\delta^{-1}.$$

2. From 1, $yR_{x^\lambda \cdot x\delta^{-1}} = yR_x^{-1} R_{x\delta^{-1}} \implies R_{x^\lambda \cdot x\delta^{-1}} = R_x^{-1} R_{x\delta^{-1}}$. Thus,

   $$(I,\ R_x^{-1} R_{x\delta^{-1}}, R_x^{-1} R_{x\delta^{-1}}) = (I, R_{(x^\lambda \cdot \delta^{-1})}, R_{(x^\lambda \cdot \delta^{-1})}) \in AUT(Q) \implies x^\lambda \cdot x\delta^{-1} \in N_\rho(Q, \cdot).$$

3. From 1, $y \cdot (x^\lambda \cdot x\delta^{-1}) = (y/x) \cdot x\delta^{-1}$. Do $y \mapsto yx$ to get

   $$yx \cdot (x^\lambda \cdot x\delta^{-1}) = (yx/x) \cdot x\delta^{-1} \implies yx \cdot (x^\lambda \cdot x\delta^{-1}) = y \cdot x\delta^{-1}.$$

4. Use $y = e$ in 3 to get $x \cdot (x^\lambda \cdot x\delta^{-1}) = x\delta^{-1}$.

5. By 3 and 4, $yx \cdot (x^\lambda \cdot x\delta^{-1}) = y \cdot x\delta^{-1} = y \cdot x(x^\lambda \cdot x\delta^{-1})$ and this implies $\big(y, x, x^\lambda \cdot x\gamma\big) = e$.

6. This is gotten from the proof of 2.

7. This is gotten from the proof of 2.

□

4.5.3 Holomorphy of Basarab loop

Corollary 4.5.9. *Let $(Q,\cdot)$ be a loop with $A(Q)$-holomorph $(H,\circ)$. Then, $(H,\circ)$ is a Basarab loop if and only if any of the following is true:*

1. $(R_{x^\rho}L_{x\alpha}, L_x, L_{x\alpha}), (R_{x\alpha}, L_{x^\lambda\alpha}R_x, R_x) \in AUT(Q,\cdot)$ *for all* $x \in Q$ *and* $\alpha \in A(Q)$.

2. $(R_x, L_{x^\lambda}R_{x\alpha^{-1}}, R_{x\alpha^{-1}}), (R_{x^\rho\alpha^{-1}}L_x, L_{x\alpha^{-1}}, L_x)$ *for all* $x \in Q$ *and for all* $\alpha \in A(Q)$.

3. $(x\alpha \cdot yx^\rho)\cdot xz = x\alpha \cdot yz$ *and* $(y \cdot x\alpha)\cdot((x^\lambda\alpha \cdot z)\cdot x) = yz \cdot x$ *for all* $x, y, z \in Q$ *and for all* $\alpha \in A(Q)$.

Proof. The result follows from Corollaries 4.5.1, 4.5.5 and Theorems 4.5.1, 4.5.2. □

Lemma 4.5.6. *Let $(Q,\cdot)$ be a Basarab loop with $A(Q)$-holomorph $(H,\circ)$. Then, $(H,\circ)$ is a Basarab loop if and only if any of the following is true:*

1. $(x\alpha \cdot (x\backslash y))\cdot z = x\alpha \cdot (x\backslash(yz))$ *and* $y \cdot ((z/x)\cdot x\delta^{-1}) = ((yz)/x)\cdot x\delta^{-1}$ *holds for for all* $x, y, z \in Q$ *and for all* $\alpha, \delta \in A(Q)$.

2. $(L_x^{-1}L_{x\alpha}, I, L_x^{-1}L_{x\alpha}), (I,\ R_x^{-1}R_{x\delta^{-1}}, R_x^{-1}R_{x\delta^{-1}}) \in AUT(Q,\cdot)$ *for all* $x \in Q$ *and for all* $\alpha, \delta \in A(Q)$.

3. $L_x^{-1}L_{x\alpha}$ *is* λ*-regular and* $R_x^{-1}R_{x\delta^{-1}}$ *is* ρ*-regular for all* $x \in Q$ *and for all* $\alpha, \delta \in A(Q)$.

4. $L_x^{-1}L_{x\alpha} \in \Lambda(Q,\cdot)$ *and* $R_x^{-1}R_{x\delta^{-1}} \in \mathcal{P}(Q,\cdot)$ *for all* $x \in Q$ *and for all* $\alpha, \delta \in A(Q)$.

Proof. This follows from Lemmas 4.5.1 and 4.5.4 □

Corollary 4.5.10. *Let $(Q,\cdot)$ be a loop with $A(Q)$-holomorph $(H,\circ)$. Then, $(H,\circ)$ is a Basarab loop if and only if $(Q,\cdot)$ is a Basarab loop and any of the following equivalent conditions holds:*

1. $L_x^{-1}L_{x\alpha} \in \Lambda(Q,\cdot)$ *and* $R_x^{-1}R_{x\delta} \in \mathcal{P}(Q,\cdot)$ *for all* $x \in Q$ *and for all* $\alpha, \delta \in A(Q)$.

2. $x\alpha \cdot x^\rho \in N_\lambda(Q,\cdot),\ x^\lambda \cdot x\delta \in N_\rho(Q,\cdot)$ *for all* $x \in Q$ *and for all* $\alpha, \delta \in A(Q)$.

3. *$\alpha = J_\rho M_n^{-1}$ and $\delta = J_\lambda M_{n'}$ for all $\alpha \in A(Q)$ and some $n \in N_\lambda(Q,\cdot),\ n' \in N_\rho(Q,\cdot)$.*

4. *$A(Q) \leq \Lambda(Q,\cdot) \cap \mathcal{P}(Q,\cdot)$.*

Proof. This follows by Corollaries 4.5.2, 4.5.6. $\square$

Remark 4.5.3. *From Corollary 4.5.10, it can be deduced that the $\Lambda(Q,\cdot)\cap\mathcal{P}(Q,\cdot)$-holomorph $H(Q,\cdot)$ of a Basarab loop $(Q,\cdot)$ is precisely a Basarab loop.*

Lemma 4.5.7. *Let $(Q,\cdot)$ be a loop with $A(Q)$-holomorph $(H,\circ)$.*

1. *Then, $(H,\circ)$ is a Basarab loop if and only if Q for all $x,y \in Q$ and $\alpha \in A(Q)$:*

$$(L_{x\alpha}R_x^{-1}, L_x, L_{x\alpha}),\ (R_{x\alpha}, R_xL_{x\alpha}^{-1}, R_x) \in AUT(Q,\cdot) \ \text{ and}$$

$$(x\alpha \cdot yx^\rho)x = x\alpha \cdot y,\ \ x\alpha \cdot (x^\lambda\alpha \cdot y)x = yx.$$

2. *$(H,\circ)$ is a Basarab loop if and only if for all $x \in Q$ and $\delta \in A(Q)$:*

$$(R_x, R_{x\delta^{-1}}L_x^{-1}, R_{x\delta^{-1}}),\ (L_xR_{x\delta^{-1}}^{-1}, L_{x\delta^{-1}}, L_x) \in AUT(Q,\cdot) \ \text{ and}$$

$$x(x^\lambda y \cdot x\delta^{-1}) = y \cdot x\delta^{-1},\ \ x(y \cdot x^\rho\delta^{-1}) \cdot x\delta^{-1} = xy.$$

Proof. 1. Let $(R_{x^\rho}L_{x\alpha}, L_x, L_{x\alpha}) \in AUT(Q,\cdot)$. Then, for all $x,y,z \in Q$,

$$yR_{x^\rho}L_{x\alpha} \cdot zL_x = (yz)L_{x\alpha} \implies (x\alpha \cdot yx^\rho)\cdot xz = x\alpha(yz) \ \ (\text{set } z = e)$$

$$\implies (x\alpha \cdot yx^\rho)\cdot x = (x\alpha)y \implies yR_{x^\rho}L_{x\alpha}R_x = yL_{x\alpha}$$

$$\implies R_{x^\rho}L_{x\alpha}R_x = L_{x\alpha} \implies R_{x^\rho}L_{x\alpha} = L_{x\alpha}R_x^{-1}$$

$$\implies R_{x^\rho}L_{x\alpha}R_x = L_{x\alpha} \implies (x\alpha \cdot yx^\rho)x = x\alpha \cdot y.$$

Thus, $(Rx^\rho L_{x\alpha}, L_x, L_{x\alpha}) \in AUT(Q) \implies (L_{x\alpha}R_x^{-1}, L_x, L_{x\alpha}) \in AUT(Q)$. Also, let

$(R_{x\alpha}, L_{x^\lambda\alpha}R_x, R_x) \in AUT(Q)$. Then for all $x, y, z \in Q$,

$$yR_{x\alpha} \cdot zL_{x^\lambda\alpha}R_x = (yz)R_x \implies (y \cdot x\alpha) \cdot (x^\lambda\alpha)z \cdot x = yz \cdot x \ \ (\text{set } y = e)$$

$$\implies x\alpha \cdot (x^\lambda\alpha \cdot z)x = zx \implies zL_{x^\lambda\alpha}R_xL_{x\alpha} = zR_x$$

$$\implies L_{x^\lambda\alpha}R_xL_{x\alpha} = R_x \implies L_{x^\lambda\alpha}R_x = R_xL_{x\alpha}^{-1}$$

$$\implies L_{x^\lambda\alpha}R_xL_{x\alpha} = R_x \implies x\alpha \cdot (x^\lambda\alpha \cdot y)x = yx$$

Thus, $(R_{x\alpha}, L_{x^\lambda\alpha}R_x, R_x) \in AUT(Q) \implies (R_{x\alpha}, R_xL_{x\alpha}^{-1}, R_x) \in AUT(Q)$.

The converse follows by doing the reverse in each case.

2. This can be gotten from 1.

□

Theorem 4.5.3. *Let $(Q,\cdot)$ be a loop with $A(Q)$-holomorph $(H,\circ)$.*

1. *$(H,\circ)$ is a Basarab loop if and only if*

   (a) *$(Q,\cdot)$ Basarab loop;*

   (b) *$(L_{x\alpha}L_x^{-1}, I, L_{x\alpha}L_x^{-1}), (R_x^{-1}R_{x\alpha}, L_xL_{x\alpha}^{-1}, I) \in AUT(Q)$; and*

   (c) *$(x\alpha \cdot yx^\rho)x = x\alpha \cdot y,\ \ x\alpha \cdot (x^\lambda\alpha \cdot y)x = yx.$*

   for all $x, y, z \in Q$ and $\alpha \in A(Q)$.

2. *$(H,\circ)$ is a Basarab loop if and only if*

   (a) *$(Q,\cdot)$ Basarab loop;*

   (b) *$(I, R_{x\delta^{-1}}R_x^{-1}, R_{x\delta^{-1}}R_x^{-1}), (R_xR_{x\delta^{-1}}^{-1}, L_x^{-1}L_{x\delta^{-1}}, I) \in AUT(Q,\cdot)$; and*

   (c) *$x(x^\lambda y \cdot x\delta^{-1}) = y \cdot x\delta^{-1},\ \ x(y \cdot x^\rho\delta^{-1}) \cdot x\delta^{-1} = xy.$*

   for all $x \in Q$ and $\delta \in A(Q)$.

Proof. 1. By Lemma 4.5.7(1), $(H, \circ)$ is a Basarab loop implies $(L_{x\alpha}R_x^{-1}, L_x, L_{x\alpha})$ and $(R_{x\alpha}, R_xL_{x\alpha}^{-1}, R_x)$ are autotopisms of Q for all $x, y, z \in Q$ and $\alpha \in A(Q)$. Consequently, $(Q, \cdot)$ is a Basarab loop, hence $(L_xR_x^{-1}, L_x, L_x)$ and $(R_x, R_xL_x^{-1}, R_x)$ are autotopisms of Q for all $x \in Q$. Let

$$C = (L_{x\alpha}R_x^{-1}, L_x, L_{x\alpha}),\ D = (R_{x\alpha}, R_xL_{x\alpha}^{-1}, R_x),\ E = (L_xR_x^{-1}, L_x, L_x),\ F = (R_x, R_xL_x^{-1}, R_x).$$

$$C \in AUT(Q) \implies C^{-1} = (R_xL_{x\alpha}^{-1}, L_x^{-1}, L_{x\alpha}^{-1}) \in AUT(Q)$$

$$D \in AUT(Q) \implies D^{-1} = (R_{x\alpha}^{-1}, L_{x\alpha}R_x^{-1}, R_x^{-1}) \in AUT(Q)$$

$$E \in AUT(Q) \implies E^{-1} = (R_xL_x^{-1}, L_x^{-1}, L_x^{-1}) \in AUT(Q)$$

$$F \in AUT(Q) \implies F^{-1} = (R_x^{-1}, L_xR_x^{-1}, R_x^{-1}) \in AUT(Q)$$

Thus, $(H, \circ)$ is a Basarab loop implies

$$F^{-1}D = (R_x^{-1}, L_xR_x^{-1}, R_x^{-1})(R_{x\alpha}, R_xL_{x\alpha}^{-1}, R_x)$$

$$= (R_x^{-1}R_{x\alpha}, L_xR_x^{-1}R_xL_{x\alpha}^{-1}, R_x^{-1}R_x) = (R_x^{-1}R_{x\alpha}, L_xL_{x\alpha}^{-1}, I)$$

$$\text{and } CE^{-1} = (L_{x\alpha}R_x^{-1}, L_x, L_{x\alpha})(R_xL_x^{-1}, L_x^{-1}, L_x^{-1})$$

$$= (L_{x\alpha}R_x^{-1}R_xL_x^{-1}, L_xL_x^{-1}, L_{x\alpha}L_x^{-1}) = (L_{x\alpha}L_x^{-1}, I, L_{x\alpha}L_x^{-1})$$

are autotopisms of Q for all $x \in Q$ and $\alpha \in A$. For the converse, do the reverse and then use Lemma 4.5.7(1).

2. This is similar to 1. We shall show some of the basic steps involved.

Going by Lemma 4.5.7(2),

$$C_* = (L_x R^{-1}_{x\delta^{-1}}, L_{x\delta^{-1}}, L_x) \text{ and } D_* = (R_x, R_{x\delta^{-1}} L^{-1}_x, R_{x\delta^{-1}}). \text{ So,}$$

$$C_* \in AUT(Q) \implies C^{-1}_* = (R_{x\delta^{-1}} L^{-1}_x, L^{-1}_{x\delta^{-1}}, L^{-1}_x) \in AUT(Q) \text{ and}$$

$$D_* \in AUT(Q) \implies D^{-1}_* = (R^{-1}_x, L_x R^{-1}_{x\delta^{-1}}, R^{-1}_{x\delta^{-1}}) \in AUT(Q).$$

Thus, $(H, \circ)$ is a Basarab loop implies that

$$D_* F^{-1} = (R_x, R_{x\delta^{-1}} L^{-1}_x, R_{x\delta^{-1}})(R^{-1}_x, L_x R^{-1}_x, R^{-1}_x)$$

$$= (R_x R^{-1}_x, R_{x\delta^{-1}} L^{-1}_x L_x R^{-1}_x, R_{x\delta^{-1}} R^{-1}_x)$$

$$= (I, R_{x\delta^{-1}} R^{-1}_x, R_{x\delta^{-1}} R^{-1}_x)$$

Also,

$$E^{-1} C_* = (R_x L^{-1}_x, L^{-1}_x, L^{-1}_x)(L_x R^{-1}_{x\delta^{-1}}, L_{x\delta^{-1}}, L_x)$$

$$= (R_x L^{-1}_x L_x R^{-1}_{x\delta^{-1}}, L^{-1}_x L_{x\delta^{-1}}, I)$$

$$= (R_x R^{-1}_{x\delta^{-1}}, L^{-1}_x L_{x\delta^{-1}}, I)$$

are autotopisms of Q for all $x \in Q$ and $\delta \in A(Q)$.

For the converse, do the reverse and then use Lemma 4.5.7(2).

□

Corollary 4.5.11. *Let $(Q, \cdot)$ be a loop with $A(Q)$-holomorph $(H, \circ)$. If $(H, \circ)$ is a Basarab loop, then*

1. *for all* $x \in Q,\ \alpha \in A(Q)$*,* $\sigma = \varphi\beta$ *and* $\kappa = \sigma\vartheta^{-1}$*,* $R_x^{-1}R_{x\alpha} \overset{\kappa}{\approxeq} L_{x\alpha}L_x^{-1}$ *so that*

$$\begin{array}{ccc} & & x^\lambda \cdot x\alpha,\ x\backslash x\alpha \\ & {}^{\sigma}\nearrow\ {}_{\omega} & \beta\big\uparrow\vartheta \\ R_x^{-1}R_{x\alpha} & \xrightarrow[\kappa]{\varphi} & L_{x\alpha}L_x^{-1} \end{array} \quad \in \quad \begin{array}{ccc} & & N(Q,\cdot) \\ & {}^{\sigma}\nearrow\ {}_{\omega} & \beta\big\uparrow\vartheta \\ \Phi(Q,\cdot)\ \mathcal{P}(Q,\cdot) & \xrightarrow[\kappa]{\varphi} & \Psi(Q,\cdot),\ \Lambda(Q,\cdot) \end{array} \tag{4.30}$$

2. *for all* $x \in Q,\ \delta \in A(Q)$*,* $\sigma = \varphi\beta$ *and* $\eta = \sigma\vartheta^{-1}$*,* $R_{x\delta}R_x^{-1} \overset{\eta}{\approxeq} L_x^{-1}L_{x\delta}$ *so that*

$$\begin{array}{ccc} & & x\delta/x\alpha\ x\delta \cdot x^\rho, \\ & {}^{\sigma}\nearrow\ {}_{\omega} & \beta\big\uparrow\vartheta \\ R_{x\delta}R_x^{-1} & \xrightarrow[\eta]{\varphi} & L_x^{-1}L_{x\delta} \end{array} \quad \in \quad \begin{array}{ccc} & & N(Q,\cdot) \\ & {}^{\sigma}\nearrow\ {}_{\omega} & \beta\big\uparrow\vartheta \\ \Phi(Q,\cdot)\ \mathcal{P}(Q,\cdot) & \xrightarrow[\eta]{\varphi} & \Psi(Q,\cdot),\ \Lambda(Q,\cdot) \end{array} \tag{4.31}$$

Proof. 1. This follows from Theorem 4.5.3(1) and Corollary 4.5.6(1,2).

2. This follows from Theorem 4.5.3(2) and Corollary 4.5.2(1,2).

□

Theorem 4.5.4. *1. Let* $(Q,\cdot)$ *be a loop with* $A(Q)$*-holomorph* $(H,\circ)$ *such that* $x^\lambda \cdot x\alpha = (x\alpha\backslash x)^\lambda = x\backslash x\alpha$ *for all* $x \in Q$ *and* $\alpha \in A(Q)$*. Then,* $(H,\circ)$ *is a Basarab loop if and only if :*

(a) $(Q,\cdot)$ *is a Basarab loop;*

(b) every automorphism of Q *is left nuclear; and*

(c) $T^{-1}_{(x,\alpha)} = R_{x^\rho}L_{x\alpha}$ *and* $T_{(x,\alpha)} = L_{x^\lambda\alpha}R_x$ *for all* $x \in Q$ *and* $\alpha \in A(Q)$*.*

2. Let $(Q,\cdot)$ *be a loop with* $A(Q)$*-holomorph* $(H,\circ)$ *such that* $x\delta^{-1} \cdot x^\rho = (x/x\delta^{-1})^\rho = x\delta^{-1}/x$ *for all* $x \in Q$ *and* $\delta \in A(Q)$*. Then,* $(H,\circ)$ *is a Basarab loop if and only if :*

(a) $(Q,\cdot)$ *is a Basarab loop;*

(b) every automorphism of Q *is right nuclear; and*

(c) $T^{-1}_{(x\delta^{-1},\delta)} = R_{x^\rho\delta^{-1}}L_x$ *and* $T_{(x\delta^{-1},\delta)} = L_{x^\lambda}R_{x\delta^{-1}}$ *for all* $x \in Q$ *and* $\delta \in A(Q)$*.*

Proof. 1. Let $(H, \circ)$ be a Basarab loop. By Lemma 4.5.7(1) and Theorem 4.5.3(1), it follows that

$$C = (L_{x\alpha}R_x^{-1}, L_x, L_{x\alpha}),\ D = (R_{x\alpha}, R_xL_{x\alpha}^{-1}, R_x) \in AUT(Q),$$
$$K = (R_x^{-1}R_{x\alpha}, L_xL_{x\alpha}^{-1}, I),\ P = (L_{x\alpha}L_x^{-1}, I, L_{x\alpha}L_x^{-1}) \in AUT(Q),$$
$$(x\alpha \cdot yx^{\rho})x = x\alpha \cdot y \text{ and } x\alpha \cdot (x^{\lambda}\alpha \cdot y)x = yx \tag{4.32}$$

From (4.32), (c) is true. On the other hand,

$$P^{-1}C = (L_xL_{x\alpha}^{-1}, I, L_xL_{x\alpha}^{-1})(L_{x\alpha}R_x^{-1}, L_x, L_{x\alpha}) = (L_xR_x^{-1}, L_x, L_x)$$
$$\text{and } DK^{-1} = (R_{x\alpha}, R_xL_{x\alpha}^{-1}, R_x)(R_{x\alpha}^{-1}R_x, L_{x\alpha}L_x^{-1}, I) = (R_x, R_xL_x^{-1}, R_x).$$

Then $P^{-1}C$ and DK^{-1} are autotopisms of $(Q, \cdot)$, hence $(Q, \cdot)$ is a Basarab loop and so (a) is true. Since $(H, \circ)$ is a Basarab loop, the autotopism $K = (R_x^{-1}R_{x\alpha}, L_xL_{x\alpha}^{-1}, I)$ implies for all $x, u, v \in Q,\ uR_x^{-1}R_{x\alpha} \cdot vL_xL_{x\alpha}^{-1} = (uv)I$.

If $v = e$, then

$$uR_x^{-1}R_{x\alpha} \cdot eL_xL_{x\alpha}^{-1} = u \implies uR_x^{-1}R_{x\alpha}R_{x\alpha\backslash x} = u \tag{4.33}$$
$$\implies R_x^{-1}R_{x\alpha} = R_{x\alpha\backslash x}^{-1} \quad (\text{by } x^{\lambda} \cdot x\alpha = (x\alpha\backslash x)^{\lambda})$$
$$\implies R_x^{-1}R_{x\alpha} = R_{(x^{\lambda}\cdot x\alpha)} \tag{4.34}$$

Also, if $u = e$ in $uR_x^{-1}R_{x\alpha} \cdot vL_xL_{x\alpha}^{-1} = (uv)I$, then

$$eR_x^{-1}R_{x\alpha} \cdot vL_xL_{x\alpha}^{-1} = vI \implies vL_xL_{x\alpha}^{-1}L_{(x^{\lambda}\cdot x\alpha)} = v \tag{4.35}$$
$$\implies L_xL_{x\alpha}^{-1} = L_{(x^{\lambda}\cdot x\alpha)}^{-1}$$
$$\implies L_{(x\alpha)}L_x^{-1} = L_{(x^{\lambda}\cdot x\alpha)} \tag{4.36}$$

Then by (4.36), the autotopism P becomes $\psi = (L_{(x^\lambda \cdot x\alpha)}, I, L_{(x^\lambda \cdot x\alpha)})$ for all $x \in Q$ and $\alpha \in A$. So,

$$(L_{(x^\lambda \cdot x\alpha)}, I, L_{(x^\lambda \cdot x\alpha)}) \implies uL_{(x^\lambda \cdot x\alpha)} \cdot vI = (uv)L_{(x^\lambda \cdot x\alpha)} \implies (x^\lambda \cdot x\alpha)u \cdot v = (x^\lambda \cdot x\alpha) \cdot (uv)$$

for all $u, v \in Q$. Thus, $x^\lambda \cdot x\alpha \in N$. Therefore, α is left nuclear, hence, (b) is true.

For the converse, assume that (a), (b) and (c) are true. Then, $(Q, \cdot)$ is a Basarab loop, $x^\lambda \cdot x\alpha \in N(Q)$ for all $x \in Q$ and $\alpha \in A(Q)$ and (4.32) is satisfied.

Now, $((x^{-1} \cdot x\alpha) \cdot u) \cdot v = (x^\lambda \cdot x\alpha) \cdot (u \cdot v)$ for all $u, v \in Q$ and so

$$\psi = (L_{(x^\lambda \cdot x\alpha)}, I, L_{(x^\lambda \cdot x\alpha)}) \in AUT(Q)$$

for all $x \in Q$. Now,

$$x^\lambda \cdot x\alpha = x^\lambda \cdot x\alpha \quad (\text{by } \; x^\lambda \cdot x\alpha = x\backslash x\alpha)$$

$$\implies x\alpha = x(x^\lambda \cdot x\alpha) \implies x\alpha \cdot y = x(x^\lambda \cdot x\alpha) \cdot y \text{ and } y \cdot x\alpha = y \cdot x(x^\lambda \cdot x\alpha) \tag{4.37}$$

$$\implies x\alpha \cdot y = x \cdot (x^\lambda \cdot x\alpha)y \text{ and } y \cdot x\alpha = yx \cdot (x^\lambda \cdot x\alpha)$$

$$\implies yL_{x\alpha} = yL_{(x^\lambda \cdot x\alpha)}L_x \text{ and } yR_{x\alpha} = yR_xR_{(x^\lambda \cdot x\alpha)}$$

$$\implies L_{x\alpha}L_x^{-1} = L_{(x^\lambda \cdot x\alpha)} \text{ and } R_x^{-1}R_{x\alpha} = R_{(x^\lambda \cdot x\alpha)} \tag{4.38}$$

Thus, by (4.38), it follows that $\psi = P = (L_{x\alpha}L_x^{-1}, I, L_{x\alpha}L_x^{-1}) \in AUT(Q)$. Also,

$$\begin{aligned}(u\cdot v)\cdot(x^\lambda\cdot x\alpha) =& u\cdot[v\cdot(x^\lambda\cdot x\alpha)] \implies (u\cdot v) = [u\cdot[v\cdot(x^\lambda\cdot x\alpha)]][(x^\lambda\cdot x\alpha)^{-1}]\\ \implies (u\cdot v) =& [u(x^\lambda\cdot x\alpha)^{-1}(x^\lambda\cdot x\alpha)\cdot[v\cdot(x^\lambda\cdot x\alpha)]][(x^\lambda\cdot x\alpha)^{-1}]\\ =& [u\cdot(x^\lambda\cdot x\alpha)^{-1}]\cdot[(x^\lambda\cdot x\alpha)\cdot[[v\cdot(x^\lambda\cdot x\alpha)](x^\lambda\cdot x\alpha)^{-1}]]\\ =& (u\cdot(x^\lambda\cdot x\alpha)^{-1})\cdot((x^\lambda\cdot x\alpha)\cdot v)\\ =& uR^{-1}_{(x^\lambda\cdot x\alpha)}\cdot vL_{(x^\lambda\cdot x\alpha)}\end{aligned}$$

This implies, $w = (R^{-1}_{(x^\lambda\cdot x\alpha)}, L_{(x^\lambda\cdot x\alpha)}, I) \in AUT(Q)$

$$\implies w^{-1} = (R_{(x^\lambda\cdot x\alpha)}, L^{-1}_{(x^\lambda\cdot x\alpha)}, I) \in AUT(Q).$$

By (4.38), it follows that $w^{-1} = (R_x^{-1}R_{x\alpha}, L_xL_{x\alpha}^{-1}, I) \in AUT(Q)$. Therefore, $(H,\circ)$ is a Basarab loop going by Theorem 4.5.3.

2. This is similar to the proof of 1 by using Lemma 4.5.7(2) and Theorem 4.5.3(2).

□

Theorem 4.5.5. *Let $(Q,\cdot)$ be a loop and $A(Q)$ be a group of automorphisms of $(Q,\cdot)$. Then, the $A(Q)$-holomorph (Q) is a Basarab loop if and only if (i) $(R_x, R_{x\delta^{-1}}L_x^{-1}, R_{x\delta^{-1}}) \in AUT(Q)$ (ii) $(L_xR^{-1}_{x\delta^{-1}}, L_{x\delta^{-1}}, L_x) \in AUT(Q)$ (iii) $L_{x^\lambda}R_{x\delta^{-1}} = R_{x\delta^{-1}}L_x^{-1}$ and $R_{x^\rho\delta^{-1}}L_x = L_xR^{-1}_{x\delta^{-1}}$ are autotopisms of Q for all $x \in Q$ and $\delta^{-1} \in A(Q)$.*

Proof. Let $(Q,\cdot)$ be a loop and $A(Q)$ be a group of automorphisms of $(Q,\cdot)$. Then, by Corollary 4.5.9, the $A(Q)$-holomorph $(H,\circ)$ of $(Q,\cdot)$ is a Basarab loop if and only if

$$(R_x, L_{x^\lambda}R_{x\delta^{-1}}, R_{x\delta^{-1}}) \text{ and } (R_{x^\rho\delta^{-1}}L_x, L_{x\delta^{-1}}, L_x)$$

are autotopisms of Q for all $x \in Q$ and $\delta^{-1} \in A(Q)$.

Then $(R_x, L_{x^\lambda}R_{x\delta^{-1}}, R_{x\delta^{-1}}) \in AUT(Q,\cdot)$ implies that for all $x, y, z \in Q$,

$$yR_x \cdot zL_{x^\lambda}R_{x\delta^{-1}} = (yz)R_{x\delta^{-1}} \implies yx \cdot ((x^\lambda)x\delta^{-1}) = (yz)x\delta^{-1}.$$

Set $y = e \implies$

$$x \cdot ((x^\lambda z)x\delta^{-1}) = zx\delta^{-1} \implies zL_{x^\lambda}R_{x\delta^{-1}}L_x = zR_{x\delta^{-1}}$$

$$L_{x^\lambda}R_{x\delta^{-1}}L_x = R_{x\delta^{-1}} \implies L_{x^\lambda}R_{x\delta^{-1}} = R_{x\delta^{-1}}L_x^{-1}.$$

Thus,

$$(R_x, L_{x^\lambda}R_{x\delta^{-1}}, R_{x\delta^{-1}}) \in AUT(Q) \implies (R_x, R_{x\delta^{-1}}L_x^{-1}, R_{x\delta^{-1}}) \in AUT(Q).$$

Also, $(R_{x^\rho\delta^{-1}}, L_{x\delta^{-1}}, L_x) \in AUT(Q)$ implies

$$yR_{x^\rho\delta^{-1}}L_x \cdot zL_{x\delta^{-1}} = (yz)L_x \implies (x \cdot yx^\rho\delta^{-1}) \cdot (x\delta^{-1} \cdot z) = x \cdot yz$$

for all $x, y, z \in Q$. Set $z = e$ then

$$(x \cdot yx^\rho\delta^{-1}) \cdot x\delta^{-1} = xy \implies yR_{x^\rho\delta^{-1}}L_xR_{x\delta^{-1}} = yL_x \implies R_{x^\rho\delta^{-1}}L_x = L_xR_{x\delta^{-1}}^{-1}.$$

Thus,

$$(R_{x^\rho\delta^{-1}}L_x, L_{x\delta^{-1}}, L_x) \in AUT(Q) \implies (L_xR_{x\delta^{-1}}^{-1}, L_{x\delta^{-1}}, L_x) \in AUT(Q).$$

□

Theorem 4.5.6. *1. Let $(Q,\cdot)$ be a loop.*

(a) If $V_1(Q,\cdot) = \{\alpha \in A(Q,\cdot) \mid P(\alpha,x) = (L_{x\alpha}L_x^{-1}, I, L_{x\alpha}L_x^{-1}) \in AUT(Q,\cdot)\}$, then $V_1(Q,\cdot) \lhd A(Q,\cdot)$.

(b) If $(Q,\cdot)$ is a Basarab loop such that $\big(x, x^\lambda, x\alpha\big) = e$ for all $x \in Q$ and $\alpha \in A(Q)$ and $V_2(Q,\cdot) = \{\alpha \in A(Q,\cdot) \mid (L_{x^\lambda\cdot x\alpha}, I, L_{x^\lambda\cdot x\alpha}) \in AUT(Q,\cdot)\} = \big\{\alpha \in A(Q,\cdot) \mid x^\lambda \cdot x\alpha \in N(Q,\cdot)\big\}$. Then $V_1(Q,\cdot) = V_2(Q,\cdot)$.

(c) If $(H,\circ)$ is a Basarab loop such that $\big(x, x^\lambda, x\alpha\big) = e$ for all $x \in Q$ and $\alpha \in A(Q)$, then $A(Q,\cdot) = V_1(Q,\cdot) = V_2(Q,\cdot)$.

2. Let $(Q,\cdot)$ be a loop.

(a) If $V_3(Q,\cdot) = \{\delta \in A(Q,\cdot) \mid B(\delta,x) = (I, R_{x\delta^{-1}}R_x^{-1}, R_{x\delta^{-1}}R_x^{-1}) \in AUT(Q,\cdot)\}$, then $V_3(Q,\cdot) \lhd A(Q,\cdot)$.

(b) If $(Q,\cdot)$ is a Basarab loop such that $(x\delta^{-1}, x^\rho, x) = e$ for all $x \in Q$ and $\delta \in A(Q)$ and $V_4(Q,\cdot) = \big\{\delta \in A(Q,\cdot) \mid \big(I, R_{(x\delta^{-1}\cdot x^\rho)}, R_{(x\delta^{-1}\cdot x^\rho)}\big) \in AUT(Q,\cdot)\big\} = \{\delta \in A(Q,\cdot) \mid x\delta^{-1}\cdot x^\rho \in N(Q,\cdot)\}$. Then $V_3(Q,\cdot) = V_4(Q,\cdot)$.

(c) If $(H,\circ)$ is a Basarab loop such that $(x\delta^{-1}, x^\rho, x) = e$ for all $x \in Q$ and $\delta \in A(Q)$, then $A(Q,\cdot) = V_3(Q,\cdot) = V_4(Q,\cdot)$.

Proof. 1. (a) Let $V_1(Q,\cdot) = \{\alpha \in A(Q,\cdot) \mid (L_{x\alpha}L_x^{-1}, I, L_{x\alpha}L_x^{-1}) \in AUT(Q,\cdot)\}$. With $\alpha = I$, $(L_{xI}L_x^{-1}, I, L_{xI}L_x^{-1}) = (I,I,I) \in AUT(Q,\cdot)$. So, $V_1(Q,\cdot) \neq \emptyset$.

Let $\alpha,\beta \in V_1(Q)$, then

$$P(\beta, x\alpha)P(\alpha,x) = (L_{x\alpha\beta}L_{x\alpha}^{-1}, I, L_{x\alpha\beta}L_{x\alpha}^{-1})(L_{x\alpha}L_x^{-1}, I, L_{x\alpha}L_x^{-1}) =$$

$$(L_{x\alpha\beta}L_{x\alpha}^{-1}L_{x\alpha}L_x^{-1}, I, L_{x\alpha\beta}L_{x\alpha}^{-1}L_{x\alpha}L_x^{-1}) = (L_{x\alpha\beta}L_x^{-1}, I, L_{x\alpha\beta}L_x^{-1}) \in AUT(Q,\cdot)$$

$$\implies \alpha\beta \in V_1(Q).$$

Next, we show that $\alpha^{-1} \in V_1(Q)$ for all $\alpha \in V_1(Q)$. Let $\alpha \in V_1(Q)$, then

$$
\begin{aligned}
P(\alpha,x) =&(L_{x\alpha}L_x^{-1}, I, L_{x\alpha}L_x^{-1}) \in AUT(Q,\cdot) \implies \\
P(\alpha,x)^{-1} =&(L_x L_{x\alpha}^{-1}, I, L_x L_{x\alpha}^{-1}) \in AUT(Q,\cdot) \implies \\
P(\alpha,x\alpha^{-1})^{-1} =& \left(L_{x\alpha^{-1}}L_{x\alpha^{-1}\alpha}^{-1}, I, L_{x\alpha^{-1}}L_{x\alpha^{-1}\alpha}^{-1}\right) \in AUT(Q,\cdot) \implies \\
P(\alpha,x\alpha^{-1})^{-1} =&(L_{x\alpha^{-1}}L_x^{-1}, I, L_{x\alpha^{-1}}L_x^{-1}) \in AUT(Q,\cdot) \implies \alpha^{-1} \in V_1(Q).
\end{aligned}
$$

If $\alpha \in V_1(Q)$ and $\theta \in A(Q)$, then for all $x, a, b \in Q$,

$$P(\alpha,x) = (L_{x\alpha}L_x^{-1}, I, L_{x\alpha}L_x^{-1}) \in AUT(Q,\cdot) \iff x\backslash(x\alpha\cdot a)\cdot b = x\backslash\,(x\alpha\cdot(ab)) \iff$$

$$(x\backslash(x\alpha\cdot a)\cdot b)\,\theta = (x\backslash\,(x\alpha\cdot(ab)))\,\theta \iff$$

$$x\theta\backslash(x\alpha\theta\cdot a\theta)\cdot b\theta = x\theta\backslash\,(x\alpha\theta\cdot(a\theta\cdot b\theta)) \quad (\text{set } x\mapsto x\theta^{-1})$$

$$\iff x\theta^{-1}\theta\backslash(x\theta^{-1}\alpha\theta\cdot a\theta^{-1}\theta)\cdot b\theta^{-1}\theta = x\theta^{-1}\theta\backslash\left(x\theta^{-1}\alpha\theta\cdot(a\theta^{-1}\theta\cdot b\theta^{-1}\theta)\right) \iff$$

$$x\backslash(x\theta^{-1}\alpha\theta\cdot a)\cdot b = x\backslash\left(x\theta^{-1}\alpha\theta\cdot(a\cdot b)\right) \iff (L_{x\theta^{-1}\alpha\theta}L_x^{-1}, I, L_{x\theta^{-1}\alpha\theta}L_x^{-1}) \in AUT(Q,\cdot)$$

$$\iff P(\theta^{-1}\alpha\theta, x) \in AUT(Q,\cdot) \iff \theta^{-1}\alpha\theta \in V_1(Q).$$

Therefore, $V_1(Q,\cdot) \lhd A(Q,\cdot)$.

(b) Let $\alpha \in V_1(Q)$, then for all $x \in Q$, $P(\alpha,x) = (L_{x\alpha}L_x^{-1}, I, L_{x\alpha}L_x^{-1}) \in AUT(Q,\cdot)$.

Consequently,

$$eL_{x\alpha}L_x^{-1}\cdot yI = (e\cdot y)L_{x\alpha}L_x^{-1} \implies (x\backslash x\alpha)y = yL_{x\alpha}L_x^{-1} \implies yL_{(x\backslash x\alpha)} = yL_{x\alpha}L_x^{-1} \implies$$

$$L_{(x\backslash x\alpha)} = L_{x\alpha}L_x^{-1} \implies L_{x\alpha}L_x^{-1} = L_{(x^\lambda\cdot x\alpha)} \implies$$

$$P(\alpha,x) = \left(L_{(x^\lambda\cdot x\alpha)}, I, L_{(x^\lambda\cdot x\alpha)}\right) \in AUT(Q,\cdot) \implies V_1(Q,\cdot) \subseteq V_2(Q,\cdot).$$

Let $\alpha \in V_2(Q)$, then for all $x \in Q$, $\left(L_{(x^\lambda\cdot x\alpha)}, I, L_{(x^\lambda\cdot x\alpha)}\right) \in AUT(Q,\cdot)$.

Consequently,

$$x^\lambda \cdot x\alpha = x^\lambda \cdot x\alpha \quad (\text{by } \ x^\lambda \cdot x\alpha = x\backslash x\alpha)$$

$$\implies x\alpha = x(x^\lambda \cdot x\alpha) \implies x\alpha \cdot y = x(x^\lambda \cdot x\alpha) \cdot y \implies x\alpha \cdot y = x \cdot (x^\lambda \cdot x\alpha)y$$

$$\implies yL_{x\alpha} = yL_{(x^\lambda \cdot x\alpha)}L_x \implies L_{x\alpha}L_x^{-1} = L_{(x^\lambda \cdot x\alpha)} \implies$$

$$P(\alpha, x) = (L_{x\alpha}L_x^{-1}, I, L_{x\alpha}L_x^{-1}) \in AUT(Q, \cdot) \implies V_2(Q, \cdot) \subseteq V_1(Q, \cdot).$$

Thus, $V_1(Q, \cdot) = V_2(Q, \cdot)$.

(c) Going by (4.36) of Theorem 4.5.4(1), $x^\lambda \cdot x\alpha \in N(Q)$ for all $x \in Q$ and $\alpha \in A(Q)$. Thus by (a) and (b), $A(Q, \cdot) = V_1(Q, \cdot) = V_2(Q, \cdot)$.

2. (a) Let $V_3(Q, \cdot) = \{\delta \in A(Q, \cdot) \mid (I, R_{x\delta^{-1}}R_x^{-1}, R_{x\delta^{-1}}R_x^{-1}) \in AUT(Q, \cdot)\}$. With $\delta = I$, $(I, R_{xI}R_x^{-1}, R_{xI}R_x^{-1}) = (I, I, I) \in AUT(Q, \cdot)$. So, $V_3(Q, \cdot) \neq \emptyset$.

Let $\delta, \beta \in V_3(Q)$, then

$$B(\beta, x\delta^{-1})B(\delta, x) = (I, R_{x\delta^{-1}\beta^{-1}}R_{x\delta^{-1}}^{-1}, R_{x\delta^{-1}\beta^{-1}}R_{x\delta^{-1}}^{-1})(I, R_{x\delta^{-1}}R_x^{-1}, R_{x\delta^{-1}}R_x^{-1}) =$$

$$(I, R_{x(\beta\delta)^{-1}}R_{x\delta^{-1}}^{-1}R_{x\delta^{-1}}R_x^{-1}, R_{x(\beta\delta)^{-1}}R_{x\delta^{-1}}^{-1}R_{x\delta^{-1}}R_x^{-1}) =$$

$$(I, R_{x(\beta\delta)^{-1}}R_x^{-1}, R_{x(\beta\delta)^{-1}}R_x^{-1}) \in AUT(Q, \cdot) \implies \beta\delta \in V_3(Q).$$

Next, we show that $\delta^{-1} \in V_3(Q)$ for all $\delta \in V_3(Q)$. Let $\delta \in V_3(Q)$, then

$$B(\delta, x) = (I, R_{x\delta^{-1}}R_x^{-1}, R_{x\delta^{-1}}R_x^{-1}) \in AUT(Q, \cdot) \implies$$

$$B(\delta, x)^{-1} = (I, R_xR_{x\delta^{-1}}^{-1}, R_xR_{x\delta^{-1}}^{-1}) \in AUT(Q, \cdot) \implies$$

$$B(\delta, x\delta)^{-1} = (I, R_{x\delta}R_{x\delta\delta^{-1}}^{-1}, R_{x\delta}R_{x\delta\delta^{-1}}^{-1}) \in AUT(Q, \cdot) \implies$$

$$B(\delta, x\delta)^{-1} = \left(I, R_{x(\delta^{-1})^{-1}}R_x^{-1}, R_{x(\delta^{-1})^{-1}}R_x^{-1}\right) \in AUT(Q, \cdot) \implies \delta^{-1} \in V_3(Q).$$

If $\delta \in V_3(Q)$ and $\theta \in A(Q)$, then for all $x, a, b \in Q$,

$$B(\delta, x) = (I, R_{x\delta^{-1}}R_x^{-1}, R_{x\delta^{-1}}R_x^{-1}) \in AUT(Q, \cdot)$$

$$\iff a \cdot \left(b \cdot x\delta^{-1}\right)/x = \left(ab \cdot x\delta^{-1}\right)/x \iff$$

$$\left(a \cdot \left(b \cdot x\delta^{-1}\right)/x\right)\theta = \left(\left(ab \cdot x\delta^{-1}\right)/x\right)\theta \iff$$

$$a \cdot \left(b \cdot x\delta^{-1}\theta\right)/x\theta = \left(ab \cdot x\delta^{-1}\theta\right)/x\theta \quad (\text{set } \ x \mapsto x\theta^{-1})$$

$$\iff a \cdot \left(b \cdot x\theta^{-1}\delta^{-1}\theta\right)/x\theta^{-1}\theta = \left(ab \cdot x\theta^{-1}\delta^{-1}\theta\right)/x\theta^{-1}\theta \iff$$

$$a \cdot \left(b \cdot x\left(\theta^{-1}\delta\theta\right)^{-1}\right)/x = \left(ab \cdot x\left(\theta^{-1}\delta\theta\right)^{-1}\right)/x \iff$$

$$\left(I, R_{x(\theta^{-1}\delta\theta)^{-1}}R_x^{-1}, R_{x(\theta^{-1}\delta\theta)^{-1}}R_x^{-1}\right) \in AUT(Q, \cdot)$$

$$\iff P(\theta^{-1}\delta\theta, x) \in AUT(Q, \cdot) \iff \theta^{-1}\delta\theta \in V_3(Q).$$

Therefore, $V_3(Q, \cdot) \lhd A(Q, \cdot)$.

(b) Let $\delta \in V_3(Q)$, then for all $x \in Q$, $B(\delta, x) = (I, R_{x\delta^{-1}}R_x^{-1}, R_{x\delta^{-1}}R_x^{-1}) \in AUT(Q, \cdot)$. Consequently,

$$y \cdot eR_{x\delta^{-1}}R_x^{-1} = (y \cdot e)R_{x\delta^{-1}}R_x^{-1} \implies y(x\delta^{-1}/x) = yR_{x\delta^{-1}}R_x^{-1} \implies$$

$$yR_{(x\delta^{-1}/x)} = yR_{x\delta^{-1}}R_x^{-1} \implies$$

$$R_{(x\delta^{-1}/x)} = R_{x\delta^{-1}}R_x^{-1} \implies R_{x\delta^{-1}}R_x^{-1} = R_{(x\delta^{-1}\cdot x^\rho)} \implies$$

$$B(\delta, x) = \left(I, R_{(x\delta^{-1}\cdot x^\rho)}, R_{(x\delta^{-1}\cdot x^\rho)}\right) \in AUT(Q, \cdot) \implies V_3(Q, \cdot) \subseteq V_4(Q, \cdot).$$

Let $\delta \in V_4(Q)$, then for all $x \in Q$, $\left(I, R_{(x\delta^{-1}\cdot x^\rho)}, R_{(x\delta^{-1}\cdot x^\rho)}\right) \in AUT(Q, \cdot)$.

Consequently,

$$x\delta^{-1} \cdot x^{\rho} = x\delta^{-1} \cdot x^{\rho}$$

$$(\text{by } \ x\delta^{-1} \cdot x^{\rho} = x\delta^{-1}/x)$$

$$\implies (x\delta^{-1} \cdot x^{\rho})x = x\delta^{-1} \implies$$

$$y \cdot (x\delta^{-1} \cdot x^{\rho})x = y \cdot x\delta^{-1} \implies y(x\delta^{-1} \cdot x^{\rho}) \cdot x = y \cdot x\delta^{-1}$$

$$\implies yR_{(x\delta^{-1} \cdot x^{\rho})}R_x = yR_{x\delta^{-1}} \implies R_{x\delta^{-1}}R_x^{-1} = R_{(x\delta^{-1} \cdot x^{\rho})} \implies$$

$$B(\delta, x) = \big(I, R_{(x\delta^{-1} \cdot x^{\rho})}, R_{(x\delta^{-1} \cdot x^{\rho})}\big) \in AUT(Q, \cdot) \implies V_4(Q, \cdot) \subseteq V_3(Q, \cdot).$$

Thus, $V_3(Q, \cdot) = V_4(Q, \cdot)$.

(c) Going by Theorem 4.5.4(2), $x\delta^{-1} \cdot x^{\rho} \in N(Q)$ for all $x \in Q$ and $\delta \in A(Q)$. Thus by (a) and (b), $A(Q, \cdot) = V_3(Q, \cdot) = V_4(Q, \cdot)$.

□

Theorem 4.5.7. *1. Let $(Q, \cdot)$ be a loop.*

(a) If $W_1(Q, \cdot) = \{\alpha \in A(Q, \cdot) \mid K(\alpha, x) = (R_x^{-1}R_{x\alpha}, L_xL_{x\alpha}^{-1}, I) \in AUT(Q, \cdot)\}$, then $W_1(Q, \cdot) \lhd A(Q, \cdot)$.

(b) If $(Q, \cdot)$ is a Basarab loop such that $x^{\lambda} \cdot x\alpha = (x\alpha \backslash x)^{\lambda} = x \backslash x\alpha$ for all $x \in Q$ and $\alpha \in A(Q)$ and $W_2(Q, \cdot) = \{\alpha \in A(Q, \cdot) \mid (R_x^{-1}R_{x\alpha}, L_xL_{x\alpha}^{-1}, I) \in AUT(Q, \cdot)\} = \big\{\alpha \in A(Q, \cdot) \mid x^{\lambda} \cdot x\alpha \in N(Q, \cdot)\big\}$. Then $W_1(Q, \cdot) = W_2(Q, \cdot)$.

(c) If $(H, \circ)$ is a Basarab loop such that $x^{\lambda} \cdot x\alpha = (x\alpha \backslash x)^{\lambda} = x \backslash x\alpha$ for all $x \in Q$ and $\alpha \in A(Q)$, then $A(Q, \cdot) = W_1(Q, \cdot) = W_2(Q, \cdot)$.

2. Let $(Q, \cdot)$ be a loop.

(a) If $W_3(Q, \cdot) = \big\{\delta \in A(Q, \cdot) \mid W(\delta, x) = (R_xR_{x\delta^{-1}}^{-1}, L_x^{-1}L_{x\delta^{-1}}, I) \in AUT(Q, \cdot)\big\}$, then $V_3(Q, \cdot) \lhd A(Q, \cdot)$.

(b) If $(Q,\cdot)$ is a Basarab loop such that

$x\delta^{-1}\cdot x^\rho=(x/x\delta^{-1})^\rho=x\delta^{-1}/x$ for all $x\in Q$

and $\delta\in A(Q)$ and $W_4(Q,\cdot)=\big\{\delta\in A(Q,\cdot)\mid(R_xR_{x\delta^{-1}}^{-1},L_x^{-1}L_{x\delta^{-1}},I)\in AUT(Q,\cdot)\big\}=$ $\{\delta\in A(Q,\cdot)\mid x\delta^{-1}\cdot x^\rho\in N(Q,\cdot)\}$. Then $W_3(Q,\cdot)=W_4(Q,\cdot)$.

(c) If $(H,\circ)$ is a Basarab loop such that $x\delta^{-1}\cdot x^\rho=(x/x\delta^{-1})^\rho=x\delta^{-1}/x$ for all $x\in Q$ and $\delta\in A(Q)$, then $A(Q,\cdot)=W_3(Q,\cdot)=W_4(Q,\cdot)$.

Proof. 1. (a) Let $W_1(Q,\cdot)=\{\alpha\in A(Q,\cdot)\mid(R_x^{-1}R_{x\alpha},L_xL_{x\alpha}^{-1},I)\in AUT(Q,\cdot)\}$. With $\alpha=I$, $(R_x^{-1}R_{xI},L_xL_{xI}^{-1},I)=(I,I,I)\in AUT(Q,\cdot)$. So, $W_1(Q,\cdot)\neq\emptyset$.

Let $\alpha,\beta\in W_1(Q)$, then

$$K(\alpha,x\alpha)K(\beta,x\alpha)=(R_x^{-1}R_{x\alpha},L_xL_{x\alpha}^{-1},I)(R_{x\alpha}^{-1}R_{x\alpha\beta},L_{x\alpha}L_{x\alpha\beta}^{-1},I)=$$
$$(R_x^{-1}R_{x\alpha}R_{x\alpha}^{-1}R_{x\alpha\beta},L_xL_{x\alpha}^{-1}L_{x\alpha}L_{x\alpha\beta}^{-1},I)=(R_x^{-1}R_{x\alpha\beta},L_xL_{x\alpha\beta}^{-1},I)\in AUT(Q,\cdot)$$
$$\implies\alpha\beta\in W_1(Q).$$

Next, we show that $\alpha^{-1}\in W_1(Q)$ for all $\alpha\in W_1(Q)$. Let $\alpha\in W_1(Q)$, then

$$\begin{aligned}
K(\alpha,x)=&(R_x^{-1}R_{x\alpha},L_xL_{x\alpha}^{-1},I)\in AUT(Q,\cdot)\implies\\
K(\alpha,x)^{-1}=&(R_{x\alpha}^{-1}R_x,L_{x\alpha}L_x^{-1},I)\in AUT(Q,\cdot)\implies\\
K(\alpha,x\alpha^{-1})=&(R_{x\alpha^{-1}}^{-1}R_{x\alpha^{-1}\alpha},L_{x\alpha^{-1}}L_{x\alpha^{-1}\alpha}^{-1},I)\\
=&(R_{x\alpha^{-1}}^{-1}R_x,L_{x\alpha^{-1}}L_x^{-1},I)\in AUT(Q,\cdot)\implies\\
K(\alpha,x\alpha^{-1})^{-1}=&(R_x^{-1}R_{x\alpha^{-1}},L_xL_{x\alpha^{-1}}^{-1},I)\in AUT(Q,\cdot)\implies\alpha^{-1}\in W_1(Q).
\end{aligned}$$

If $\alpha \in W_1(Q)$ and $\theta \in A(Q)$, then for all $x, a, b \in Q$,

$$K(\alpha, x) = (R_x^{-1}R_{x\alpha}, L_xL_{x\alpha}^{-1}, I) \in AUT(Q,\cdot) \iff (a/x \cdot x\alpha) \cdot x\alpha\backslash(xb) = ab \iff$$
$$((a/x \cdot x\alpha) \cdot x\alpha\backslash(xb))\,\theta = (ab)\theta \iff$$
$$(a\theta/x\theta \cdot x\alpha\theta) \cdot x\alpha\theta\backslash(x\theta \cdot b\theta) = a\theta \cdot b\theta \quad (\text{set } x \mapsto x\theta^{-1})$$
$$\iff (a\theta/x\theta^{-1}\theta \cdot x\theta^{-1}\alpha\theta) \cdot x\theta^{-1}\alpha\theta\backslash(x\theta^{-1}\theta \cdot b\theta) = a\theta \cdot b\theta \iff$$
$$(a\theta/x \cdot x\theta^{-1}\alpha\theta) \cdot x\theta^{-1}\alpha\theta\backslash(x \cdot b\theta) = a\theta \cdot b\theta \iff$$
$$K(\theta^{-1}\alpha\theta, x) = (R_x^{-1}R_{x\theta^{-1}\alpha\theta}, L_xL_{x\theta^{-1}\alpha\theta}^{-1}, I) \in AUT(Q,\cdot)$$
$$\iff K(\theta^{-1}\alpha\theta, x) \in AUT(Q,\cdot) \iff \theta^{-1}\alpha\theta \in W_1(Q).$$

Therefore, $W_1(Q,\cdot) \lhd A(Q,\cdot)$.

(b) Let $\alpha \in W_1(Q)$, then for all $x \in Q,\ K(\alpha, x) = (R_x^{-1}R_{x\alpha}, L_xL_{x\alpha}^{-1}, I) \in AUT(Q,\cdot)$. With the condition $x^\lambda \cdot x\alpha = (x\alpha\backslash x)^\lambda$ and the arguments from (4.33) to (4.34) of the proof of Theorem 4.5.4(1), $R_x^{-1}R_{x\alpha} = R_{(x^\lambda \cdot x\alpha)}$. Also, using the arguments from (4.35) to (4.36) of the proof of Theorem 4.5.4(1), $L_xL_{x\alpha}^{-1} = L_{(x^\lambda \cdot x\alpha)}^{-1}$. So,

$$K(\alpha, x) = (R_x^{-1}R_{x\alpha}, L_xL_{x\alpha}^{-1}, I) = \Big(R_{(x^\lambda \cdot x\alpha)}, L_{(x^\lambda \cdot x\alpha)}^{-1}, I\Big) \in AUT(Q,\cdot) \implies$$
$$W_1(Q,\cdot) \subseteq W_2(Q,\cdot).$$

Let $\alpha \in W_2(Q)$, then for all $x \in Q,\ \Big(R_{(x^\lambda \cdot x\alpha)}, L_{(x^\lambda \cdot x\alpha)}^{-1}, I\Big) \in AUT(Q,\cdot)$. With the condition $x^\lambda \cdot x\alpha = x\backslash x\alpha$ and the arguments from (4.37) to (4.38) of the proof of Theorem 4.5.4(1),

$$L_{x\alpha}L_x^{-1} = L_{(x^\lambda \cdot x\alpha)} \text{ and } R_x^{-1}R_{x\alpha} = R_{(x^\lambda \cdot x\alpha)} \implies$$
$$\Big(R_{(x^\lambda \cdot x\alpha)}, L_{(x^\lambda \cdot x\alpha)}^{-1}, I\Big) = (R_x^{-1}R_{x\alpha}, L_xL_{x\alpha}^{-1}, I)W_2(Q,\cdot) \subseteq W_1(Q,\cdot).$$

Thus, $W_1(Q,\cdot) = W_2(Q,\cdot)$.

(c) Going by (4.36) of Theorem 4.5.4(1), $x^\lambda \cdot x\alpha \in N(Q)$ for all $x \in Q$ and $\alpha \in A(Q)$. Thus by (a) and (b), $A(Q,\cdot) = V_1(Q,\cdot) = V_2(Q,\cdot)$.

2. This is similar to the proof of 1.

□

4.6 Characterization of Some Subloops of a Basarab loop

In this section, properties of some functions defined on a Basarab loop are established. Some subloops of a Basarab loop which are characterized by permutations are obtained, by application of these functions. The middle inner mapping is expressed in terms of a left (right) translation mapping and the middle translation mapping. Using some functions, necessary and sufficient conditions for a loop to be a left (right) Basarab loop, and Basarab loop are fine-tuned in the cases of a left (right) CC-loop, and CC-loop, respectively. Necessary and sufficient condition for a Basarab loop to be associative is given in terms of the middle inner mapping. Also, necessary and sufficient condition for the left and right inner mappings of a Basarab loop to coincide is established. With consideration of these functions defined on a Basarab loop, it is proved that a Basarab loop $(Q,\cdot)$ is a cross inverse property loop if and only if Q is commutative or an abelian group.

4.6.1 Properties of Some functions in a Basarab loop

Lemma 4.6.1. A loop $(Q,\cdot)$ is a Basarab loop if and only if there exist functions $g, h : Q\times Q \longrightarrow Q$ defined $g(x,\ y) = x\cdot yx^\rho$ and $h(x,z) = x^\lambda z\cdot x$ such that $L_x^{-1}L_yL_x = L_{g(x,y)}$ and $R_x^{-1}R_yR_x = R_{h(x,y)}$.

Proof. Let $(Q,\cdot)$ be a Basarab loop. Consider the Basarab identity $x \cdot y(xJ_\rho) \cdot xz = x \cdot$

$yz, \forall\, x, y, z \in Q$. We have that

$$zL_xL_{(x\cdot y(xJ_\rho))} = zL_yL_x \implies zL_xL(yR_{xJ_\rho}L_x) = zL_yL_x \implies L_xL_{(yR_{xJ_\rho}L_x)} = L_yL_x$$

$$\implies L_xL_g(x,y) = L_yL_x \implies L_{g(x,y)} = L_x^{-1}L_yL_x \implies L_{(x\cdot yx^\rho)} = L_x^{-1}L_yL_x.$$

Also, consider the Basarab identity $(yx)\cdot((xJ_\lambda z)\cdot x) = yz\cdot x$. This means

$$yR_xR_{((xJ_\lambda z)\cdot x)} = yR_zR_x \implies R_xR_{(zL_{xJ_\lambda}R_x)} = R_zR_x$$

$$\implies R_xR_{h(x,z)} = R_zR_x \implies R_{h(x,z)} = R_x^{-1}R_zR_x.$$

Thus, for all $x, y \in Q,\ R_{h(x,y)} = R_x^{-1}R_yR_x$. □

Lemma 4.6.2. Let $(Q,\ \cdot)$ be a Basarab loop. Then there are functions $g, h : Q\times Q \longrightarrow Q$ such that any of the following holds for all $x, y, z \in Q$:

1. $x\cdot yz = g(x,y)\cdot xz$.

2. $zy\cdot x = zx\cdot h(x,y)$.

3. $x = g(x,y)\cdot xy^\rho$: $CIP \iff g(x,y) = y \iff$ commmutative or abelian group.

4. $x\cdot y = g(x,y)\cdot x$: Q is commutative $\iff g(x,y) = y$.

5. $x = y^\lambda x\cdot h(x,y) : CIP \iff h(x,y) = y$.

6. $yx = x\cdot h(x,y)$: Q is commutative $\iff h(x,y) = y$.

7. $x\cdot yg(x,y) = g(x,y)\cdot xg(x,y)$.

8. $h(x,z) = g(x,z) \iff xh(x,y)\cdot g(x,z) = yz\cdot x$.

9. $g(x,y) = h(x,y) \iff h(x,y)\cdot(g(x,z)\cdot x) = x\cdot yz$.

10. $xh(x,y)\cdot((xz)/x) = yz\cdot x \iff g(x,z) = h(x,z)$.

11. $(x\backslash(yx))\cdot(g(x,z)\cdot x) = x\cdot yz \iff g(x,y) = h(x,y)$.

Proof. 1. From Lemma 4.6.1, a loop $(Q,\cdot)$ is a Basarab loop if and only if there are functions $g,h : Q^2 \longrightarrow Q$ so that $L_x^{-1}L_yL_x = L_{g(x,y)}$ and $R_x^{-1}R_yR_x = R_{h(x,y)}$ where $xy = (x\cdot yx^\rho)x$ and $yx = x(x^\lambda y\cdot x)$ hold for all $x,y \in Q$. Thus, $L_x^{-1}L_yL_x = L_{g(x,y)}$ implies that $L_yL_x = L_xL_{g(x,y)}$, so that for all $z \in Q,\ zL_yL_x = zL_xL_{g(x,y)}$ gives $x\cdot yz = g(x,y)\cdot xz$ for all $x,y,z \in Q$.

2. Also, $R_x^{-1}R_yR_x = R_{h(x,y)}$ implies $R_yR_x = R_xR_{h(x,y)}$. It follows that for every $z \in Q,\ zR_yR_x = zR_xR_{h(x,y)}$ implies $zy\cdot x = zx\cdot h(x,y)$.

3. Substitute $z = y^\rho$ in 1.

4. Substitute $z = e$ in 1.

5. Substitute $z = y^\lambda$ in 2.

6. Substitute $z = e$ in 2.

7. Substitute $z = g(x,y)$ in 1.

8. From the Basarab identity $(yx)\cdot(x^\lambda z\cdot x) = yz\cdot x$, it follows that $yx\cdot h(x,z) = yz\cdot x$ for all $x,y,z \in Q$. Replace $h(x,z)$ with $g(x,z)$ and consider yx using 6, then it follows that $h(x,z) = g(x,z)$ if and only if $xh(x,y)\cdot g(x,z) = yz\cdot x$ for all $x,y,z \in Q$.

9. From the Basarab identity $(x\cdot yx^\rho)\cdot xz = x\cdot yz,\ g(x,y)\cdot xz = x\cdot yz$ holds for all $x,y,z \in Q$. Then, replacing $g(x,y)$ with $h(x,y)$ and applying 4, it follows that $g(x,y) = h(x,y)$ if and only if $h(x,y)\cdot(g(x,z)\cdot x) = x\cdot yz$ for all $x,y,z \in Q$.

10. From 4, $xz = g(x,z)\cdot x$ implies $(xz)/x = g(x,z)$ for all $x,z \in Q$. Putting $g(x,z) = (xz)/x$ in 8 gives

$$xh(x,y)\cdot((xz)/x) = yz\cdot x \Leftrightarrow g(x,z) = h(x,z).$$

11. Using 6, $yx = x \cdot h(x, y)$ implies $h(x, y) = x\backslash(yx)$ for every $x, y \in Q$. Then 9 becomes

$$(x\backslash(yx)) \cdot g(x, z) \cdot x = x \cdot yz \Leftrightarrow g(x, y) = h(x, y).$$

□

Lemma 4.6.3. Let $(Q, \ \cdot)$ be a Basarab loop. Then, there are functions $g, h, m, n : Q \times Q \longrightarrow Q$ defined by $h(x, y) = x^{\lambda}y \cdot x, \ g(x, y) = x \cdot xy^{\rho}, \ m_z(x, y) = (zx)\backslash(zy \cdot x), \ n_z(x, y) = (x \cdot yz)/(xz)$ such that any of the following holds for all $x, y, z \in Q$:

1. $h(x, z) = g(x, z) \iff yx \cdot ((xz)/x) = yz \cdot x \iff g(x, z) = m_y(x, z).$
2. $h(x, y) = g(x, y) \iff (x\backslash(yx)) \cdot xz = x \cdot yz \iff h(x, y) = n_z(x, y).$
3. If $h(x, z) = g(x, z),$ then $x \cdot ((xz)/x) = zx.$
4. If $h(x, y) = g(x, y), \ (x\backslash(yx)) \cdot x = xy.$
5. If $h(x, z) = g(x, z),$ then $(xz)/x = x^{\lambda}z \cdot x.$
6. If $h(x, y) = g(x, y),$ then $x\backslash(yx) = x \cdot yx^{\rho}.$
7. If $h(x, y) = g(x, y),$ then $(xy)/x = x\backslash(yx)$ or $T_x = T_x^{-1}$ or $|T_x| = 2.$
8. If $h(x, z) = g(x, z),$ then $x\backslash(zx) = x^{\lambda}z \cdot x.$
9. If $h(x, y) = g(x, y),$ then $(xy)/x = x \cdot yx^{\rho}.$

Proof. 1. From 10 of Lemma 4.6.2,

$$h(x, z) = g(x, z) \iff xh(x, y) \cdot ((xz)/x) = yz \cdot x.$$

Now, $yx \cdot ((xz)/x) = yz \cdot x$ for all $x, y, z \in Q$ by 6 of Lemma 4.6.2. So, with

$$g(x, z) = ((xz)/x) \ \text{ and } m_y(x, z) = yx\backslash(yz \cdot x),$$

$$\text{we have } yx \cdot ((xz)/x) = yz \cdot x \iff g(x,z) = m_y(x,z).$$

2. Applying 11 of Lemma 4.6.2,

$$g(x,y) = h(x,y) \iff (x\backslash(yx)) \cdot (g(x,z) \cdot x) = x \cdot yz.$$

   Now, $(x\backslash(yx)) \cdot xz = x \cdot yz$ by 6 of Lemma 4.6.2, for all $x, y, z \in Q$. So, with $h(x,y) = (x\backslash(yx))$ and $n_z(x,y) = (x \cdot yz)/xz$. Then, we have

$$(x\backslash(yx)) \cdot xz = x \cdot yz \iff h(x,y) = n_z(x,y).$$

3. With $h = g$, set $y = e$ in 1, then $x \cdot ((xz)/x) = zx$.

4. With $h = g$, set $z = e$ in 2, then $(x\backslash(yx)) \cdot = xy$.

5. With $h = g$, set $y = x^\lambda$ in 1, then $(xz)/x = x^\lambda z \cdot x$.

6. With $h = g$, set $z = x^\rho$ in 2, then $x\backslash(yx) = x \cdot yx^\rho$.

7. From 3 or 4, $zL_xR_x^{-1}L_x = zR_x \implies L_xR_x^{-1} = R_xL_x^{-1} \implies yL_xR_x^{-1} = yR_xL_x^{-1}$ for all $x, y \in Q$. Then $(xy)/x = x\backslash(yx)$ for all $x, y \in Q$.

8. Using 7 in 5, then $x\backslash(zx) = x^\lambda z \cdot x$.

9. Using 7 in 6, then $(xy)/x = x \cdot yx^\rho$.

□

Lemma 4.6.4. Let $(Q, \cdot)$ be a Basarab loop. There are mappings $G, H : Q \longrightarrow SYM(Q)$ such that the following hold: $T_x^{-1} = G_x = J_\rho L_x M_x^{-1} = R_{x^\rho} L_x = L_x R_x^{-1}$ and $T_x = H_x = J_\lambda R_x M_x = L_{x^\lambda} R_x = R_x L_x^{-1}$ for all $x \in Q$.

Proof. From 3 of Lemma 4.6.2,

$$x = g(x,y)\cdot xy^\rho \implies g(x,y) = x/(xy^\rho) \implies g(x,y) = (xy^\rho)M_x^{-1}$$

$$\implies g(x,y) = y^\rho L_x M_x^{-1} \implies g(x,y) = yJ_\rho L_x M_x^{-1} \implies x{\cdot}yx^\rho = yJ_\rho L_x M_x^{-1} \text{ (by Lemma 4.6.1)}$$

$$\implies yR_{x^\rho}L_x = yJ_\rho L_x M_x^{-1} \implies R_{x^\rho}L_x = J_\rho L_x M_x^{-1} := G_x.$$

Also, from 5 of Lemma 4.6.2,

$$x = y^\lambda x\cdot h(x,y) \implies h(x,y) = (y^\lambda x)\backslash x \implies h(x,y) = (y^\lambda x)M_x$$

$$\implies h(x,y) = y^\lambda R_x M_x \implies h(x,y) = yJ_\lambda R_x M_x \implies x^\lambda y{\cdot}x = yJ_\lambda R_x M_x \text{ (by Lemma 4.6.1)}$$

$$\implies yL_{x^\lambda}R_x = yJ_\lambda R_x M_x \implies L_{x^\lambda}R_x = J_\lambda R_x M_x := H_x$$

□

Lemma 4.6.5. Let $(Q,\cdot)$ be a Basarab loop. Let $W_x = R_x M_x$, and $U'_x = L_x M_x^{-1}$. Then any two of the following implies the third:

1. $J_\lambda = J_\rho$
2. $G_x = H_x$
3. $U'_x = W_x$.

Proof. Let Q be a Basarab loop. Consider $G_x = J_\rho L_x M_x^{-1} = L_x R_x^{-1}$, $H_x = J_\lambda R_x M_x$, $W_x = R_x M_x$ and $U'_x = L_x M_x^{-1}$.

(i) $J_\rho = J_\lambda \iff G_x M_x L_x^{-1} = H_x M_x^{-1} R_x^{-1} \iff G_x(L_x M_x^{-1})^{-1} = H_x(R_x M_x)^{-1} \iff G_x U_x'^{-1} = H_x W_x^{-1} \iff I = H_x W_x^{-1} U'_x G_x^{-1} \iff W_x H_x^{-1} = U'_x G_x^{-1} \iff W_x G_x = U'_x H_x$. Thus, $J_\rho = J_\lambda$ and $G_x = H_x$ implies $U'_x = W_x$. Also, $J_\rho = J_\lambda$ and $U'_x = W_x$ implies $G_x = H_x$.

(ii) $G_x = H_x \iff J_\rho L_x M_x^{-1} = J_\lambda R_x M_x \iff J_\rho U_x' = J_\lambda W_x$. It follows that, $G_x = H_x$ and $J_\rho = J_\lambda$ implies $U_x' = W_x$. Also, $G_x = H_x$ and $U_x' = W_x$ implies $J_\rho = J_\lambda$.

(iii) $U_x' = W_x \iff R_x M_x = L_x M_x^{-1} \iff J_\rho H_x = J_\lambda G_x$. Thus, U_x' and $J_\rho = J_\lambda$ implies $H_x = G_x$. Also, $U_x' = W_x$ and $H_x = G_x$ implies $J_\rho = J_\lambda$. □

Lemma 4.6.6. Let $(Q, \cdot)$ be a Basarab loop. There are functions $g, h : Q \times Q \longrightarrow Q$ such that the following hold:

$h(x, y) = (y^\lambda x)\backslash x = x^\lambda y \cdot x = x\backslash(yx)$ and $g(x, y) = x/(xy^\rho) = x \cdot yx^\rho = (xy)/x$ for all $x, y \in Q$.

Proof. The proof follows from Lemma 4.6.1. □

Corollary 4.6.1. A loop is right Basarab if and only if it is RCC and $h(x, y) = n(x, y)$.

Proof. This holds by Corollary 4.3.4 and Lemma 4.6.3. □

Corollary 4.6.2. A loop is left Basarab if and only if it is LCC and $g(x, y) = n_z(x, y)$.

Proof. This holds by Corollary 4.3.3 and Lemma 4.6.3. □

Corollary 4.6.3. A loop $(Q, \cdot)$ is a Basarab loop if and only if it is a CC-loop and $g(x, y) = n_z(x, y)$ and $h(x, y) = m_z(x, y)$.

Proof. This is true by Theorem 4.3.5 and Lemma 4.6.3. □

Corollary 4.6.4. Let $(Q, \cdot)$ be a CC-loop. Then

$L_x^{-1} L_y L_x = L_{(xy)/x} = L_{n_e(x,y)}$ and $R_x^{-1} R_y R_x = R_{x\backslash(yx)} = R_{m_e(x,y)}$.

Proof. By CC-loop translations and Lemma 4.6.3 the result follows. □

Theorem 4.6.1. Let $(Q, \cdot)$ be a Basarab loop.

1. $(Q, \cdot)$ is associative if and only if $T_{xy} = T_x T_y$.

2. $|L_{(x,y)}| = 2$ if and only if $|R_{(x,y)}| = 2$ if and only if $|T_{(x,y)}| = 2$ for all $x, y \in Q$.

3. $R_{(x,y)} = L_{(y,x)}$ if and only if $(T_xT_y)^2 = T_{xy}^2$. Hence $(T_xT_y)^{2n} = T_{xy}^{2n}$.

4. Q is a CIPL if and only if Q is commutative or abelian group if and only if $g(x,y) = y$ if and only if $h(x,y) = y$.

5. Right and left distributive laws with respect to $g(\cdot,\cdot)$ and $h(\cdot,\cdot)$: (i) $h(x,z) = g(x,z)$ if and only if $yz \cdot x = xh(x,y) \cdot g(x,z)$ (ii) $h(x,z) = g(x,z)$ if and only if $x \cdot yz = h(x,y) \cdot (g(x,z) \cdot x)$.

6. $h(x,y) = g(x,y)$ or $m_e(x,y) = n_e(x,y)$ implies $|T_x| = 2$.

7. $T_xL_{(x,y)} = L_{(x,y)}T_x$ if and only if $L_{(x,y)}^{T_x} = L_{(x,y)}$ if and only if $T_x^{T_yT_x} = T_x^{T_{yx}} \iff [L_{(x,y)}, T_x] = I$ if and only if $T_x^{T_y} = T_x^{T_{yx}}$ if and only if ${}^{T_x}T_y^{-1}T_{yx} = T_y^{-1}T_{yx}$. Hence, (i) $[T_x, T_y] = I \iff [T_x, T_{xy}] = I$ (ii) $[T_x, T_y] = I \iff [T_x, T_{yx}] = I$.

8. Let $T_xL_{(y,x)} = L_{(y,x)}T_x$. Then, (i) $[T_x, T_{xy}] = I$ if and only if $[T_x, T_y^{-1}] = I$ if and only if $L_{(y,x)} = L_{(x,y)}$ (ii) $[T_x, T_y] = I \iff [T_x, T_{xy}] = I$.

9. $T_xR_{(x,y)} = R_{(x,y)}T_x$ if and only if $R_{(x,y)} = {}^{T_x}R_{(x,y)} \iff [R_{(x,y),T_x}] = I$ if and only if $T_yT_{xy}^{-1T_x} = T_yT_{xy}^{-1} \iff {}^{T_y}T_x = {}^{T_{xy}}T_x$. Hence, $|T_x| = I \iff {}^{T_y}T_x^{-1} = {}^{T_{xy}}T_x$ and $[T_x, T_y] = I \iff [T_x, T_{xy}] = I$.

10. $T_xR_{(y,x)} = R_{(y,x)}T_x$ if and only if $R_{(y,x)} = {}^{T_x}R_{(y,x)} \iff {}^{T_y}T_xT_{yx}^{-1T_x} = T_xT_{yx}^{-1}$. Hence, $[T_x, T_y] = I$ if and only if $[T_x, T_{yx}^{-1}] = I$.

11. $T_xT_yT_{xy}^{-1} = T_yT_xT_{yx}^{-1}$, $T_xT_y = T_yT_x$ if and only if $T_{xy} = T_{yx}$ if and only if $[T_x, T_y] = I$.

Proof. Let $(Q,\cdot)$ be a Basarab loop, then by Corollary 4.3.20,

$$L_{(x,y)} = T_x^{-1}T_y^{-1}T_{yx}, \ R_{(x,y)} = T_xT_yT_{xy}^{-1}.$$

1. Q is associative if and only if

$$L_{(x,y)} = I \iff R_{(x,y)} = I \iff T_{xy} = T_x T_y.$$

2.

$$|L_{(x,y)}| = 2 \iff L^2_{(x,y)} = I \iff T_x^{-1} T_y^{-1} T_{yx} = T_{yx}^{-1} T_y T_x$$

$$\iff (T_y T_x)^{-1} T_{yx} = T_{yx}^{-1} T_y T_x \iff T_{yx} = T_y T_x T_{yx}^{-1} T_y T_x$$

$$\iff T_y T_x T_{yx}^{-1} T_x T_{yx}^{-1} = I \iff |T_y T_x T_{yx}^{-1}| = 2 \iff |T_{(x,y)}| = 2.$$

$$|R_{(x,y)}| = 2 \iff R^2_{(x,y)} = I \iff T_x T_y T_{xy}^{-1} = T_{xy} T_y^{-1} T_x^{-1} \iff$$

$$T_x T_y T_{xy}^{-1} T_x T_y = T_{xy} \iff T_x T_y T xy^{-1} T_x T_y T_{xy}^{-1} = I \iff$$

$$|T_x T y T_{xy}^{-1}| = 2 \iff |T_{(x,y)}| = 2.$$

So, $|L_{(x,y)}| = 2$ if and only if $|R_{(y,x)}| = 2$ if and only if $|T_{(x,y)}| = 2$.

3. Furthermore,

$$R_{(x,y)} = L_{(y,x)} \iff T_x T_y T_{xy}^{-1} = T_y^{-1} T_x^{-1} T_{xy} \iff T_x T_y = T_y^{-1} T_x^{-1} T_{xy}^2$$

$$\iff T_x T_y T_x T_y = T_{xy}^{-2} \iff (T_x T_y)^2 = T_{xy}^2.$$

So, $R_{(x,y)} = L_{(y,x)} \iff (T_x T_y)^2 = T_{xy}^2$. The remaining part of the proof follows from Lemma 4.6.2 and Lemma 4.6.3.

4. Use 3,4,5,6 of Lemma 4.6.2.

5. Use 8,9,10,11 of Lemma 4.6.2.

6. Use 3 or 7 of Lemma 4.6.3.

7.

$$T_xL_{(x,y)} = L_{(x,y)}T_x \iff T_x(T_yT_x)^{-1}T_{yx} = (T_yT_x)^{-1}T_{yx}T_x$$

$$\iff (T_yT_x)T_x(T_yT_x)^{-1} = T_{yx}T_xT_{yx}^{-1} \Leftrightarrow T_x^{T_yT_x} = T_x^{T_{yx}}.$$

Let $T_xL_{(x,y)} = L_{(x,y)}T_x$, then

$$[T_x, T_{yx}] = I \iff T_x = T_{yx}T_xT_{xy}^{-1} \iff T_xT_{xy} = T_{yx}T_x \iff [T_x, T_{xy}] = I.$$

So, $[T_x, T_{yx}] = I \iff T_yT_xT_y^{-1} = T_x \iff T_yT_x = T_xT_y \iff [T_y, T_x] = I$. Hence,

$$T_xL_{(y,x)} = L_{(y,x)}T_x \iff L_{(y,x)} = {}^{T_x}L_{(y,x)} \iff [T_x, T_{xy}] = I.$$

8.

$$T_xL_{(y,x)} = L_{(y,x)}T_x \iff T_xT_y^{-1}T_x^{-1}T_{xy} = T_y^{-1}T_x^{-1}T_{xy}T_x$$

$$\iff T_x(T_xT_y)^{-1}T_{xy} = (T_xT_y)^{-1}T_{xy}T_x$$

$$\iff (T_xT_y)T_x(T_xT_y)^{-1} = T_{xy}T_xT_{xy}^{-1} \iff T_x^{T_xT_y} = T_x^{T_{xy}}.$$

Also, $T_xT_y^{-1}T_x^{-1} = T_y^{-1}T_x^{-1}T_{xy}T_xT_{xy}^{-1} \iff T_yT_xT_y^{-1}T_x^{-1} = T_x^{-1}T_{xy}T_xT_y^{-1} \iff$

$$T_yT_xT_y^{-1} = T_x^{-1}T_{xy}T_x(T_x^{-1}T_{xy})^{-1} \iff T_x^{T_y} = T_x^{T_x^{-1}T_{xy}}.$$

9.

$$T_xR_{(x,y)} = R_{y,x}T_x \iff T_xT_xT_yT_{xy}^{-1} = T_xT_yT_{xy}^{-1}T_x$$

$$\iff T_xT_yT_{xy}^{-1} = T_yT_{xy}^{-1}T_x \iff T_y^{-1}T_xT_yT_{xy}^{-1} = T_{xy}^{-1}T_x$$

$$\iff T_y^{-1}T_xT_y = T_{xy}^{-1}T_xT_{xy} \iff {}^{T_y}T_x^{=}\ {}^{T_{xy}}T_x.$$

Hence, (i) $|T_x| = 2 \iff T_yT_{xy}^{-1} = T_xT_yT_{xy}^{-1}T_x \iff T_y = T_xT_yT_{xy}^{-1}T_xT_{xy}$

$$\iff T_y^{-1}T_x^{-1}T_y = T_{xy}^{-1}T_xT_{xy} \iff {}^{T_y}T_x^{-1} = {}^{T_{xy}}T_x.$$

(ii) $[T_x, T_y] = I \iff T_yT_xT_{xy}^{-1} = T_yT_{xy}^{-1}T_x \iff T_xT_{xy}^{-1} = T_{xy}^{-1}T_x$

$$\iff T_{xy}T_x = T_xT_{xy} \iff [T_x, T_{xy}] = I.$$

10. $T_xR_{(y,x)} = R_{(y,x)}T_x \iff T_xT_yT_xT_{yx}^{-1} = T_yT_xT_{yx}^{-1}T_x$

$$\iff T_y^{-1}T_xT_yT_xT_{yx}^{-1}T_x^{-1} = T_xT_{yx}^{-1} \iff {}^{T_y}T_xT_{yx}^{-1\,T_x} = T_xT_{yx}^{-1}.$$

So,

$$[T_x, T_y] = I \iff T_yT_{yx}^{-1}T_x = T_yT_xT_{yx}^{-1} \iff T_{yx}^{-1}T_x = T_xT_{yx}^{-1} \iff [T_x, T_{yx}^{-1}] = I.$$

Hence, $[T_x, T_y] = I$ if and only if $[T_x, T_{yx}^{-1}] = I$.

11. A Basarab loop is a CCL. In a CCL, $R_{(x,y)} = R_{(y,x)}$. So, $T_xT_yT_{xy}^{-1} = T_yT_xT_{yx}^{-1}$.

□

4.6.2 Characterization of a Subloop of a Basarab loop by Middle Inner Mapping

Let $(Q, \cdot)$ be a Basarab loop. Then $Q' \ := \{x \in Q : G_x = H_x\} = \{x \in Q : T_x = T_x^{-1}\} = \{x \in Q : |T_x| = 2\}$ and

$$Q'_* := \{x \in Q : U'_x = W_x,\ U'_x = L_x M_x^{-1},\ W_x = R_x M_x\}.$$

From Lemma 4.6.4,

$$G_x = J_\rho L_x M_x^{-1} = R_{x^\rho} L_x = L_x R_x^{-1} \tag{4.39}$$

$$\text{and} \ \ H_x = J_\lambda R_x M_x = L_{x^\lambda} R_x = R_x L_x^{-1} \tag{4.40}$$

Theorem 4.6.2. Let $(Q, \cdot)$ be a Basarab loop. Then Q' is a subloop of Q, if

1. $R_{(x,y)} = L_{(y,x)}$ or $(T_x T_y)^2 = T_{xy}^2$ for all $x, y \in Q'$ and
2. $[T_x, T_y] = I$ for all $x, y \in Q'$.

Proof. $Q' \neq \emptyset$ because $T_e^2 = I \Rightarrow T_e = T_e^{-1} \implies e \in Q'$. Let $x, y \in Q'$, then $T_x^2 = T_y^2 = I$, and $T_{xy}^2 = (T_x T_y)^2 = T_x T_y T_x T_y = T_x^2 T_y^2 = II = I$. So, $xy \in Q'$. Therefore, Q' is a subloop of Q. □

Theorem 4.6.3. Let $(Q, \cdot)$ be a Basarab loop. Then for every $x \in Q$, x is contained in Q' or $|T_x| = 2$ if and only if:

1. $L_x M_x^{-1} L_{yx} = J_\lambda L_y R_x$
2. $M_x^{-1} R_x^{-1} = L_x^{-1} J_\lambda L_{x^\lambda}$ or $W_x^{-1} = L_x^{-1} J_\lambda L_{x^\lambda}$
3. $J_\rho L_x M_x^{-1} = L_y R_x L_{yx}^{-1}$

4. $T_x^{-1} = J_\lambda T_x U_x$.

   Hence,

5. $R_x^{-1} L_{x^\lambda}^{-1} R_x = T_x^{-1} L_{x^\lambda}^{-1} T_x$

6. $J_\rho L_{x^\lambda} R_x = L_{x^\lambda}^{-1} R_x L_x^{-1} M_x$

7. right Basarab law holds

Proof. Let $(Q, \cdot)$ be a Basarab loop. Using Lemma 4.6.4, we apply equations 4.39 and 4.40 as follows:

1. For every $x \in Q,\ x \in Q' \iff J_\rho L_x M_x^{-1} = L_{x^\lambda} R_x$

$$\iff L_x M_x^{-1} = J_\rho^{-1} L_{x^\lambda} R_x \iff L_x M_x^{-1} = J_\lambda L_{x^\lambda} R_x.$$

   Then, by Lemma 4.3.6, $L_x M_x^{-1} = J_\lambda L_{x^\lambda} R_x$

$$\iff L_x M_x^{-1} = J_\lambda L_y R_x L_{yx}^{-1} \iff L_x M_x^{-1} L_{yx} = J_\lambda L_y R_x,$$

   for all $y \in Q$.

2. For every $x \in Q,\ x \in Q' \iff J_\rho L_x M_x^{-1} = L_{x^\lambda} R_x$

$$\iff L_x M_x^{-1} = J_\lambda L_{x^\lambda} R_x \iff M_x^{-1} = L_x^{-1} J_\lambda L_{x^\lambda} R_x \iff M_x^{-1} R_x^{-1} = L_x^{-1} J_\lambda L_{x^\lambda}$$

$$\iff (R_x M_x)^{-1} = L_x^{-1} J_\lambda L_{x^\lambda} \iff W_x^{-1} = L_x^{-1} J_\lambda L_{x^\lambda}$$

3. For every $x \in Q,\ x \in Q' \iff J_\rho L_x M_x^{-1} = L_{x^\lambda} R_x$. Then, by Lemma 4.3.6,

$$J_\rho L_x M_x^{-1} = L_{x^\lambda} R_x \iff J_\rho L_x M_x^{-1} = L_y R_x L_{yx}^{-1} R_x^{-1} R_x$$

$$\iff J_\rho L_x M_x^{-1} = L_y R_x L_{yx}^{-1}$$

4. For every $x \in Q,\ x \in Q' \iff J_\rho L_x M_x^{-1} = R_x L_x^{-1}$

$$\iff L_x = J_\lambda T_x M_x \iff L_x R_x^{-1} = J_\lambda T_x M_x R_x^{-1} \iff T_x^{-1} = J_\lambda T_x U_x$$

5. For every $x \in Q,\ x \in Q' \iff J_\rho L_x M_x^{-1} = R_x L_x^{-1}$. Then, by Lemma 4.3.6,

$$J_\rho L_x M_x^{-1} = R_x L_x^{-1} = R_x L_x^{-1} \iff J_\rho L_x M_x^{-1} = L_{x^\lambda}^{-1} L_y R_x L_{yx}^{-1} L_x^{-1}$$

$$\iff J_\rho L_x M_x^{-1} L_x = L_{x^\lambda}^{-1} L_y R_x L_{yx}^{-1} \iff L_y R_x L_{yx}^{-1} L_x = L_{x^\lambda}^{-1} L_y R_x L_{yx}^{-1}.$$

Also, from proof of (1), $L_x M_x^{-1} = J_\lambda L_y R_x L_{yx}^{-1}$ Then

$$L_y R_x L_{yx}^{-1} L_x = L_{x^\lambda}^{-1} L_y R_x L_{yx}^{-1} \iff L_y R_x L_{yx}^{-1} L_x = L_{x^\lambda}^{-1} J_\rho L_x M_x^{-1}.$$

Also, from (3), $L_y R_x L_{yx}^{-1} = L_{x^\lambda}^{-1} \iff J_\rho L_x M_x^{-1} L_x = L_{x^\lambda}^{-1} J_\rho L_x M_x^{-1}$. Since, $J_\rho L_x M_x^{-1} = T_x$ by proof of (4), it follows that

$$J_\rho L_x M_x^{-1} L_x = L_{x^\lambda}^{-1} J_\rho L_x M_x^{-1} \iff T_x L_x = L_{x^\lambda}^{-1} T_x.$$

By (4), $L_x = R_x^{-1} L_{x^\lambda}^{-1} R_x$. Then, $T_x L_x = L_{x^\lambda}^{-1} T_x \iff T_x R_x^{-1} L_{x^\lambda}^{-1} R_x = L_{x^\lambda}^{-1} T_x$

$$\iff R_x^{-1} L_{x^\lambda}^{-1} R_x = T_x^{-1} L_{x^\lambda}^{-1} T_x.$$

6. For every $x \in Q,\ x \in Q' \iff J_\rho L_x M_x^{-1} = J_\lambda R_x M_x$. Then, by proof of (4),

$J_\rho L_x M_x^{-1} = T_x$ is obtained when (3) is applied. It follows that

$$J_\rho L_x M_x^{-1} = J_\lambda R_x M_x \iff T_x = J_\lambda R_x M_x \iff L_y R_x L_{yx}^{-1} = J_\lambda R_x M_x$$

by proof of (4). Next, we apply Lemma 4.3.6 on $L_y R_x L_{yx}^{-1} = J_\lambda R_x M_x$ and get

$$L_{x^\lambda} R_x L_{yx} L_{yx}^{-1} = J_\lambda R_x M_x \iff L_{x^\lambda} R_x = J_\lambda R_x M_x \iff J_\rho L_{x^\lambda} R_x = R_x M_x.$$

Then, by proof of (4), $R_x = L_{x^\lambda}^{-1} R_x L_x^{-1}$. So, $J_\rho L_{x^\lambda} R_x = R_x M_x$

$$\iff J_\rho L_{x^\lambda} R_x = L_{x^\lambda}^{-1} R_x L_x^{-1} M_x.$$

7. For every $x \in Q,\ x \in Q' \iff J_\rho L_x M_x^{-1} = J_\lambda R_x M_x$. Then, by (3), $L_y R_x L_{yx}^{-1} = J_\lambda R_x M_x$. By the proof of (6), $L_{x^\lambda} R_x L_{yx} L_{yx}^{-1} = J_\lambda R_x M_x$. Then,

$$L_{x^\lambda} R_x L_{yx} = J_\lambda R_x M_x \iff L_y R_x L_{yx}^{-1} = L_{x^\lambda} R_x L_{yx} L_{yx}^{-1} \iff L_y R_x = L_{x^\lambda} R_x L_{yx}.$$

Hence, right Basarab law holds by Lemma 4.3.6

□

Theorem 4.6.4. Let $(Q, \cdot)$ be a Basarab loop. Then for every $x \in Q$, x is contained in Q' or $|T_x| = 2$ if and only if:

1. $W_x R_{xz} = J_\rho R_z L_x$
2. $U_x T_x = R_x^{-1} J_\rho R_{x^\rho}$
3. $M_x^{-1} L_x^{-1} L_x R_x^{-1} J_\rho = R_{xz} L_x^{-1} R_z^{-1}$
4. $W_x = J_\rho T_x^{-1}$

Hence,

5. $L_x^{-1}R_{x^\rho}^{-1}L_x = T_xR_{x^\rho}^{-1}T_x^{-1}$

6. $R_{x^\rho}^{-1}L_xR_x^{-1}M_x^{-1} = J_\lambda R_{x^\rho}L_x$

7. the left Basarab law holds.

Proof. Let $(Q,\cdot)$ be a Basarab loop. Using Lemma 4.6.4, we apply equations 4.39 and 4.40 as follows:

1. For every $x \in Q,\ x \in Q' \iff J_\lambda R_xM_x = R_{x^\rho}L_x$

   $$\iff R_xM_x = J_\lambda^{-1}R_{x^\rho}L_x \iff R_xM_x = J_\rho R_{x^\rho}L_x.$$

   Then, by Lemm 4.3.6,

   $$R_xM_x = J_\rho R_{x^\rho}L_x \iff R_xM_x = J_\rho R_zL_xR_{xz}^{-1} \iff R_xM_xR_{xz} = J_\rho R_zL_x,$$

   for all $x,z \in Q \iff W_xR_{xz} = J_\rho R_zL_x$.

2. For every $x \in Q,\ x \in Q' \iff J_\lambda R_xM_x = R_{x^\rho}L_x$

   $$\iff R_xM_x = J_\rho R_{x^\rho}L_x \iff M_x = R_x^{-1}J_\rho R_{x^\rho}L_x \iff M_xL_x^{-1} = J_\rho R_{x^\rho},$$

   for all $x \in Q \iff M_xR_x^{-1}RxL_x^{-1} = R_x^{-1}J_\rho R_{x^\rho} \iff U_xT_x = R_x^{-1}J_\rho R_{x^\rho}$.

3. For every $x \in Q,\ x \in Q' \iff J_\lambda R_xM_x = R_{x^\rho}L_x$. Then, by Lemma 4.3.6,

   $$R_{x^\rho} = R_zL_xR_{xz}^{-1}L_x^{-1}.$$

   Thus, $J_\lambda R_xM_x = R_{x^\rho}L_x \iff J_\lambda R_xM_x = R_zL_xR_{xz}^{-1}L_x^{-1}L_x \iff J_\lambda R_xM_x = R_zL_xR_{xz}^{-1}$

$$\iff M_x^{-1}R_x^{-1}J_\rho = R_{xz}L_x^{-1}R_z^{-1} \iff M_x^{-1}L_x^{-1}L_xR_x^{-1}J_\rho = R_{xz}L_x^{-1}R_z^{-1}$$

4. For every $x \in Q,\ x \in Q' \iff J_\lambda R_xM_x = L_xR_x^{-1} \iff R_xM_x = J_\lambda^{-1}L_xR_x^{-1}$

$$\iff R_xM_x = J_\rho L_xR_x^{-1} \iff W_x = J_\rho T_x^{-1}.$$

5. For every $x \in Q,\ x \in Q' \iff J_\lambda R_xM_x = L_xR_x^{-1}$. By Lemma 4.3.6,

$$L_x = R_{x^\rho}^{-1}R_zL_xR_{xz}^{-1}.$$

Thus,$J_\lambda R_xM_x = L_xR_x^{-1} \iff J_\lambda R_xM_x = R_{x^\rho}^{-1}R_zL_xR_{xz}^{-1}R_x^{-1}$

$$\iff J_\lambda R_xM_xR_x = R_{x^\rho}^{-1}R_zL_xR_{xz}^{-1}.$$

By (3), it follows that,

$$J_\lambda R_xM_xR_x = R_{x^\rho}^{-1}R_zL_xR_{xz}^{-1} \iff R_zL_xR_{xz}^{-1}R_x = R_{x^\rho}^{-1}R_zL_xR_{xz}^{-1}.$$

From (1), $J_\lambda R_xM_x = R_zL_xR_{xz}^{-1}$. Then, $R_zL_xR_{xz}^{-1}R_x = R_{x^\rho}^{-1}R_xL_xR_{xz}^{-1}$

$$\iff R_zL_xR_{xz}^{-1}R_x = R_{x^\rho}^{-1}J_\lambda R_xM_x.$$

Using (3), $R_zL_xR_{xz}^{-1}R_x = R_{x^\rho}^{-1}J_\lambda R_xM_x \iff J_\lambda R_xM_xR_x = R_{x^\rho}^{-1}J_\lambda R_xM_x$

$$\iff T_x^{-1}R_x = R_{x^\rho}^{-1}T_x^{-1},$$

since $J_\lambda R_xM_x = T_x^{-1}$ in the proof of (4). Then, $T_x^{-1}R_x = R_{x^\rho}^{-1}T_x^{-1} \iff$

$T_x^{-1}L_x^{-1}R_{x^\rho}^{-1}L_x = R_{x^\rho}^{-1}T_x^{-1}$, using (4). Hence,

$$T_x^{-1}L_x^{-1}R_{x^\rho}^{-1}L_x = R_{x^\rho}^{-1}T_x^{-1} \iff L_x^{-1}R_{x^\rho}^{-1}L_x = T_xR_{x^\rho}^{-1}T_x^{-1}.$$

6. For every $x \in Q,\ x \in Q' \iff J_\rho L_x M_x^{-1} = J_\lambda R_x M_x$. From proof of (4), $J_\lambda R_x M_x = T_x^{-1}$, it follows that

$$J_\rho L_x M_x^{-1} = J_\lambda R_x M_x \iff J_\rho L_x M_x^{-1} = T_x^{-1} \iff J_\rho L_x M_x^{-1} = R_z L_x R_{xz}^{-1},$$

by using $T_x^{-1} = R_z L_x R_{xz}^{-1}$ in proof of (4). Then

$$J_\rho L_x M_x^{-1} = T_x^{-1} \iff J_\rho L_x M_x^{-1} = R_z L_x R_{xz}^{-1}.$$

By Lemma 4.3.6, $R_{x^\rho}L_xR_{xz} = R_zL_x$, it follows that

$$J_\rho L_x M_x^{-1} = R_z L_x R_{xz}^{-1} \iff J_\rho L_x M_x^{-1} = R_{x^\rho}L_xR_{xz}R_{xz}^{-1} \iff J_\rho L_x M_x^{-1} = R_{x^\rho}L_x$$

$$\iff L_x M_x^{-1} = J_\lambda R_{x^\rho} L_x.$$

By (4), $L_x M_x^{-1} = J_\lambda R_{x^\rho} L_x \iff R_{x^\rho}^{-1} L_x R_x^{-1} M_x^{-1} = J_\lambda R_{x^\rho} L_x$.

7. For every $x \in Q,\ x \in Q' \iff J_\rho L_x M_x^{-1} = J_\lambda R_x M_x \iff J_\rho L_x M_x^{-1} = R_z L_x R_{xz}^{-1}$. From proof of (6), $J_\rho L_x M_x^{-1} = R_{x^\rho} L_x$. Then

$$R_{x^\rho} L_x = R_z L_x R_{xz}^{-1} \iff R_{x^\rho} L_x R_{xz} = R_z L_x.$$

Hence, the left Basarab law holds.

□

Theorem 4.6.5. Let $(Q, \cdot)$ be a Basarab loop. Then $Q'_* \neq \emptyset$ if and only if $J_\lambda = J_\rho$.

Proof. Given $Q'_* := \{x \in Q : U'_x = W_x,\ U'_x = L_x M_x^{-1},\ W_x = R_x M_x\}$,

$$Q'_* \neq \emptyset \Leftrightarrow e \in Q'_* \Leftrightarrow U'_e = W_e$$

$$\Leftrightarrow L_e M_e^{-1} = R_e M_e \Leftrightarrow M_e^{-1} = M_e \Leftrightarrow J_\lambda = J_\rho.$$

Therefore, Q'_* is a subloop of Q. □

Theorem 4.6.6. Let $(Q, \cdot)$ be a Basarab loop. Then for every $x \in Q$, x is contained in Q'_* if and only if:

1. $W_x R_{xz} = J_\lambda R_z L_x$
2. $J_\lambda G_x = W_x$
3. $M_x = R_x^{-1} J_\lambda T_x^{-1}$
4. $J_\rho L_y R_x L_{yx}^{-1} = U'_x$
5. $U'_x = J_\rho L_{x^\lambda} R_x$ or $W_x = J_\lambda^{-1} L_x R_x^{-1}$
6. $M_x^{-1} = L_x^{-1} J_\rho T_x$
7. $R_{x^\rho} = L_{x^\rho} T_x$
8. $L_y R_x = L_x R_x^{-1} L_{yx}$
9. $G_x = R_x L_x^{-1}$
10. $R_x M_x L_x^{-1} = J_\rho R_{x^\rho}$.

Proof. Let $(Q, \cdot)$ be a Basarab loop, by applying the mappings $U'_x = L_x M_x^{-1}$ and $W_x = R_x M_x$ to Lemma 4.6.4, it follows that:

$G_x = J_\rho U'_x = R_{x^\rho} L_x = L_x R_x^{-1}$ and $H_x = J_\lambda W_x = L_{x^\lambda} R_x = R_x L_x^{-1}$. This implies that

$$J_\lambda G_x = U'_x = J_\lambda R_{x^\rho} L_x = J_\lambda L_x R_x^{-1} \tag{4.41}$$

$$\text{and } J_\rho H_x = W_x = J_\rho L_{x^\lambda} R_x = J_\rho R_x L_x^{-1}. \tag{4.42}$$

By equations 4.41 and 4.42, it follows that:

1. For every $x \in Q,\ x \in Q'_* \iff J_\rho H_x = J_\lambda R_{x^\rho} L_x \iff J_\rho H_x L_x^{-1} = J_\lambda R_{x^\rho}$

$$\iff J_\rho J_\lambda R_x M_x L_x^{-1} = J_\lambda R_{x^\rho} \iff R_x M_x L_x^{-1} = J_\lambda R_{x^\rho}$$

$$\iff W_x L_x^{-1} = J_\lambda R_{x^\rho} \iff W_x = J_\lambda R_{x^\rho} L_x$$

   , since $H_x = J_\lambda R_x M_x$. By Lemma 4.3.6, $R_{x^\rho} L_x = R_z L_x R_{xz}^{-1}$. It follows that, $x \in Q'_* \iff W_x = J_\lambda R_{x^\rho} L_x \iff W_x = J_\lambda R_z L_x R_{xz}^{-1} \iff W_x R_{xz} = J_\lambda R_z L_x$.

2. For every $x \in Q,\ x \in Q'_* \iff J_\lambda G_x = J_\rho H_x \iff G_x = J_\lambda^{-1} J_\rho H_x$

$$\iff G_x = J_\rho^2 H_x \iff J_\rho L_x M_x^{-1} = J_\rho J_\rho H_x \iff U'_x = J_\rho H_x,$$

   since $G_x = J_\rho L_x M_x^{-1}$. Also, for every $x \in Q,\ x \in Q'_* \iff J_\lambda G_x = J_\rho H_x \iff$

$$J_\lambda G_x = J_\rho J_\lambda R_x M_x \iff J_\lambda G_x = R_x M_x \iff J_\lambda G_x = W_x.$$

3. For every $x \in Q,\ x \in Q'_* \iff J_\rho H_x = J_\lambda L_x R_x^{-1} \iff J_\rho H_x R_x = J_\lambda L_x$

$$\iff J_\rho J_\lambda R_x M_x R_x = J_\lambda L_x \iff R_x M_x R_x = J_\lambda L_x$$

   $\iff W_x R_x = J_\lambda L_x \iff W_x = J_\lambda L_x R_x^{-1} \iff W_x = J_\lambda T_x^{-1}$, since $H_x = J_\lambda R_x M_x$.

Also, from the preceding part,

$$x \in Q'_* \iff R_x M_x R_x = J_\lambda L_x \iff M_x R_x = R_x^{-1} J_\lambda L_x$$

$$\iff M_x = R_x^{-1} J_\lambda L_x R_x^{-1} \iff M_x = R_x^{-1} J_\lambda R_{x^\lambda} L_x \iff M_x = R_x^{-1} J_\lambda T_x^{-1}.$$

4. For every $x \in Q,\ x \in Q'_*$

$$\iff J_\rho L_{x^\lambda} R_x = J_\lambda G_x \iff L_{x^\lambda} R_x = J_\rho^{-1} J_\lambda G_x \iff L_{x^\lambda} R_x = J_\lambda^2 G_x$$

. By Lemma 4.3.6, $L_{x^\lambda} R_x = J_\lambda^2 G_x \iff L_y R_x L_{yx}^{-1} = J_\lambda^2 G_x$

$$\iff R_x L_{yx}^{-1} = L_y^{-1} J_\lambda^2 G_x \iff L_y R_x = J_\lambda^2 G_x L_{yx} \iff J_\rho L_y R_x = J_\lambda G_x L_{yx}$$

$$\iff J_\rho L_y R_x = J_\lambda J_\rho L_x M_x^{-1} L_{yx} \iff J_\rho L_y R_x = L_x M_x^{-1} L_{yx}$$

$$\iff J_\rho L_y R_x = U'_x L_{yx} \iff J_\rho L_y R_x L_{yx}^{-1} = U'_x$$

5. For every $x \in Q$,

$$x \in Q'_* \iff J_\lambda R_{x^\rho} L_x = J_\rho L_{x^\lambda} R_x.$$

Using Lemma 4.6.4, $R_{x^\rho} = J_\rho L_x M_x^{-1} L_x^{-1}$. Then $J_\lambda R_{x^\rho} L_x = J_\rho L_{x^\lambda} R_x$

$$\iff J_\lambda J_\rho L_x M_x^{-1} L_x^{-1} L_x = J_\rho L_{x^\lambda} R_x \iff L_x M_x^{-1} = J_\rho L_{x^\lambda} R_x \iff U'_x = J_\rho L_{x^\lambda} R_x.$$

Next, for every $x \in Q,\ x \in Q'_* \iff J_\rho L_{x^\lambda} R_x = J_\rho L_x R_x^{-1}$

$$\iff L_{x^\lambda} = J_\rho^{-1} J_\lambda L_x R_x^{-1} R_x^{-1} \iff L_{x^\lambda} = J_\lambda^2 L_x R_x^{-2}.$$

By Lemma 4.6.4,

$$L_{x^\lambda} = J_\lambda R_x M_x R_x^{-1},\ L_{x^\lambda} = J_\lambda^2 L_x R_x^{-2} \iff J_\lambda R_x M_x R_x^{-1} = J_\lambda^2 L_x R_x^{-2}$$

$$\iff R_x M_x R_x^{-1} = J_\lambda^{-1} L_x R_x^{-2} \iff R_x M_x = J_\lambda^{-1} L_x R_x^{-1} \iff W_x = J_\lambda^{-1} L_x R_x^{-1}$$

6. For every $x \in Q$,

$$x \in Q'_* \iff J_\lambda G_x = J_\rho R_x L_x^{-1} \iff J_\lambda G_x L_x = J_\rho R_x$$

$$\iff J_\lambda J_\rho M_x^{-1} L_x = J_\rho R_x \iff M_x^{-1} L_x = L_x^{-1} J_\rho R_x \iff M_x^{-1} = L_x^{-1} J_\rho R_x L_x^{-1}$$

$$\iff M_x^{-1} = L_x^{-1} J_\rho L_{x^\lambda} \iff M_x^{-1} = L_x^{-1} J_\rho T_x.$$

7. For every $x \in Q$,

$$x \in Q'_* \iff J_\lambda R_{x^\rho} L_x = J_\rho L_{x^\lambda} R_x \iff J_\lambda R_{x^\rho} = J_\rho L_{x^\lambda} R_x L_x^{-1}$$

$$\iff J_\lambda R_{x^\rho} = J_\rho L_{x^\lambda} L_{x^\lambda} R_x \iff J_\lambda R_{x^\rho} = J_\rho L_{x^\lambda} T_x.$$

8. For every $x \in Q,\ x \in Q'_* \iff J_\rho L_{x^\lambda} R_x = J_\lambda L_x R_x^{-1}$

$$\iff J_\rho L_{x^\lambda} = J_\lambda L_x R_x^{-2} \iff J_\rho = J_\lambda L_x R_x^{-2} L_{x^\lambda}^{-1}.$$

By Lemma 4.3.6, $L_{x^\lambda}^{-1} = R_x L_{yx} R_x^{-1} L_y^{-1}$, it follows that,

$$J_\rho = J_\lambda L_x R_x^{-2} L_{x^\lambda}^{-1} \iff J_\rho = J_\lambda L_x R_x^{-2} R_x L_{yx} R_x^{-1} L_y^{-1}$$

$$\iff J_\rho = J_\lambda L_x R_x^{-1} L_{yx} R_x^{-1} L_y^{-1} \iff J_\rho L_y R_x = J_\lambda L_x R_x^{-1} L_{yx}$$

$$\iff L_yR_x = L_xR_x^{-1}L_{yx}.$$

9. For every $x \in Q,\ x \in Q'_* \iff J_\lambda G_x = J_\rho R_x L_x^{-1}$

$$\iff G_x = J_\lambda^{-1} J_\rho R_x L_x^{-1} \iff G_x = J_\rho^2 R_x L_x^{-1} \iff G_x = R_x L_x^{-1}$$

10. For every $x \in Q$,

$$x \in Q'_* \iff J_\rho H_x = J_\lambda R_{x^\rho} L_x \iff H_x = J_\rho^{-1} J_\lambda R_{x^\rho} L_x \iff H_x = J_\lambda J_\lambda R_{x^\rho} L_x$$

$$\iff J_\lambda R_x M_x = J_\lambda^2 R_{x^\rho} L_x \iff R_x M_x = J_\lambda R_{x^\rho} L_x \iff R_x M_x L_x^{-1} = J_\lambda R_{x^\rho}$$

$$\iff R_x M_x L_x^{-1} = J_\rho R_{x^\rho}.$$

□

4.6.3 Characterization of a Subloop of a Basarab loop by Inverse Translation Mappings

Let $(Q, \cdot)$ be a Basarab loop, and let Q^{ii} be the set for all $x \in Q$, such that the set $\{x \in Q : xJ_\lambda = xJ_\rho\}$ is satisfied. The elements of $(Q, \cdot)$ containing in Q^{ii} are characterized. From Lemma 4.6.4,

$$G_x M_x L_x^{-1} = J_\rho = R_{x^\rho} L_x M_x L_x^{-1} = L_x R_x^{-1} M_x L_x^{-1} \quad (4.43)$$

$$\text{and } H_x M_x^{-1} R_x^{-1} = J_\lambda = L_{x^\lambda} R_x M_x^{-1} R_x^{-1} = R_x L_x^{-1} M_x^{-1} R_x^{-1}. \quad (4.44)$$

Theorem 4.6.7. Let $(Q, \cdot)$ be a Basarab loop. Then Q^{ii} is a subloop of Q, if Q has AIP or $AAIP$.

Proof. $Q^{ii} \neq \emptyset \iff e^\lambda = e^\rho$. By AIP, $(xy)^\rho = x^\rho y^\rho = x^\lambda y^\lambda = (xy)^\lambda \iff (xy)^\rho = (xy)^\lambda$ for all $x, y \in Q^{ii}$. By $AAIP$, $(xy)^\rho = y^\rho x^\rho = y^\lambda x^\lambda = (xy)^\lambda \iff (xy)^\rho = (xy)^\lambda$ for all $x, y \in Q^{ii}$. Let $x, y \in Q^{ii}$, then $x^\rho = x^\lambda$ and $y^\rho = y^\lambda$. □

Theorem 4.6.8. Let $(Q, \cdot)$ be a Basarab loop. Then for every $x \in Q,\ x$ is contained in Q^{ii} if and only if any of the following conditions hold :

1. $W_x G_x^2 = U'_x$ 2. $G_x = J_\lambda U'_x$ 3. $H_x = J_\rho W_x$.

Proof. By equations 4.43 and 4.44, it follows that:

1. For every $x \in Q$,

$$x \in Q^{ii} \Leftrightarrow G_x M_x L_x^{-1} = H_x M_x^{-1} R_x^{-1} \Leftrightarrow G_x M_x = H_x M_x^{-1} R_x^{-1} L_x$$

$$\Leftrightarrow H_x^{-1} G_x M_x = M_x^{-1} R_x^{-1} L_x \Leftrightarrow G_x G_x M_x = M_x^{-1} R_x^{-1} L_x$$

$$\Leftrightarrow G_x^2 M_x = M_x^{-1} R_x^{-1} L_x \Leftrightarrow G_x^2 = (R_x M_x)^{-1} L_x M_x^{-1} \Leftrightarrow G_x^2 = W_x^{-1} U'_x \Leftrightarrow W_x G_x^2 = U'_x.$$

2. For every $x \in Q$,

$$x \in Q^{ii} \Leftrightarrow L_x R_x^{-1} M_x L_x^{-1} = R_x L_x^{-1} M_x^{-1} R_x^{-1} \Leftrightarrow R_x^{-1} M_x L_x^{-1} R_x = L_x^{-1} R_x L_x^{-1} M_x^{-1}$$

$$\Leftrightarrow R_x^{-1} M_x L_x^{-1} R_x = L_x^{-1} H_x M_x^{-1} \Leftrightarrow R_x^{-1} M_x L_x^{-1} R_x = L_x^{-1} J_\lambda R_x M_x M_x^{-1}$$

$$\Leftrightarrow R_x^{-1} M_x L_x^{-1} R_x = L_x^{-1} J_\lambda R_x \Leftrightarrow R_x^{-1} M_x L_x^{-1} = L_x^{-1} J_\lambda \Leftrightarrow R_x^{-1} M_x = L_x^{-1} J_\lambda L_x$$

$$\Leftrightarrow I = H_x J_\lambda U'_x \Leftrightarrow G_x = J_\lambda U'_x.$$

3. For every $x \in Q$,

$$x \in Q^{ii} \Leftrightarrow L_x R_x^{-1} M_x L_x^{-1} = R_x L_x^{-1} M_x^{-1} R_x^{-1}$$

$$\Leftrightarrow G_xM_xL_x^{-1} = R_xL_x^{-1}M_x^{-1}R_x^{-1} \Leftrightarrow J_\rho L_xM_x^{-1}M_xL_x^{-1} = R_xL_x^{-1}M_x^{-1}R_x^{-1}$$

$$\Leftrightarrow J_\rho = R_xL_x^{-1}M_x^{-1}R_x^{-1} \Leftrightarrow R_x^{-1}J_\rho R_x = L_x^{-1}M_x^{-1} \Leftrightarrow H_x = J_\rho W_x.$$

□

Theorem 4.6.9. Let $(Q,\cdot)$ be a Basarab loop. Then for every $x \in Q,\ x$ is contained in Q^{ii} if and only if any of the following conditions hold :

1. $G_x^2M_x = M_x^{-1}R_x^{-1}L_x$
2. $M_xL_x^{-1}R_x = H_x^2M_x^{-1}$
3. $R_xM_xL_x = L_xM_x^{-1}H_xR_x$
4. $R_xM_xG_xL_x = L_xM_x^{-1}R_x$
5. $R_x^{-1}L_xR_x^{-1} = (R_xM_xL_x)^{-1}L_xM_x^{-1}$
6. $R_x^{-1}M_xL_x^{-1}R_xM_x = L_x^{-1}R_xL_x^{-1}$
7. $R_{x^\rho}^{-1}H_xM_x^{-1} = L_xM_xL_x^{-1}R_x$
8. $M_x^{-1}R_x^{-1}L_x = H_x^{-1}R_{x^\rho}L_xM_x$
9. $L_x^{-1}R_xM_xL_xR_x^{-1} = (R_{x^\rho}L_xM_x)^{-1}$
10. $L_x^{-1}R_{x^\rho}^{-1}H_x = M_xL_x^{-1}R_xM_x$
11. $L_x^{-1}M_x^{-1}R_x^{-1}L_x = R_x^{-1}R_{x^\rho}L_xM_x$
12. $L_x^{-1}H_xM_x^{-1}R_x^{-1} = R_x^{-1}M_xL_x^{-1}$
13. $L_xR_x^{-1}L_xR_x^{-1}M_x = M_x^{-1}R_x^{-1}L_x$
14. $M_xL_x^{-1}R_xM_xL_x = R_xL_x^{-1}R_x$

15. $L_{x^\lambda}^{-1}L_xR_x^{-1}M_x = R_xM_x^{-1}R_x^{-1}L_x$

16. $L_xM_xL_x^{-1} = R_{x^\rho}^{-1}R_xL_x^{-1}M_x^{-1}R_x^{-1}$

17. $R_x^{-1}M_x = L_x^{-1}J_\lambda L_x$

18. $R_x^{-1}J_\rho R_x = L_x^{-1}M_x^{-1}$

Proof. Let $(Q,\cdot)$ be a Basarab loop. Using Lemma 4.6.4, we apply equations 4.43 and 4.44 as follows:

1. For every $x \in Q$,

$$x \in Q^{ii} \iff G_xM_xL_x^{-1} = H_xM_x^{-1}R_x^{-1} \iff G_xM_x = H_xM_x^{-1}R_x^{-1}L_x$$

$$\iff H_x^{-1}G_xM_x = M_x^{-1}R_x^{-1}L_x \iff G_xG_xM_x = M_x^{-1}R_x^{-1}L_x$$

$$\iff G_x^2M_x = M_x^{-1}R_x^{-1}L_x$$

2. For every $x \in Q$,

$$x \in Q^{ii} \iff G_xM_xL_x^{-1} = H_xM_x^{-1}R_x^{-1}$$

$$\iff G_xM_xL_x^{-1}R_x = H_xM_x^{-1} \iff M_xL_x^{-1}R_x = G_x^{-1}H_xM_x^{-1}$$

$$\iff M_xL_x^{-1}R_x = H_xH_xM_x^{-1} \iff M_xL_x^{-1}R_x = H_x^2M_x^{-1}$$

3. For every $x \in Q$,

$$x \in Q^{ii} \iff G_xM_xL_x^{-1} = H_xM_x^{-1}R_x^{-1}$$

$$\iff M_xL_x^{-1} = G_x^{-1}H_xM_x^{-1}R_x^{-1} \iff M_xL_x^{-1} = G_x^{-1}H_x(R_xM_x)^{-1}$$

$$\iff M_xL_x^{-1}R_xM_x = G_x^{-1}H_x \iff L_x^{-1}R_xM_x = M_x^{-1}G_x^{-1}H_x$$

$$\iff L_x^{-1}R_xM_x = M_x^{-1}H_xR_xL_x^{-1} \iff L_x^{-1}R_xM_xL_x = M_x^{-1}H_xR_x$$

$$\iff R_xM_xL_x = L_xM_x^{-1}H_xR_x$$

4. For every $x \in Q,\ x \in Q^{ii} \iff G_xM_xL_x^{-1} = H_xM_x^{-1}R_x^{-1}$

$$\iff H_x^{-1}G_xM_xL_x^{-1} = M_x^{-1}R_x^{-1} \iff H_x^{-1}G_x = M_x^{-1}R_x^{-1}L_xM_x^{-1}$$

$$\iff M_xH_x^{-1}G_x = R_x^{-1}L_xM_x^{-1} \iff M_xH_x^{-1}L_xR_x^{-1} = R_x^{-1}L_xM_x^{-1}$$

$$\iff M_xG_xL_x = R_x^{-1}L_xM_x^{-1}R_x \iff R_xM_xG_xL_x = L_xM_x^{-1}R_x$$

5. For every $x \in Q,\ x \in Q^{ii} \iff G_xM_xL_x^{-1} = H_xM_x^{-1}R_x^{-1}$

$$\iff G_xM_xL_x^{-1} = R_xL_x^{-1}M_x^{-1}R_x^{-1} \iff R_x^{-1}G_xM_xL_x^{-1} = L_x^{-1}M_x^{-1}R_x^{-1}$$

$$\iff R_x^{-1}G_xM_x = L_x^{-1}M_x^{-1}R_x^{-1}L_x \iff R_x^{-1}L_xR_x^{-1}M_x = L_x^{-1}M_x^{-1}R_x^{-1}L_x$$

$$\iff R_x^{-1}L_xR_x^{-1} = L_x^{-1}M_x^{-1}R_x^{-1}L_xM_x^{-1} \iff R_x^{-1}L_xR_x^{-1} = (R_xM_xL_x)^{-1}L_xM_x^{-1}$$

6. For every $x \in Q$,

$$x \in Q^{ii} \iff G_xM_xL_x^{-1} = H_xM_x^{-1}R_x^{-1} \iff L_xR_x^{-1}M_xL_x^{-1} = R_xL_x^{-1}M_x^{-1}R_x^{-1}$$

$$\iff R_x^{-1}M_xL_x^{-1} = L_x^{-1}H_xM_x^{-1}R_x^{-1} \iff R_x^{-1}M_xL_x^{-1} = L_x^{-1}R_xL_x^{-1}M_x^{-1}R_x^{-1}$$

$$\iff R_x^{-1}M_xL_x^{-1}R_x = L_x^{-1}R_xL_x^{-1}M_x^{-1} \iff R_x^{-1}M_xL_x^{-1}R_xM_x = L_x^{-1}R_xL_x^{-1}$$

7. For every $x \in Q$,

$$x \in Q^{ii} \iff H_xM_x^{-1}R_x^{-1} = R_{x^\rho}L_xM_xL_x^{-1}$$

$$\iff R_{x^\rho}^{-1}H_xM_x^{-1}R_x^{-1} = L_xM_xL_x^{-1} \iff R_{x^\rho}^{-1}H_xM_x^{-1} = L_xM_xL_x^{-1}R_x$$

8. For every $x \in Q$,

$$x \in Q^{ii} \iff H_xM_x^{-1}R_x^{-1} = R_{x^\rho}L_xM_xL_x^{-1}$$

$$\iff M_x^{-1}R_x^{-1} = H_x^{-1}R_{x^\rho}L_xM_xL_x^{-1} \iff M_x^{-1}R_x^{-1}L_x = H_x^{-1}R_{x^\rho}L_xM_x$$

9. For every $x \in Q$,

$$x \in Q^{ii} \iff H_xM_x^{-1}R_x^{-1} = R_{x^\rho}L_xM_xL_x^{-1}$$

$$\iff R_xL_x^{-1}M_x^{-1}R_x^{-1} = R_{x^\rho}L_xM_xL_x^{-1} \iff L_x^{-1}M_x^{-1}R_x^{-1} = R_x^{-1}R_{x^\rho}L_xM_xL_x^{-1}$$

$$\iff (R_xM_xL_x)^{-1} = R_x^{-1}R_{x^\rho}L_xM_xL_x^{-1} \iff (R_xM_xL_x)^{-1} = (L_xM_x^{-1}L_x^{-1}R_{x^\rho}^{-1}R_x)^{-1}$$

$$\iff R_xM_xL_x = L_xM_x^{-1}L_x^{-1}R_{x^\rho}^{-1}R_x \iff R_xM_xL_x = L_x(R_{x^\rho}L_xM_x)^{-1}R_x$$

$$\iff L_x^{-1}R_xM_xL_xR_x^{-1} = (R_{x^\rho}L_xM_x)^{-1}$$

10. For every $x \in Q$,

$$x \in Q^{ii} \iff H_xM_x^{-1}R_x^{-1} = R_{x^\rho}L_xM_xL_x^{-1} \iff H_x = R_{x^\rho}L_xM_xL_x^{-1}R_xM_x$$

$$\iff R_{x^\rho}^{-1}H_x = L_xM_xL_x^{-1}R_xM_x \iff L_x^{-1}R_{x^\rho}^{-1}H_x = M_xL_x^{-1}R_xM_x$$

11. For every $x \in Q$,

$$x \in Q^{ii} \iff H_xM_x^{-1}R_x^{-1} = R_{x^\rho}L_xM_xL_x^{-1}$$

$$\iff M_x^{-1}R_x^{-1} = H_x^{-1}R_{x^\rho}L_xM_xL_x^{-1} \iff M_x^{-1}R_x^{-1} = G_xR_{x^\rho}L_xM_xL_x^{-1}$$

$$\iff M_x^{-1}R_x^{-1} = L_xR_x^{-1}R_{x^\rho}L_xM_xL_x^{-1} \iff L_x^{-1}M_x^{-1}R_x^{-1} = R_x^{-1}R_{x^\rho}L_xM_xL_x^{-1}$$

$$\iff L_x^{-1}M_x^{-1}R_x^{-1}L_x = R_x^{-1}R_{x^\rho}L_xM_x$$

12. For every $x \in Q$,

$$x \in Q^{ii} \iff H_xM_x^{-1}R_x^{-1} = R_{x^\rho}L_xM_xL_x^{-1}$$

$$\iff H_xM_x^{-1}R_x^{-1} = G_xM_xL_x^{-1} \iff H_xM_x^{-1} = G_xM_xL_x^{-1}R_x$$

$$\iff G_x^{-1}H_xM_x^{-1} = M_xL_x^{-1}R_x \iff H_x^2M_x^{-1} = M_xL_x^{-1}R_x$$

$$\iff H_x^2M_x^{-1}R_x^{-1} = M_xL_x^{-1} \iff R_xL_x^{-1}H_xM_x^{-1}R_x^{-1} = M_xL_x^{-1}$$

$$\iff L_x^{-1}H_xM_x^{-1}R_x^{-1} = R_x^{-1}M_xL_x^{-1}$$

13. For every $x \in Q$,

$$x \in Q^{ii} \iff L_xR_x^{-1}M_xL_x^{-1} = R_xL_x^{-1}M_x^{-1}R_x^{-1}$$

$$\iff L_xR_x^{-1}M_x = R_xL_x^{-1}M_x^{-1}R_x^{-1}L_x \iff R_x^{-1}L_xR_x^{-1}M_x = L_x^{-1}M_x^{-1}R_x^{-1}L_x$$

$$\iff L_xR_x^{-1}L_xR_x^{-1}M_x = M_x^{-1}R_x^{-1}L_x$$

14. For every $x \in Q$,

$$x \in Q^{ii} \iff L_xR_x^{-1}M_xL_x^{-1} = R_xL_x^{-1}M_x^{-1}R_x^{-1}$$

$$\iff L_xR_x^{-1}M_xL_x^{-1}R_xM_x = R_xL_x^{-1} \iff M_xL_x^{-1}R_xM_x = R_xL_x^{-1}R_xL_x^{-1}$$

$$\iff M_x L_x^{-1} R_x M_x L_x = R_x L_x^{-1} R_x$$

15. For every $x \in Q$,

$$x \in Q^{ii} \iff L_x R_x^{-1} M_x L_x^{-1} = R_x L_x^{-1} M_x^{-1} R_x^{-1} \iff L_x R_x^{-1} M_x L_x^{-1} = L_{x^\lambda} R_x M_x^{-1} R_x^{-1}$$

$$\iff L_{x^\lambda}^{-1} L_x R_x^{-1} M_x = R_x M_x^{-1} R_x^{-1} L_x$$

16. For every $x \in Q$,

$$x \in Q^{ii} \iff L_x R_x^{-1} M_x L_x^{-1} = R_x L_x^{-1} M_x^{-1} R_x^{-1}$$

$$\iff R_{x^\rho} L_x M_x L_x^{-1} = R_x L_x^{-1} M_x^{-1} R_x^{-1} \iff L_x M_x L_x^{-1} = R_{x^\rho}^{-1} R_x L_x^{-1} M_x^{-1} R_x^{-1}$$

17. For every $x \in Q$,

$$x \in Q^{ii} \iff L_x R_x^{-1} M_x L_x^{-1} = R_x L_x^{-1} M_x^{-1} R_x^{-1} \iff R_x^{-1} M_x L_x^{-1} R_x = L_x^{-1} R_x L_x^{-1} M_x^{-1}$$

$$\iff R_x^{-1} M_x L_x^{-1} R_x = L_x^{-1} H_x M_x^{-1} \iff R_x^{-1} M_x L_x^{-1} R_x = L_x^{-1} J_\lambda R_x M_x M_x^{-1}$$

$$\iff R_x^{-1} M_x L_x^{-1} R_x = L_x^{-1} J_\lambda R_x \iff R_x^{-1} M_x L_x^{-1} = L_x^{-1} J_\lambda \iff R_x^{-1} M_x = L_x^{-1} J_\lambda L_x$$

18. For every $x \in Q$,

$$x \in Q^{ii} \iff L_x R_x^{-1} M_x L_x^{-1} = R_x L_x^{-1} M_x^{-1} R_x^{-1}$$

$$\iff G_x M_x L_x^{-1} = R_x L_x^{-1} M_x^{-1} R_x^{-1} \iff J_\rho L_x M_x^{-1} M_x L_x^{-1} = R_x L_x^{-1} M_x^{-1} R_x^{-1}$$

$$\iff J_\rho = R_x L_x^{-1} M_x^{-1} R_x^{-1} \iff R_x^{-1} J_\rho R_x = L_x^{-1} M_x^{-1}$$

□

Theorem 4.6.10. Let $(Q,\cdot)$ be a Basarab loop and let $G_x = J_\rho L_x M_x^{-1}$ and $H_x = J_\lambda R_x M_x$ be defined on Q. Then for every $x \in Q,\ x$ is contained in Q^{ii} if and only if any of the following conditions hold :

1. $L_x M_x^{-1} J_\rho = J_\lambda M_x^{-1} R_x^{-1}$
2. $M_x L_x^{-1} J_\rho = J_\lambda R_x M_x$
3. $R_x^{-1} J_\rho M_x L_x^{-1} R_x = M_x R_x L_x^{-1} M_x^{-1}$
4. $L_x^{-1} R_x M_x J_\rho L_x = M_x^{-1} R_x L_x^{-1} M_x$
5. $R_{x^\rho}^{-1} J_\lambda = L_x M_x L_x^{-1}$
6. $L_x M_x^{-1} L_x^{-1} = J_\lambda^{-1} R_{x^\rho}$
7. $L_x^{-1} R_{x^\rho}^{-1} = M_x L_x^{-1} J_\rho$
8. $L_x^{-1} J_\lambda L_x = R_x^{-1} M_x$

Proof. From Theorem 4.6.9,

1. consider $G_x^2 M_x = M_x^{-1} R_x^{-1} L_x$. Put $G_x = J_\rho L_x M_x^{-1}$. Then

$$G_x G_x M_x = M_x^{-1} R_x^{-1} L_x \iff J_\rho L_x M_x^{-1} J_\rho L_x M_x^{-1} M_x = M_x^{-1} R_x^{-1} L_x$$

$$\iff J_\rho L_x M_x^{-1} J_\rho L_x = M_x^{-1} R_x^{-1} L_x \iff J_\rho L_x M_x^{-1} J_\rho = M_x^{-1} R_x^{-1}$$

$$\iff L_x M_x^{-1} J_\rho = J_\lambda M_x^{-1} R_x^{-1}$$

2. Consider $M_x L_x^{-1} R_x = H_x^2 M_x^{-1}$. Put $H_x = J_\lambda R_x M_x$. Then

$$M_x L_x^{-1} R_x = H_x^2 M_x^{-1} \iff M_x L_x^{-1} R_x = J_\lambda R_x M_x J_\lambda R_x M_x M_x^{-1}$$

$$\iff M_xL_x^{-1}R_x = J_\lambda R_xM_xJ_\lambda R_x \iff M_xL_x^{-1} = J_\lambda R_xM_xJ_\lambda$$

$$\iff M_xL_x^{-1}J_\rho = J_\lambda R_xM_x$$

3. Consider, $R_xM_xL_x = L_xM_x^{-1}H_xR_x$. Put $H_x = J_\lambda R_xM_x$. Then

$$R_xM_xL_x = L_xM_x^{-1}J_\lambda R_xM_xR_x \iff M_xL_x^{-1}R_xM_xL_x = J_\lambda R_xM_xR_x$$

$$\iff M_xL_x^{-1}R_x = J_\lambda R_xM_xR_xL_x^{-1}M_x^{-1} \iff R_x^{-1}J_\rho M_xL_x^{-1}R_x = M_xR_xL_x^{-1}M_x^{-1}$$

4. Consider $R_xM_xG_xL_x = L_xM_x^{-1}R_x$. Put $G_x = J_\rho L_xM_x^{-1}$. Then

$$R_xM_xG_xL_x = L_xM_x^{-1}R_x \iff R_xM_xJ_\rho L_xM_x^{-1}L_x = L_xM_x^{-1}R_x$$

$$\iff R_xM_xJ_\rho L_x = L_xM_x^{-1}R_xL_x^{-1}M_x \iff L_x^{-1}R_xM_xJ_\rho L_x = M_x^{-1}R_xL_x^{-1}M_x$$

5. consider $R_{x^\rho}^{-1}H_xM_x^{-1} = L_xM_xL_x^{-1}R_x$. Put $H_x = J_\lambda R_xM_x$. Then

$$R_{x^\rho}^{-1}H_xM_x^{-1} = L_xM_xL_x^{-1}R_x \iff R_{x^\rho}^{-1}J_\lambda R_xM_xM_x^{-1} = L_xM_xL_x^{-1}R_x$$

$$\iff R_{x^\rho}^{-1}J_\lambda R_x = L_xM_xL_x^{-1}R_x \iff R_{x^\rho}^{-1}J_\lambda = L_xM_xL_x^{-1}.$$

6. consider $M_x^{-1}R_x^{-1}L_x = H_x^{-1}R_{x^\rho}L_xM_x$. Put $H_x = J_\lambda R_xM_x$. Then,

$$M_x^{-1}R_x^{-1}L_x = H_x^{-1}R_{x^\rho}L_xM_x \iff M_x^{-1}R_x^{-1}L_x = (J_\lambda R_xM_x)^{-1}R_{x^\rho}L_xM_x$$

$$\iff M_x^{-1}R_x^{-1}L_x = M_x^{-1}R_x^{-1}J_\lambda^{-1}R_{x^\rho}L_xM_x \iff M_x^{-1}R_x^{-1}L_xM_x^{-1} = M_x^{-1}R_x^{-1}J_\lambda^{-1}R_{x^\rho}L_x$$

$$\iff L_xM_x^{-1} = J_\lambda^{-1}R_{x^\rho}L_x \iff L_xM_x^{-1}L_x^{-1} = J_\lambda^{-1}R_{x^\rho}$$

7. consider $L_x^{-1}R_{x^\rho}^{-1}H_x = M_xL_x^{-1}R_xM_x$. Put $H_x = J_\lambda R_xM_x$. Then,

$$L_x^{-1}R_{x^\rho}^{-1}H_x = M_xL_x^{-1}R_xM_x \iff L_x^{-1}R_{x^\rho}^{-1}J_\lambda R_xM_x = M_xL_x^{-1}R_xM_x$$

$$\iff L_x^{-1}R_{x^\rho}^{-1}J_\lambda = M_xL_x^{-1} \iff L_x^{-1}R_{x^\rho}^{-1} = M_xL_x^{-1}J_\rho$$

8. consider $L_x^{-1}H_xM_x^{-1}R_x^{-1} = R_x^{-1}M_xL_x^{-1}$. Put $H_x = J_\lambda R_xM_x$. Then,

$$L_x^{-1}H_xM_x^{-1}R_x^{-1} = R_x^{-1}M_xL_x^{-1} \iff L_x^{-1}J_\lambda R_xM_xM_x^{-1}R_x^{-1} = R_x^{-1}M_xL_x^{-1}$$

$$\iff L_x^{-1}J_\lambda R_xR_x^{-1} = R_x^{-1}M_xL_x^{-1} \iff L_x^{-1}J_\lambda = R_x^{-1}M_xL_x^{-1} \iff L_x^{-1}J_\lambda L_x = R_x^{-1}M_x$$

□

Theorem 4.6.11. Let $(Q,\cdot)$ be a Basarab loop and let $G_x = R_{x^\rho}L_x$ and $H_x = L_{x^\lambda}R_x$ be defined on Q. Then for every $x \in Q,\ x$ is contained in Q^{ii} if and only if any of the following conditions hold :

1. $L_xR_{x^\rho}L_xM_x = R_{x^\rho}^{-1}M_x^{-1}R_x^{-1}L_x$
2. $L_{x^\lambda}^{-1}M_xL_x^{-1}R_x = R_xL_{x^\lambda}R_xM_x^{-1}$
3. $R_xM_xL_xR_x^{-2} = L_xM_x^{-1}L_{x^\lambda}$
4. $R_xM_xR_{x^\rho} = L_xM_x^{-1}R_xL_x^{-2}$
5. $M_x^{-1}L_x^{-1}R_{x^\rho}^{-1}L_{x^\lambda} = L_x^{-1}R_xM_xR_x^{-1}$
6. $R_xM_x^{-1}R_x^{-1}L_x = L_{x^\lambda}^{-1}R_{x^\rho}L_xM_x$
7. $M_x^{-1}L_x^{-1}R_{x^\rho}^{-1}L_{x^\lambda} = L_x^{-1}R_xM_xR_x^{-1}$

Proof. From Theorem 4.6.9,

1. consider $G_x^2 M_x = M_x^{-1} R_x^{-1} L_x$. Put $G_x = R_{x^\rho} L_x$. Then

$$G_x^2 M_x = M_x^{-1} R_x^{-1} L_x \iff R_{x^\rho} L_x R_{x^\rho} L_x M_x = M_x^{-1} R_x^{-1} L_x$$

$$\iff L_x R_{x^\rho} L_x M_x = R_{x^\rho}^{-1} M_x^{-1} R_x^{-1} L_x$$

2. consider $M_x L_x^{-1} R_x = H_x^2 M_x^{-1}$. Put $H_x = L_{x^\lambda} R_x$. Then

$$M_x L_x^{-1} R_x = H_x^2 M_x^{-1} \iff M_x L_x^{-1} R_x = L_{x^\lambda} R_x L_{x^\lambda} R_x M_x^{-1}$$

$$\iff L_{x^\lambda}^{-1} M_x L_x^{-1} R_x = R_x L_{x^\lambda} R_x M_x^{-1}$$

3. consider $R_x M_x L_x = L_x M_x^{-1} H_x R_x$. Put $H_x = L_{x^\lambda} R_x$. Then

$$R_x M_x L_x = L_x M_x^{-1} H_x R_x \iff R_x M_x L_x = L_x M_x^{-1} L_{x^\lambda} R_x R_x$$

$$\iff R_x M_x L_x R_x^{-2} = L_x M_x^{-1} L_{x^\lambda}$$

4. consider $R_x M_x G_x L_x = L_x M_x^{-1} R_x$ Put $G_x = R_{x^\rho} L_x$. Then

$$R_x M_x G_x L_x = L_x M_x^{-1} R_x \iff R_x M_x R_{x^\rho} L_x L_x = L_x M_x^{-1} R_x$$

$$\iff R_x M_x R_{x^\rho} = L_x M_x^{-1} R_x L_x^{-2}$$

5. consider $R_{x^\rho}^{-1} H_x M_x^{-1} = L_x M_x L_x^{-1} R_x$. Put $H_x = L_{x^\lambda} R_x$. Then

$$R_{x^\rho}^{-1} H_x M_x^{-1} = L_x M_x L_x^{-1} R_x \iff R_{x^\rho}^{-1} L_{x^\lambda} R_x M_x^{-1} = L_x M_x L_x^{-1} R_x$$

$$\iff R_{x^\rho}^{-1} L_{x^\lambda} = L_x M_x L_x^{-1} R_x M_x R_x^{-1} \iff M_x^{-1} L_x^{-1} R_{x^\rho}^{-1} L_{x^\lambda} = L_x^{-1} R_x M_x R_x^{-1}$$

6. consider $M_x^{-1}R_x^{-1}L_x = H_x^{-1}R_{x^\rho}L_xM_x$. Put $H_x = L_{x^\lambda}R_x$. Then

$$M_x^{-1}R_x^{-1}L_x = H_x^{-1}R_{x^\rho}L_xM_x \iff M_x^{-1}R_x^{-1}L_x = (L_{x^\lambda}R_x)^{-1}R_{x^\rho}L_xM_x$$

$$\iff M_x^{-1}R_x^{-1}L_x = R_x^{-1}L_{x^\lambda}^{-1}R_{x^\rho}L_xM_x \iff R_xM_x^{-1}R_x^{-1}L_x = L_{x^\lambda}^{-1}R_{x^\rho}L_xM_x$$

7. consider $L_x^{-1}R_{x^\rho}^{-1}H_x = M_xL_x^{-1}R_xM_x$. Put $H_x = L_{x^\lambda}R_x$. Then

$$L_x^{-1}R_{x^\rho}^{-1}H_x = M_xL_x^{-1}R_xM_x \iff L_x^{-1}R_{x^\rho}^{-1}L_{x^\lambda}R_x = M_xL_x^{-1}R_xM_x$$

$$\iff M_x^{-1}L_x^{-1}R_{x^\rho}^{-1}L_{x^\lambda} = L_x^{-1}R_xM_xR_x^{-1}$$

□

Theorem 4.6.12. Let $(Q,\cdot)$ be a Basarab loop and let $G_x = R_{x^\rho}L_x$ and $H_x = L_{x^\lambda}R_x$ be defined on Q. Then for every $x, y, z \in Q$, x is contained in Q^{ii} if and only if :

1. $G_x^2M_xR_{xz}L_x^{-1} = M_x^{-1}R_x^{-1}R_{x^\rho}^{-1}R_z$
2. $M_xL_x^{-1}L_y^{-1}L_{x^\lambda} = H_x^2M_x^{-1}L_{yx}^{-1}R_x^{-1}$
3. $R_xL_{yx}M_xL_xL_{yx}^{-1}R_x^{-1} = L_{x^\lambda}^{-1}L_yL_xM_x^{-1}H_xL_y^{-1}L_{x^\lambda}$
4. $R_z^{-1}R_{x^\rho}R_xM_xG_xL_x = L_xR_{xz}^{-1}M_x^{-1}R_x$
5. $R_{x^\rho}^{-1}H_xM_x^{-1}L_{yx}^{-1}R_x^{-1} = L_xM_xL_x^{-1}L_y^{-1}L_{x^\lambda}$
6. $R_x^{-1}L_{x^\lambda}^{-1}L_yM_x = L_{yx}L_x^{-1}J_\lambda L_x$
7. $R_{x^\rho}^{-1}R_zL_xR_{xz}^{-1} = M_x^{-1}R_x^{-1}J_\lambda R_x$.

Proof. From Theorem 4.6.9,

1. consider, $G_x^2 M_x = M_x^{-1} R_x^{-1} L_x$. Then by Lemma 4.3.6,

$$L_x R_{xz} = R_{x^\rho}^{-1} R_z L_x \implies L_x = R_{x^\rho}^{-1} R_z L_x R_{xz}^{-1}.$$

Thus, $G_x^2 M_x = M_x^{-1} R_x^{-1} L_x \iff G_x^2 M_x = M_x^{-1} R_x^{-1} R_{x^\rho}^{-1} R_z L_x R_{xz}^{-1}$

$$\iff G_x^2 M_x R_{xz} L_x^{-1} = M_x^{-1} R_x^{-1} R_{x^\rho}^{-1} R_z$$

2. consider $M_x L_x^{-1} R_x = H_x^2 M_x^{-1}$. Then by Lemma 4.3.6,

$$R_x L_{yx} = L_{x^\lambda}^{-1} L_y R_x \implies L_y^{-1} L_{x^\lambda} R_x L_{yx} = R_x.$$

Thus, $M_x L_x^{-1} R_x = H_x^2 M_x^{-1} \iff M_x L_x^{-1} L_y^{-1} L_{x^\lambda} R_x L_{yx} = H_x^2 M_x^{-1}$

$$\iff M_x L_x^{-1} L_y^{-1} L_{x^\lambda} = H_x^2 M_x^{-1} L_{yx}^{-1} R_x^{-1}$$

3. consider $R_x M_x L_x = L_x M_x^{-1} H_x R_x$. Then by Lemma 4.3.6, $L_y^{-1} L_{x^\lambda} R_x L_{yx} = R_x$. Thus, $R_x M_x L_x = L_x M_x^{-1} H_x R_x \iff L_y^{-1} L_{x^\lambda} R_x L_{yx} M_x L_x = L_x M_x^{-1} H_x L_y^{-1} L_{x^\lambda} R_x L_{yx}$

$$\iff R_x L_{yx} M_x L_x L_{yx}^{-1} R_x^{-1} = L_{x^\lambda}^{-1} L_y L_x M_x^{-1} H_x L_y^{-1} L_{x^\lambda}$$

4. consider $R_x M_x G_x L_x = L_x M_x^{-1} R_x$. Then by Lemma 4.3.6, $L_x = R_{x^\rho}^{-1} R_z L_x R_{xz}^{-1}$. Thus,

$$R_x M_x G_x L_x = L_x M_x^{-1} R_x \iff R_x M_x G_x L_x = R_{x^\rho}^{-1} R_z L_x R_{xz}^{-1} M_x^{-1} R_x$$

$$\iff R_z^{-1} R_{x^\rho} R_x M_x G_x L_x = L_x R_{xz}^{-1} M_x^{-1} R_x$$

5. consider $R_{x^\rho}^{-1} H_x M_x^{-1} = L_x M_x L_x^{-1} R_x$. Then by Lemma 4.3.6, $L_y^{-1} L_{x^\lambda} R_x L_{yx} = R_x$.

Thus,

$$R_{x^\rho}^{-1} H_x M_x^{-1} = L_x M_x L_x^{-1} R_x \iff R_{x^\rho}^{-1} H_x M_x^{-1} = L_x M_x L_x^{-1} L_y^{-1} L_{x^\lambda} R_x L_{yx}$$

$$\iff R_{x^\rho}^{-1} H_x M_x^{-1} L_{yx}^{-1} R_x^{-1} = L_x M_x L_x^{-1} L_y^{-1} L_{x^\lambda}$$

6. consider $R_x^{-1} M_x = L_x^{-1} J_\lambda L_x$. Then by Lemma 4.3.6, $L_y^{-1} L_{x^\lambda} R_x L_{yx} = R_x$. Thus,

$$R_x^{-1} M_x = L_x^{-1} J_\lambda L_x \iff M_x = R_x L_x^{-1} J_\lambda L_x$$

$$\iff M_x = L_y^{-1} L_{x^\lambda} R_x L_{yx} L_x^{-1} J_\lambda L_x \iff R_x^{-1} L_{x^\lambda}^{-1} L_y M_x = L_{yx} L_x^{-1} J_\lambda L_x$$

7. consider $R_x^{-1} J_\rho R_x = L_x^{-1} M_x^{-1}$. Then by Lemma 4.3.6, $L_x = R_{x^\rho}^{-1} R_z L_x R_{xz}^{-1}$. Thus,

$$R_x^{-1} J_\rho R_x = L_x^{-1} M_x^{-1} \iff R_x^{-1} J_\rho R_x = L_x^{-1} M_x^{-1} \iff L_x R_x^{-1} J_\rho R_x = M_x^{-1}$$

$$\iff R_{x^\rho}^{-1} R_z L_x R_{xz}^{-1} R_x^{-1} J_\rho R_x = M_x^{-1} \iff R_{x^\rho}^{-1} R_z L_x R_{xz}^{-1} = M_x^{-1} R_x^{-1} J_\lambda R_x.$$

□

Theorem 4.6.13. Let $(Q, \cdot)$ be a Basarab loop and let $G_x = J_\rho L_x M_x^{-1}$ and $H_x = J_\lambda R_x M_x$ be defined on Q. Then for every $x, y, z \in Q$, x is contained in Q^{ii} if and only if any of the following conditions hold :

1. $L_x R_{xz}^{-1} M_x^{-1} J_\rho = R_z^{-1} R_{x^\rho}^{-1} J_\lambda M_x^{-1} R_x^{-1}$

2. $L_y J_\rho M_x L_x^{-1} J_\rho = L_{x^\lambda} R_x L_{yx} M_x$

3. $R_z^{-1} J_\lambda L_x = L_x R_{xz}^{-1} M_x$

4. $R_{xz}^{-1} M_x^{-1} L_x^{-1} = L_x^{-1} R_z^{-1} R_{x^\rho} J_\rho^{-1} R_{x^\rho}$

5. $L_x^{-1}R_{x^\rho}^{-1}J_\lambda R_{x^\rho}^{-1} = M_x R_{xz} L_x^{-1} R_z^{-1}$

6. $L_y^{-1}L_{x^\lambda}R_x L_{yx} = M_x L_x^{-1} J_\rho L_x$

Proof. From Theorem 4.6.10,

1. consider $L_x M_x^{-1} J_\rho = J_\lambda M_x^{-1} R_x^{-1}$. Then by Lemma 4.3.6, $L_x = R_{x^\rho}^{-1} R_z L_x R_{xz}^{-1}$. Thus,

$$L_x M_x^{-1} J_\rho = J_\lambda M_x^{-1} R_x^{-1} \iff R_{x^\rho}^{-1} R_z L_x R_{xz}^{-1} M_x^{-1} J_\rho = J_\lambda M_x^{-1} R_x^{-1}$$

$$\iff L_x R_{xz}^{-1} M_x^{-1} J_\rho = R_z^{-1} R_{x^\rho}^{-1} J_\lambda M_x^{-1} R_x^{-1}$$

2. consider $M_x L_x^{-1} J_\rho = J_\lambda R_x M_x$. Then by Lemma 4.3.6, $L_y^{-1} L_{x^\lambda} R_x L_{yx} = R_x$. Thus,

$$M_x L_x^{-1} J_\rho = J_\lambda R_x M_x \iff L_y J_\rho M_x L_x^{-1} J_\rho = L_{x^\lambda} R_x L_{yx} M_x$$

3. consider $R_{x^\rho}^{-1} J_\lambda = L_x M_x L_x^{-1}$. Then by Lemma 4.3.6, $L_x = R_{x^\rho}^{-1} R_z L_x R_{xz}^{-1}$. Thus,

$$R_{x^\rho}^{-1} J_\lambda = L_x M_x L_x^{-1} \iff R_{x^\rho}^{-1} J_\lambda = R_{x^\rho}^{-1} R_z L_x R_{xz}^{-1} M_x L_x^{-1}$$

$$\iff J_\lambda = R_z L_x R_{xz}^{-1} M_x L_x^{-1} \iff R_z^{-1} J_\lambda L_x = L_x R_{xz}^{-1} M_x$$

4. consider $L_x M_x^{-1} L_x^{-1} = J_\rho^{-1} R_{x^\rho}$. Then by Lemma 4.3.6, $L_x = R_{x^\rho}^{-1} R_z L_x R_{xz}^{-1}$. Thus,

$$L_x M_x^{-1} L_x^{-1} = J_\rho^{-1} R_{x^\rho} \iff R_{x^\rho}^{-1} R_z L_x R_{xz}^{-1} M_x^{-1} L_x^{-1} = J_\rho^{-1} R_{x^\rho}$$

$$\iff R_{xz}^{-1} M_x^{-1} L_x^{-1} = L_x^{-1} R_z^{-1} R_{x^\rho} J_\rho^{-1} R_{x^\rho}$$

5. $L_x^{-1} R_{x^\rho}^{-1} = M_x L_x^{-1} J_\rho$. Then by Lemma 4.3.6, $L_x = R_{x^\rho}^{-1} R_z L_x R_{xz}^{-1}$. Thus,

$$L_x^{-1} R_{x^\rho}^{-1} = M_x L_x^{-1} J_\rho \iff L_x^{-1} R_{x^\rho}^{-1} J_\lambda L_x = M_x$$

$$\iff L_x^{-1}R_{x^\rho}^{-1}J_\lambda R_{x^\rho}^{-1} = M_xR_{xz}L_x^{-1}R_z^{-1}$$

6. $L_x^{-1}J_\lambda L_x = R_x^{-1}M_x$. Then by Lemma 4.3.6, $L_y^{-1}L_{x^\lambda}R_xL_{yx} = R_x$. Thus,

$$L_x^{-1}J_\lambda L_x = R_x^{-1}M_x \iff R_xL_x^{-1}J_\lambda L_x = M_x \iff L_y^{-1}L_{x^\lambda}R_xL_{yx}L_x^{-1}J_\lambda L_x = M_x$$

$$\iff L_y^{-1}L_{x^\lambda}R_xL_{yx} = M_xL_x^{-1}J_\rho L_x$$

□

Theorem 4.6.14. Let $(Q,\cdot)$ be a Basarab loop and let $G_x = R_{x^\rho}L_x$ and $H_x = L_{x^\lambda}R_x$ be defined on Q. Then for every $x \in Q,\ x$ is contained in Q^{ii} if and only if any of the following conditions hold :

1. $L_x^{-1}R_z^{-1}H_xM_x^{-1} = R_{xz}^{-1}M_xL_x^{-1}R_x$
2. $H_xM_x^{-1}R_x^{-1}L_x = R_zL_xR_{xz}^{-1}M_x$
3. $R_xL_x^{-1}M_x^{-1}R_x^{-1} = R_zL_xR_{xz}^{-1}M_xL_x^{-1}$
4. $R_x^{-1}L_y^{-1}L_xR_x^{-1}M_x = L_{yx}^{-1}M_x^{-1}R_x^{-1}L_x$
5. $M_xL_x^{-1}R_xM_xL_x^{-1} = R_{xz}L_x^{-1}R_z^{-1}R_x$

Proof. From Theorem 4.6.9,

1. consider $R_{x^\rho}^{-1}H_xM_x^{-1} = L_xM_xL_x^{-1}R_x$. Then by Lemma 4.3.6, $R_{x^\rho}^{-1} = L_xR_{xz}L_x^{-1}R_z^{-1}$. Thus,

$$R_{x^\rho}^{-1}H_xM_x^{-1} = L_xM_xL_x^{-1}R_x \iff L_xR_{xz}L_x^{-1}R_z^{-1}H_xM_x^{-1} = L_xM_xL_x^{-1}R_x$$

$$\iff L_x^{-1}R_z^{-1}H_xM_x^{-1} = R_{xz}^{-1}M_xL_x^{-1}R_x$$

2. consider $M_x^{-1}R_x^{-1}L_x = H_x^{-1}R_{x^\rho}L_xM_x$. Then by Lemma 4.3.6, $R_{x^\rho} = R_zL_xR_{xz}^{-1}L_x^{-1}$. Thus,

$$M_x^{-1}R_x^{-1}L_x = H_x^{-1}R_{x^\rho}L_xM_x \iff M_x^{-1}R_x^{-1}L_x = H_x^{-1}R_zL_xR_{xz}^{-1}L_x^{-1}L_xM_x$$

$$\iff M_x^{-1}R_x^{-1}L_x = H_x^{-1}R_zL_xR_{xz}^{-1}M_x \iff H_xM_x^{-1}R_x^{-1}L_x = R_zL_xR_{xz}^{-1}M_x$$

3. consider $L_x^{-1}M_x^{-1}R_x^{-1}L_x = R_x^{-1}R_{x^\rho}L_xM_x$. Then by Lemma 4.3.6, $R_{x^\rho} = R_zL_xR_{xz}^{-1}L_x^{-1}$. Thus,

$$L_x^{-1}M_x^{-1}R_x^{-1}L_x = R_x^{-1}R_{x^\rho}L_xM_x \iff L_x^{-1}M_x^{-1}R_x^{-1}L_x = R_x^{-1}R_zL_xR_{xz}^{-1}L_x^{-1}L_xM_x$$

$$\iff L_x^{-1}M_x^{-1}R_x^{-1}L_x = R_x^{-1}R_zL_xR_{xz}^{-1}M_x \iff R_xL_x^{-1}M_x^{-1}R_x^{-1} = R_zL_xR_{xz}^{-1}M_xL_x^{-1}$$

4. consider $L_{x^\lambda}^{-1}L_xR_x^{-1}M_x = R_xM_x^{-1}R_x^{-1}L_x$. Then by Lemma 4.3.6,
$L_{x^\lambda}^{-1} = R_xL_{yx}R_x^{-1}L_y^{-1}$. Thus,

$$L_{x^\lambda}^{-1}L_xR_x^{-1}M_x = R_xM_x^{-1}R_x^{-1}L_x \iff R_xL_{yx}R_x^{-1}L_y^{-1}L_xR_x^{-1}M_x = R_xM_x^{-1}R_x^{-1}L_x$$

$$\iff L_{yx}R_x^{-1}L_y^{-1}L_xR_x^{-1}M_x = M_x^{-1}R_x^{-1}L_x \iff R_x^{-1}L_y^{-1}L_xR_x^{-1}M_x = L_{yx}^{-1}M_x^{-1}R_x^{-1}L_x$$

5. consider $L_xM_xL_x^{-1} = R_{x^\rho}^{-1}R_xL_x^{-1}M_x^{-1}R_x^{-1}$ Then by Lemma 4.3.6, $R_{x^\rho}^{-1} = L_xR_{xz}L_x^{-1}R_z^{-1}$. Thus,

$$L_xM_xL_x^{-1} = R_{x^\rho}^{-1}R_xL_x^{-1}M_x^{-1}R_x^{-1} \iff L_xM_xL_x^{-1} = L_xR_{xz}L_x^{-1}R_z^{-1}R_xL_x^{-1}M_x^{-1}R_x^{-1}$$

$$\iff M_xL_x^{-1}R_xM_xL_x^{-1} = R_{xz}L_x^{-1}R_z^{-1}R_x$$

□

Theorem 4.6.15. Let $(Q, \cdot)$ be a Basarab loop and let $G_x = R_{x^\rho} L_x$ and $H_x = L_{x^\lambda} R_x$ be defined on Q. Then for every $x \in Q,\ x$ is contained in Q^{ii} if and only if any of the following conditions hold :

1. $R_{xz}^{-1} R_z L_x R_{xz}^{-1} M_x = L_x^{-1} R_z M_x^{-1} R_x^{-1} L_x$
2. $R_x^{-1} L_y^{-1} M_x L_x^{-1} R_x M_x = L_{yx}^{-1} L y R_x L_{yx}^{-1}$
3. $R_x M_x L_x R_x^{-1} L_{yx} = L_x M_x^{-1} L y R_x$
4. $R_x M_x R_z L_x = L_x M_x^{-1} R_x L_x^{-1} R_{xz}$
5. $L_{yx}^{-1} M_x^{-1} R_x^{-1} L_x = R_x^{-1} L_y^{-1} R_{x^\rho} L_x M_x$
6. $L_{yx}^{-1} M_x^{-1} R_x^{-1} L_x = R_x^{-1} L_y^{-1} R_{x^\rho} L_x M_x$
7. $L_{yx}^{-1} M_x^{-1} R_x^{-1} L_x = R_x^{-1} L_y^{-1} L_x R_x^{-1} M_x$

Proof. From Theorem 4.6.9,

1. consider $G_x^2 M_x = M_x^{-1} R_x^{-1} L_x$. Then by Lemma 4.3.6,

   $L_x R_{xz} = R_{x^\rho}^{-1} R_z L_x \implies R_{x^\rho} L_x = R_z L_x R_{xz}^{-1}$. Thus,

$$G_x^2 M_x = M_x^{-1} R_x^{-1} L_x \iff R_z L_x R_{xz}^{-1} R_z L_x R_{xz}^{-1} M_x = M_x^{-1} R_x^{-1} L_x$$

$$\iff R_{xz}^{-1} R_z L_x R_{xz}^{-1} M_x = L_x^{-1} R_z M_x^{-1} R_x^{-1} L_x$$

2. consider $M_x L_x^{-1} R_x = H_x^2 M_x^{-1}$. Then by Lemma 4.3.6,

   $R_x L_{yx} = L_{x^\lambda}^{-1} L_y R_x \iff L_{x^\lambda} R_x = L y R_x L_{yx}^{-1}$. Thus,

$$M_x L_x^{-1} R_x = H_x^2 M_x^{-1} \iff M_x L_x^{-1} R_x = L_{x^\lambda} R_x L_{x^\lambda} R_x M_x^{-1}$$

$$\iff M_x L_x^{-1} R_x = L y R_x L_{yx}^{-1} L y R_x L_{yx}^{-1} M_x^{-1}$$

$$\iff R_x^{-1}L_y^{-1}M_xL_x^{-1}R_x = L_{yx}^{-1}LyR_xL_{yx}^{-1}M_x^{-1}$$

$$\iff R_x^{-1}L_y^{-1}M_xL_x^{-1}R_xM_x = L_{yx}^{-1}LyR_xL_{yx}^{-1}$$

3. consider $R_xM_xL_x = L_xM_x^{-1}H_xR_x$. Then by Lemma 4.3.6, $L_{x^\lambda}R_x = LyR_xL_{yx}^{-1}$. Thus,

$$R_xM_xL_x = L_xM_x^{-1}H_xR_x \iff R_xM_xL_x = L_xM_x^{-1}LyR_xL_{yx}^{-1}R_x$$

$$\iff R_xM_xL_xR_x^{-1}L_{yx} = L_xM_x^{-1}LyR_x$$

4. consider $R_xM_xG_xL_x = L_xM_x^{-1}R_x$. Then by Lemma 4.3.6, $L_xR_{xz} = R_{x^\rho}^{-1}R_zL_x \implies R_{x^\rho}L_x = R_zL_xR_{xz}^{-1}$. Thus,

$$R_xM_xG_xL_x = L_xM_x^{-1}R_x \iff R_xM_xR_{x^\rho}L_xL_x = L_xM_x^{-1}R_x$$

$$\iff R_xM_xR_zL_xR_{xz}^{-1}L_x = L_xM_x^{-1}R_x \iff R_xM_xR_zL_x = L_xM_x^{-1}R_xL_x^{-1}R_{xz}$$

5. consider $R_{x^\rho}^{-1}H_xM_x^{-1} = L_xM_xL_x^{-1}R_x$. Then by Lemma 4.3.6, $L_{x^\lambda}R_x = LyR_xL_{yx}^{-1}$. Thus,

$$R_{x^\rho}^{-1}H_xM_x^{-1} = L_xM_xL_x^{-1}R_x \iff R_{x^\rho}^{-1}L_{x^\lambda}R_xM_x^{-1} = L_xM_xL_x^{-1}R_x$$

$$\iff R_{x^\rho}^{-1}L_yR_xL_{yx}^{-1}M_x^{-1} = L_xM_xL_x^{-1}R_x$$

$$\iff L_{yx}^{-1}M_x^{-1} = R_x^{-1}L_y^{-1}R_{x^\rho}L_xM_xL_x^{-1}R_x \iff L_{yx}^{-1}M_x^{-1}R_x^{-1}L_x = R_x^{-1}L_y^{-1}R_{x^\rho}L_xM_x$$

6. consider $M_x^{-1}R_x^{-1}L_x = H_x^{-1}R_{x^\rho}L_xM_x$. Then by Lemma 4.3.6, $L_{x^\lambda}R_x = LyR_xL_{yx}^{-1}$. Thus,

$$M_x^{-1}R_x^{-1}L_x = H_x^{-1}R_{x^\rho}L_xM_x \iff H_xM_x^{-1}R_x^{-1}L_x = R_{x^\rho}L_xM_x$$

$$\iff L_{x^\lambda}R_xM_x^{-1}R_x^{-1}L_x = R_{x^\rho}L_xM_x \iff$$

$$LyR_xL_{yx}^{-1}M_x^{-1}R_x^{-1}L_x = R_{x^\rho}L_xM_x \iff L_{yx}^{-1}M_x^{-1}R_x^{-1}L_x = R_x^{-1}L_y^{-1}R_{x^\rho}L_xM_x$$

7. consider $L_x^{-1}H_xM_x^{-1}R_x^{-1} = R_x^{-1}M_xL_x^{-1}$. By Lemma 4.3.6, $L_{x^\lambda}R_x = L_yR_xL_{yx}^{-1}$. Then

$$L_x^{-1}H_xM_x^{-1}R_x^{-1} = R_x^{-1}M_xL_x^{-1} \iff L_x^{-1}L_{x^\lambda}R_xM_x^{-1}R_x^{-1} = R_x^{-1}M_xL_x^{-1}$$

$$\iff L_x^{-1}L_yR_xL_{yx}^{-1}M_x^{-1}R_x^{-1} = R_x^{-1}M_xL_x^{-1}$$

$$\iff R_xL_{yx}^{-1}M_x^{-1}R_x^{-1} = L_y^{-1}L_xR_x^{-1}M_xL_x^{-1} \iff L_{yx}^{-1}M_x^{-1}R_x^{-1}L_x = R_x^{-1}L_y^{-1}L_xR_x^{-1}M_x$$

□

Theorem 4.6.16. Let $(Q,\cdot)$ be a Basarab loop and let $G_x = R_{x^\rho}L_x$ and $H_x = L_{x^\lambda}R_x$ be defined on Q such that $R_xM_x = L_xM_x^{-1}H_xR_xL_x^{-1}$. Then for every $x \in Q,\ x$ is contained in Q^{ii} if and only if any of the following conditions hold :

1. $H_xG_x = I$

2. $G_xH_x = I$

Proof. From Theorem 4.6.9,

1. consider $R_xM_xG_xL_x = L_xM_x^{-1}R_x$. By Theorem 4.6.9 (3), $R_xM_x = L_xM_x^{-1}H_xR_xL_x^{-1}$. Then

$$R_xM_xG_xL_x = L_xM_x^{-1}R_x \iff H_xR_xL_x^{-1}G_xL_x = R_x \iff H_xL_{x^\lambda}R_xG_xL_x = R_x$$

$$\iff H_xH_xG_x = R_xL_x^{-1} \iff H_xH_xG_x = H_x \iff H_xG_x = I$$

2. consider $R_x^{-1}L_xR_x^{-1} = (R_xM_xL_x)^{-1}L_xM_x^{-1}$ By Theorem 4.6.9(3), $R_xM_x =$

$L_xM_x^{-1}H_xR_xL_x^{-1}$. Then

$$R_x^{-1}L_xR_x^{-1} = (R_xM_xL_x)^{-1}L_xM_x^{-1} \iff R_x^{-1}L_xR_x^{-1} = (L_xM_x^{-1}H_xR_x)^{-1}L_xM_x^{-1}$$

$$\iff R_x^{-1}L_xR_x^{-1} = R_x^{-1}H_x^{-1}M_xL_x^{-1}L_xM_x^{-1} \iff R_x^{-1}L_xR_x^{-1} = R_x^{-1}H_x^{-1}M_xM_x^{-1}$$

$$\iff R_x^{-1}L_xR_x^{-1} = R_x^{-1}H_x^{-1} \iff L_xR_x^{-1} = H_x^{-1} \iff G_x = H_x^{-1} \iff G_xH_x = I$$

□

Corollary 4.6.5. Let $(Q,\cdot)$ be a Basarab loop and let $G_x = R_{x^\rho}L_x$ and $H_x = L_{x^\lambda}R_x$ be defined on Q such that $R_xM_x = L_xM_x^{-1}H_xR_xL_x^{-1}$. Then for every $x \in Q,\ x$ is contained in Q^{ii} if and only if $H_xG_x = G_xH_x = I$

Proof. From (1) and (2) of Theorem 4.6.16, $H_xG_x = G_xH_x = I$.

□

Theorem 4.6.17. Let $(Q,\cdot)$ be a Basarab loop and let $G_x = R_{x^\rho}L_x$ and $H_x = L_{x^\lambda}R_x$ be defined on Q such that $R_xM_x = L_xM_x^{-1}H_xR_xL_x^{-1}$. Then for every $x \in Q,\ x$ is contained in Q^{ii} if and only if any of the following conditions hold :

1. $R_{xz}^{-1}R_zL_x = L_x^{-1}R_zL_xG_x^2R_{xz}$
2. $L_y^{-1}M_xM_x^{-1}H_x^2L_{yx} = R_xL_{yx}^{-1}L_yR_x$

Proof. From Theorem 4.6.15,

1. consider $R_{xz}^{-1}R_zL_xR_{xz}^{-1}M_x = L_x^{-1}R_zM_x^{-1}R_x^{-1}L_x$. Put $R_xM_x = L_xM_x^{-1}H_xR_xL_x^{-1}$. Then

$$R_{xz}^{-1}R_zL_xR_{xz}^{-1}M_x = L_x^{-1}R_zM_x^{-1}R_x^{-1}L_x$$

$$\iff R_{xz}^{-1}R_zL_xR_{xz}^{-1}M_x = L_x^{-1}R_z(L_xM_x^{-1}H_xR_xL_x^{-1})^{-1}L_x$$

$$\iff R_{xz}^{-1}R_zL_xR_{xz}^{-1}M_x = L_x^{-1}R_zL_xR_x^{-1}H_x^{-1}M_xL_x^{-1}L_x$$

$$\iff R_{xz}^{-1}R_zL_xR_{xz}^{-1}M_x = L_x^{-1}R_zL_xR_x^{-1}H_x^{-1}M_x$$

$$\iff R_{xz}^{-1}R_zL_xR_{xz}^{-1} = L_x^{-1}R_zL_xR_x^{-1}H_x^{-1}$$

$$\iff R_{xz}^{-1}R_zL_xR_{xz}^{-1} = L_x^{-1}R_zL_xG_x^2 \iff R_{xz}^{-1}R_zL_x = L_x^{-1}R_zL_xG_x^2R_{xz}$$

2. consider $R_x^{-1}L_y^{-1}M_xL_x^{-1}R_xM_x = L_{yx}^{-1}LyR_xL_{yx}^{-1}$. Put $R_xM_x = L_xM_x^{-1}H_xR_xL_x^{-1}$. Then

$$R_x^{-1}L_y^{-1}M_xL_x^{-1}R_xM_x = L_{yx}^{-1}LyR_xL_{yx}^{-1}$$

$$\iff R_x^{-1}L_y^{-1}M_xL_x^{-1}L_xM_x^{-1}H_xR_xL_x^{-1} = L_{yx}^{-1}LyR_xL_{yx}^{-1}$$

$$\iff R_x^{-1}L_y^{-1}M_xM_x^{-1}H_xR_xL_x^{-1} = L_{yx}^{-1}LyR_xL_{yx}^{-1}$$

$$\iff R_x^{-1}L_y^{-1}M_xM_x^{-1}H_x^2 = L_{yx}^{-1}LyR_xL_{yx}^{-1}$$

$$\iff R_x^{-1}L_y^{-1}M_xM_x^{-1}H_x^2L_{yx} = L_{yx}^{-1}LyR_x \iff L_y^{-1}M_xM_x^{-1}H_x^2L_{yx} = R_xL_{yx}^{-1}LyR_x$$

□

Theorem 4.6.18. Let $(Q,\cdot)$ be a Basarab loop and let $G_x = R_{x^\rho}L_x$ and $H_x = L_{x^\lambda}R_x$ be defined on Q such that $R_xM_x = L_xM_x^{-1}H_xR_xL_x^{-1}$. Then for every $x\in Q,\ x$ is contained in Q^{ii}if and only if any of the following conditions hold :

1. $L_x^{-1}R_{x^\rho} = R_x^{-1}L_x^{-1}$
2. $R_x^{-1}L_{x^\lambda} = L_x^{-1}R_x^{-1}$

Proof. From Theorem 4.6.11,

1. consider $R_xM_xR_{x^\rho} = L_xM_x^{-1}R_xL_x^{-2}$. Put $R_xM_x = L_xM_x^{-1}H_xR_xL_x^{-1}$. Then

$$R_xM_xR_{x^\rho} = L_xM_x^{-1}R_xL_x^{-2} \iff L_xM_x^{-1}H_xR_xL_x^{-1}R_{x^\rho} = L_xM_x^{-1}R_xL_x^{-2}$$

$$\iff H_xR_xL_x^{-1}R_{x^\rho} = R_xL_x^{-2} \iff R_xL_x^{-1}R_xL_x^{-1}R_{x^\rho} = R_xL_x^{-1}L_x^{-1}$$

$$\iff R_xL_x^{-1}R_{x^\rho} = L_x^{-1} \iff L_x^{-1}R_{x^\rho} = R_x^{-1}L_x^{-1}$$

2. consider $M_x^{-1}L_x^{-1}R_{x^\rho}^{-1}L_{x^\lambda} = L_x^{-1}R_xM_xR_x^{-1}$. Put $R_xM_x = L_xM_x^{-1}H_xR_xL_x^{-1}$. Then

$$M_x^{-1}L_x^{-1}R_{x^\rho}^{-1}L_{x^\lambda} = L_x^{-1}R_xM_xR_x^{-1} \iff M_x^{-1}L_x^{-1}R_{x^\rho}^{-1}L_{x^\lambda} = L_x^{-1}L_xM_x^{-1}H_xR_xL_x^{-1}R_x^{-1}$$

$$\iff M_x^{-1}L_x^{-1}R_{x^\rho}^{-1}L_{x^\lambda} = M_x^{-1}H_xR_xL_x^{-1}R_x^{-1} \iff L_x^{-1}R_{x^\rho}^{-1}L_{x^\lambda} = H_xR_xL_x^{-1}R_x^{-1}$$

$$\iff (R_{x^\rho}L_x)^{-1}L_{x^\lambda} = H_xR_xL_x^{-1}R_x^{-1} \iff G_x^{-1}L_{x^\lambda} = H_xR_xL_x^{-1}R_x^{-1}$$

$$\iff H_xL_{x^\lambda} = H_xR_xL_x^{-1}R_x^{-1} \iff L_{x^\lambda} = R_xL_x^{-1}R_x^{-1} \iff R_x^{-1}L_{x^\lambda} = L_x^{-1}R_x^{-1}$$

□

4.6.4 Characterization of a Subloop of a Basarab loop by Middle Translation Mapping

Let $(Q, \cdot)$ be a Basarab loop. Then the elements of
$(Q, \cdot)$ defined as $Q^{iii} := \{x \in Q : M_x = M_x^{-1},\ G_x, H_x \in SYM(Q)\}$ are investigated. From Lemma 4.6.4,

$$L_x^{-1}J_\lambda G_x = M_x^{-1} = L_x^{-1}J_\lambda R_{x^\rho}L_x = L_x^{-1}J_\lambda L_xR_x^{-1} \tag{4.45}$$

$$\text{and}\ \ R_x^{-1}J_\rho H_x = M_x = R_x^{-1}J_\rho L_{x^\lambda}R_x = R_x^{-1}J_\rho R_xL_x^{-1}. \tag{4.46}$$

Theorem 4.6.19. Let $(Q, \cdot)$ be a Basarab loop. Then Q^{iii} is a subloop of Q, if

1. $J_\rho = J_\lambda,\ M_{xy}^2 = (M_xM_y)^2$ for all $x, y \in Q^{iii}$; and

2. $[M_x, M_y] = I$ for all $x, y \in Q^{iii}$.

Proof. Obviously, $Q^{iii} \neq \emptyset \iff M_e^2 = I \iff J_\rho^2 = I \iff |J_\rho| = 2$

$$\iff J_\rho J_\rho = I \iff J_\rho J_\rho = J_\rho J_\lambda \iff J_\rho = J_\lambda.$$

Next, $x, y \in Q^{iii} \iff M_x^2 = I = M_y^2$. Thus, $M_{xy}^2 = (M_x M_y)^2 = M_x M_y M_x M_y = M_x^2 M_y^2 = II = I$. □

Theorem 4.6.20. Let $(Q, \cdot)$ be a Basarab loop. Then for every $x \in Q,\ x$ is contained in Q^{iii} if and only if any of the following conditions hold :

1. $J_\lambda G_x^2 = G_x J_\rho$
2. $H_x J_\lambda = J_\rho H_x^2$
3. $R_x^{-1} G_x J_\lambda = L_x^{-1} J_\rho G_x$
4. $R_x^{-1} J_\lambda H_x = L_x^{-1} H_x J_\rho$
5. $H_x J_\lambda = J_\rho H_x R_x L_x^{-1}$
6. $J_\lambda G_x L_x = G_x J_\rho R_x$
7. $H_x J_\lambda R_{x^\rho} = J_\rho H_x L_x^{-1}$
8. $L_x H_x^{-1} J_\lambda R_x = R_\rho^{-1} J_\rho L_x$
9. $J_\lambda R_{x^\rho} L_x^2 = G_x J_\rho R_x$
10. $L_x^{-1} J_\lambda G x R_x^{-1} = R_x^{-1} J_\rho L_{x^\lambda}$
11. $H_x R_x = J_\lambda R_x L_x^{-1} J_\lambda L_x$
12. $R_x^{-1} J_\rho R_x = L_x^{-1} J_\lambda L_x R_x^{-1} L_x$

Proof. By equations 4.45 and 4.46, it follows that:

1. For every $x \in Q$,

$$x \in Q^{iii} \iff L_x^{-1} J_\lambda G_x = R_x^{-1} J_\rho H_x \iff L_x^{-1} J_\lambda G_x H_x^{-1} = R_x^{-1} J_\rho$$

$$\iff L_x^{-1} J_\lambda G_x G_x = R_x^{-1} J_\rho \iff L_x^{-1} J_\lambda G_x^2 = R_x^{-1} J_\rho$$

$$\iff J_\lambda G_x^2 = L_x R_x^{-1} J_\rho \iff J_\lambda G_x^2 = G_x J_\rho$$

2. For every $x \in Q$,

$$x \in Q^{iii} \iff L_x^{-1} J_\lambda G_x = R_x^{-1} J_\rho H_x \iff L_x^{-1} J_\lambda = R_x^{-1} J_\rho H_x G_x^{-1}$$

$$\iff L_x^{-1} J_\lambda = R_x^{-1} J_\rho H_x H_x \iff L_x^{-1} J_\lambda = R_x^{-1} J_\rho H_x^2$$

$$\iff R_x L_x^{-1} J_\lambda = J_\rho H_x^2 \iff H_x J_\lambda = J_\rho H_x^2$$

3. For every $x \in Q$,

$$x \in Q^{iii} \iff L_x^{-1} J_\lambda G_x = R_x^{-1} J_\rho H_x \iff L_x^{-1} J_\lambda G_x H_x^{-1} = R_x^{-1} J_\rho$$

$$\iff G_x H_x^{-1} = J_\lambda^{-1} L_x R_x^{-1} J_\rho \iff G_x H_x^{-1} J_\lambda = J_\rho L_x R_x^{-1}$$

$$\iff L_x R_x^{-1} G_x J_\lambda = J_\rho L_x R_x^{-1} \iff R_x^{-1} G_x J_\lambda = L_x^{-1} J_\rho G_x$$

4. For every $x \in Q$,

$$x \in Q^{iii} \iff L_x^{-1} J_\lambda G_x = R_x^{-1} J_\rho H_x \iff L_x^{-1} J_\lambda = R_x^{-1} J_\rho H_x G_x^{-1}$$

$$\iff J_\rho^{-1} R_x L_x^{-1} J_\lambda = H_x G_x^{-1} \iff J_\lambda H_x J_\lambda = R_x L_x^{-1} H_x \iff R_x^{-1} J_\lambda H_x = L_x^{-1} H_x J_\rho$$

5. For every $x \in Q$,

$$x \in Q^{iii} \iff L_x^{-1} J_\lambda G_x = R_x^{-1} J_\rho H_x \iff L_x^{-1} J_\lambda L_x R_x^{-1} = R_x^{-1} J_\rho H_x$$

$$\iff L_x^{-1} J_\lambda L_x = R_x^{-1} J_\rho H_x R_x \iff R_x L_x^{-1} J_\lambda L_x = J_\rho R_x L_x^{-1} R_x$$

$$\iff H_x J_\lambda = J_\rho H_x R_x L_x^{-1}$$

6. For every $x \in Q$,

$$x \in Q^{iii} \iff L_x^{-1} J_\lambda G_x = R_x^{-1} J_\rho H_x \iff L_x^{-1} J_\lambda L_x R_x^{-1} = R_x^{-1} J_\rho R_x L_x^{-1}$$

$$\iff L_x^{-1} J_\lambda L_x R_x^{-1} L_x = R_x^{-1} J_\rho R_x \iff J_\lambda L_x R_x^{-1} L_x = L_x R_x^{-1} J_\rho R_x$$

$$\iff J_\lambda G_x L_x = G_x J_\rho R_x$$

7. For every $x \in Q$,

$$x \in Q^{iii} \iff L_x^{-1} J_\lambda R_{x^\rho} L_x = R_x^{-1} J_\rho H_x \iff L_x^{-1} J_\lambda R_{x^\rho} = R_x^{-1} J_\rho H_x L_x^{-1}$$

$$\iff R_x L_x^{-1} J_\lambda R_{x^\rho} = J_\rho H_x L_x^{-1} \iff H_x J_\lambda R_{x^\rho} = J_\rho H_x L_x^{-1}$$

8. For every $x \in Q$,

$$x \in Q^{iii} \iff L_x^{-1} J_\rho R_{x^\rho} L_x = R_x^{-1} J_\rho H_x \iff L_x^{-1} J_\lambda R_{x^\rho} L_x H_x^{-1} = R_x^{-1} J_\rho$$

$$\iff R_{x^\rho} L_x H_x^{-1} = J_\rho L_x R_x^{-1} J_\rho \iff L_x H_x^{-1} J_\lambda R_x = R_\rho^{-1} J_\rho L_x$$

9. For every $x \in Q$,

$$x \in Q^{iii} \iff L_x^{-1}J_\lambda G_x = R_x^{-1}J_\rho H_x \iff L_x^{-1}J_\lambda R_{x^\rho}L_x = R_x^{-1}J_\rho R_x L_x^{-1}$$

$$\iff L_x^{-1}J_\lambda R_{x^\rho}L_x^2 = R_x^{-1}J_\rho R_x \iff J_\lambda R_{x^\rho}L_x^2 = G_x J_\rho R_x$$

10. For every $x \in Q$,

$$x \in Q^{iii} \iff L_x^{-1}J_\lambda G_x = R_x^{-1}J_\rho H_x \iff L_x^{-1}J_\lambda G_x = R_x^{-1}J_\rho H_x$$

$$\iff L_x^{-1}J_\lambda G_x = R_x^{-1}J_\rho L_{x^\lambda}R_x \iff L_x^{-1}J_\lambda GxR_x^{-1} = R_x^{-1}J_\rho L_{x^\lambda}$$

11. For every $x \in Q$,

$$x \in Q^{iii} \iff R_x^{-1}J_\rho H_x = L_x^{-1}J_\lambda L_x R_x^{-1} \iff J_\rho H_x = R_x L_x^{-1}J_\lambda L_x R_x^{-1}$$

$$\iff J_\rho H_x R_x = R_x L_x^{-1}J_\lambda L_x \iff H_x R_x = J_\lambda R_x L_x^{-1}J_\lambda L_x$$

12. For every $x \in Q$,

$$x \in Q^{iii} \iff R_x^{-1}J_\rho H_x = L_x^{-1}J_\lambda L_x R_x^{-1} \iff R_x^{-1}J_\rho R_x L_x^{-1} = L_x^{-1}J_\lambda L_x R_x^{-1}$$

$$\iff R_x^{-1}J_\rho R_x = L_x^{-1}J_\lambda L_x R_x^{-1}L_x.$$

□

4.6.5 Characterization of a Subloop of a Basarab loop by Inverse elements

Let $(Q,\cdot)$ be a Basarab loop. Then the elements of $(Q,\cdot)$ defined as
$Q^{iv} := \{x \in Q : R_{x^\rho} = L_{x^\lambda}\}$ are studied. From Lemma 4.6.4,

$$G_xL_x^{-1} = J_\rho L_xM_x^{-1}L_x^{-1} = R_{x^\rho} = L_xR_x^{-1}L_x^{-1} \tag{4.47}$$

$$\text{and } H_xR_x^{-1} = J_\lambda R_xM_xR_x^{-1} = L_{x^\lambda} = R_xL_x^{-1}R_x^{-1}. \tag{4.48}$$

Theorem 4.6.21. Let $(Q,\cdot)$ be a Basarab loop. Then Q^{iv} is a subloop of Q, if Q is an abelian group.

Proof. If $Q^{iv} \neq \emptyset$. Then $R_{e^\rho} = L_{e^\lambda} \iff R_e = L_e \iff e \in Q$.
If Q is an abelian group. Then for all $x, y, z \in Q,\ z \cdot x^\rho y^\rho = zx^\rho \cdot y^\rho = z \cdot x^\lambda y^\lambda = zx^\lambda \cdot y^\lambda$

$$\iff z \cdot x^\rho y^\rho = zx^\rho \cdot y^\rho = x^\lambda y^\lambda \cdot z = y^\lambda \cdot x^\lambda z$$

$$\iff zR_{(x^\rho y^\rho)} = zR_{x^\rho}R_{y^\rho} = zL_{(x^\lambda y^\lambda)} = zL_{x^\lambda}L_{y^\lambda}$$

$$\iff zR_{(xy)^\rho} = zR_{x^\rho}R_{y^\rho} = zL_{(xy)^\lambda} = zL_{x^\lambda}L_{y^\lambda} \text{ (by AIP)}$$

$$\iff zR_{(xy)^\rho} = zL_{(xy)^\lambda} \iff R_{(xy)^\rho} = L_{(xy)^\lambda}$$

$$\iff R_{u^\rho} = L_{u^\lambda} \text{ for all } u = xy \iff u \in Q^{iv}.$$

□

Theorem 4.6.22. Let $(Q,\cdot)$ be a Basarab loop. Then for every $x \in Q,\ x$ is contained in Q^{iv} if and only if any of the following conditions hold :

1. $(L_xR_x)^{-1} = L_x^{-1}H_xR_x^{-1}$

2. $L_x^{-1}R_xL_x = R_xL_x^{-1}R_x$

3. $R_x^{-1}G_xL_x^{-1} = (R_xL_x)^{-1}$

4. $L_x^{-1}J_\lambda R_xL_x^{-1} = M_x^{-1}L_x^{-1}R_x$

5. $R_x^{-1}L_xM_x = H_x^{-1}J_\rho L_x$

6. $R_x^{-1}L_x = G_x^2$

7. $R_x^{-1}L_xR_x = L_xR_x^{-1}L_x$

8. $L_x^{-1}R_x = H_x^2$

9. $R_x^{-1}J_\rho L_x = M_xR_x^{-1}L_xR_x$

10. $L_x^{-1}R_xM_x^{-1} = H_xJ_\lambda R_x$

11. $R_x^{-1}J_\rho^2L_x = M_xR_x^{-1}L_xM_x$

12. $R_x^{-1}L_x^{-1}R_x = L_x^{-1}R_xL_x^{-1}$.

Proof. 1. For every $x \in Q,\ x \in Q^{iv} \iff G_xL_x^{-1} = H_xR_x^{-1}$

$$\iff L_xR_x^{-1}L_x^{-1} = H_xR_x^{-1} \iff R_x^{-1}L_x^{-1} = H_xR_x^{-1} \iff (L_xR_x)^{-1} = L_x^{-1}H_xR_x^{-1}$$

2. For every $x \in Q,\ x \in Q^{iv} \iff G_xL_x^{-1} = H_xR_x^{-1} \iff G_xL_x^{-1}R_x = H_x$

$$\iff L_xR_x^{-1}L_x^{-1}R_x = H_x \iff L_xR_x^{-1}L_x^{-1}R_x = R_xL_x^{-1}$$

$$\iff R_x^{-1}L_x^{-1}R_x = L_x^{-1}R_xL_x^{-1} \iff L_x^{-1}R_xL_x = R_xL_x^{-1}R_x$$

3. For every $x \in Q,\ x \in Q^{iv} \iff G_xL_x^{-1} = H_xR_x^{-1} \iff G_xL_x^{-1}R_x = H_x$

$$\iff G_xL_x^{-1} = R_xL_x^{-1}R_x^{-1} \iff R_x^{-1}G_xL_x^{-1} = L_x^{-1}R_x^{-1} \iff R_x^{-1}G_xL_x^{-1} = (R_xL_x)^{-1}$$

4. For every $x \in Q,\ x \in Q^{iv} \iff H_xR_x^{-1} = J_\rho L_x M_x^{-1} L_x^{-1}$

$$\iff R_xL_x^{-1}R_x^{-1} = J_\rho L_x M_x^{-1} L_x^{-1} \iff L_x^{-1} J_\lambda R_x L_x^{-1} R_x^{-1} = M_x^{-1} L_x^{-1}$$

$$\iff L_x^{-1} J_\lambda R_x L_x^{-1} = M_x^{-1} L_x^{-1} R_x$$

5. For every $x \in Q,\ x \in Q^{iv} \iff H_xR_x^{-1} = J_\rho L_x M_x^{-1} L_x^{-1}$

$$\iff R_x^{-1} = H_x^{-1} = J_\rho L_x M_x^{-1} L_x^{-1} \iff R_x^{-1} L_x M_x = H_x^{-1} J_\rho L_x$$

6. For every $x \in Q,\ x \in Q^{iv} \iff H_xR_x^{-1} = L_xR_x^{-1}L_x^{-1}$

$$\iff R_xL_x^{-1}R_x^{-1} = L_xR_x^{-1}L_x^{-1} \iff L_x^{-1}R_x^{-1}L_x = R_x^{-1}L_xR_x^{-1}$$

$$\iff L_x^{-1}R_x^{-1}L_x = R_x^{-1}G_x \iff R_x^{-1}L_x = L_xR_x^{-1}G_x$$

$$\iff R_x^{-1}L_x = G_xG_x \iff R_x^{-1}L_x = G_x^2$$

7. For every $x \in Q,\ x \in Q^{iv} \iff H_xR_x^{-1} = L_xR_x^{-1}L_x^{-1}$

$$\iff H_xR_x^{-1}L_x = L_xR_x^{-1} \iff R_x^{-1}L_x = H_x^{-1}L_xR_x^{-1}$$

$$\iff R_x^{-1}L_xR_x = (R_xL_x^{-1})L_x \iff R_x^{-1}L_xR_x = L_xR_x^{-1}L_x$$

8. For every $x \in Q,\ x \in Q^{iv} \iff G_xL_x^{-1} = R_xL_x^{-1}R_x^{-1}$

$$\iff L_xR_x^{-1}L_x^{-1} = R_xL_x^{-1}R_x^{-1} \iff R_x^{-1}L_xR_x = L_x^{-1}R_xL_x^{-1}$$

$$\iff R_x^{-1}L_x^{-1}R_x = L_x^{-1}H_x \iff L_x^{-1}R_x = R_xL_x^{-1}H_x$$

$$\iff L_x^{-1}R_x = H_xH_x \iff L_x^{-1}R_x = H_x^2$$

9. For every $x \in Q,\ x \in Q^{iv} \iff G_xL_x^{-1} = J_\lambda R_xM_xR_x^{-1}$

$$\iff L_xR_x^{-1}L_x^{-1} = J_\lambda R_xM_xR_x^{-1} \iff R_x^{-1}J_\rho L_xR_x^{-1} = M_xR_x^{-1}L_x$$

$$\iff R_x^{-1}J_\rho L_x = M_xR_x^{-1}L_xR_x$$

10. For every $x \in Q,\ x \in Q^{iv} \iff G_xL_x^{-1} = J_\lambda R_xM_xR_x^{-1} \iff G_xL_x^{-1}R_xM_x^{-1} = J_\lambda R_x$

$$\iff L_x^{-1}R_xM_x^{-1} = G_x^{-1}J_\lambda R_x \iff L_x^{-1}R_xM_x^{-1} = H_xJ_\lambda R_x$$

11. For every $x \in Q,\ x \in Q^{iv} \iff J_\rho L_xM_x^{-1}L_x^{-1} = J_\lambda R_xM_xR_x^{-1}$

$$\iff J_\rho J_\rho L_xM_x^{-1}L_x^{-1} = R_xM_xR_x^{-1} \iff J_\rho^2L_xM_x^{-1} = R_xM_xR_x^{-1}L_x$$

$$\iff J_\rho^2L_x = R_xM_xR_x^{-1}L_xM_x \iff J_\rho^2L_x = R_xM_xR_x^{-1}L_xM_x$$

$$\iff R_x^{-1}J_\rho^2L_x = M_xR_x^{-1}L_xM_x$$

12. For every $x \in Q$,

$$x \in Q^{iv} \iff L_xR_x^{-1}L_x^{-1} = R_xL_x^{-1}R_x^{-1} \iff R_x^{-1}L_x^{-1}R_x = L_x^{-1}R_xL_x^{-1}.$$

□

4.6.6 Characterization of a Subloop of a Basarab loop by Commutativity

Let $(Q, \cdot)$ be a Basarab loop. Then the elements of $(Q, \cdot)$ defined as

$Q^v := \{x \in Q : L_x = R_x\} = \{x \in G : xz = zx,\ \forall\ z \in Q\} = C(Q) = \text{centrum (Q)}$ are examined. From Lemma 4.6.4,

$$J_\lambda G_x M_x = L_x = J_\lambda R_{x^\rho} L_x M_x = J_\lambda L_x R_x^{-1} M_x \tag{4.49}$$

$$\text{and } J_\rho H_x M_x^{-1} = R_x = J_\rho L_{x^\lambda} R_x M_x^{-1} = J_\rho R_x L_x^{-1} M_x^{-1}. \tag{4.50}$$

Theorem 4.6.23. Let $(Q, \cdot)$ be a Basarab loop. Then Q^v is a subloop of Q, if Q^v is commutative.

Proof. If $Q^v \neq \emptyset$, then $L_e = R_e \implies e \in Q$. Let $x, y \in Q^v$, then

$R_x = L_x \implies z \in Q,\ zR_x = zL_x \implies zx = xz \implies$ commutativity.

Also, $R_{xy} = L_{xy} \implies zR_{xy} = zL_{xy}\ \forall z \in Q \implies z \cdot xy = xy \cdot z \implies$ commutativity. □

Theorem 4.6.24. Let $(Q, \cdot)$ be a Basarab loop. Then for every $x \in Q,\ x$ is contained in Q^v if and only if any of the following conditions hold :

1. $J_\rho^2 H_x = G_x M_x^2$
2. $R_{x^\rho}^{-1} J_\rho^2 = L_x M_x^2 G_x$
3. $H_x M_x^{-2} = J_\lambda^2 L_x R_x^{-1}$
4. $R_x M_x^{-2} L_x^{-1} = L_{x^\lambda}^{-1} J_\lambda^2 R_{x^\rho}$
5. $L_x^{-1} J_\rho^2 L_{x^\lambda} = R_x^{-1} M_x^2 R_x^{-1}$
6. $L_x^{-1} M_x^{-2} R_x = R_x^{-1} J_\lambda^2 J_\lambda L_x$
7. $L_x M_x^2 L_x = R_{x^\rho}^{-1} J_\rho^2 R_x$
8. $L_x^{-1} M_x^{-2} L_x^{-1} = R_x^{-1} J_\lambda^2 R_{x^\rho}$
9. $R_x^{-1} J_\lambda^2 G_x = R_x L_x^{-1} M_x^{-2}$

Proof. 1. For every $x \in Q,\ x \in Q^v \iff J_\rho H_x M_x^{-1} = J_\lambda G_x M_x$

$$\iff J_\lambda^{-1} J_\rho H_x = G_x M_x M_x \iff J_\rho^2 H_x = G_x M_x^2$$

2. For every $x \in Q,\ x \in Q^v \iff J_\rho H_x M_x^{-1} = J_\lambda R_{x^\rho} L_x M_x$

$$\iff J_\lambda^{-1} J_\rho H_x = R_{x^\rho} L_x M_x M_x \iff J_\rho^2 H_x = R_{x^\rho} L_x M_x^2 \iff R_{x^\rho}^{-1} J_\rho^2 H_x = L_x M_x M_x$$

$$\iff R_{x^\rho}^{-1} J_\rho^2 = L_x M_x^2 H_x^{-1} \iff R_{x^\rho}^{-1} J_\rho^2 = L_x M_x^2 G_x$$

3. For every $x \in Q,\ x \in Q^v \iff J_\rho H_x M_x^{-1} = J_\lambda L_x R_x^{-1} M_x$

$$\iff H_x M_x^{-1} M_x^{-1} = J_\rho^{-1} J_\lambda L_x R_x^{-1} \iff H_x M_x^{-2} = J_\lambda^2 L_x R_x^{-1}$$

4. For every $x \in Q,\ x \in Q^v \iff J_\rho L_{x^\lambda} R_x M_x^{-1} = J_\lambda R_{x^\rho} L_x M_x$

$$\iff L_{x^\lambda} R_x M_x^{-1} M_x^{-1} = J_\rho^{-1} J_\lambda R_{x^\rho} \iff R_x M_x^{-2} L_x^{-1} = L_{x^\lambda}^{-1} J_\lambda^2 R_{x^\rho}$$

5. For every $x \in Q,\ x \in Q^v \iff J_\rho L_{x^\lambda} R_x M_x^{-1} = J_\lambda L_x R_x^{-1} M_x$

$$\iff L_x^{-1} J_\lambda^{-1} J_\rho L_{x^\lambda} R_x M_x^{-1} = R_x^{-1} M_x \iff L_x^{-1} J_\rho^2 L_{x^\lambda} = R_x^{-1} M_x M_x R_x^{-1}$$

$$\iff L_x^{-1} J_\rho^2 L_{x^\lambda} = R_x^{-1} M_x^2 R_x^{-1}$$

6. For every $x \in Q,\ x \in Q^v \iff J_\rho R_x L_x^{-1} M_x^{-1} = J_\lambda L_x R_x^{-1} M_x$

$$\iff L_x^{-1} M_x^{-1} M_x^{-1} R_x = R_x^{-1} J_\rho^{-1} J_\lambda L_x \iff L_x^{-1} M_x^{-2} R_x = R_x^{-1} J_\lambda^2 J_\lambda L_x$$

7. For every $x \in Q,\ x \in Q^v \iff J_\lambda R_{x^\rho} L_x M_x = J_\rho R_x L_x^{-1} M_x^{-1}$

$$\iff L_x M_x M_x L_x = R_{x^\rho}^{-1} J_\lambda^{-1} J_\rho R_x \iff L_x M_x^2 L_x = R_{x^\rho}^{-1} J_\rho^2 R_x$$

8. For every $x \in Q,\ x \in Q^v \iff J_\rho R_x L_x^{-1} M_x^{-1} = J_\lambda R_{x^\rho} L_x M_x$

$$\iff L_x^{-1} M_x^{-1} M_x^{-1} L_x^{-1} = R_x^{-1} J_\rho^{-1} J_\lambda R_{x^\rho} \iff L_x^{-1} M_x^{-2} L_x^{-1} = R_x^{-1} J_\lambda^2 R_{x^\rho}$$

9. For every $x \in Q,\ x \in Q^v \iff J_\lambda G_x M_x = J_\rho R_x L_x^{-1} M_x^{-1}$

$$\iff R_x^{-1} J_\rho^{-1} J_\lambda G_x M_x = R_x L_x^{-1} M_x^{-1} \iff R_x^{-1} J_\lambda^2 G_x = R_x L_x^{-1} M_x^{-2}$$

□

4.7 Basarab loop and centrum-Abelian inner mappings loop

In this section, it is proved that a Basarab loop $(Q,\cdot)$ is a centrum-abelian inner mappings loop. In Basarab loop $(Q,\cdot)$, the following are shown to be true: the center $Z(Q,\cdot)$ of a Basarab loop is normal; the quotient $Q/Z(Q,\cdot)$ is an abelian group; the centrum $C(Q,\cdot)$ is normal; and the quotient $Q/C(Q,\cdot)$ is an abelian group.

Definition 4.7.1. *Let $(Q,\cdot)$ be loop and $A \subset Q$. then (i) $\alpha\ \beta \in SYM(Q)$ are said to commute on A written $[A,B]_H = I$ or $AB|_H = BA$ if $hAB = hBA$ for all $h \in H$, (ii) $\alpha\ \beta \in SYM(Q)$ are said to be inverse of each other over H if $AB|_H = I = BA|_H$.*

Theorem 4.7.1. Let $(Q,\cdot)$ be a Basarab loop. Then for every z in the centrum $C(Q,\cdot)$, $zT_xT_y = zT_yT_x$ holds for all $x,y \in Q$ or $T_xT_y|_{C(Q,\cdot)} = T_yT_x|_{C(Q,\cdot)}$ or $[T_x, T_y]$.

Proof. For every $z \in C$,

$$zT_xT_y \implies zR_xL_x^{-1}R_yL_y^{-1} \implies y\backslash((x\backslash zx)y) \implies y\backslash((x\backslash xz)y)$$

$$\implies y\backslash zy \implies y\backslash yz \implies z.$$

Also, $zT_yT_x \implies zR_yL_y^{-1}R_xL_x^{-1} \implies x\backslash((y\backslash zy)x) \implies x\backslash((y\backslash yz)x)$

$$\implies x\backslash(zx) \implies x\backslash(xz) \implies z.$$

Then for a fixed $z \in C,\ zT_xT_y \implies z$ and $zT_yT_x \implies z$ yields $zT_xT_y = zT_yT_x$. $\square$

Remark 4.7.1. *Note that, $zT_x|_{C(Q,\cdot)} = z$ for all $z \in C(Q,\cdot)$. So, $T_x|_{C(Q,\cdot)} = I$ for all $z \in C(Q,\cdot)$. Also, $zT_x^{-1} = z$ then $T_x^{-1}|_{C(Q,\cdot)} = I$.*

Theorem 4.7.2. Let $(Q,\cdot)$ be a Basarab loop. Then for every v in the centrum $C(Q,\cdot),\ vL_{(x,y)}L_{(z,w)} = vL_{(z,w)}L_{(x,y)}$ holds for all $x,y,z,w \in Q$.

Proof. For every $v \in C$,

$$vL_{(x,y)}L_{(z,w)} = vT_x^{-1}T_y^{-1}T_{yx}T_z^{-1}T_w^{-1}T_{wz} = vL_xR_x^{-1}L_yR_y^{-1}R_{yx}L_{yx}^{-1}L_zR_z^{-1}L_wR_w^{-1}R_{wz}L_{wz}^{-1}$$

$$= (wz)\backslash(((w((z((yx)\backslash(((y((xv)/x))/y)(yx))))/z))/w)(wz))$$

$$= (wz)(((w)((z(yx\backslash(((yv)/y)yx)))/z))/w)wz)$$

$$= (wz)\backslash(((w((z(yx\backslash(v\cdot yx)))/z))/w)wz)$$

$$= (wz)\backslash(((w((zv)/z))/w)wz) = wz\backslash(((wv)/w)wz)$$

$$= (wz)\backslash(v\cdot wz) = (wz)\backslash(wz\cdot v) = v.$$

Also, for every $v \in C,\ vL_{(z,w)}L_{(x,y)} = vT_z^{-1}T_w^{-1}T_{wz}T_x^{-1}T_y^{-1}T_{yx}$

$$= vL_zR_z^{-1}L_wR_w^{-1}R_{wz}L_{wz}^{-1}L_xR_x^{-1}L_yR_y^{-1}R_{yx}L_{yx}^{-1}$$

$$= (yx)\backslash(((y((x((wz)\backslash(((w(zv/z))/w)wz)))/x))/y)(yx))$$

$$= (yx)\backslash(((y((x((wz)\backslash(((wv)/w)wz)))/x))/y)(yx))$$

$$= (yx)\backslash(((y((x))wz)\backslash(v\cdot wz)))/x))/y)(yx))$$

$$= (yx)\backslash(((y((xv)/x))/y)(yx)) = (yx)\backslash(((yv)/y)(yx))$$

$$= (yx)\backslash(v\cdot yx) = v.$$

Hence, for a fixed $v \in C,\ vL_{(x,y)}L_{(z,w)} = vL_{(z,w)}L_{(x,y)}$ □

Theorem 4.7.3. Let $(Q,\cdot)$ be a Basarab loop. Then for every u in the centrum $C(Q,\cdot),\ uR_{(x,y)}R_{(z,w)} = uR_{(z,w)}R_{(x,y)}$ holds for all $x,y,z,w \in Q$.

Proof. For every $u \in C$,

$uR_{(x,y)}R_{(z,w)} = uT_xT_yT_{xy}^{-1}T_zT_wT_{zw}^{-1} = uR_xL_x^{-1}R_yL_y^{-1}L_{xy}R_{xy}^{-1}R_zL_z^{-1}R_wL_w^{-1}L_{zw}R_{zw}^{-1}$

$$= ((zw)(w\backslash((z\backslash(((xy(y\backslash((x\backslash(ux))y)))/xy)z))w)))/(zw)$$

$$= ((zw)(w\backslash((z\backslash(((xy(y\backslash(uy)))/xy)z))w)))/(zw)$$

$$= ((zw)(w\backslash((z\backslash(((xy\cdot u)/xy)z))w)))/(zw)$$

$$= ((zw)(w\backslash((z\backslash uz)w)))/(zw)$$

$$= ((zw)(w\backslash(uw)))/(zw)$$

$$= (zw\cdot u)/(zw) = u$$

Also, for every $u \in C,\ uR_{(z,w)}R_{(x,y)} = uT_zT_wT_{zw}^{-1}T_xT_yT_{xy}^{-1}$

$$= uR_zL_z^{-1}R_wL_w^{-1}L_{zw}R_{zw}^{-1}R_xL_x^{-1}R_yL_y^{-1}L_{xy}R_{xy}^{-1}$$

$$= ((xy)(y\backslash((x\backslash(((zw(\backslash((z\backslash(uz))w)))/(zw))x))y)))/(xy)$$

$$= (xy(y\backslash((x\backslash(((zw(w\backslash(uw)))\backslash(zw))x))y)))\backslash(xy)$$

$$= (xy(y\backslash((x\backslash(((zw \cdot u)/(zw))x))y)))/(xy)$$

$$= (xy(y\backslash((x\backslash(ux))y)))\backslash(xy) = (xy(y\backslash(uy)))/(xy)$$

$$= (xy \cdot u)/(xy) = u.$$

Hence, for a fixed $u \in C,\ uR_{(x,y)}R_{(z,w)} = uR_{(z,w)}R_{(x,y)}$ holds for all $x, y, z, w \in Q$. $\square$

Theorem 4.7.4. Let $(Q, \cdot)$ be a Basarab loop. Then for every v in the centrum $C(Q, \cdot),\ vL_{(x,y)}T_z = vT_zL_{(x,y)}$ holds for all $x, y, z, w \in Q$.

Proof. For every $v \in C,\ vL_(x, y)T_z = vT_x^{-1}T_y^{-1}T_{yx}T_x$

$$= vL_xR_x^{-1}L_yR_y^{-1}R_{yx}L_{yx}^{-1}R_zL_z^{-1} = z\backslash((yx\backslash(((y((xv)/x))/y)yx))z)$$

$$= z\backslash((yx\backslash(((yv)/y)yx))z) = z\backslash((yx\backslash(v \cdot yx))z)$$

$$= z\backslash(vz) = v.$$

Also, for every $v \in C$,

$$vT_zL_{(x,y)} = vTzT_x^{-1}T_y^{-1}T_{yx}$$

$$= vR_zL_z^{-1}L_xR_x^{-1}L_yR_y^{-1}R_{yx}L_{yx}^{-1} = yx\backslash(((y((z\backslash(vz)))/x))/y)yx)$$

$$= yx\backslash(((y((xv)/x))/y)yx) = (yx)\backslash(((yv)/y)yx)$$

$$= (yx)\backslash(v \cdot yx) = v.$$

Hence, $vL_{(x,y)}T_z = vT_zL_{(x,y)}$ for a fixed $v \in C$. $\square$

Theorem 4.7.5. Let $(Q, \cdot)$ be a Basarab loop. Then for every u in the centrum $C(Q, \cdot),\ uR_{(x,y)}T_z = uT_zR_{(x,y)}$ holds for all $x, y, z \in Q$.

Proof. For every $v \in C,\ uR_{(x,y)}T_z = uT_xT_yT_{xy}^{-1}T_z$

$$= uR_xL_x^{-1}R_yL_y^{-1}L_{xy}R_{xy}^{-1}R_zL_z^{-1} = z\backslash(((xy(y\backslash((x\backslash(ux))y)))/xy)z)$$

$$= z\backslash(((xy(y\backslash(uy)))/xy)z) = z\backslash(((xy \cdot u)/xy)z)$$

$$= z\backslash(uz) = u.$$

Next, for every $u \in C,\ uT_zR_{(x,y)} = uT_zT_yT_{xy}^{-1}$

$$uR_zL_z^{-1}R_xL_x^{-1}R_yL_y^{-1}L_{xy}R_{xy}^{-1} = (xy(y\backslash((x\backslash((z\backslash(uz))x))y)))/(xy)$$

$$= (xy(y\backslash((x\backslash(ux))y)))/(xy) = (xy(y\backslash(uy)))/(xy) = (xy \cdot u)/(xy) = u.$$

Therefore, for a fixed $u \in C$, $uR_{(x,y)}T_z = uT_zR_{(x,y)}$. □

Theorem 4.7.6. Let $(Q,\cdot)$ be a Basarab loop. Then for every u in the centrum $C(Q,\cdot),\ uL_{(x,y)}R_{(z,w)} = uR_{(z,w)}L_{(x,y)}$ holds for all $x, y, z, w \in Q$.

Proof. For every $u \in C$,

$$uL_{(x,y)}R_{(z,w)} = uT_x^{-1}T_y^{-1}T_{yx}T_zT_wT_{zw}^{-1}$$

$$= uL_xR_x^{-1}L_yR_y^{-1}R_{yx}L_{yx}^{-1}R_xL_z^{-1}R_wL_w^{-1}L_{zw}R_{zw}^{-1}$$

$$= (zw(w\backslash((z\backslash((yx\backslash(((y((xu)/x))/y)yx))z))w)))/(zw)$$

$$= (zw(w\backslash((z\backslash((yx\backslash(((yu)/y)yx))z))w)))/(zw)$$

$$= (zw(w\backslash((z\backslash((yx\backslash(u \cdot yx))z))w)))/(zw)$$

$$= (zw(w\backslash((z\backslash(uz))w)))/(zw)$$

$$= (zw(w\backslash((z\backslash(uz))w)))/(zw)$$

$$= (zw(w\backslash(uw)))/(zw) = (zw \cdot u)/(zw) = u.$$

Also, for every $u \in C(Q,\cdot),\ uR_{(z,w)}L_{(x,y)} = uT_zT_wT_{zw}^{-1}T_x^{-1}T_y^{-1}T_{yx}$

$$= (yx)\backslash(((y((x((zw(w\backslash((z\backslash(uz))w)))/zw))/w))/y)yx)$$

$$= (yx)\backslash(((y((x((zw)(w\backslash(uw)))/zw))/x))/y)yx)$$

$$= (yx)\backslash(((y((x((zw \cdot u)/zw))/x))/y)yx)$$

$$= (yx)\backslash(((y((xu)/x))/y)yx) = (yx)\backslash(((yu)/y)yx)$$

$$= (yx)\backslash(u \cdot yx) = u.$$

Therefore, for a fixed $u \in C$, $uL_{(x,y)}R_{(z,w)} = uR_{(z,w)}L_{(x,y)}$ holds for all $x, y, z, w \in Q$. □

Corollary 4.7.1. Let $(Q,\cdot)$ be a Basarab loop. Then for every fixed element of the centrum $C(Q,\cdot)$, the following are true for all $x, y, z \in Q$:

$$T_xT_y|_{C(Q,\cdot)} = T_yT_x|_{C(Q,\cdot)};\ L(x,y)L_{(z,w)}|_{C(Q,\cdot)} = L_{(z,w)}L_{(x,y)}|_{C(Q,\cdot)};$$

$$R_{(x,y)}R_{(z,w)}|_{C(Q,\cdot)} = R_{(z,w)}R_{(x,y)}|_{C(Q,\cdot)};\ L_{(x,y)}T_x|_{C(Q,\cdot)} = T_zL_{(x,y)}|_{C(Q,\cdot)};$$

$$R_{(x,y)}T_z|_{C(Q,\cdot)} = T_zR_{(x,y)}|_{C(Q,\cdot)};\ L_{(x,y)}R_{(z,w)}|_{C(Q,\cdot)} = R_{(z,w)}L_{(x,y)}|_{C(Q,\cdot)}.$$

Proof. The result follows from Theorems 4.7.1, 4.7.2, 4.7.3, 4.7.4, 4.7.5, and 4.7.6. □

Theorem 4.7.7. Let $(Q,\cdot)$ be a Basarab loop. Then for every z in the centrum $C(Q,\cdot)$, the following are true for all $x, y, z \in Q$:

1. $zT_x^{-1}T_y = zT_yT_x^{-1} = z$
2. $zT_x^{-1}T_y|_{C(Q,\cdot)} = zT_yT_x^{-1}|_{C(Q,\cdot)}$

3. $[T_x^{-1}, T_y]_{C(Q,\cdot)} = I$

4. every $z \in C(Q, \cdot)$ is a fixed point of $T_x^{-1}T_y$ and $T_yT_x^{-1}$.

Proof. For $z \in z \in C(Q, \cdot)$,

$$zT_x^{-1}T_y = zL_xR_x^{-1}R_yL_y^{-1} = y\backslash(((xz)/x)y) = z$$

$$zT_yT_x^{-1} = zR_yL_y^{-1}L_xR_x^{-1} = (x(y\backslash(zy)))/x = (xz)/x = z$$

Hence, the result follows. □

Theorem 4.7.8. A Basarab loop $(Q, \cdot)$ is a $C(Q, \cdot)$- Abelian inner mappings loop (AIML), that is, centrum-Abelian inner mappings loop.

Proof. This follows from Lemma 3.8.4 and Corollary 4.7.1. □

Theorem 4.7.9. Let $(Q, \cdot)$ be a Basarab loop. Then the following are true:

1. $Z(Q, \cdot)$ is normal

2. the quotient $Q/Z(Q, \cdot)$ is an abelian group

3. $C(Q, \cdot)$ is normal

4. the quotient $Q/C(Q, \cdot)$ is an abelian group.

Proof. Let $(Q, \cdot)$ be a Basarab loop, then the Nucleus $N(Q, \cdot)$ is normal and the quotient $Q/N(Q, \cdot)$ is an abelian group, and $Z(Q, \cdot) \leq N(Q, \cdot)$. It follows that $Z(Q, \cdot)$ is a normal and the quotient $Q/Z(Q, \cdot)$ is also an abelian group. Also, since $Z(Q, \cdot) = C(Q, \cdot)$ in $(Q, \cdot)$, then $C(Q, \cdot)$ is normal and $Q/C(Q, \cdot)$ is an abelian group. □

4.8 Algebraic Properties of Associators of a Basarab loop

In this section, some associators in the nucleus and center of a Basarab loop are examined. Relationship between associators and inner mappings of a Basarab loop is defined. Some special cases like, associator with right inverse component, and associator with a left inverse component are considered. It is shown that the associator of any three elements of a Basarab loop is contained in the center and centrum of a Basarab loop. Some expressions for an associator of a loop are obtained for when: one component of the associator is a product of the loop and its nucleus; one component of the associator is a product of the loop and its nucleus which is a normal subloop of the loop; one component of the associator is a product of the loop and its center; two components of the associator are products of the loop and its nucleus; two components of the associator are product of the loop and its nucleus which is a normal subloop of the loop; two components of the associator are products of the loop and its center; three components of the associator are products of the loop and its nucleus; three components of the associator are products of the loop and its nucleus which is a normal subloop of the loop; and three components of the associator are products of the loop and its center.

4.8.1 Some Associators in the Nucleus and Center of a Basarab loop

Theorem 4.8.1. Let $(Q,\cdot)$ be a Basarab loop. Then

$$Z = (a,b,c)Z = Z[a,b,c]$$

for all $a,b,c \in Q$

Proof. For every $a,b,c \in Q$,

$$Z(Q,\cdot) \trianglelefteq (Q,\cdot) \text{ implies} (aZ \cdot bZ) \cdot cZ = aZ(bZ \cdot cZ)$$

$$\implies (ab)Z \cdot cZ = aZ \cdot (bc)Z \implies (ab \cdot c)Z = (a \cdot bc)Z \implies (a \cdot bc)\backslash((ab \cdot c)Z) = Z$$

$$\implies Z = ((a \cdot bc)\backslash(ab \cdot c))Z \implies Z = (a, b, c)Z.$$

Also, for every $a, b, c \in Q$,

$$(Za \cdot Zb) \cdot Zc = Za(Zb \cdot Zc)Z(ab) \cdot Zc = Za \cdot Z(ab)$$

$$\implies Z(ab \cdot c) = Z(a \cdot bc) \implies Z = (Z(a \cdot bc))/(ab \cdot c)$$

$$\implies Z = Z((a \cdot bc)/(ab \cdot c)) \implies Z = Z[a, b, c].$$

Hence, for every fixed $a, b, c \in Q$,

$$Z = (a, b, c)Z = Z[a, b, c] = Z.$$

□

Corollary 4.8.1. Let $(Q, \cdot)$ be a Basarab loop. Then

$$C = (a, b, c)C = C[a, b, c]$$

for all $a, b, c \in Q$.

Proof. The result follows immediately from Theorems 4.7.9 and 4.8.1. □

Lemma 4.8.1. Let $(Q, \cdot)$ be a loop with a nucleus $N(Q, \cdot)$, and let (x, y, z) be the associator of elements $x, y, z \in Q$. Then the following are true for $c \in N$:

1. $(x, y, zc) = c^{-1}(x, y, z)c$
2. $(x, yc, z) = (x, y, cz)$
3. $(cx, y, z) = (x, y, z)$

4. $(xc, y, z) = (x, cy, z)$

Proof. Let $(Q, \cdot)$ be a loop with a nucleus $N(Q, \cdot)$, and let (x, y, z) be the associator of elements $x, y, z \in Q$ and $c \in N$. Then :

1.

$$(x, y, zc) = (x \cdot (y \cdot zc))\backslash(xy \cdot zc)$$

$$(x \cdot (y \cdot zc))(x, y, zc) = (xy \cdot zc)$$

$$(x \cdot (yz \cdot c))(x, y, zc) = (xy \cdot z)c$$

$$((x \cdot yz)c)(x, y, zc) = (xy \cdot z)c$$

$$(x \cdot yz) \cdot c(x, y, zc) = (xy \cdot z)c$$

$$c(x, y, zc) = (x \cdot yz)\backslash((xy \cdot z)c)$$

$$c(x, y, zc) = ((x \cdot yz)\backslash(xy \cdot z))c$$

$$c(x, y, zc) = (x, y, z)c$$

$$(x, y, zc) = c^{-1}(x, y, z)c$$

2.

$$(x, yc, z) = (x \cdot (yc \cdot z))\backslash((x \cdot yc) \cdot z)$$

$$(x \cdot (yc \cdot z))(x, yc, z) = (x \cdot (yc \cdot z))\backslash((x \cdot yc) \cdot z)$$

$$(x \cdot (y \cdot cz))(x, yc, z) = ((xy \cdot c) \cdot z)(x \cdot (y \cdot cz))(x, yc, z) = (xy \cdot cz)$$

$$(x, yc, z) = (x \cdot (y \cdot cz))\backslash(xy \cdot cz)$$

$$(x, yc, z) = (x, y, cz)$$

3.

$$(cx, y, z) = (cx \cdot yz)\backslash((cx \cdot y)z)$$

$$(cx \cdot yz)(cx, y, z) = ((cx \cdot y)z)$$

$$(cx \cdot yz)(cx, y, z) = (c \cdot xy)z$$

$$(c \cdot (x \cdot yz))(cx, y, z) = c(xy \cdot z)$$

$$c \cdot [(x \cdot yz)(cx, y, z)] = c(xy \cdot z)$$

$$(x \cdot yz)(cx, y, z) = (xy \cdot z)$$

$$(cx, y, z) = (x \cdot yz)\backslash(xy \cdot z)$$

$$(cx, y, z) = (x, y, z)$$

4.

$$(xc, y, z) = (xc \cdot yz)\backslash((xc \cdot y)z)$$

$$(xc \cdot yz)(xc, y, z) = ((xc \cdot y)z)$$

$$(x \cdot (cy \cdot z))(xc, y, z) = ((x \cdot cy)z)$$

$$(zc, y, z) = (x \cdot (cy \cdot z))\backslash((x \cdot cy)z)$$

$$(xc, y, z) = (x, cy, z)$$

□

Lemma 4.8.2. Let $(Q, \cdot)$ be a loop with a nontrivial nucleus $N(Q, \cdot)$, which is a normal subloop of $(Q, \cdot)$, and let (x, y, z) be the associator of elements $x, y, z \in Q$. If $q \in N$, then the following hold:

1. $(x, yq, z) = (x, y, z)$

2. $(xq, y, z) = (x, y, z)$

3. $(x, y, qz) = (x, y, z)$

4. $q^{-1}(x, y, z)q = (x, y, z)$

5. $(x, qy, z) = (x, y, z)$

Proof. Let $(Q, \cdot)$ be a loop with a nontrivial nucleus $N(Q, \cdot)$, and let (x, y, z) be the associator of elements $x, y, z \in Q$ and $q \in N$. Since $N(Q, \cdot)$ is normal subloop of $(Q, \cdot)$, then for every $a \in Q,\ aq = ra$ where $q, r \in N$

1.

$$(x, yq, z) = (x \cdot (yq \cdot z))\backslash((x \cdot yq)z)$$

$$(x \cdot (yq \cdot z))(x, yq, z) = ((x \cdot yq) \cdot z)$$

$$(x \cdot (ry \cdot z))(x, yq, z) = ((x \cdot ry) \cdot z)$$

$$(x \cdot (r \cdot yz))(x, yq, z) = ((xr \cdot y) \cdot z)$$

$$(xr \cdot yz)(x, yq, z) = ((xr \cdot y) \cdot z)$$

$$(sx \cdot yz)(x, yq, z) = ((sx \cdot y) \cdot z)$$

$$(s \cdot (x \cdot yz))(x, yq, z) = ((s \cdot xy) \cdot z)$$

$$(s \cdot (x \cdot yz))(x, yq, z) = s \cdot (xy \cdot z)$$

$$s((x \cdot yz)(x, yq, z)) = s \cdot (xy \cdot z)$$

$$(x \cdot yz)(x, yq, z) = (xy \cdot z)$$

$$(x, yq, z) = (x \cdot yz)\backslash(xy \cdot z)$$

$$(x, yq, z) = (x, y, z)$$

2.

$$(xq, y, z) = (xq \cdot yz)\backslash((xq \cdot y) \cdot z)$$

$$(xq \cdot yz)(xq, y, z) = ((xq \cdot y) \cdot z)$$

$$(rx \cdot yz)(xq, y, z) = ((rx \cdot y) \cdot z)$$

$$(r \cdot (x \cdot yz))(xq, y, z) = ((r \cdot xy) \cdot z)$$

$$r((x \cdot yz)(xq, y, z)) = r \cdot (xy \cdot z)$$

$$(x \cdot yz)(xq, y, z) = (xy \cdot z)$$

$$(xq, y, z) = (x \cdot yz)\backslash(xy \cdot z)$$

$$(xq, y, z) = (x, y, z)$$

3.

$$(x, y, qz) = (x \cdot (y \cdot qz))\backslash(xy \cdot qz)$$

$$(x \cdot (y \cdot qz))(x, y, qz) = (xy \cdot qz)$$

$$(x \cdot (yq \cdot z))(x, y, qz) = ((xy \cdot q)z)$$

$$(x \cdot (yq \cdot z))(x, y, qz) = ((x \cdot yq)z)$$

$$(x \cdot (ry \cdot z))(x, y, qz) = ((x \cdot ry)z)$$

$$(x \cdot (r \cdot yz))(x, y, qz) = ((xr \cdot y)z$$

$$(xr \cdot yz)(x, y, qz) = ((xr \cdot y)z)(sx \cdot yz)(x, y, qz) = ((sx \cdot y)z)$$

$$(s \cdot (x \cdot yz))(x, y, qz) = ((s \cdot xy)z)$$

$$s((x \cdot yz))(x, y, qz) = q(xy \cdot z)$$

$$(x \cdot yz)(x, y, az) = (xy \cdot z)$$

$$(x, y, qz) = (x \cdot yz)\backslash(xy \cdot z)$$

$$(x, y, qz) = (x, y, z)$$

4. By (1) of Lemma 4.8.1, $q^{-1}(x, y, z)q = (x, y, zq)$. Thus,

$$(x, y, zq) = (x \cdot (y \cdot zq))\backslash(xy \cdot zq)$$

$$(x \cdot (y \cdot zq))(x, y, zq) = (xy \cdot zq)$$

$$(x \cdot (y \cdot rz))(x, y, zq) = (xy \cdot rz)$$

$$(x \cdot (yr \cdot z))(x, y, zq) = ((xy \cdot r)z)$$

$$(x \cdot (yr \cdot z))(x, y, zq) = ((x \cdot yr)z)$$

$$(x \cdot (sy \cdot z))(x, y, zq) = ((x \cdot sy)z)$$

$$(x \cdot (s \cdot yz))(x, y, zq) = ((xs \cdot y)z)$$

$$(xs \cdot yz)(x, y, zq) = (xs \cdot y)z$$

$$(tx \cdot yz)(x, y, zq) = (tx \cdot y)z$$

$$(t(x \cdot yz))(x, y, zq) = (t \cdot xy)z$$

$$t((x \cdot yz)(x, y, zq)) = t(xy \cdot z)$$

$$(x \cdot yz)(x, y, zq) = (xy \cdot z)$$

$$(x, y, zq) = (x \cdot yz)\backslash(xy \cdot z)$$

$$q^{-1}(x, y, z)q = (x, y, z) \text{ by (1) of Lemma 4.8.1.}$$

5.

$$(x, qy, z) = (x \cdot (qy \cdot))\backslash((x \cdot qy) \cdot z)$$

$$(x \cdot (qy \cdot z))(x, qy, z) = ((x \cdot qy) \cdot z)$$

$$(x \cdot (q \cdot yz))(x, qy, z) = ((xq \cdot y) \cdot z)$$

$$(xq \cdot yz)(x, qy, z) = ((xq \cdot y) \cdot z)$$

$$(rx \cdot yz)(x, qy, z) = (rx \cdot y) \cdot z$$

$$(r \cdot (x \cdot yz))(x, qy, z) = (r \cdot xy) \cdot z$$

$$r \cdot ((x \cdot yz)(x, qy, z)) = r \cdot (xy \cdot z)$$

$$(x \cdot yz)(x, qy, z) = (xy \cdot z)$$

$$(x, qy, z) = (x \cdot yz)\backslash(xy \cdot z)$$

$$(x, qy, z) = (x, y, z)$$

□

Corollary 4.8.2. Let $(Q, \cdot)$ be a Basarab loop with a nontrivial nucleus $N(Q, \cdot)$, and let (x, y, z) be the associator of elements $x, y, z \in Q$. If $q \in N$, then the following hold:

1. $(x, yq, z) = (x, y, z)$

2. $(xq, y, z) = (x, y, z)$

3. $(x, y, qz) = (x, y, z)$

4. $q^{-1}(x, y, z)q = (x, y, z)$

5. $(x, qy, z) = (x, y, z)$

Proof. The nucleus $N(Q, \cdot)$ of Basarab loop $(Q, \cdot)$ is a normal subloop. Hence, the result. □

Lemma 4.8.3. Let $(Q, \cdot)$ be a loop with a center $Z(Q, \cdot)$, and let (x, y, z) be the associator of elements $x, y, z \in Q$ .Then the following are true for $a \in Z$:

1. $(ax, y, z) = a^{-1}(x, y, z)a = (x, y, z)$

2. $a(x, ay, z)a^{-1} = (x, ay, z) = (x, y, z)$

3. $(x, y, az)a = (x, y, z)a$

4. $(xa, y, z) = (x, y, z)$

5. $(x, ya, z) = (x, y, z)$

6. $(x, y, za) = (x, y, z)$

Proof. Let $a \in N \cap C$:

1.

$$(ax, y, z) = (ax \cdot yz)\backslash((ax \cdot y)\cdot)$$

$$(ax \cdot yz)(ax, y, z) = ((ax \cdot y) \cdot z)$$

$$(xa \cdot yz)(ax, y, z) = ((xa \cdot y) \cdot z)$$

$$(x \cdot (a \cdot yz))(ax, y, z) = ((x \cdot ay) \cdot z)$$

$$(x \cdot (ay \cdot z))(ax, y, z) = ((x \cdot ay) \cdot z)$$

$$(x \cdot (ya \cdot z))(ax, y, z) = ((x \cdot ya) \cdot z)$$

$$(x \cdot (y \cdot az))(ax, y, z) = ((xy \cdot a) \cdot z)$$

$$(x \cdot (y \cdot az))(ax, y, z) = (xy \cdot az)$$

$$(x \cdot (y \cdot za))(ax, y, z) = (xy \cdot za)$$

$$(x \cdot (yz \cdot a))(ax, y, z) = (xy \cdot z)a$$

$$((x \cdot yz) \cdot a)(ax, y, z) = (xy \cdot z)a$$

$$(x \cdot yz)(a \cdot (ax, y, z)) = (xy \cdot z)a$$

$$a(ax, y, z) = (x \cdot yz)\backslash(xy \cdot z)a$$

$$a(ax, y, z) = ((x \cdot yz)\backslash(xy \cdot z))a$$

$$a(ax, y, z) = (x, y, z)a$$

$$(ax, y, z) = a^{-1}(x, y, z)a = (x, y, z)$$

2.

$$(x, ay, z) = (x \cdot (ay \cdot z))\backslash((x \cdot ay) \cdot z)$$

$$(x \cdot (ay \cdot z))(x, ay, z) = ((x \cdot ay) \cdot z)$$

$$(x \cdot (ya \cdot z))(x, ay, z) = ((x \cdot ya) \cdot z)$$

$$(x \cdot (y \cdot az))(x, ay, z) = ((xy \cdot a) \cdot z)$$

$$(x \cdot (y \cdot za))(x, ay, z) = (xy \cdot az)$$

$$(x \cdot (yz \cdot a))(x, ay, z) = (xy \cdot za)$$

$$((x \cdot yz)a)(x, ay, z) = (xy \cdot z)a$$

$$(x \cdot yz)(a(x, ay, z)) = (xy \cdot z)a$$

$$a(x, ay, z) = (x \cdot yz)\backslash((xy \cdot z)a)$$

$$a(x, ay, z) = ((x \cdot yz)\backslash(xy \cdot z))a$$

$$a(x, ay, z) = (x, y, z)a$$

$$a(x, ay, z)a^{-1} = (x, y, z)$$

$$a(x, ay, z)a^{-1} = (x, ay, z) = (x, y, z)$$

3.

$$(x, y, az) = (x \cdot (y \cdot az))\backslash(xy \cdot az)$$

$$(x \cdot (y \cdot az))(x, y, az) = (xy \cdot az)$$

$$(x \cdot (y \cdot za))(x, y, az) = (xy \cdot za)$$

$$(x \cdot (yz \cdot a))(x, y, az) = (xy \cdot z)a$$

$$((x \cdot yz) \cdot a)(x, y, az) = (xy \cdot z)a$$

$$(x \cdot yz)(a(x, y, az)) = (xy \cdot z)a$$

$$a(x, y, az) = (x \cdot yz)\backslash((xy \cdot z)a)$$

$$a(x, y, az) = ((x \cdot yz)\backslash(xy \cdot z))a$$

$$(x, y, az)a = (x, y, z)a$$

$$(x, y, az) = (x, y, z)$$

4.

$$(xa, y, z) = (xa \cdot yz)\backslash((xa \cdot y) \cdot z)$$

$$(xa \cdot yz)(xa, y, z) = ((xa \cdot y) \cdot z)$$

$$(ax \cdot yz)(xa, y, z) = ((ax \cdot y) \cdot z)$$

$$(a \cdot (x \cdot yz))(xa, y, z) = ((a \cdot xy) \cdot z)$$

$$(a \cdot (x \cdot yz))(xa, y, z) = a(xy \cdot z)$$

$$a \cdot ((x \cdot yz)(xa, y, z)) = a(xy \cdot z)$$

$$((x \cdot yz)(xa, y, z)) = (xy \cdot z)$$

$$(xa, y, z) = (x \cdot yz)\backslash(xy \cdot z)$$

$$(xa, y, z) = (x, y, z)$$

5.

$$(x, ya, z) = (x \cdot (ya \cdot z))\backslash((x \cdot ya) \cdot z)$$

$$(x \cdot (ya \cdot z))(x, ya, z) = ((x \cdot ya) \cdot z)$$

$$(x \cdot (ay \cdot z))(x, ya, z) = ((x \cdot ay) \cdot z)$$

$$((x \cdot ay)z)(x, ya, z) = ((xa \cdot y) \cdot z)$$

$$((xa \cdot y)z)(x, ya, z) = ((ax \cdot y) \cdot z)$$

$$((ax \cdot y)z)(x, ya, z) = ((a \cdot xy) \cdot z)$$

$$((a \cdot xy)z)(x, ya, z) = (a \cdot (xy \cdot z))$$

$$(a \cdot (xy \cdot z))(x, ya, z) = a \cdot (xy \cdot z)$$

$$a \cdot ((xy \cdot z)(x, ya, z)) = a \cdot (xy \cdot z)$$

$$(xy \cdot z)(x, ya, z) = (xy \cdot z)$$

$$(x, ya, z) = (xy \cdot z)\backslash(xy \cdot z)$$

$$(x, ya, z) = (x, y, z)$$

6.

$$(x, y, za) = (x \cdot (y \cdot) za))\backslash(xy \cdot) za)$$

$$(x \cdot (y \cdot) za))(x, y, za) = (xy \cdot) za)$$

$$(x \cdot (y \cdot az))(x, y, za) = (xy \cdot az)$$

$$(x \cdot (ya \cdot z))(x, y, za) = ((xy \cdot a)z)$$

$$(x \cdot (ay \cdot z))(x, y, za) = ((x \cdot ya)z)$$

$$(x \cdot (a \cdot yz))(x, y, za) = ((x \cdot ay)z)$$

$$(xa \cdot yz)(x, y, za) = (xa \cdot y)z$$

$$(ax \cdot yz)(x, y, za) = (ax \cdot y)z$$

$$(a \cdot (x \cdot yz))(x, y, za) = (a \cdot xy)z$$

$$a \cdot ((x \cdot yz)(x, y, za)) = a(xy \cdot z)$$

$$((x \cdot yz)(x, y, za)) = (xy \cdot z)$$

$$(x, y, za) = (x \cdot yz)\backslash(xy \cdot z)$$

$$(x, y, za) = (x, y, z)$$

□

Lemma 4.8.4. Let $(Q, \cdot)$ be a loop with a nucleus $N(Q, \cdot)$, and let $[x, y, z]$ be the associator of elements $x, y, z \in Q$. Then the following hold are true for $m \in N$:

1. $[x, y, zm] = [x, y, z]$
2. $[x, ym, z] = [x, y, mz]$
3. $[xm, y, z] = [x, my, z]$
4. $[mx, y, z] = m[x, y, z]m^{-1}$

Proof. 1.

$$[x, y, zm] = (x \cdot (y \cdot zm))/(xy \cdot zm)$$

$$[x, y, zm](xy \cdot zm) = (x \cdot (y \cdot zm))$$

$$[x, y, zm]((xy \cdot z)m) = (x \cdot (yz \cdot m))$$

$$([x, y, zm](xy \cdot z))m = (x \cdot yz)m$$

$$[x, y, zm](xy \cdot z) = (x \cdot yz)$$

$$[x, y, zm] = (x \cdot yz)/(xy \cdot z)$$

$$[x, y, zm] = [x, y, z]$$

2.

$$[x, ym, z] = (x \cdot (ym \cdot z))/((x \cdot ym) \cdot z)$$

$$[x, ym, z]((x \cdot ym) \cdot z) = (x \cdot (ym \cdot z))$$

$$[x, ym, z]((xy \cdot m) \cdot z) = (x \cdot (y \cdot mz))$$

$$[x, ym, z]((xy \cdot mz)) = (x \cdot (y \cdot mz))$$

$$[x, ym, z] = (x \cdot (y \cdot mz))/((x \cdot y) \cdot mz)$$

$$[x, ym, z] = [x, y, mz]$$

3.

$$[xm, y, z] = (xm \cdot yz)/((xm \cdot y) \cdot z)$$

$$[xm, y, z]((xm \cdot y) \cdot z) = (xm \cdot yz)$$

$$[xm, y, z]((x \cdot my) \cdot z) = (x \cdot (m \cdot yz))$$

$$[xm, y, z]((x \cdot my) \cdot z) = (x \cdot (my \cdot z))$$

$$[xm, y, z] = (x \cdot (my \cdot z)/((x \cdot my) \cdot z)$$

$$[xm, y, z] = [x, my, z]$$

4.

$$[mx, y, z] = (mx \cdot yz)/((mx \cdot y) \cdot z)$$

$$[mx, y, z]((mx \cdot y) \cdot z) = (mx \cdot yz)$$

$$[mx, y, z]((m \cdot xy) \cdot z) = m(x \cdot yz)$$

$$[mx, y, z](m(xy \cdot z)) = m(x \cdot yz)$$

$$[mx, y, z]m \cdot (xy \cdot z) = m(x \cdot yz)$$

$$[mx, y, z]m = (m(x \cdot yz))/(xy \cdot z)$$

$$[mx, y, z]m = m((x \cdot yz)/(xy \cdot z))$$

$$[mx, y, z]m = m[x, y, z]$$

$$[mx, y, z] = m[x, y, z]m^{-1}$$

$\square$

Lemma 4.8.5. Let $(Q, \cdot)$ be a loop with a nucleus $N(Q, \cdot)$, and let (x, y, z) be the associator of elements $x, y, z \in Q$. Then the following are true for $u, w \in N$:

1. $(x, uy, wz) = (xu, yw, z)$

2. $(ux, wy, z) = (xw, y, z)$

3. $(ux, y, wz) = (x, yw, z)$

Proof. 1.

$$(x, uy, wz) = (x{\cdot}(uy{\cdot}wz))\backslash((x{\cdot}uy){\cdot}wz) \iff (x{\cdot}(uy{\cdot}wz))(x, uy, wz) = ((x{\cdot}uy){\cdot}wz)$$

$$\iff (x \cdot ((uy \cdot w) \cdot z))(x, uy, wz) = ((xu \cdot y)w \cdot z)$$

$$\iff (x \cdot ((u \cdot yw)z))(x, uy, wz) = ((xu \cdot yw) \cdot z)$$

$$\iff (x \cdot ((u \cdot (yw \cdot z))))(x, uy, wz) = ((xu \cdot yw) \cdot z)$$

$$\iff (xu \cdot (yw \cdot z))(x, uy, wz) = ((xu \cdot yw) \cdot z)$$

$$\iff (x, uy, wz) = (xu \cdot (yw \cdot z))\backslash((xu \cdot yw) \cdot z)$$

$$\iff (x, uy, wz) = (xu, yw, z)$$

2.

$$(ux, wy, z) = (ux{\cdot}(wy{\cdot}z))\backslash((ux{\cdot}wy){\cdot}z) \iff (ux{\cdot}(wy{\cdot}z))(ux, wy, z) = ((ux{\cdot}wy){\cdot}z)$$

$$\iff (ux \cdot (w \cdot yz))(ux, wy, z) = (((u \cdot (x \cdot wy)) \cdot z))$$

$$\iff (u \cdot (x(w \cdot yz)))(ux, wy, z) = u \cdot ((x \cdot wy) \cdot z)$$

$$\iff u \cdot ((x(w \cdot yz)))(ux, wy, z)) = u \cdot ((x \cdot wy) \cdot z)$$

$$\iff ((x(w \cdot yz))(ux, wy, z)) = u \cdot ((x \cdot wy) \cdot z)$$

$$\iff ((x(w \cdot yz))(ux, wy, z)) = (x \cdot wy) \cdot z)$$

$$\iff (xw \cdot yz)(ux, wy, z) = ((xw \cdot y) \cdot z)$$

$$\iff (ux, wy, z) = (xw \cdot yz)\backslash((xw \cdot y) \cdot z) \iff (ux, wy, z) = (xw, y, z).$$

3.

$$(ux, y, wz) = (ux{\cdot}(y{\cdot}wz))\backslash((ux{\cdot}y){\cdot}wz) \iff (ux{\cdot}(y{\cdot}wz))(ux, y, wz) = ((ux{\cdot}y){\cdot}wz)$$

$$\iff (u{\cdot}x(y{\cdot}wz))(ux, y, wz) = ((u{\cdot}xy){\cdot}wz) \iff u{\cdot}(x(y{\cdot}wz)(ux, y, wz)) = u{\cdot}(xy{\cdot}wz)$$

$$\iff (x(y \cdot wz)(ux, y, wz)) = (xy \cdot wz) \iff (x(yw \cdot z)(ux, y, wz)) = ((xy \cdot w)z)$$

$$\iff (x(yw \cdot z)(ux, y, wz)) = ((x \cdot yw)z) \iff (ux, y, wz) = (x(yw \cdot z))\backslash((x \cdot yw)z)$$

$$\iff (ux, y, wz) = (x, yw, z).$$

□

Lemma 4.8.6. Let $(Q, \cdot)$ be a loop with a nontrivial nucleus $N(Q, \cdot)$, which is a normal subloop of $(Q, \cdot)$ and let (x, y, z) be the associator of elements $x, y, z \in Q$ and $q \in N$. If $u, w \in N$, then the following are true:

1. $(x, uy, wz) = (x, y, z)$
2. $(ux, wy, z) = (x, y, z)$
3. $(ux, y, wz) = (x, y, z)$

Proof. The proof is similarly to the proof of Lemma 4.8.2. □

Corollary 4.8.3. Let $(Q,\cdot)$ be a Basarab loop with a nontrivial nucleus $N(Q,\cdot)$, and let (x,y,z) be the associator of elements $x,y,z\in Q$ and $q\in N$. If $u,w\in N$, then the following are true:

1. $(x,uy,wz)=(x,y,z)$

2. $(ux,wy,z)=(x,y,z)$

3. $(ux,y,wz)=(x,y,z)$

Proof. The nucleus $N(Q,\cdot)$ of Basarab loop $(Q,\cdot)$ is a normal subloop. Hence, the result. □

Lemma 4.8.7. Let $(Q,\cdot)$ be a loop with a nontrivial nucleus $N(Q,\cdot)$, and let (x,y,z) be the associator of elements $x,y,z\in Q$ and $u,w\in N$. Then the following are true:

1. $(x,yu,zw)=w^{-1}(x,yu,z)w$

2. $(xu,yw,z)=(x,uy,wz)$

3. $(xu,y,zw)=w^{-1}(x,uy,z)w$

Proof. 1.

$$(x,yu,zw)=(x\cdot(yu\cdot zw))\backslash((x\cdot yu)\cdot zw) \iff (x\cdot(yu\cdot zw))(x,yu,zw)=((x\cdot yu)\cdot zw)$$

$$\iff (x\cdot((yu\cdot z)w))(x,yu,zw)=((x\cdot yu)z\cdot w) \iff ((x\cdot(yu\cdot z))w)(x,yu,zw)=(x\cdot yu)z\cdot w$$

$$\iff (x\cdot(yu\cdot z))\cdot w(x,yu,zw)=(x\cdot yu)z\cdot w \iff w(x,yu,zw)=(x\cdot(yu\cdot z))\backslash((x\cdot yu)z\cdot w)$$

$$\iff w(x,yu,zw)=((x\cdot(yu\cdot z))\backslash(x\cdot yu)z)w \iff w(x,yu,zw)=(x,yu,z)w$$

$$\iff (x,yu,zw)=w^{-1}(x,yu,z)w$$

2.

$$(xu, yw, z) = (xu\cdot(yw\cdot)z))\backslash((xuxu\cdot yw)\cdot z) \iff (xu\cdot(yw\cdot z))(xu, yw, z) = ((xu\cdot yw)\cdot z)$$

$$\iff (xu\cdot(y\cdot wz))(xu, yw, z) = ((xu\cdot y)\cdot w)\cdot z \iff (xu\cdot(y\cdot wz))(xu, yw, z) = ((xu\cdot y)\cdot wz)$$

$$\iff (xu, yw, z) = (xu\cdot(y\cdot wz))\backslash((xu\cdot y)\cdot wz) \iff (xu, yw, z) = (x\cdot u(y\cdot wz))\backslash((x\cdot uy)\cdot wz)$$

$$\iff (xu, yw, z) = (x \cdot (uy \cdot wz))\backslash((x \cdot uy) \cdot wz) \iff (xu, yw, z) = (x, uy, wz)$$

3.

$$(xu, y, zw) = (xu\cdot(y\cdot zw))\backslash((xu\cdot y)\cdot zw) \iff (xu\cdot(y\cdot zw))(xu, y, zw) = ((xu\cdot y)\cdot zw)$$

$$\iff (x\cdot u(y\cdot zw))(xu, y, zw) = ((x\cdot uy)\cdot zw) \iff (x\cdot(uy\cdot zw))(xu, y, zw) = ((x\cdot uy)z\cdot w)$$

$$\iff (x\cdot(uy\cdot zw))(xu, y, zw) = ((x\cdot uy)z)w \iff (x\cdot((uy\cdot z)w))(xu, y, zw) = ((x\cdot uy)z)w$$

$$\iff (x(uy\cdot z))w(xu, y, zw) = ((x\cdot uy)z)w \iff w(xu, y, zw) = ((x(uy\cdot z))\backslash((x\cdot uy)z))w$$

$$\iff w(xu, y, zw) = (x, uy, z)w \iff (xu, y, zw) = w^{-1}(x, uy, z)w$$

□

Lemma 4.8.8. Let $(Q, \cdot)$ be a loop with a nontrivial nucleus $N(Q, \cdot)$, which is a normal subloop and let (x, y, z) be the associator of elements $x, y, z \in Q$. If $u, w \in N$, then the following hold:

1. $(x, yu, zw) = (x, y, z)$
2. $(xu, yw, z) = (x, y, z)$
3. $(xu, y, zw) = (x, y, z)$

Proof. The proof is similarly to the proof of Lemma 4.8.2.

□

Corollary 4.8.4. Let $(Q,\cdot)$ be a Basarab loop with a nontrivial nucleus $N(Q,\cdot)$, and let (x,y,z) be the associator of elements $x,y,z\in Q$. If $u,w\in N$, then the following hold:

1. $(x,yu,zw)=(x,y,z)$
2. $(xu,yw,z)=(x,y,z)$
3. $(xu,y,zw)=(x,y,z)$

Proof. The nucleus $N(Q,\cdot)$ of Basarab loop $(Q,\cdot)$ is a normal subloop. Hence, the result. □

Lemma 4.8.9. Let $(Q,\cdot)$ be a loop with a nontrivial nucleus $N(Q,\cdot)$, and let (x,y,z) be the associator of elements $x,y,z\in Q$. If $u,v,w\in N$, then the following hold:

1. $(xu,yv,zw)=w^{-1}(xu,yv,z)w$
2. $(ux,vy,wz)=(x,vy,wz)$

Proof. 1.

$$(xu,yv,zw)=(xu\cdot(yv\cdot zw))\backslash((xu\cdot yv)\cdot zw)$$

$$\iff (xu\cdot(yv\cdot zw))(xu,yv,zw)=((xu\cdot yv)\cdot zw)$$

$$\iff (xu\cdot((yv\cdot z)w))(xu,yv,zw)=(xu\cdot yv)z\cdot w$$

$$\iff (xu\cdot(yv\cdot z))w(xu,yv,zw)=(xu\cdot yv)z\cdot w$$

$$\iff w(xu,yv,zw)=(xu\cdot(yv\cdot z))\backslash((xu\cdot yv)z)w$$

$$\iff w(xu,yv,zw)=((xu\cdot(yv\cdot z))\backslash)((xu\cdot yv)z))w$$

$$\iff w(xu,yv,zw)=(xu,yv,z)w \iff (xu,yv,zw)=w^{-1}(xu,yv,z)w$$

2.

$$(ux,vy,wz)=(ux\cdot(vy\cdot wz))\backslash((ux\cdot vy)\cdot wz)$$

$$\iff (ux\cdot(vy\cdot wz))(ux,vy,wz)=((ux\cdot vy)\cdot wz)$$

$$\iff (u\cdot x(vy\cdot wz))(ux,vy,wz)=((u\cdot(x\cdot vy))\cdot wz)$$

$$\iff u\cdot(x(vy\cdot wz)(ux,vy,wz))=u\cdot((x\cdot vy)\cdot wz)$$

$$\iff (x(vy\cdot wz)(ux,vy,wz)=((x\cdot vy)\cdot wz)$$

$$\iff (ux,vy,wz)=(x(vy\cdot wz))\backslash((x\cdot vy)\cdot wz) \iff (ux,vy,wz)=(x,vy,wz)$$

□

Lemma 4.8.10. Let $(Q,\cdot)$ be a loop with a nontrivial nucleus $N(Q,\cdot)$, which is a normal subloop of $(Q,\cdot)$ and let (x,y,z) be the associator of elements $x,y,z\in Q$. If $u,v,w\in N$, then the following hold:

1. $(xu,yv,zw)=(x,y,z)$
2. $(ux,vy,wz)=(x,y,z)$

Proof. The proof is similar to the proof of Lemma 4.8.2.

□

Corollary 4.8.5. Let $(Q,\cdot)$ be a loop with a nontrivial nucleus $N(Q,\cdot)$, which is a normal subloop of $(Q,\cdot)$ and let (x,y,z) be the associator of elements $x,y,z\in Q$. If $u,v,w\in N$, then the following hold:

1. $(xu,yv,zw)=(x,y,z)$
2. $(ux,vy,wz)=(x,y,z)$

Proof. The nucleus $N(Q,\cdot)$ of Basarab loop $(Q,\cdot)$ is a normal subloop. Hence, the result. □

Lemma 4.8.11. Let $(Q,\cdot)$ be a Basarab loop with a center $Z(Q,\ \cdot)$, and let (x,y,z) be the associator of elements $x,y,z\in Q$. Then the following are true for $c\in Z$:

1. $(y^{\lambda},y,zc)=e$

2. $(x,x^{\rho},zc)=(x,x^{\rho},z)$

3. $(y^{\lambda},yc,z)=e$

4. $(x,x^{\rho}c,z)=(x,x^{\rho},z)$

Proof. 1. From Lemma 4.8.1(1), set $x=y^{\lambda}$ on the left, and let $c\in N\cap C$. Then

$$(y^{\lambda},y,zc)=(y^{\lambda}\cdot(y\cdot zc))\backslash(y^{\lambda}y\cdot zc)\iff(y^{\lambda}\dot{(}y\cdot zc))(y^{\lambda},y,zc)=(y^{\lambda}y\cdot zc)$$

$$\iff(y^{\lambda}\cdot yz)(y^{\lambda},y,zc)c=zc\iff(y^{\lambda}\cdot yz)(y^{\lambda},y,zc)=z$$

$$\iff(y^{\lambda},y,zc)=(y^{\lambda}\cdot yz)\backslash z\iff(y^{\lambda},y,zc)=(y^{\lambda}\cdot yz)\backslash y^{\lambda}y\cdot z=z\backslash z=e$$

$$\iff(y^{\lambda},y,zc)=e$$

2. From Lemma 4.8.1(1), set $y=x^{\rho}$ on the left, and let $c\in N\cap C$. Then

$$(x,x^{\rho},zc)=(x\cdot(x^{\rho}\cdot zc))\backslash(xx^{\rho}\cdot zc)\iff(x,x^{\rho},zc)=(x\cdot(x^{\rho}\cdot zc))\backslash zc$$

$$\iff(x,x^{\rho},zc)=(x\cdot(x^{\rho}z\cdot c))\backslash zc\iff(x,x^{\rho},zc)=((x\cdot x^{\rho}z)c)\backslash zc$$

$$\iff((x\cdot x^{\rho}z)c)(x,x^{\rho},zc)=zc\iff(x\cdot x^{\rho}z)(x,x^{\rho},zc)c=zc,\ (\text{since } c\in N\cap C)$$

$$\iff(x\cdot x^{\rho}z)(x,x^{\rho},zc)=z\iff(x,x^{\rho},zc)=(xx^{\rho}z)\backslash z=(x\cdot x^{\rho}z)\backslash(xx^{\rho}\cdot z)=(x,x^{\rho},z)$$

$$\iff(x,x^{\rho},zc)=(x,x^{\rho},z)$$

3. From Lemma 4.8.1(2), set $x = y^\lambda$ on the left, and let $c \in N \cap C$. Then

$$(y^\lambda, yc, z) = (y^\lambda \cdot (yc \cdot z))\backslash((y^\lambda \cdot yc) \cdot z) \iff (y^\lambda \cdot (yc \cdot z))(y^\lambda, yc, z) = ((y^\lambda \cdot yc) \cdot z)$$

$$\iff (y^\lambda \cdot (y \cdot cz))(y^\lambda, yc, z) = ((y^\lambda y \cdot c) \cdot z) \iff (y^\lambda \cdot (y \cdot cz))(y^\lambda, yc, z) = (cz)$$

$$\iff (y^\lambda \cdot (yz)c)(y^\lambda, yc, z) = cz \iff (y^\lambda \cdot yz)c(y^\lambda, yc, z) = cz \iff zc(y^\lambda, yc, z) = cz$$

$$\iff zc(y^\lambda, yc, z) = zc \iff (y^\lambda, yc, z) = e$$

4. From Lemma 4.8.1(2), set $y = x^\rho$ on the left, and let $c \in N \cap C$. Then

$$(x, x^\rho c, z) = (x \cdot (x^\rho c \cdot z))\backslash((x \cdot x^\rho c) \cdot z) \iff (x \cdot (x^\rho c \cdot z))(x, x^\rho c, z) = ((x \cdot x^\rho c) \cdot z)$$

$$\iff (x \cdot (x^\rho \cdot cz))(x, x^\rho c, z) = ((xx^\rho \cdot c) \cdot z) \iff (x \cdot (x^\rho \cdot zc))(x, x^\rho c, z) = cz$$

$$\iff (x \cdot (x^\rho z \cdot c))(x, x^\rho c, z = cz) \iff (x \cdot x^\rho z)c(x, x^\rho c, z) = cz$$

$$\iff c(x \cdot x^\rho z)(x, x^\rho c, z) = cz \iff (x \cdot x^\rho z)(x, x^\rho c, z) = z$$

$$\iff (x, x^\rho c, z) = (x \cdot x^\rho z)\backslash z = (x \cdot x^\rho z)\backslash(xx^\rho \cdot z)$$

$$\iff (x, x^\rho c, z) = (x, x^\rho, z)$$

□

Lemma 4.8.12. Let $(Q, \cdot)$ be a Basarab loop with a nucleus $N(Q, \cdot)$, and let (x, y, z) be the associator of elements $x, y, z \in Q$. Then the following are true for $c \in N$:

1. $(cx, z^\lambda, z) = (x, z^\lambda, z)$
2. $(cx, y, y^\rho) = e$
3. $(xc, z^\lambda, z) = (x, z^\lambda, z)$

4. $(xc, y, y^\rho) = (x, y, y^\rho)$

Proof. 1. From Lemma 4.8.1(3), set $y = z^\lambda$ on the left, and let $c \in N$. Then

$$(cx, z^\lambda, z) = (cx \cdot z^\lambda z)\backslash((cx \cdot z^\lambda)z) \iff (cx \cdot z^\lambda z)(cx, z^\lambda, z) = ((cx \cdot z^\lambda)z)$$

$$\iff (cx)(cx, z^\lambda, z) = ((cx \cdot z^\lambda)z) \iff (cx)(cx, z^\lambda, z) = ((c \cdot xz^\lambda)z)$$

$$\iff (cx)(cx, z^\lambda, z) = c \cdot (xz^\lambda \cdot z) \iff x(cx, z^\lambda, z) = (xz^\lambda \cdot z)$$

$$\iff (cx, z^\lambda, z) = x\backslash(xz^\lambda \cdot z) \iff (cx, z^\lambda, z) = x \cdot z^\lambda z\backslash(xz^\lambda \cdot z)$$

$$\iff (cx, z^\lambda, z) = (x, z^\lambda, z).$$

2. From Lemma 4.8.1(3), set $z = y^\rho$ on the left, and let $c \in N$. Then

$$(cx, y, y^\rho) = (cx \cdot yy^\rho)\backslash((cx \cdot y)y^\rho) \iff (cx \cdot yy^\rho)(cx, y, y^\rho) = ((cx \cdot y)y^\rho)$$

$$\iff (cx)(cx, y, y^\rho) = (cxy^\rho y)y^\rho \iff (cx)(cx, y, y^\rho) = (cy^\rho xy)y^\rho$$

$$\iff c \cdot x(cx, y, y^\rho) = c \cdot (xy \cdot y^\rho) \iff x(cx, y, y^\rho) = x$$

$$\iff (cx, y, y^\rho) = e$$

3. From Lemma 4.8.1(4), set $y = z^\lambda$ on the left, and let $c \in N$. Then

$$(xc, z^\lambda, z) = (xc \cdot z^\lambda)\backslash((xc \cdot z^\lambda)z) \iff (xc \cdot z^\lambda)(xc, z^\lambda, z) = ((xc \cdot z^\lambda)z)$$

$$\iff (xc)(xc, z^\lambda, z) = ((xc \cdot z^\lambda)z) \iff cx(xc, z^\lambda, z) = ((cx \cdot z^\lambda)z)$$

$$\iff cx(xc, z^\lambda, z) = ((c \cdot xz^\lambda)z) \iff cx(xc, z^\lambda, z) = c \cdot (xz^\lambda \cdot z) \implies x(xc, z^\lambda, z) = xz^\lambda \cdot z$$

$$\iff (xc, z^\lambda, z) = x\backslash xz^\lambda \cdot z \implies (xc, z^\lambda, z) = x \cdot z^\lambda \backslash xz^\lambda \cdot z \iff (xc, z^\lambda, z) = (x, z^\lambda, z)$$

4. From Lemma 4.8.1(4), set $z = y^\rho$ on the left, and let $c \in N$. Then

$$(xc, y, y^\rho) = (xc \cdot yy^\rho)\backslash((xc \cdot y)y^\rho) \iff (xc \cdot yy^\rho)(xc, y, y^\rho) = ((xc \cdot y)y^\rho)$$

$$\iff (xc)(xc, y, y^\rho) = ((xc \cdot y)y^\rho) \iff (xc)(xc, y, y^\rho) = ((cx \cdot y)y^\rho)$$

$$\iff (cx)(xc, y, y^\rho) = ((c \cdot xy)y^\rho) \iff c \cdot x(xc, y, y^\rho) = c(xy \cdot y^\rho)$$

$$\iff x(xc, y, y^\rho) = xy \cdot y^\rho \iff (xc, y, y^\rho) = x\backslash xy \cdot y^\rho$$

$$\iff (xc, y, y^\rho) = x \cdot yy^\rho \backslash xy \cdot y^\rho \iff (xc, y, y^\rho) = (x, y, y^\rho)$$

□

4.8.2 Relationship between Associators and Inner Mappings of a Basarab loop

Lemma 4.8.13. Let $(Q, \cdot)$ be a loop, and $A(Q)$ its associator subloop. Then for every $x, y, z, \in Q,\ (x, e, z) = e,\ (e, y, z) = e, (x, y, e) = e, [e, y, z] = e, [x, e, z] = e,$ and $[x, y, e] = e$, where e is an identity element of Q.

Proof. Consider the following, $(xy \cdot z) = (x \cdot yz)(x, y, z) \implies (x, y, z) = (x \cdot yz)\backslash(xy \cdot z)$ Set $x = e,\ (e, y, z) = (e\cdot)\backslash(ey \cdot z) \implies (e, y, z) = \backslash yz \implies (e, y, z) = e$. Next, set $y = e,\ (x, y, z) = (x \cdot yz)\backslash(xy \cdot z) \implies (x, e, z) = (x \cdot ez)\backslash(xe \cdot z) \implies (x, e, z) = (x \cdot z)\backslash(x \cdot z) \implies (x, e, z) = e$. Next, set $z = e,\ (x, y, z) = (x \cdot yz)\backslash(xy \cdot z) \implies (x, y, e) = (x \cdot ye)\backslash(xy \cdot e) \implies (x, y, e) = (x \cdot z)\backslash(x \cdot z) \implies (x, y, e) = e$.

Similarly, consider $(x \cdot yz) = [x, y, z](xy \cdot z)$, and set $x = e$. Then $(x \cdot yz)/(xy \cdot z) = [x, y, z] \implies (e \cdot yz)/(ey \cdot z) = [e, y, z] \implies (y \cdot z)/(y \cdot z) = [e, y, z] \implies e = [e, y, z]$.

Next set $z = e$, then $(x \cdot yz)/(xy \cdot z) = [x, y, z] \implies (x \cdot ye)/(xy \cdot e) = [x, y, e] \implies (x \cdot y)/(x \cdot y) = [x, y, e] \implies e = [x, y, e]$. □

Corollary 4.8.6. Let $(Q, \cdot)$ be a Basarab loop. Then the associator of any three elements of Q is an identity element if any of these three elements is an identity element, that is, $(a, b, e) = (a, e, b) = (e, a, b) = e$ for all $a, b \in Q$.

Proof. The proof follows immediately from Lemma 4.8.13. □

Theorem 4.8.2. Let $(Q, \cdot)$ be a Basarab loop. Then $wR_{(x,y)} = w$ if $(w, x, y) = e$.

Proof. Let $(Q, \cdot)$ be a Basarab loop. Then

$$wR_{(x,y)} = wR_xR_yR_{xy}^{-1} \implies wR_{(x,y)} = (wx \cdot y)/xy \implies wR_{(x,y)} = (w \cdot xy)(w, x, y)/xy.$$

Let $(w, x, y) = e$. Then $wR_{(x,y)} = (w \cdot xy)/xy = w \implies wR_{(x,y)} = w$. □

Theorem 4.8.3. Let $(Q, \cdot)$ be a Basarab loop. Then $vL_{(x,y)} = v$ if $[y, x, v] = e$.

Proof. Let $(Q, \cdot)$ be a Basarab loop. Then

$$vL_{(x,y)} = vL_xL_yL_{yx}^{-1} \implies vL_{(x,y)} = (yx)\backslash(y \cdot xv) \implies vL_{(x,y)} = (yx)\backslash[y, x, v](yx \cdot v).$$

Let $[y, x, v] = e$. Then $vL_{(x,y)} = (yx)\backslash(yx \cdot v) = v \implies vL_{(x,y)} = v$. □

Theorem 4.8.4. Let $(Q, \cdot)$ be a Basarab loop, and $A(Q)$ its associator subloop. Then for every $x, y, z, w \in Q$:

1. $[x, y, z]R_{w^\rho}L_w = [w, x, yz][wx, y, z][w, x, y]^{-1}[w, xy, z]^{-1}$
2. $(x, y, z)L_{w^\lambda}R_w = (x, yz, w)^{-1}(y, z, w)^{-1}(x, y, zw)(xy, z, w)$

Proof. Let $(Q, \cdot)$ be a Basarab loop, then every associator of three elements contains in $N(Q)$.

1. For all $x, y, z, w \in Q$, $w[x, y, z](xy \cdot z) = w(x \cdot yx) = [w, x, yz] \cdot (wx \cdot yz)$

$\iff w[x, y, z](xy \cdot z) = [w, x, yz][wx, y, z](wx \cdot y)z$

$\iff w[x, y, z](xy \cdot z) = [w, x, yz][wx, y, z][w, x, y]^{-1}(w \cdot xy)z$

$\iff w[x, y, z](xy \cdot z) = [w, x, yz][wx, y, z][w, x, y]^{-1}[w, xy, z]^{-1}w(xy \cdot z)$

$\iff w[x, y, z] = [w, x, yz][wx, y, z][w, x, y]^{-1}[w, xy, z]^{-1}w$

$\iff [x, y, z]L_w = [w, x, yz][wx, y, z][w, x, y]^{-1}[w, xy, z]^{-1}R_w$

$\iff [x, y, z]L_w R_w^{-1} = [w, x, yz][wx, y, z][w, x, y]^{-1}[w, xy, z]^{-1}$

$\iff [x, y, z]R_{w^\rho} L_w = [w, x, yz][wx, y, z][w, x, y]^{-1}[w, xy, z]^{-1}$

2. For all $x, y, z, w \in Q$, $(x \cdot yz)(x, y, z)w = (xy \cdot w) = (xy \cdot zw)(xy, z, w)$

$\iff (x \cdot yz)(x, y, z)w = (x \cdot (y \cdot zw))(x, y, zw)(xy, z, w)$

$\iff (x \cdot yz)(x, y, z)w = x(yz \cdot w)(y, z, w)^{-1}(x, y, zw)(xy, z, w)$

$\iff (x \cdot yz)(x, y, z)w = (x \cdot yz)w(x, yz, w)^{-1}(y, z, w)^{-1}(x, y, zw)(xy, z, w)$

$\iff (x, y, z)w = w(x, yz, w)^{-1}(y, z, w)^{-1}(x, y, zw)(xy, z, w)$

$\iff (x, y, z)R_w = (x, yz, w)^{-1}(y, z, w)^{-1}(x, y, zw)(xy, z, w)L_w$

$\iff (x, y, z)R_w L_w^{-1} = (x, yz, w)^{-1}(y, z, w)^{-1}(x, y, zw)(xy, z, w)$

$\iff (x, y, z)L_{w^\lambda} R_w = (x, yz, w)^{-1}(y, z, w)^{-1}(x, y, zw)(xy, z, w)$

□

Theorem 4.8.5. Let $(Q, \cdot)$ be a Basarab loop, and $A(Q)$ its associator subloop. Then for every $x, y, z, w \in Q$:

1. $[y, x, z]R_{w^\rho} L_w = [w, y, xz][wy, x, z][w, y, x]^{-1}[w, yx, z]^{-1}$

2. $[z, y, x]R_{w^\rho} L_w = [w, z, yx][wz, y, x][w, z, y]^{-1}[w, zy, x]^{-1}$

3. $[x, z, y]R_{w^\rho} L_w = [w, x, zy][wx, z, y][w, x, z]^{-1}[w, xz, y]^{-1}$

Proof. Let $(Q, \cdot)$ be a Basarab loop, then every associator of three elements contains in $N(Q)$.

1. For all $x, y, z, w \in Q$, $w[y, xz](yx \cdot z) = w(y \cdot xz) = [w, yxz] \cdot (wy \cdot xz)$

$\iff w[y, xz](yx \cdot z) = [w, y, xz][wy, x, z][w, y, x]^{-1}(w \cdot yx)z$

$\iff w[y, xz](yx \cdot z) = [w, y, xz][wy, x, z][w, y, x]^{-1}[w, yx, z]^{-1}w(yx \cdot z)$

$\iff w[y, x, z] = [w, y, xz][wy, x, z][w, y, x]^{-1}[w, yx, z]^{-1}w$

$\iff [y, x, z]L_w = [w, y, xz][wy, x, z][w, y, x]^{-1}[w, yx, z]^{-1}R_x^{-1}$

$\iff [y, x, z]L_wR_w^{-1} = [w, y, xz][wy, x, z][w, y, x]^{-1}[w, yx, z]^{-1}$

$\iff [y, x, z]R_{w^\rho}L_w = [w, y, xz][wy, x, z][w, y, x]^{-1}[w, yx, z]^{-1}$.

2. For all $x, y, z, w \in Q$, $w[z, y, x](zy \cdot x) = w(z \cdot yx) = [w, z, yx] \cdot (wz \cdot yx)$

$\iff w[z, y, x](zy \cdot x) = [w, z, yx][wz, y, x](wz \cdot y)x$

$\iff w[z, y, x](zy \cdot x) = [w, z, yx][wz, y, x][w, z, y]^{-1}(w \cdot zy)x$

$\iff w[z, y, x](zy \cdot x) = [w, z, yx][wz, y, x][w, z, y]^{-1}[w, zy, x]^{-1}w(zy \cdot x)$

$\iff w[z, y, x] = [w, z, yx][wz, y, x][w, z, y]^{-1}[w, zy, x]^{-1}w$

$\iff [z, y, x]L_w = [w, z, yx][wz, y, x][w, z, y]^{-1}[w, zy, x]^{-1}R_w$

$\iff [z, y, x]R_{w^\rho}L_w = [w, z, yx][wz, y, x][w, z, y]^{-1}[w, zy, x]^{-1}$

3. For all $x, y, z, w \in Q$, $w[x, z, y](xz \cdot y) = w(x \cdot zy) = [w, x, zy] \cdot (wx \cdot zy)$

$\iff w[x, z, y](xz \cdot y) = [w, x, zy][wx, z, y](wx \cdot z)y$

$\iff w[x, z, y](xz \cdot y) = [w, x, zy][wx, z, y][w, x, z]^{-1}(w \cdot xz)y$

$\iff w[x, z, y](xz \cdot y) = [w, x, zy][wx, z, y][w, x, z]^{-1}[w, xz, y]^{-1}w(xz \cdot y)$

$\iff w[x, z, y] = [w, x, zy][wx, z, y][w, x, z]^{-1}[w, xz, y]^{-1}w$

$\iff [x, z, y]L_w = [w, x, zy][wx, z, y][w, x, z]^{-1}[w, xz, y]^{-1}R_w$

$\iff [x, z, y]R_{w^\rho}L_w = [w, x, zy][wx, z, y][w, x, z]^{-1}[w, xz, y]^{-1}$

□

Theorem 4.8.6. Let $(Q, \cdot)$ be a Basarab loop, and $A(Q)$ its associator subloop. Then for every $x, y, z, w \in Q$:

1. $(y, x, z)L_{w^\lambda}R_w = (y, xz, w)^{-1}(x, z, w)^{-1}(y, xzw)(yx, z, w)$

2. $(z,y,x)L_{w^\lambda}R_w = (z,yx,w)^{-1}(y,x,w)^{-1}(z,y,xw)(zy,x,w)$

3. $(x,z,y)L_{w^\lambda}R_w = (x,zy,w)^{-1}(z,y,w)^{-1}(x,z,yw)(xz,y,w)$

Proof. Let $(Q,\cdot)$ be a Basarab loop, then every associator of three elements contains in $N(Q)$.

1. For all $x,y,z,w \in Q$, $(y\cdot xz)(y,x,z)w = (yx\cdot z)w = (yx\cdot zw)(yx,z,w)$

$\iff (y\cdot xz)(y,x,z)w = (y\cdot(x\cdot zw))(y,xzw)(yx,z,w)$

$\iff (y\cdot xz)(y,x,z)w = y(xz\cdot w)(x,z,w)^{-1}(y,x,zw)(yx,z,w)$

$\iff (y\cdot xz)(y,x,z)w = (y\cdot xz)w(y,xz,w)^{-1}(x,z,w)^{-1}(y,x,zw)(yx,z,w)$

$\iff (y,x,z)w = w(y,xz,w)^{-1}(x,z,w)^{-1}(y,x,zw)(yx,z,w)$

$\iff (y,x,z)R_w = (y,xz,w)^{-1}(x,z,w)^{-1}(y,x,zw)(yx,z,w)L_w$

$\iff (y,x,z)L_{w^\lambda}R_w = (y,xz,w)^{-1}(x,z,w)^{-1}(y,x,zw)(yx,z,w)$

2. For all $x,y,z,w \in Q$, $(z\cdot yx)(z,y,x)w = (zy\cdot x)w = (zy\cdot xw)(zy,x,w)$

$\iff (z\cdot yx)(z,y,x)w = (z\cdot(y\cdot xw))(z,y,xw)(zy,x,w)$

$\iff (z\cdot yx)(z,y,x)w = z(yx\cdot w)(y,x,w)^{-1}(z,y,xw)(zy,x,w)$

$\iff (z\cdot yx)(z,y,x)w = (z\cdot yx)w(z,yx,w)^{-1}(y,x,w)^{-1}(z,y,xw)(zy,x,w)$

$\iff (z,y,x)w = w(z,yx,w)^{-1}(y,x,w)^{-1}(z,y,xw)(zy,x,w)$

$\iff (z,y,x)R_w = (z,yx,w)^{-1}(y,x,w)^{-1}(z,y,xw)(zy,x,w)L_w$

$\iff (z,y,x)L_{w^\lambda}R_w = (z,yx,w)^{-1}(y,x,w)^{-1}(z,y,xw)(zy,x,w)$

3. For all $x,y,z,w \in Q$, $(x\cdot zy)(x,z,y)w = (xz\cdot y)w = (xz\cdot yw)(xz,y,w)$

$\iff (x\cdot zy)(x,z,y)w = (xz\cdot y)w = (x\cdot(z\cdot yw))(x,z,yw)(xz,y,w)$

$\iff (x\cdot zy)(x,z,y)w = (xz\cdot y)wx(zy\cdot w)(z,y,w)^{-1}(x,z,yw)(xz,y,w)$

$\iff (x\cdot zy)(x,z,y)w = (x\cdot zy)w(x,zy,w)^{-1}(z,y,w)^{-1}(x,z,yw)(xz,y,w)$

$\iff (x,z,y)w = w(x,zy,w)^{-1}(z,y,w)^{-1}(x,z,yw)(xz,y,w)$

$\iff (x,z,y)R_w = (x,zy,w)^{-1}(z,y,w)^{-1}(x,z,yw)(xz,y,w)L_w$

$\iff (x,z,y)L_{w^\lambda}R_w = (x,zy,w)^{-1}(z,y,w)^{-1}(x,z,yw)(xz,y,w)$

□

Theorem 4.8.7. Let $(Q,\cdot)$ be a Basarab loop, and $A(Q)$ its associator subloop. Then for every $x,y,z,w\in Q$:

1. $[w,y,z]R_{x^\rho}L_x=[x,w,yz][xw,y,z][x,w,y]^{-1}[x,wy,z]^{-1}$
2. $(w,y,z)L_{x^\lambda}R_x=(w,yz,x)^{-1}(y,z,x)^{-1}(w,y,zx)(wy,z,x)$
3. $[y,w,z]R_{x^\rho}L_x=[x,y,wz][xy,w,z][x,y,w]^{-1}[x,yw,z]^{-1}$
4. $[z,y,w]R_{x^\rho}L_x=[x,z,yw][xz,y,w][x,z,y]^{1}[x,zy,w]^{-1}$
5. $[w,z,y]R_{x^\rho}L_x=[x,w,zy][xw,z,y][x,w,z]^{-1}[x,wz,y]^{-1}$
6. $(y,w,z)L_{x^\lambda}R_x=(y,wz,x)^{-1}(w,z,x)^{-1}(y,w,zx)(yw,z,x)$
7. $(z,y,w)L_{x^\lambda}R_x=(z,yw,x)^{-1}(y,w,x)^{-1}(z,y,wx)(zy,w,x)$
8. $(w,z,y)L_{x^\lambda}R_x=(w,zy,x)^{-1}(z,y,x)^{-1}(w,z,yx)(wz,y,x)$

Proof. Let $(Q,\cdot)$ be a Basarab loop, then every associator of three elements contains in $N(Q)$.

1. For all $x,y,z,w\in Q$, $x[w,y,z](wy\cdot z)=x(w\cdot yz)=[x,w,yz]\cdot(xw\cdot yz)$

   $\iff x[w,y,z](wy\cdot z)=[x,w,yz][xw,y,z](xw\cdot y)z$

   $\iff x[w,y,z](wy\cdot z)=[x,w,yz][xw,y,z][x,w,y]^{-1}(x\cdot wy)z$

   $\iff x[w,y,z](wy\cdot z)=[x,w,yz][xw,y,z][x,w,y]^{-1}[x,wy,z]^{-1}x(wy\cdot z)$

   $\iff x[w,y,z]=[x,w,yz][xw,y,z][x,w,y]^{-1}[x,wy,z]^{-1}x$

   $\iff [w,y,z]L_x=[x,w,yz][xw,y,z][x,w,y]^{-1}[x,wy,z]^{-1}R_x$

   $\iff [w,y,z]R_{x^\rho}L_x=[x,w,yz][xw,y,z][x,w,y]^{-1}[x,wy,z]^{-1}$

2. For all $x,y,z,w\in Q$, $(w\cdot yz)(w,y,z)x=(wy\cdot z)x=(wy\cdot zx)(wy,z,x)$

   $\iff (w\cdot yz)(w,y,z)x=(w\cdot(y\cdot zx))(w,y,zx)(wy,z,x)$

   $\iff (w\cdot yz)(w,y,z)x=w(yz\cdot x)(y,z,x)^{-1}(w,y,zx)(wy,z,x)$

$$\iff (w \cdot yz)(w,y,z)x = (w \cdot yz)x(w,yz,x)^{-1}(y,z,x)^{-1}(w,y,zx)(wy,z,x)$$

$$\iff (w,y,z)x = x(w,yz,x)^{-1}(y,z,x)^{-1}(w,y,zx)(wy,z,x)$$

$$\iff (w,y,z)R_x = (w,yz,x)^{-1}(y,z,x)^{-1}(w,y,zx)(wy,z,x)L_x$$

$$\iff (w,y,z)L_{x^\lambda}R_x = (w,yz,x)^{-1}(y,z,x)^{-1}(w,y,zx)(wy,z,x)$$

3. For all $x,y,z,w \in Q,\ x[y,w,z](yw \cdot z) = x(y \cdot wz) = [x,y,wz] \cdot (xy \cdot wz)$

$$\iff x[y,w,z](yw \cdot z) = x(y \cdot wz) = [x,y,wz][xy,w,z][x,y,w]^{-1}(x \cdot yw)z$$

$$\iff x[y,w,z](yw \cdot z) = [x,y,wz][xy,w,z][x,y,w]^{-1}[x,yw,z]^{-1}x(yw \cdot z)$$

$$\iff x[y,w,z] = [x,y,wz][xy,w,z][x,y,w]^{-1}[x,yw,z]^{-1}x$$

$$\iff [y,w,z]L_x = [x,y,wz][xy,w,z][x,y,w]^{-1}[x,yw,z]^{-1}R_x$$

$$\iff [y,w,z]L_xR_x^{-1} = [x,y,wz][xy,w,z][x,y,w]^{-1}[x,yw,z]^{-1}$$

$$\iff [y,w,z]R_{x^\rho}L_x = [x,y,wz][xy,w,z][x,y,w]^{-1}[x,yw,z]^{-1}$$

4. For all $x,y,z,w \in Q,\ x[z,y,w](zy \cdot w) = x(z \cdot yx) = [x,z,yw] \cdot (xz \cdot yw)$

$$\iff x[z,y,w](zy \cdot w) = [x,z,yw][xz,y,w](xz \cdot y)w$$

$$\iff x[z,y,w](zy \cdot w) = [x,z,yw][xz,y,w][x,z,y]^{1}(x \cdot zy)w$$

$$\iff x[z,y,w](zy \cdot w) = [x,z,yw][xz,y,w][x,z,y]^{1}[x,zy,w]^{-1}x(zy \cdot w)$$

$$\iff x[z,y,w][x,z,yw][xz,y,w][x,z,y]^{1}[x,zy,w]^{-1}x$$

$$\iff [z,y,w]L_x = [x,z,yw][xz,y,w][x,z,y]^{1}[x,zy,w]^{-1}R_x$$

$$\iff [z,y,w]R_{x^\rho}L_x = [x,z,yw][xz,y,w][x,z,y]^{1}[x,zy,w]^{-1}$$

5. For all $x,y,z,w \in Q,\ x[w,z,y](wz \cdot y) = x(w \cdot zy) = [x,w,zy] \cdot (xw \cdot zy)$

$$\iff x[w,z,y](wz \cdot y) = [x,w,zy][xw,z,y](xw \cdot z)y$$

$$\iff x[w,z,y](wz \cdot y) = [x,w,zy][xw,z,y][x,w,z]^{-1}(x \cdot wz)y$$

$$\iff x[w,z,y](wz \cdot y) = [x,w,zy][xw,z,y][x,w,z]^{-1}[x,wz,y]^{-1}x(wz \cdot y)$$

$$\iff x[w,z,y] = [x,w,zy][xw,z,y][x,w,z]^{-1}[x,wz,y]^{-1}x$$

$$\iff [w,z,y]L_x = [x,w,zy][xw,z,y][x,w,z]^{-1}[x,wz,y]^{-1}R_x$$

$$\iff [w,z,y]R_{x^\rho}L_x = [x,w,zy][xw,z,y][x,w,z]^{-1}[x,wz,y]^{-1}$$

6. For all $x, y, z, w \in Q,\ (y \cdot wz)(y, w, z)x = (yw \cdot z)x = (yw \cdot zx)(yw, z, x)$

$\iff (y \cdot wz)(y, w, z)x(y \cdot (w \cdot zx))(y, w, zx)(yw, z, x)$

$\iff (y \cdot wz)(y, w, z)x = y(wz \cdot x)(w, z, x)^{-1}(y, w, zx)(yw, z, x)$

$\iff (y \cdot wz)(y, w, z)x = (y \cdot wz)x(y, wz, x)^{-1}(w, z, x)^{-1}(y, w, zx)(yw, z, x)$

$\iff (y, w, z)x = x(y, wz, x)^{-1}(w, z, x)^{-1}(y, w, zx)(yw, z, x)$

$\iff (y, w, z)R_x = (y, wz, x)^{-1}(w, z, x)^{-1}(y, w, zx)(yw, z, x)L_x$

$\iff (y, w, z)L_{x^\lambda}R_x = (y, wz, x)^{-1}(w, z, x)^{-1}(y, w, zx)(yw, z, x)$

7. For all $x, y, z, w \in Q,\ (z \cdot yw)(z, y, w)x = (zy \cdot wx)(zy, w, x)$

$\iff (z \cdot yw)(z, y, w)x = (z \cdot (y \cdot wx))(z, y, wx)(zy, w, x)$

$\iff (z \cdot yw)(z, y, w)x = z(yw \cdot x)(y, w, x)^{-1}(z, y, wx)(zy, w, x)$

$\iff (z \cdot yw)(z, y, w)x = (z \cdot yw)x(z, yw, x)^{-1}(y, w, x)^{-1}(z, y, wx)(zy, w, x)$

$\iff ((z, y, w)x = x(z, yw, x)^{-1}(y, w, x)^{-1}(z, y, wx)(zy, w, x)$

$\iff (z, y, w)R_x = (z, yw, x)^{-1}(y, w, x)^{-1}(z, y, wx)(zy, w, x)L_x$

$\iff (z, y, w)L_{x^\lambda}R_x = (z, yw, x)^{-1}(y, w, x)^{-1}(z, y, wx)(zy, w, x).$

8. For all $x, y, z, w \in Q,\ (w \cdot zy)(w, z, y)x = (wz \cdot y)x = (wz \cdot yx)(wz, y, x)$

$\iff (w \cdot zy)(w, z, y)x = (w \cdot (zcdotyx))(w, z, yx)(wz, y, x)$

$\iff (w \cdot zy)(w, z, y)x = w(zy \cdot x)(z, y, x)^{-1}(w, z, yx)(wz, y, x)$

$\iff (w \cdot zy)(w, z, y)x = (w \cdot zy)x(w, zy, x)^{-1}(z, y, x)^{-1}(w, z, yx)(wz, y, x)$

$\iff (w, z, y)x = x(w, zy, x)^{-1}(z, y, x)^{-1}(w, z, yx)(wz, y, x)$

$\iff (w, z, y)R_x = (w, zy, x)^{-1}(z, y, x)^{-1}(w, z, yx)(wz, y, x)L_x$

$\iff (w, z, y)L_{x^\lambda}R_x = (w, zy, x)^{-1}(z, y, x)^{-1}(w, z, yx)(wz, y, x).$

□

Theorem 4.8.8. Let $(Q, \cdot)$ be a Basarab loop, and $A(Q)$ its associator subloop. Then for every $x, y, z, w \in Q$:

1. $[x, w, z]R_{y^\rho}L_y = [y, x, wz][yx, w, z][y, x, w]^{-1}[y, xw, z]^{-1}$

2. $(x,w,z)L_{y^\lambda}R_y = (x,wz,y)^{-1}(w,z,y)^{-1}(x,w,zy)(xw,z,y)$

3. $[w,x,z]R_{y^\rho}L_y = [y,w,xz][yw,x,z][y,w,x]^{-1}[y,wx,z]^{-1}$

4. $[z,w,x]R_{y^\rho}L_y = [y,z,wx][yz,w,x][y,z,w]^{-1}[y,zw,x]^{-1}$

5. $[x,z,w]R_{y^\rho}L_y = [y,x,zw][yx,z,w][y,x,z]^{-1}[y,xz,w]^{-1}$

6. $(w,x,z)L_{y^\lambda}R_y = (w,xz,y)^{-1}(x,z,y)^{-1}(w,x,zy)(wx,z,y)$

7. $(z,w,x)L_{y^\lambda}R_y = (z,wx,y)^{-1}(w,x,y)^{-1}(z,w,xy)(zw,x,y)$

8. $(x,z,w)L_{y^\lambda}R_y = (x,zw,y)^{-1}(z,w,y)^{-1}(x,z,wy)(xz,w,y)$

Proof. Let $(Q,\cdot)$ be a Basarab loop, then every associator of three elements contains in $N(Q)$.

1. For all $x,y,z,w \in Q$, $y[x,w,z](xw\cdot z) = y(x\cdot wz) = [y,x,wz]\cdot(yx\cdot wz)$

   $\iff y[x,w,z](xw\cdot z) = [y,x,wz][yx,w,z](yx\cdot w)z$

   $\iff y[x,w,z](xw\cdot z) = [y,x,wz][yx,w,z][y,x,w]^{-1}(y\cdot xw)z$

   $\iff y[x,w,z](xw\cdot z) = [y,x,wz][yx,w,z][y,x,w]^{-1}[y,xw,z]^{-1}y(xw\cdot z)$

   $\iff y[x,w,z] = [y,x,wz][yx,w,z][y,x,w]^{-1}[y,xw,z]^{-1}y$

   $\iff [x,w,z]L_y = [y,x,wz][yx,w,z][y,x,w]^{-1}[y,xw,z]^{-1}R_y$

   $\iff [x,w,z]L_yR_y^{-1} = [y,x,wz][yx,w,z][y,x,w]^{-1}[y,xw,z]^{-1}$

   $\iff [x,w,z]R_{y^\rho}L_y = [y,x,wz][yx,w,z][y,x,w]^{-1}[y,xw,z]^{-1}$

2. For all $x,y,z,w \in Q$, $(x\cdot wz)(x,w,z)y = (xw\cdot z)y = (xw\cdot zy)(xw,z,y)$

   $\iff (x\cdot wz)(x,w,z)y = (x(w\cdot zy))(x,w,zy)(xw,z,y)$

   $\iff (x\cdot wz)(x,w,z)y = x(wz\cdot y)(w,z,y)^{-1}(x,w,zy)(xw,z,y)$

   $\iff (x\cdot wz)(x,w,z)y = (x\cdot wz)y(x,wz,y)^{-1}(w,z,y)^{-1}(x,w,zy)(xw,z,y)$

   $\iff (x,w,z)y = y(x,wz,y)^{-1}(w,z,y)^{-1}(x,w,zy)(xw,z,y)$

   $\iff \quad (x,w,z)R_x \;=\; (x,wz,y)^{-1}(w,z,y)^{-1}(x,w,zy)(xw,z,y)L_y(x,w,z)R_yL_y^{-1} \;=$

$(x, wz, y)^{-1}(w, z, y)^{-1}(x, w, zy)(xw, z, y)$

$\iff (x, w, z)L_{y^\lambda}R_y = (x, wz, y)^{-1}(w, z, y)^{-1}(x, w, zy)(xw, z, y)$

3. For all $x, y, z, w \in Q$, $y[w, x, z](wx \cdot z) = y(w \cdot xz) = [y, w, xz] \cdot (yw \cdot xz)$

$\iff y[w, x, z](wx \cdot z) = [y, w, xz][yw, x, z][y, w, x]^{-1}(y \cdot wx)z$

$\iff y[w, x, z](wx \cdot z) = [y, w, xz][yw, x, z][y, w, x]^{-1}[y, wx, z]^{-1}y(wx \cdot z)$

$\iff y[w, x, z] = [y, w, xz][yw, x, z][y, w, x]^{-1}[y, wx, z]^{-1}y$

$\iff [w, x, z]L_y = [y, w, xz][yw, x, z][y, w, x]^{-1}[y, wx, z]^{-1}R_y$

$\iff [w, x, z]L_yR_x^{-1} = [y, w, xz][yw, x, z][y, w, x]^{-1}[y, wx, z]^{-1}$

$\iff [w, x, z]R_{y^\rho}L_y = [y, w, xz][yw, x, z][y, w, x]^{-1}[y, wx, z]^{-1}$

4. For all $x, y, z, w \in Q$, $y[z, w, x](zw \cdot x) = y(z \cdot wx) = [y, z, wx] \cdot (yz \cdot wx)$

$\iff y[z, w, x](zw \cdot x) = [y, z, wx][yz, w, x](yz \cdot w)x$

$\iff y[z, w, x](zw \cdot x) = [y, z, wx][yz, w, x][y, z, w]^{-1}(y \cdot zw)x$

$\iff y[z, w, x](zw \cdot x) = [y, z, wx][yz, w, x][y, z, w]^{-1}[y, zw, x]^{-1}y(zw \cdot x)$

$\iff y[z, w, x] = [y, z, wx][yz, w, x][y, z, w]^{-1}[y, zw, x]^{-1}y$

$\iff [z, w, x]L_y = [y, z, wx][yz, w, x][y, z, w]^{-1}[y, zw, x]^{-1}R_y$

$\iff [z, w, x]R_{y^\rho}L_y = [y, z, wx][yz, w, x][y, z, w]^{-1}[y, zw, x]^{-1}$

5. For all $x, y, z, w \in Q$, $y[x, z, w](xz \cdot w) = y(x \cdot zw) = [y, x, zw] \cdot (yx \cdot zw)$

$\iff y[x, z, w](xz \cdot w) = [y, x, zw][yx, z, w](yx \cdot z)w$

$\iff y[x, z, w](xz \cdot w) = [y, x, zw][yx, z, w][y, x, z]^{-1}(y \cdot xz)w$

$\iff y[x, z, w](xz \cdot w) = [y, x, zw][yx, z, w][y, x, z]^{-1}[y, xz, w]^{-1}y(xz \cdot w)$

$\iff y[x, z, w] = [y, x, zw][yx, z, w][y, x, z]^{-1}[y, xz, w]^{-1}y$

$\iff [x, z, w]L_y = [y, x, zw][yx, z, w][y, x, z]^{-1}[y, xz, w]^{-1}R_y$

$\iff [x, z, w]R_{y^\rho}L_y = [y, x, zw][yx, z, w][y, x, z]^{-1}[y, xz, w]^{-1}$

6. For all $x, y, z, w \in Q$, $(w \cdot xz)(w, x, z)y = (wx \cdot z)y = (wx \cdot zy)(wx, z, y)$

$\iff (w \cdot xz)(w, x, z)y = (w \cdot (x \cdot zy))(w, x, zy)(wx, z, y)$

$\iff (w \cdot xz)(w,x,z)y = w(xz \cdot y)(x,z,y)^{-1}(w,x,zy)(wx,z,y)$

$\iff (w \cdot xz)(w,x,z)y = (w \cdot xz)y(w,xz,y)^{-1}(x,z,y)^{-1}(w,x,zy)(wx,z,y)$

$\iff (w,x,z)y = y(w,xz,y)^{-1}(x,z,y)^{-1}(w,x,zy)(wx,z,y)$

$\iff (w,x,zR_)y = (w,xz,y)^{-1}(x,z,y)^{-1}(w,x,zy)(wx,z,y)L_y$

$\iff (w,x,z)L_{y^\lambda}R_y = (w,xz,y)^{-1}(x,z,y)^{-1}(w,x,zy)(wx,z,y)$

7. For all $x,y,z,w \in Q,\ (z \cdot wx)(z,w,x)y = (zw \cdot x)y = (zw \cdot xy)(zw,x,y)$

$\iff (z \cdot wx)(z,w,x)y = (z \cdot (w \cdot xy))(z,w,xy)(zw,x,y)$

$\iff (z \cdot wx)(z,w,x)y = z(wx \cdot y)(w,x,y)^{-1}(z,w,xy)(zw,x,y)$

$\iff (z \cdot wx)(z,w,x)y = (z \cdot wx)y(z,wx,y)^{-1}(w,x,y)^{-1}(z,w,xy)(zw,x,y)$

$\iff (z,w,x)y = y(z,wx,y)^{-1}(w,x,y)^{-1}(z,w,xy)(zw,x,y)$

$\iff (z,w,x)R_y = (z,wx,y)^{-1}(w,x,y)^{-1}(z,w,xy)(zw,x,y)L_y$

$\iff (z,w,x)L_{y^\lambda}R_y = (z,wx,y)^{-1}(w,x,y)^{-1}(z,w,xy)(zw,x,y)$

8. For all $x,y,z,w \in Q,\ (x \cdot zw)(x,z,w)y = (xz \cdot w) = (xz \cdot wy)(xz,w,y)$

$\iff (x \cdot zw)(x,z,w)y = (x \cdot (z \cdot wy))(x,z,wy)(xz,w,y)$

$\iff (x \cdot zw)(x,z,w)y = x(zx \cdot y)(z,w,y)^{-1}(x,z,wy)(xz,w,y)$

$\iff (x \cdot zw)(x,z,w)y = (x \cdot zw)y(x,zw,y)^{-1}(z,w,y)^{-1}(x,z,wy)(xz,w,y)$

$\iff (x,z,w)y = y(x,zw,y)^{-1}(z,w,y)^{-1}(x,z,wy)(xz,w,y)$

$\iff (x,z,w)R_y = (x,zw,y)^{-1}(z,w,y)^{-1}(x,z,wy)(xz,w,y)L_y$

$\iff (x,z,w)L_{y^\lambda}R_y = (x,zw,y)^{-1}(z,w,y)^{-1}(x,z,wy)(xz,w,y)$

$\square$

Theorem 4.8.9. Let $(Q,\cdot)$ be a Basarab loop, and $A(Q)$ its associator subloop. Then for every $x,y,z,w \in Q$:

1. $[x,y,w]R_{z^\rho}L_z = [z,x,yw][zx,y,w][z,x,y]^{-1}[z,xy,w]^{-1}$

2. $(x,y,w)L_{z^\lambda}R_z = (x,yw,z)^{-1}(y,w,z)^{-1}(x,y,wz)(xy,w,z)$

3. $[y,x,w]R_{z^\rho}L_z = [z,y,xw][zy,x,w][z,y,x]^{-1}[z,yx,w]^{-1}$

4. $[w,y,x]R_{z^\rho}L_z = [z,w,yx][zw,y,x][z,w,y]^{-1}[z,wy,x]^{-1}$

5. $[x,w,y]R_{z^\rho}L_x = [z,x,wy][zx,w,y][z,x,w]^{-1}[z,xw,y]^{-1}$

6. $(y,x,w)L_{z^\lambda}R_z = (y,xw,z)^{-1}(x,w,z)^{-1}(y,x,wz)(yx,w,z)$

7. $(w,y,x)L_{z^\lambda}R_z = (w,yx,z)^{-1}(y,x,z)^{-1}(w,y,xz)(wy,x,z)$

8. $(x,w,y)L_{z^\lambda}R_z = (x,wy,z)^{-1}(w,y,z)^{-1}(x,w,yz)(xw,y,z)$

Proof. Let $(Q,\cdot)$ be a Basarab loop, then every associator of three elements contains in $N(Q)$.

1. For all $x,y,z,w\in Q,\ z[x,y,w](xy\cdot w) = z(x\cdot yw) = [z,x,yw]\cdot(zx\cdot yw)$

$\iff z[x,y,w](xy\cdot w) = [z,x,yw][zx,y,w](zx\cdot y)w$

$\iff z[x,y,w](xy\cdot w) = [z,x,yw][zx,y,w][z,x,y]^{-1}(z\cdot xy)w$

$\iff z[x,y,w](xy\cdot w) = [z,x,yw][zx,y,w][z,x,y]^{-1}[z,xy,w]^{-1}z(xy\cdot w)$

$\iff z[x,y,w] = [z,x,yw][zx,y,w][z,x,y]^{-1}[z,xy,w]^{-1}z$

$\iff [x,y,w]L_z = [z,x,yw][zx,y,w][z,x,y]^{-1}[z,xy,w]^{-1}R_z$

$\iff [x,y,w]L_zR_z^{-1} = [z,x,yw][zx,y,w][z,x,y]^{-1}[z,xy,w]^{-1}$

$\iff [x,y,w]R_{z^\rho}L_z = [z,x,yw][zx,y,w][z,x,y]^{-1}[z,xy,w]^{-1}$

2. For all $x,y,z,w\in Q,\ (x\cdot yw)(x,y,w)z = (xy\cdot w)z = (xy\cdot wz)(xy,w,z)$

$\iff (x\cdot yw)(x,y,w)z = (x\cdot(y\cdot wz))(x,y,wz)(xy,w,z)$

$\iff (x\cdot yw)(x,y,w)z = x(yw\cdot z)(y,w,z)^{-1}(x,y,wz)(xy,w,z)$

$\iff (x\cdot yw)(x,y,w)z = (x\cdot yw)z(x,yw,z)^{-1}(y,w,z)^{-1}(x,y,wz)(xy,w,z)$

$\iff (x,y,w)z = z(x,yw,z)^{-1}(y,w,z)^{-1}(x,y,wz)(xy,w,z)$

$\iff (x,y,w)R_z = (x,yw,z)^{-1}(y,w,z)^{-1}(x,y,wz)(xy,w,z)L_z$

$\iff (x,y,w)R_zL_z^{-1} = (x,yw,z)^{-1}(y,w,z)^{-1}(x,y,wz)(xy,w,z)$

$\iff (x,y,w)L_{z^\lambda}R_z = (x,yw,z)^{-1}(y,w,z)^{-1}(x,y,wz)(xy,w,z)$

3. For all $x, y, z, w \in Q$, $z[y, x, w](yx \cdot w) = z(y \cdot xw) = [z, y, xw] \cdot (zy \cdot xw)$

$\iff z[y, x, w](yx \cdot w) = [z, y, xw][zy, x, w][z, y, x]^{-1}(z \cdot yx)w$

$\iff z[y, x, w](yx \cdot w) = [z, y, xw][zy, x, w][z, y, x]^{-1}[z, yx, w]^{-1}z(yx \cdot w)$

$\iff z[y, x, w] = [z, y, xw][zy, x, w][z, y, x]^{-1}[z, yx, w]^{-1}z$

$\iff [y, x, w]L_z = [z, y, xw][zy, x, w][z, y, x]^{-1}[z, yx, w]^{-1}R_z$

$\iff [y, x, w]L_z R_z^{-1} = [z, y, xw][zy, x, w][z, y, x]^{-1}[z, yx, w]^{-1}$

$\iff [y, x, w]R_{z^\rho}L_z = [z, y, xw][zy, x, w][z, y, x]^{-1}[z, yx, w]^{-1}$

4. For all $x, y, z, w \in Q$, $z[w, y, x](wy \cdot x) = z(w \cdot yx) = [z, w, yx] \cdot (zw \cdot yx)$

$\iff z[w, y, x](wy \cdot x) = [z, w, yx][zw, y, x](zw \cdot y)x$

$\iff z[w, y, x](wy \cdot x) = [z, w, yx][zw, y, x][z, w, y]^{-1}(z \cdot wy)x$

$\iff z[w, y, x](wy \cdot x) = [z, w, yx][zw, y, x][z, w, y]^{-1}[z, wy, x]^{-1}z(wy \cdot x)$

$\iff z[w, y, x] = [z, w, yx][zw, y, x][z, w, y]^{-1}[z, wy, x]^{-1}z$

$\iff [w, y, x]L_z = [z, w, yx][zw, y, x][z, w, y]^{-1}[z, wy, x]^{-1}R_z$

$\iff [w, y, x]R_{z^\rho}L_z = [z, w, yx][zw, y, x][z, w, y]^{-1}[z, wy, x]^{-1}$

5. For all $x, y, z, w \in Q$, $z[x, w, y](xw \cdot y) = [z, x, wy][zx, w, y](zx \cdot w)y$

$\iff z[x, w, y](xw \cdot y) = [z, x, wy][zx, w, y](zx \cdot w)y$

$\iff z[x, w, y](xw \cdot y) = [z, x, wy][zx, w, y][z, x, w]^{-1}(z \cdot xw)y$

$\iff z[x, w, y](xw \cdot y) = [z, x, wy][zx, w, y][z, x, w]^{-1}[z, xw, y]^{-1}z(xw \cdot y)$

$\iff z[x, w, y] = [z, x, wy][zx, w, y][z, x, w]^{-1}[z, xw, y]^{-1}z$

$\iff [x, w, y]L_z = [z, x, wy][zx, w, y][z, x, w]^{-1}[z, xw, y]^{-1}R_z$

$\iff [x, w, y]R_{z^\rho}L_z = [z, x, wy][zx, w, y][z, x, w]^{-1}[z, xw, y]^{-1}$

6. For all $x, y, z, w \in Q$, $(y \cdot xw)(y, x, w)z = (yx \cdot w)z = (yx \cdot wz)(yx, w, z)$

$\iff (y \cdot xw)(y, x, w)z = (y \cdot (x \cdot wz))(y, x, wz)(yx, w, z)$

$\iff (y \cdot xw)(y, x, w)z = y(xw \cdot z)(x, w, z)^{-1}(y, x, wz)(yx, w, z)$

$\iff (y \cdot xw)(y, x, w)z = (y \cdot xw)z(y, xw, z)^{-1}(x, w, z)^{-1}(y, x, wz)(yx, w, z)$

$$\iff (y,x,w)z = z(y,xw,z)^{-1}(x,w,z)^{-1}(y,x,wz)(yx,w,z)$$

$$\iff (y,x,w)R_z = (y,xw,z)^{-1}(x,w,z)^{-1}(y,x,wz)(yx,w,z)L_z$$

$$\iff (y,x,w)L_{z^\lambda}R_z = (y,xw,z)^{-1}(x,w,z)^{-1}(y,x,wz)(yx,w,z)$$

7. For all $x,y,z,w \in Q$, $(w\cdot yx)(w,y,x)z = (wy\cdot x)z = (wy\cdot xz)(wy,x,z)$

$$\iff (w\cdot yx)(w,y,x)z = (w\cdot(y\cdot xz))(w,y,xz)(wy,x,z)$$

$$\iff (w\cdot yx)(w,y,x)z = w(yx\cdot z)(y,x,z)^{-1}(w,y,xz)(wy,x,z)$$

$$\iff (w\cdot yx)(w,y,x)z = (w\cdot yx)z(w,yx,z)^{-1}(y,x,z)^{-1}(w,y,xz)(wy,x,z)$$

$$\iff (w,y,x)z = z(w,yx,z)^{-1}(y,x,z)^{-1}(w,y,xz)(wy,x,z)$$

$$\iff (w,y,x)R_z = (w,yx,z)^{-1}(y,x,z)^{-1}(w,y,xz)(wy,x,z)L_z$$

$$\iff (w,y,x)L_{z^\lambda}R_z = (w,yx,z)^{-1}(y,x,z)^{-1}(w,y,xz)(wy,x,z)$$

8. For all $x,y,z,w \in Q$, $(x\cdot wy)(x,w,y)z = (xw\cdot y)z = (xw\cdot yz)(xw,y,z)$

$$\iff (x\cdot wy)(x,w,y)z = (x\cdot(w\cdot yz))(x,w,yz)(xw,y,z)$$

$$\iff (x\cdot wy)(x,w,y)z = x(wy\cdot z)(w,y,z)^{-1}(x,w,yz)(xw,y,z)$$

$$\iff (x\cdot wy)(x,w,y)z = (x\cdot wy)z(x,wy,z)^{-1}(w,y,z)^{-1}(x,w,yz)(xw,y,z)$$

$$\iff (x,w,y)z = z(x,wy,z)^{-1}(w,y,z)^{-1}(x,w,yz)(xw,y,z)$$

$$\iff (x,w,y)R_z = (x,wy,z)^{-1}(w,y,z)^{-1}(x,w,yz)(xw,y,z)L_z$$

$$\iff (x,w,y)L_{z^\lambda}R_z = (x,wy,z)^{-1}(w,y,z)^{-1}(x,w,yz)(xw,y,z).$$

□

4.8.3 Relationship between Associators and Inner Mappings of a Basarab loop with a right Inverse component

Theorem 4.8.10. Let $(Q,\cdot)$ be a Basarab loop, and $A(Q)$ its associator subloop. Then for every $x,z,w \in Q$:

1. $[x,x^\rho,z]R_{w^\rho}L_w = [w,x,x^\rho z][wx,x^\rho,z][w,x,x^\rho]^{-1}$

2. $(x,x^\rho,z)L_{w^\lambda}R_w = (x,x^\rho z,w)^{-1}(x^\rho,z,w)^{-1}(x,x^\rho,zw)$

Proof. Let $(Q,\cdot)$ be a Basarab loop. Then

1. For all $x,y,z,w \in Q$, $w[x,x^\rho,z](xx^\rho \cdot z) = w[x \cdot x^\rho z] = [w,x,x^\rho z] \cdot (wx \cdot x^\rho z)$

$\iff w[x,x^\rho,z](xx^\rho \cdot z) = [w,x,x^\rho z][wx,x^\rho,z](wx \cdot x^\rho)z$

$\iff w[x,x^\rho,z](xx^\rho \cdot z) = [w,x,x^\rho z][wx,x^\rho,z][w,x,x^\rho]^{-1}(w \cdot xx^\rho)z$

$\iff w[x,x^\rho,z](xx^\rho \cdot z) = [w,x,x^\rho z][wx,x^\rho,z][w,x,x^\rho]^{-1}[w,xx^\rho,z]^{-1}w(xx^\rho \cdot z)$

$\iff w[x,x^\rho,z] = [w,x,x^\rho z][wx,x^\rho,z][w,x,x^\rho]^{-1}[w,xx^\rho,z]^{-1}w$

$\iff w[x,x^\rho,z] = [w,x,x^\rho z][wx,x^\rho,z][w,x,x^\rho]^{-1}ew$

$\iff [x,x^\rho,z]L_w = [w,x,x^\rho z][wx,x^\rho,z][w,x,x^\rho]^{-1}R_w$

$\iff [x,x^\rho,z]L_wR_w^{-1} = [w,x,x^\rho z][wx,x^\rho,z][w,x,x^\rho]^{-1}$

$\iff [x,x^\rho,z]R_{w^\rho}L_w = [w,x,x^\rho z][wx,x^\rho,z][w,x,x^\rho]^{-1}$

2. For all $x,y,z,w \in Q$, $(x \cdot x^\rho z)(x,x^\rho,z)w = (xx^\rho \cdot z)w = (xx^\rho \cdot zw)(xx^\rho,z,w)$

$\iff (x \cdot x^\rho z)(x,x^\rho,z)w = (x \cdot (x^\rho \cdot zw))(x,x^\rho,zw)e$

$\iff (x \cdot x^\rho z)(x,x^\rho,z)w = x(x^\rho z \cdot w)(x^\rho,z,w)^{-1}(x,x^\rho,zw)$

$\iff (x \cdot x^\rho z)(x,x^\rho,z)w = (x \cdot x^\rho z)w(x,x^\rho z,w)^{-1}(x^\rho,z,w)^{-1}(x,x^\rho,zw)$

$\iff (x,x^\rho,z)w = w(x,x^\rho z,w)^{-1}(x^\rho,z,w)^{-1}(x,x^\rho,zw)$

$\iff (x,x^\rho,z)R_w = (x,x^\rho z,w)^{-1}(x^\rho,z,w)^{-1}(x,x^\rho,zw)L_w$

$\iff (x,x^\rho,z)R_wL_w^{-1} = (x,x^\rho z,w)^{-1}(x^\rho,z,w)^{-1}(x,x^\rho,zw)$

$\iff (x,x^\rho,z)L_{w^\lambda}R_w = (x,x^\rho z,w)^{-1}(x^\rho,z,w)^{-1}(x,x^\rho,zw)$

□

Theorem 4.8.11. Let $(Q,\cdot)$ be a Basarab loop, and $A(Q)$ its associator subloop. Then for every $x,z,w \in Q$:

1. $[x^\rho,x,z]R_{w^\rho}L_w = [w,x^\rho,xz][wx^\rho,x,z][w,x^\rho,x]^{-1}[w,x^\rho x,z]^{-1}$

2. $[z,x^\rho,x]R_{w^\rho}L_w = [w,z,x^\rho][wz,x^\rho,x][w,z,x^\rho]^{-1}[w,zx^\rho,x]^{-1}$

3. $w[x, z, x^{\rho}]R_{w^{\rho}} = [w, x, zx^{\rho}][wx, z, x^{\rho}][w, x, z]^{-1}[w, xz, x^{\rho}]^{-1}$

Proof. Let $(Q, \cdot)$ be a Basarab loop, then every associator of three elements contains in $N(Q)$.

1. For all $x, y, z, w \in Q,\ w[x^{\rho}, x, z](x^{\rho}x \cdot z) = w[x^{\rho} \cdot xz] = [w, x^{\rho}, xz] \cdot (wx^{\rho} \cdot xz)$

$\iff w[x^{\rho}, x, z](x^{\rho}x \cdot z) = [w, x^{\rho}, xz][wx^{\rho}, x, z](wx^{\rho} \cdot x)z$

$\iff w[x^{\rho}, x, z](x^{\rho}x \cdot z) = [w, x^{\rho}, xz][wx^{\rho}, x, z][w, x^{\rho}, x]^{-1}(w \cdot x^{\rho}x)z$

$\iff w[x^{\rho}, x, z](x^{\rho}x \cdot z) = [w, x^{\rho}, xz][wx^{\rho}, x, z][w, x^{\rho}, x]^{-1}[w, x^{\rho}x, z]^{-1}w(x^{\rho}x \cdot z)$

$\iff w[x^{\rho}, x, z] = [w, x^{\rho}, xz][wx^{\rho}, x, z][w, x^{\rho}, x]^{-1}[w, x^{\rho}x, z]^{-1}w$

$\iff [x^{\rho}, x, z]L_w R_w^{-1} = [w, x^{\rho}, xz][wx^{\rho}, x, z][w, x^{\rho}, x]^{-1}[w, x^{\rho}x, z]^{-1}$

$\iff [x^{\rho}, x, z]R_{w^{\rho}}L_w = [w, x^{\rho}, xz][wx^{\rho}, x, z][w, x^{\rho}, x]^{-1}[w, x^{\rho}x, z]^{-1}$

2. For all $x, y, z, w \in Q,\ w[z, x^{\rho}, x](zx^{\rho} \cdot x) = w[z \cdot x^{\rho}x] = [w, z, x^{\rho}x] \cdot (wz \cdot x^{\rho}x)$

$\iff w[z, x^{\rho}, x](zx^{\rho} \cdot x) = [w, z, x^{\rho}][wz, x^{\rho}, x](wz \cdot x^{\rho})x$

$\iff w[z, x^{\rho}, x](zx^{\rho} \cdot x) = [w, z, x^{\rho}][wz, x^{\rho}, x][w, z, x^{\rho}]^{-1}(w \cdot zx^{\rho})x$

$\iff w[z, x^{\rho}, x](zx^{\rho} \cdot x) = [w, z, x^{\rho}][wz, x^{\rho}, x][w, z, x^{\rho}]^{-1}[w, zx^{\rho}, x]^{-1}w(zx^{\rho} \cdot x)$

$\iff w[z, x^{\rho}, x] = [w, z, x^{\rho}][wz, x^{\rho}, x][w, z, x^{\rho}]^{-1}[w, zx^{\rho}, x]^{-1}w$

$\iff [z, x^{\rho}, x]L_w R_x^{-1} = [w, z, x^{\rho}][wz, x^{\rho}, x][w, z, x^{\rho}]^{-1}[w, zx^{\rho}, x]^{-1}$

$\iff [z, x^{\rho}, x]R_{w^{\rho}}L_w = [w, z, x^{\rho}][wz, x^{\rho}, x][w, z, x^{\rho}]^{-1}[w, zx^{\rho}, x]^{-1}$

3. For all $x, y, z, w \in Q,\ w[x, z, x^{\rho}](xz \cdot x^{\rho}) =$

$\iff w[x, z, x^{\rho}](xz \cdot x^{\rho}) = w[x \cdot zx^{\rho}] = [w, x, zx^{\rho}] \cdot (wx \cdot zx^{\rho})$

$\iff w[x, z, x^{\rho}](xz \cdot x^{\rho}) = [w, x, zx^{\rho}][wx, z, x^{\rho}](wx \cdot z)x^{\rho}$

$\iff w[x, z, x^{\rho}](xz \cdot x^{\rho}) = [w, x, zx^{\rho}][wx, z, x^{\rho}][w, x, z]^{-1}(w \cdot xz)x^{\rho}$

$\iff w[x, z, x^{\rho}](xz \cdot x^{\rho}) = [w, x, zx^{\rho}][wx, z, x^{\rho}][w, x, z]^{-1}[w, xz, x^{\rho}]^{-1}w(xz \cdot x^{\rho})$

$\iff w[x, z, x^{\rho}] = [w, x, zx^{\rho}][wx, z, x^{\rho}][w, x, z]^{-1}[w, xz, x^{\rho}]^{-1}w$

$\iff [x, z, x^{\rho}]L_w R_w^{-1} = [w, x, zx^{\rho}][wx, z, x^{\rho}][w, x, z]^{-1}[w, xz, x^{\rho}]^{-1}$

$\iff w[x, z, x^{\rho}]R_{w^{\rho}} = [w, x, zx^{\rho}][wx, z, x^{\rho}][w, x, z]^{-1}[w, xz, x^{\rho}]^{-1}$

□

Theorem 4.8.12. Let $(Q,\cdot)$ be a Basarab loop, and $A(Q)$ its associator subloop. Then for every $x,z,w\in Q$:

1. $(x^\rho,xz)L_{w^\lambda R_w}=(x^\rho,xz,w)^{-1}(x,z,w)^{-1}(x^\rho,x,zw)(x^\rho x,z,w)$
2. $(z,x^\rho,x)L_{w^\rho}R_w=(z,x^\rho x,w)^{-1}(x^\rho,x,w)^{-1}(z,x^\rho,xw)(zx^\rho,x,w)$
3. $(x,z,x^\rho)L_{w^\rho}R_w=(x,zx^\rho,w)^{-1}(z,x^\rho,w)^{-1}(x,z,x^\rho w)(xz,x^\rho,w)$

Proof. Let $(Q,\cdot)$ be a Basarab loop. Then

1. For all $x,y,z,w\in Q$, $(x^\rho\cdot xz)(x^\rho,xz)w=(x^\rho x\cdot z)w=(x^\rho x\cdot zw)(x^\rho x,z,w)$

$\iff (x^\rho\cdot xz)(x^\rho,xz)w=(x^\rho\cdot(x\cdot zw))(x^\rho,x,zw)(x^\rho x,z,w)$

$\iff (x^\rho\cdot xz)(x^\rho,xz)w=x^\rho(xz\cdot w)(x,z,w)^{-1}(x^\rho,x,zw)(x^\rho x,z,w)$

$\iff (x^\rho\cdot xz)(x^\rho,xz)w=(x^{\rho\cdot xz})w(x^\rho,xz,w)^{-1}(x,z,w)^{-1}(x^\rho,x,zw)(x^\rho x,z,w)$

$\iff (x^\rho\cdot xz)(x^\rho,xz)w=w(x^\rho,xz,w)^{-1}(x,z,w)^{-1}(x^\rho,x,zw)(x^\rho x,z,w)$

$\iff (x^\rho,xz)R_w=(x^\rho,xz,w)^{-1}(x,z,w)^{-1}(x^\rho,x,zw)(x^\rho x,z,w)L_w$

$\iff (x^\rho,xz)R_wL_w^{-1}=(x^\rho,xz,w)^{-1}(x,z,w)^{-1}(x^\rho,x,zw)(x^\rho x,z,w)$

$\iff (x^\rho,xz)L_{w^\lambda R_w}=(x^\rho,xz,w)^{-1}(x,z,w)^{-1}(x^\rho,x,zw)(x^\rho x,z,w)$

2. For all $x,y,z,w\in Q$, $(z\cdot x^\rho x)(z,x^\rho,x)w=(zx^\rho\cdot x)w=(zx^\rho\cdot xw)(zx^\rho,x,w)$

$\iff (z\cdot x^\rho x)(z,x^\rho,x)w=(z\cdot(x^\rho\cdot xw))(z,x^\rho,xw)(zx^\rho,x,w)$

$\iff (z\cdot x^\rho x)(z,x^\rho,x)w=z(x^\rho x\cdot w)(x^\rho,x,w)^{-1}(z,x^\rho,xw)(zx^\rho,x,w)$

$\iff (z\cdot x^\rho x)(z,x^\rho,x)w=(z\cdot x^\rho x)w(z,x^\rho x)^{-1}(x^\rho,x,w)^{-1}(z,x^\rho,xw)(zx^\rho,x,w)$

$\iff (z,x^\rho,x)w=w(z,x^\rho x)^{-1}(x^\rho,x,w)^{-1}(z,x^\rho,xw)(zx^\rho,x,w)$

$\iff (z,x^\rho,x)R_wL_w^{-1}=(z,x^\rho x)^{-1}(x^\rho,x,w)^{-1}(z,x^\rho,xw)(zx^\rho,x,w)$

$\iff (z,x^\rho,x)L_{w^\rho}R_w=(z,x^\rho x,w)^{-1}(x^\rho,x,w)^{-1}(z,x^\rho,xw)(zx^\rho,x,w)$

3. For all $x,y,z,w\in Q$, $(x\cdot zx^\rho)(x,z,x^\rho)w=(xz\cdot x^\rho)w=(xz\cdot x^\rho w)(xz,x^\rho,w)$

$\iff (x\cdot zx^\rho)(x,z,x^\rho)w=(x\cdot(z\cdot x^\rho w))(x,z,x^\rho w)(xz,x^\rho,w)$

$\iff (x\cdot zx^\rho)(x,z,x^\rho)w=x(zx^\rho\cdot w)(z,x^\rho,w)^{-1}(x,z,x^\rho w)(xz,x^\rho,w)$

$$\iff (x\cdot zx^\rho)(x,z,x^\rho)w = (x\cdot zx^\rho)w(x,zx^\rho,w)^{-1}(z,x^\rho,w)^{-1}(x,z,x^\rho w)(xz,x^\rho,w)$$

$$\iff (x,z,x^\rho)w = w(x,zx^\rho,w)^{-1}(z,x^\rho,w)^{-1}(x,z,x^\rho w)(xz,x^\rho,w)$$

$$\iff (x,z,x^\rho)R_wL_w^{-1} = (x,zx^\rho,w)^{-1}(z,x^\rho,w)^{-1}(x,z,x^\rho w)(xz,x^\rho,w)$$

$$\iff (x,z,x^\rho)L_{w^\rho}R_w = (x,zx^\rho,w)^{-1}(z,x^\rho,w)^{-1}(x,z,x^\rho w)(xz,x^\rho,w)$$

□

Theorem 4.8.13. Let $(Q,\cdot)$ be a Basarab loop, and $A(Q)$ its associator subloop. Then for every $x,y,w\in Q$:

1. $[x,y,x^\rho]R_{w^\rho}L_w = [w,x,yx^\rho][wx,y,x^\rho][w,x,y]^{-1}[w,xy,x^\rho]^{-1}$
2. $(x,y,x^\rho)L_{w^\lambda}R_w = (x,yx^\rho,w)^{-1}(y,x^\rho,w)^{-1}(x,y,x^\rho w)(xy,x^\rho,w)$

Proof. Let $(Q,\cdot)$ be a Basarab loop, then every associator of three elements contains in $N(Q)$.

1. For all $x,y,z,w\in Q,\ w[x,y,x^\rho](xy\cdot x^\rho) = w(x\cdot yx^\rho) = [w,x,yx^\rho](wx\cdot yx^\rho)$

$$\iff w[x,y,x^\rho](xy\cdot x^\rho) = [w,x,yx^\rho][wx,y,x^\rho](wx\cdot y)x^\rho$$

$$\iff w[x,y,x^\rho](xy\cdot x^\rho) = [w,x,yx^\rho][wx,y,x^\rho][w,x,y]^{-1}(w\cdot xy)x^\rho$$

$$\iff w[x,y,x^\rho](xy\cdot x^\rho) = [w,x,yx^\rho][wx,y,x^\rho][w,x,y]^{-1}[w,xy,x^\rho]^{-1}w(xy\cdot x^\rho)$$

$$\iff w[x,y,x^\rho](xy\cdot x^\rho) = [w,x,yx^\rho][wx,y,x^\rho][w,x,y]^{-1}[w,xy,x^\rho]^{-1}w$$

$$\iff [x,y,x^\rho]L_w = [w,x,yx^\rho][wx,y,x^\rho][w,x,y]^{-1}[w,xy,x^\rho]^{-1}R_x$$

$$\iff [x,y,x^\rho]L_wR_w^{-1} = [w,x,yx^\rho][wx,y,x^\rho][w,x,y]^{-1}[w,xy,x^\rho]^{-1}$$

$$\iff [x,y,x^\rho]R_{w^\rho}L_w = [w,x,yx^\rho][wx,y,x^\rho][w,x,y]^{-1}[w,xy,x^\rho]^{-1}$$

2. For all $x,y,z,w\in Q,\ (x\cdot yx^\rho)(x,y,x^\rho)w = (xy\cdot x^\rho)w = (xy\cdot x^\rho w)(xy,x^\rho,w)$

$$\iff (x\cdot yx^\rho)(x,y,x^\rho)w = (xx^\rho(yx^\rho\cdot x^\rho))(x,y,x^\rho w)(xy,x^\rho,w)$$

$$\iff (x\cdot yx^\rho)(x,y,x^\rho)w = x(yx^\rho\cdot w)(y,x^\rho,w)^{-1}(x,y,x^\rho w)(xy,x^\rho,w)$$

$$\iff (x\cdot yx^\rho)(x,y,x^\rho)w = (x\cdot yx^\rho w(x,yx^\rho,w)^{-1}(y,x^\rho,w)^{-1}(x,y,x^\rho w)(xy,x^\rho,w)$$

$$\iff (x,y,x^\rho)w = (x,yx^\rho,w)^{-1}(y,x^\rho,w)^{-1}(x,y,x^\rho w)(xy,x^\rho,w)$$

$$\iff (x,y,x^\rho)R_w = (x,yx^\rho,w)^{-1}(y,x^\rho,w)^{-1}(x,y,x^\rho w)(xy,x^\rho,w)L_w$$

$$\iff (x,y,x^\rho)R_wL_w^{-1} = (x,yx^\rho,w)^{-1}(y,x^\rho,w)^{-1}(x,y,x^\rho w)(xy,x^\rho,w)$$

$$\iff (x,y,x^\rho)L_{w^\lambda}R_w = (x,yx^\rho,w)^{-1}(y,x^\rho,w)^{-1}(x,y,x^\rho w)(xy,x^\rho,w)$$

□

Theorem 4.8.14. Let $(Q,\cdot)$ be a Basarab loop, and $A(Q)$ its associator subloop. Then for every $x,y,w \in Q$:

1. $[y,x,x^\rho]R_{w^\rho}L_w = [wy,x,x^\rho][w,y,x]^{-1}[w,yx,x^\rho]^{-1}$
2. $[x^\rho,y,x]R_{w^\rho}L_w = [w,x^\rho,yx][wx^\rho,y,x][w,x^\rho,y]^{-1}[w,x^\rho y,x]^{-1}$
3. $[x,x^\rho,y]R_{w^\rho}L_w = [w,x,x^\rho y][wx,x^\rho,y][w,x,x^\rho]^{-1}[w,xx^\rho,y]^{-1}$

Proof. 1. For all $x,y,w \in Q$, $w[y,x,x^\rho](yx\cdot x^\rho) = w(y\cdot xx^\rho) = [w,y,xx^\rho]\cdot(wy\cdot xx^\rho)$

$$\iff w[y,x,x^\rho](yx\cdot x^\rho) = [w,y,xx^\rho][wy,x,x^\rho][w,y,x]^{-1}(w\cdot yx)x^\rho$$

$$\iff w[y,x,x^\rho](yx\cdot x^\rho) = [w,y,xx^\rho][wy,x,x^\rho][w,y,x]^{-1}w(yx\cdot x^\rho)$$

$$\iff w[y,x,x^\rho] = [w,y,xx^\rho][wy,x,x^\rho][w,y,x]^{-1}[w,yx,x^\rho]^{-1}ww$$

$$\iff w[y,x,x^\rho] = e\cdot[wy,x,x^\rho][w,y,x]^{-1}[w,yx,x^\rho]^{-1}w$$

$$\iff [y,x,x^\rho]L_w = [wy,x,x^\rho][w,y,x]^{-1}[w,yx,x^\rho]^{-1}R_x$$

$$\iff [y,x,x^\rho]L_wR_x^{-1} = [wy,x,x^\rho][w,y,x]^{-1}[w,yx,x^\rho]^{-1}$$

$$\iff [y,x,x^\rho]R_{w^\rho}L_w = [wy,x,x^\rho][w,y,x]^{-1}[w,yx,x^\rho]^{-1}$$

2. For all $x,y,w \in Q$, $w[x^\rho,y,x](x^\rho y\cdot x) = w(x^{rho}\cdot yx) = [w,x^\rho,yx]\cdot(wx^\rho\cdot yx)$

$$\iff w[x^\rho,y,x](x^\rho y\cdot x) = [w,x^\rho,yx][wx^\rho,y,x](wx^\rho\cdot y)x$$

$$\iff w[x^\rho,y,x](x^\rho y\cdot x) = [w,x^\rho,yx][wx^\rho,y,x][w,x^\rho,y]^{-1}(w\cdot x^\rho y)x$$

$$\iff w[x^\rho,y,x](x^\rho y\cdot x) = [w,x^\rho,yx][wx^\rho,y,x][w,x^\rho,y]^{-1}[w,x^\rho y,x]^{-1}w(x^\rho y\cdot x)$$

$$\iff w[x^\rho,y,x] = [w,x^\rho,yx][wx^\rho,y,x][w,x^\rho,y]^{-1}[w,x^\rho y,x]^{-1}w$$

$$\iff [x^\rho,y,x]L_w = [w,x^\rho,yx][wx^\rho,y,x][w,x^\rho,y]^{-1}[w,x^\rho y,x]^{-1}R_w$$

$$\iff [x^\rho,y,x]R_{w^\rho}L_w = [w,x^\rho,yx][wx^\rho,y,x][w,x^\rho,y]^{-1}[w,x^\rho y,x]^{-1}$$

3. For all $x, y, w \in Q,\ w[x, x^\rho, y](xx^\rho \cdot y) = w(x \cdot x^\rho y) = [w, x, x^\rho y] \cdot (wx \cdot x^\rho y)$

$$\iff w[x, x^\rho, y](xx^\rho \cdot y) = [w, x, x^\rho y][wx, x^\rho, y](wx \cdot x^\rho)y$$

$$\iff w[x, x^\rho, y](xx^\rho \cdot y) = [w, x, x^\rho y][wx, x^\rho, y][w, x, x^\rho]^{-1}(w \cdot xx^\rho)y$$

$$\iff w[x, x^\rho, y](xx^\rho \cdot y) = [w, x, x^\rho y][wx, x^\rho, y][w, x, x^\rho]^{-1}[w, xx^\rho, y]^{-1}w(xx^\rho \cdot y)$$

$$\iff w[x, x^\rho, y] = [w, x, x^\rho y][wx, x^\rho, y][w, x, x^\rho]^{-1}[w, xx^\rho, y]^{-1}w$$

$$\iff [x, x^\rho, y]L_w R_w = [w, x, x^\rho y][wx, x^\rho, y][w, x, x^\rho]^{-1}[w, xx^\rho, y]^{-1}$$

$$\iff [x, x^\rho, y]R_{w^\rho} L_w = [w, x, x^\rho y][wx, x^\rho, y][w, x, x^\rho]^{-1}[w, xx^\rho, y]^{-1}$$

□

Theorem 4.8.15. Let $(Q, \cdot)$ be a Basarab loop, and $A(Q)$ its associator subloop. Then for every $x, y, w \in Q$:

1. $(y, x, x^\rho)L_{w^\lambda}R_w = (y, xx^\rho, w)^{-1}(x, x^\rho, w)^{-1}(y, x, x^\rho w)(yx, x^\rho, w)$

2. $(x^\rho, y, x)L_{w^\lambda}R_x = (x^\rho, yx, w)^{-1}(y, x, w)^{-1}(x^\rho, y, xw)(x^\rho y, x, w)$

3. $(x, x^\rho, y)L_{w^\rho}R_w = (x, x^\rho y, w)^{-1}(x^\rho, y, w)^{-1}(x, x^\rho, yw)$

Proof. Let $(Q, \cdot)$ be a Basarab loop, then every associator of three elements contains in $N(Q)$.

1. For all $x, y, w \in Q,\ (y \cdot xx^\rho)(y, x, x^\rho)w = (yx \cdot x^\rho)w = (yx \cdot x^\rho w)(yx, x^\rho, w)$

$$\iff (y \cdot xx^\rho)(y, x, x^\rho)w = (y \cdot (x \cdot x^\rho))(y, x, x^\rho w)(yx, x^\rho, w)$$

$$\iff (y \cdot xx^\rho)(y, x, x^\rho)w = y(xx^\rho \cdot w)(x, x^\rho, w)^{-1}(y, x, x^\rho w)(yx, x^\rho, w)$$

$$\iff (y \cdot xx^\rho)(y, x, x^\rho)w = (y \cdot xx^\rho)w(y, xx^\rho, w)^{-1}(x, x^\rho, w)^{-1}(y, x, x^\rho w)(yx, x^\rho, w)$$

$$\iff (y, x, x^\rho)w = w(y, xx^\rho, w)^{-1}(x, x^\rho, w)^{-1}(y, x, x^\rho w)(yx, x^\rho, w)$$

$$\iff (y, x, x^\rho)R_w = (y, xx^\rho, w)^{-1}(x, x^\rho, w)^{-1}(y, x, x^\rho w)(yx, x^\rho, w)L_w$$

$$\iff (y, x, x^\rho)L_{w^\lambda}R_w = (y, xx^\rho, w)^{-1}(x, x^\rho, w)^{-1}(y, x, x^\rho w)(yx, x^\rho, w)$$

2. For all $x, y, w \in Q,\ (x^\rho \cdot yx)(x^\rho, y, x)w = (x^\rho y \cdot x)w = (x^\rho y \cdot xw)(x^\rho y, x, w)$

$$\iff (x^\rho \cdot yx)(x^\rho, y, x)w = (x^\rho \cdot (y \cdot xw))(x^\rho, y, xw)(x^\rho y, x, w)$$

$$\iff (x^\rho \cdot yx)(x^\rho, y, x)w = x^\rho(yx \cdot w)(y, x, w)^{-1}(x^\rho, y, xw)(x^\rho y, x, w)$$

$$\iff (x^\rho \cdot yx)(x^\rho, y, x)w = (x^\rho \cdot yx)w(x^\rho, yx, w)^{-1}(y, x, w)^{-1}(x^\rho, y, xw)(x^\rho y, x, w)$$

$$\iff (x^\rho, y, x)w = w(x^\rho, yx, w)^{-1}(y, x, w)^{-1}(x^\rho, y, xw)(x^\rho y, x, w)$$

$$\iff (x^\rho, y, x)R_w = (x^\rho, yx, w)^{-1}(y, x, w)^{-1}(x^\rho, y, xw)(x^\rho y, x, w)L_w$$

$$\iff (x^\rho, y, x)L_{w^\lambda}R_x = (x^\rho, yx, w)^{-1}(y, x, w)^{-1}(x^\rho, y, xw)(x^\rho y, x, w)$$

3. For all $x, y, w \in Q$, $(x \cdot x^\rho y)(x, x^\rho, y)w = (xx^\rho \cdot y)w = (xx^\rho \cdot yw)(xx^\rho, y, w)$

$$\iff (x \cdot x^\rho y)(x, x^\rho, y)w = (x \cdot (x^\rho \cdot yw))(x, x^\rho, yw)(xx^\rho, y, w)$$

$$\iff (x \cdot x^\rho y)(x, x^\rho, y)w = x(x^\rho y \cdot w)(x^\rho, y, w)^{-1}(x, x^\rho, yw)(xx^\rho, y, w)$$

$$\iff (x \cdot x^\rho y)(x, x^\rho, y)w = (x \cdot x^\rho y)w(x, x^\rho y, w)^{-1}(x^\rho, y, w)^{-1}(x, x^\rho, yw)(xx^\rho, y, w)$$

$$\iff)(x, x^\rho, y)w = w(x, x^\rho y, w)^{-1}(x^\rho, y, w)^{-1}(x, x^\rho, yw)(xx^\rho, y, w)$$

$$\iff (x, x^\rho, y)w = (x, x^\rho y, w)^{-1}(x^\rho, y, w)^{-1}(x, x^\rho, yw) \cdot e$$

$$\iff (x, x^\rho, y)R_w = (x, x^\rho y, w)^{-1}(x^\rho, y, w)^{-1}(x, x^\rho, yw)L_w$$

$$\iff (x, x^\rho, y)L_{w^\rho}R_w = (x, x^\rho y, w)^{-1}(x^\rho, y, w)^{-1}(x, x^\rho, yw)$$

□

Theorem 4.8.16. Let $(Q, \cdot)$ be a Basarab loop, and $A(Q)$ its associator subloop. Then for every $y, z, w \in Q$:

1. $[y^\rho, y, z]R_{w^\rho}L_w = [w, y^\rho, yz][wy^\rho, y, z][w, y^\rho, y]^{-1}[w, y^\rho y, z]^{-1}$

2. $(y^\rho, y, z)L_{w^\lambda}R_w = (y^\rho, yz, w)^{-1}(y, z, w)^{-1}(y^\rho, y, zw)(y^\rho y, z, w)$

Proof. Let $(Q, \cdot)$ be a Basarab loop, then every associator of three elements contains in $N(Q)$.

1. Set $x = y^\rho$ in Theorem 4.8.4(1),

   $[x, y, z]R_{w^\rho}L_w = [w, x, yz][wx, y, z][w, x, y]^{-1}[w, xy, z]^{-1}$. Then

   $[y^\rho, y, z]R_{w^\rho}L_w = [w, y^\rho, yz][wy^\rho, y, z][w, y^\rho, y]^{-1}[w, y^\rho y, z]^{-1}$

2. Set $x = y^\rho$ in Theorem 4.8.4(2),

   $(x, y, z)L_{w^\lambda}R_w = (x, yz, w)^{-1}(y, z, w)^{-1}(x, y, zw)(xy, z, w)$. Then

   $(y^\rho, y, z)L_{w^\lambda}R_w = (y^\rho, yz, w)^{-1}(y, z, w)^{-1}(y^\rho, y, zw)(y^\rho y, z, w)$

□

Theorem 4.8.17. Let $(Q,\cdot)$ be a Basarab loop, and $A(Q)$ its associator subloop. Then for every $x, y, w \in Q$:

1. $[y, y^\rho, z]R_{w^\rho}L_w = [w, y, y^\rho z][wy, y^\rho, z][w, y, y^\rho]^{-1}$
2. $[z, y, y^\rho]R_{w^\rho}L_w = [wz, y, y^\rho][w, z, y]^{-1}[w, zy, y^\rho]^{-1}$
3. $[y^\rho, z, y]R_{w^\rho}L_w = [w, y^\rho, zy][wy^\rho, z, y][w, y^\rho, z]^{-1}[w, y^\rho z, y]^{-1}$

Proof. Let $(Q,\cdot)$ be a Basarab loop, and $A(Q)$ its associator subloop. Then for every $x, y, z, w \in Q$:

1. Set $x = y^\rho$ in Theorem 4.8.5(1),

   $[y, x, z]R_{w^\rho}L_w = [w, y, xz][wy, x, z][w, y, x]^{-1}[w, yx, z]^{-1}$

   $\implies [y, y^\rho, z]R_{w^\rho}L_w = [w, y, y^\rho z][wy, y^\rho, z][w, y, y^\rho]^{-1}[w, yy^\rho, z]^{-1}$

   $\implies [y, y^\rho, z]R_{w^\rho}L_w = [w, y, y^\rho z][wy, y^\rho, z][w, y, y^\rho]^{-1}$

2. Set $x = y^\rho$ in Theorem 4.8.5(2),

   $[z, y, x]R_{w^\rho}L_x = [w, z, yx][wz, y, x][w, z, y]^{-1}[w, zy, x]^{-1}$

   $\implies [z, y, y^\rho]R_{w^\rho}L_w = [w, z, yy^\rho][wz, y, y^\rho][w, z, y]^{-1}[w, zy, y^\rho]^{-1}$

   $\implies [z, y, y^\rho]R_{w^\rho}L_w = [wz, y, y^\rho][w, z, y]^{-1}[w, zy, y^\rho]^{-1}$

3. Set $x = y^\rho$ in Theorem 4.8.5(3),

   $[x, z, y]R_{w^\rho}L_x = [w, x, zy][wx, z, y][w, x, z]^{-1}[w, xz, y]^{-1}$

   $\implies [y^\rho, z, y]R_{w^\rho}L_w = [w, y^\rho, zy][wy^\rho, z, y][w, y^\rho, z]^{-1}[w, y^\rho z, y]^{-1}$

□

Theorem 4.8.18. Let $(Q,\cdot)$ be a Basarab loop, and $A(Q)$ its associator subloop. Then for every $y, z, w \in Q$:

1. $(y,y^{\rho},z)L_{w^{\lambda}}R_w=(y,y^{\rho}z,w)^{-1}(y^{\rho},z,w)^{-1}(y,y^{\rho},zw)$

2. $(z,y,y^{\rho})L_{w^{\lambda}}R_w=(y,y^{\rho},w)^{-1}(z,y,y^{\rho}w)(zy,y^{\rho},w)$

3. $(y^{\rho},z,y)L_{w^{\lambda}}R_w=(y^{\rho},zy,w)^{-1}(z,y,w)^{-1}(y^{\rho},z,yw)(y^{\rho}z,y,w)$

Proof. Let $(Q,\cdot)$ be a Basarab loop, then every associator of three elements contains in $N(Q)$.

1. Set $x=y^{\rho}$ in Theorem 4.8.6(1),

   $(y,x,z)L_{w^{\lambda}}R_w=(y,xz,w)^{-1}(x,z,w)^{-1}(y,x,zw)(yx,z,w)$

   $\implies (y,y^{\rho},z)L_{w^{\lambda}}R_w=(y,y^{\rho}z,w)^{-1}(y^{\rho},z,w)^{-1}(y,y^{\rho},zw)(yy^{\rho},z,w)$

   $\implies (y,y^{\rho},z)L_{w^{\lambda}}R_w=(y,y^{\rho}z,w)^{-1}(y^{\rho},z,w)^{-1}(y,y^{\rho},zw)$

2. Set $x=y^{\rho}$ in Theorem 4.8.6(2),

   $(z,y,x)L_{w^{\lambda}}R_w=(z,yx,w)^{-1}(y,x,w)^{-1}(z,y,xw)(zy,x,w)$

   $\implies (z,y,y^{\rho})L_{w^{\lambda}}R_w=(z,yy^{\rho},w)^{-1}(y,y^{\rho},w)^{-1}(z,y,y^{\rho}w)(zy,y^{\rho},w)$

   $\implies (z,y,y^{\rho})L_{w^{\lambda}}R_w=(y,y^{\rho},w)^{-1}(z,y,y^{\rho}w)(zy,y^{\rho},w)$

3. Set $x=y^{\rho}$ in Theorem 4.8.6(3),

   $(x,z,y)L_{w^{\lambda}}R_w=(x,zy,w)^{-1}(z,y,w)^{-1}(x,z,yw)(xz,y,w)$

   $\implies (y^{\rho},z,y)L_{w^{\lambda}}R_w=(y^{\rho},zy,w)^{-1}(z,y,w)^{-1}(y^{\rho},z,yw)(y^{\rho}z,y,w)$

□

Theorem 4.8.19. Let $(Q,\cdot)$ be a Basarab loop, and $A(Q)$ its associator subloop. Then for every $x,z,w\in Q$:

1. $[x,z^{\rho},z]R_{w^{\rho}}L_w=[w,x,z^{\rho}z][wx,z^{\rho},z][w,x,z^{\rho}]^{-1}[w,xz^{\rho},z]^{-1}$

2. $(x,z^{\rho},z)L_{w^{\lambda}}R_w=(x,z^{\rho}z,w)^{-1}(z^{\rho},z,w)^{-1}(x,z^{\rho},zw)(xz^{\rho},z,w)$

Proof. Let $(Q,\cdot)$ be a Basarab loop, then every associator of three elements contains in $N(Q)$.

1. Set $y = z^\rho$ in Theorem 4.8.4(1),

   $[x, y, z]R_{w^\rho}L_w = [w, x, yz][wx, y, z][w, x, y]^{-1}[w, xy, z]^{-1}$

   $\implies [x, z^\rho, z]R_{w^\rho}L_w = [w, x, z^\rho z][wx, z^\rho, z][w, x, z^\rho]^{-1}[w, xz^\rho, z]^{-1}$

2. Set $y = z^\rho$ in Theorem 4.8.4(2),

   $(x, z^\rho, z)L_{w^\lambda}R_w = (x, z^\rho z, w)^{-1}(z^\rho, z, w)^{-1}(x, z^\rho, zw)(xz^\rho, z, w)$

   $\implies (x, z^\rho, z)L_{w^\lambda}R_w = (x, z^\rho z, w)^{-1}(z^\rho, z, w)^{-1}(x, z^\rho, zw)(xz^\rho, z, w)$

□

Theorem 4.8.20. Let $(Q, \cdot)$ be a Basarab loop, and $A(Q)$ its associator subloop. Then for every $x, z, w \in Q$:

1. $[z^\rho, x, z]R_{w^\rho}L_w = [w, z^\rho, xz][wz^\rho, x, z][w, z^\rho, x]^{-1}[w, z^\rho x, z]^{-1}$

2. $[z, z^\rho, x]R_{w^\rho}L_w = [w, z, z^\rho x][wz, z^\rho, x][w, z, z^\rho]^{-1}$

3. $[x, z, z^\rho]R_{w^\rho}L_w = [wx, z, z^\rho][w, x, z]^{-1}[w, xz, z^\rho]^{-1}$

Proof. Let $(Q, \cdot)$ be a Basarab loop, then every associator of three elements contains in $N(Q)$.

1. Set $y = z^\rho$ in Theorem 4.8.5(1),

   $[y, x, z]R_{w^\rho}L_w = [w, y, xz][wy, x, z][w, y, x]^{-1}[w, yx, z]^{-1}$

   $\implies [z^\rho, x, z]R_{w^\rho}L_w = [w, z^\rho, xz][wz^\rho, x, z][w, z^\rho, x]^{-1}[w, z^\rho x, z]^{-1}$

2. Set $y = z^\rho$ in Theorem 4.8.5(2),

   $[z, y, x]R_{w^\rho}L_w = [w, z, yx][wz, y, x][w, z, y]^{-1}[w, zy, x]^{-1}$

   $\implies [z, z^\rho, x]R_{w^\rho}L_w = [w, z, z^\rho x][wz, z^\rho, x][w, z, z^\rho]^{-1}[w, zz^\rho, x]^{-1}$

   $\implies [z, z^\rho, x]R_{w^\rho}L_w = [w, z, z^\rho x][wz, z^\rho, x][w, z, z^\rho]^{-1}$

3. Set $y = z^\rho$ in Theorem 4.8.5(3),

   $[x, z, y]R_{w^\rho}L_w = [w, x, zy][wx, z, y][w, x, z]^{-1}[w, xz, y]^{-1}$

$\implies [x, z, z^\rho]R_{w^\rho}L_w = [w, x, zz^\rho][wx, z, z^\rho][w, x, z]^{-1}[w, xz, z^\rho]^{-1}$

$\implies [x, z, z^\rho]R_{w^\rho}L_w = [wx, z, z^\rho][w, x, z]^{-1}[w, xz, z^\rho]^{-1}$

□

Theorem 4.8.21. Let $(Q, \cdot)$ be a Basarab loop, and $A(Q)$ its associator subloop. Then for every $x, z, w \in Q$:

1. $(z^\rho, x, z)L_{w^\lambda}R_w = (z^\rho, xz, w)^{-1}(x, z, w)^{-1}(z^\rho, xzw)(z^\rho x, z, w)$

2. $(z, z^\rho, x)L_{w^\lambda}R_w = (z, z^\rho x, w)^{-1}(z^\rho, x, w)^{-1}(z, z^\rho, xw)$

3. $(x, z, z^\rho)L_{w^\lambda}R_w = (z, z^\rho, w)^{-1}(x, z, z^\rho w)(xz, z^\rho, w)$

Proof. Let $(Q, \cdot)$ be a Basarab loop, then every associator of three elements contains in $N(Q)$.

1. Set $y = z^\rho$ in Theorem 4.8.6(1),

   $(y, x, z)L_{w^\lambda}R_w = (y, xz, w)^{-1}(x, z, w)^{-1}(y, xzw)(yx, z, w)$

   $\implies (z^\rho, x, z)L_{w^\lambda}R_w = (z^\rho, xz, w)^{-1}(x, z, w)^{-1}(z^\rho, xzw)(z^\rho x, z, w)$

2. Set $y = z^\rho$ in Theorem 4.8.6(2),

   $(z, y, x)L_{w^\lambda}R_w = (z, yx, w)^{-1}(y, x, w)^{-1}(z, y, xw)(zy, x, w)$

   $\implies (z, z^\rho, x)L_{w^\lambda}R_w = (z, z^\rho x, w)^{-1}(z^\rho, x, w)^{-1}(z, z^\rho, xw)(zz^\rho, x, w)$

   $\implies (z, z^\rho, x)L_{w^\lambda}R_w = (z, z^\rho x, w)^{-1}(z^\rho, x, w)^{-1}(z, z^\rho, xw)$

3. Set $y = z^\rho$ in Theorem 4.8.6(3),

   $(x, z, y)L_{w^\lambda}R_w = (x, zy, w)^{-1}(z, y, w)^{-1}(x, z, yw)(xz, y, w)$

   $\implies (x, z, z^\rho)L_{w^\lambda}R_w = (x, zz^\rho, w)^{-1}(z, z^\rho, w)^{-1}(x, z, z^\rho w)(xz, z^\rho, w)$

   $\implies (x, z, z^\rho)L_{w^\lambda}R_w = (z, z^\rho, w)^{-1}(x, z, z^\rho w)(xz, z^\rho, w)$

□

4.8.4 Relationship between Associators and Inner Mappings of a Basarab loop with a left Inverse component

Theorem 4.8.22. Let $(Q, \cdot)$ be a Basarab loop, and $A(Q)$ its associator subloop. Then for every $y, z, w \in Q$:

1. $[z^\lambda, y, z]R_{w^\rho}L_w = [w, z^\lambda, yz][wz^\lambda, y, z][w, z^\lambda, y]^{-1}[w, z^\lambda y, z]^{-1}$

2. $(z^\lambda, y, z)L_{w^\lambda}R_w = (z^\lambda, yz, w)^{-1}(y, z, w)^{-1}(z^\lambda, y, zw)(z^\lambda y, z, w)$

Proof. Let $(Q, \cdot)$ be a Basarab loop, then every associator of three elements contains in $N(Q)$.

1. Set $x = z^\lambda$ in Theorem 4.8.4(1),

   $[x, y, z]R_{w^\rho}L_w = [w, x, yz][wx, y, z][w, x, y]^{-1}[w, xy, z]^{-1}$

   $\implies [z^\lambda, y, z]R_{w^\rho}L_w = [w, z^\lambda, yz][wz^\lambda, y, z][w, z^\lambda, y]^{-1}[w, z^\lambda y, z]^{-1}$

2. Set $y = z^\rho$ in Theorem 4.8.4(2),

   $(x, y, z)L_{w^\lambda}R_w = (x, yz, w)^{-1}(y, z, w)^{-1}(x, y, zw)(xy, z, w)$

   $\implies (z^\lambda, y, z)L_{w^\lambda}R_w = (z^\lambda, yz, w)^{-1}(y, z, w)^{-1}(z^\lambda, y, zw)(z^\lambda y, z, w)$

□

Theorem 4.8.23. Let $(Q, \cdot)$ be a Basarab loop, and $A(Q)$ its associator subloop. Then for every $x, z, w \in Q$:

1. $[y, z^\lambda, z]R_{w^\rho}L_w = [w, y, z^\lambda z][wy, z^\lambda, z][w, y, z^\lambda]^{-1}[w, yz^\lambda, z]^{-1}$

2. $[z, y, z^\lambda]R_{w^\rho}L_w = [w, z, yz^\lambda][wz, y, z^\lambda][w, z, y]^{-1}[w, zy, z^\lambda]^{-1}$

3. $[z^\lambda, z, y]R_{w^\rho}L_w = [w, z^\lambda, zy][wz^\lambda, z, y][w, z^\lambda, z]^{-1}$

Proof. Let $(Q, \cdot)$ be a Basarab loop, then every associator of three elements contains in $N(Q)$.

1. Set $x = z^\lambda$ in Theorem 4.8.5(1),

   $[y, x, z]R_{w^\rho}L_w = [w, y, xz][wy, x, z][w, y, x]^{-1}[w, yx, z]^{-1}$

   $\implies [y, z^\lambda, z]R_{w^\rho}L_w = [w, y, z^\lambda z][wy, z^\lambda, z][w, y, z^\lambda]^{-1}[w, yz^\lambda, z]^{-1}$

2. Set $x = z^\lambda$ in Theorem 4.8.5(2),

   $[z, y, x]R_{w^\rho}L_w = [w, z, yx][wz, y, x][w, z, y]^{-1}[w, zy, x]^{-1}$

   $\implies [z, y, z^\lambda]R_{w^\rho}L_w = [w, z, yz^\lambda][wz, y, z^\lambda][w, z, y]^{-1}[w, zy, z^\lambda]^{-1}$

3. Set $x = z^\lambda$ in Theorem 4.8.5(3),

   $[z^\lambda, z, y]R_{w^\rho}L_w = [w, z^\lambda, zy][wz^\lambda, z, y][w, z^\lambda, z]^{-1}[w, z^\lambda z, y]^{-1}$

   $\implies [z^\lambda, z, y]R_{w^\rho}L_w = [w, z^\lambda, zy][wz^\lambda, z, y][w, z^\lambda, z]^{-1}[w, z^\lambda z, y]^{-1}$

   $\implies [z^\lambda, z, y]R_{w^\rho}L_w = [w, z^\lambda, zy][wz^\lambda, z, y][w, z^\lambda, z]^{-1}$

□

Theorem 4.8.24. Let $(Q, \cdot)$ be a Basarab loop, and $A(Q)$ its associator subloop. Then for every $y, z, w \in Q$:

1. $(y, z^\lambda, z)L_{w^\lambda}R_w = (z^\lambda, z, w)^{-1}(y, z^\lambda, zw)(yz^\lambda, z, w)$

2. $(z, y, z^\lambda)L_{w^\lambda}R_w = (z, yz^\lambda, w)^{-1}(y, z^\lambda, w)^{-1}(z, y, z^\lambda w)(zy, z^\lambda, w)$

3. $(z^\lambda, z, y)L_{w^\lambda}R_w = (z^\lambda, zy, w)^{-1}(z, y, w)^{-1}(z^\lambda, z, yw)$

Proof. Let $(Q, \cdot)$ be a Basarab loop, then every associator of three elements contains in $N(Q)$.

1. Set $x = z^\lambda$ in Theorem 4.8.6(1),

   $(y, x, z)L_{w^\lambda}R_w = (y, xz, w)^{-1}(x, z, w)^{-1}(y, xzw)(yx, z, w)$

   $\implies (y, z^\lambda, z)L_{w^\lambda}R_w = (y, z^\lambda z, w)^{-1}(z^\lambda, z, w)^{-1}(y, z^\lambda, zw)(yx, z, w)$

   $\implies (y, z^\lambda, z)L_{w^\lambda}R_w = (z^\lambda, z, w)^{-1}(y, z^\lambda, zw)(yx, z, w)$

2. Set $x = z^\lambda$ in Theorem 4.8.6(2),

   $(z, y, x)L_{w^\lambda}R_w = (z, yx, w)^{-1}(y, x, w)^{-1}(z, y, xw)(zy, x, w)$

   $\implies (z, y, z^\lambda)L_{w^\lambda}R_w = (z, yz^\lambda, w)^{-1}(y, z^\lambda, w)^{-1}(z, y, z^\lambda w)(zy, z^\lambda, w)$

3. Set $x = z^\lambda$ in Theorem 4.8.6(3),

$$(x, z, y)L_{w^\lambda}R_w = (x, zy, w)^{-1}(z, y, w)^{-1}(x, z, yw)(xz, y, w)$$

$$\implies (z^\lambda, z, y)L_{w^\lambda}R_w = (z^\lambda, zy, w)^{-1}(z, y, w)^{-1}(z^\lambda, z, yw)(z^\lambda z, y, w)$$

$$\implies (z^\lambda, z, y)L_{w^\lambda}R_w = (z^\lambda, zy, w)^{-1}(z, y, w)^{-1}(z^\lambda, z, yw)$$

□

Theorem 4.8.25. Let $(Q, \cdot)$ be a Basarab loop, and $A(Q)$ its associator subloop. Then for every $x, y, w \in Q$:

1. $[x, y, x^\lambda]R_{w^\rho}L_w = [w, x, yx^\lambda][wx, y, x^\lambda][w, x, y]^{-1}[w, xy, x^\lambda]^{-1}$

2. $(x, y, x^\lambda)L_{w^\lambda}R_w = (x, yx^\lambda, w)^{-1}(y, x^\lambda, w)^{-1}(x, y, x^\lambda w)(xy, x^\lambda, w)$

Proof. Let $(Q, \cdot)$ be a Basarab loop, then every associator of three elements contains in $N(Q)$.

1. Set $z = x^\lambda$ in Theorem 4.8.4(1),

$$[x, y, z]R_{w^\rho}L_w = [w, x, yz][wx, y, z][w, x, y]^{-1}[w, xy, z]^{-1}$$

$$\implies [x, y, x^\lambda]R_{w^\rho}L_w = [w, x, yx^\lambda][wx, y, x^\lambda][w, x, y]^{-1}[w, xy, x^\lambda]^{-1}$$

2. Set $z = x^\lambda$ in Theorem 4.8.4(2),

$$(x, y, z)L_{w^\lambda}R_w = (x, yz, w)^{-1}(y, z, w)^{-1}(x, y, zw)(xy, z, w)$$

$$\implies (x, y, x^\lambda)L_{w^\lambda}R_w = (x, yx^\lambda, w)^{-1}(y, x^\lambda, w)^{-1}(x, y, x^\lambda w)(xy, x^\lambda, w)$$

Theorem 4.8.26. Let $(Q, \cdot)$ be a Basarab loop, and $A(Q)$ its associator subloop. Then for every $x, y, w \in Q$:

1. $[y, x, x^\lambda]R_{w^\rho}L_w = [w, y, xx^\lambda][wy, x, x^\lambda][w, y, x]^{-1}[w, yx, x^\lambda]^{-1}$

2. $[x^\lambda, y, x]R_{w^\rho}L_w = [w, x^\lambda, yx][wx^\lambda, y, x][w, x^\lambda, y]^{-1}[w, x^\lambda y, x]^{-1}$

3. $[x, x^\lambda, y]R_{w^\rho}L_w = [w, x, x^\lambda y][wx, x^\lambda, y][w, x, x^\lambda]^{-1}[w, xx^\lambda, y]^{-1}$

Proof. Let $(Q, \cdot)$ be a Basarab loop, then every associator of three elements contains in $N(Q)$.

1. Set $z = x^\lambda$ in Theorem 4.8.5(1),

   $[y,x,z]R_{w^\rho}L_w = [w,y,xz][wy,x,z][w,y,x]^{-1}[w,yx,z]^{-1}$

   $\implies [y,x,x^\lambda]R_{w^\rho}L_w = [w,y,xx^\lambda][wy,x,x^\lambda][w,y,x]^{-1}[w,yx,x^\lambda]^{-1}$

2. Set $y = z^\rho$ in Theorem 4.8.5(2),

   $[z,y,x]R_{w^\rho}L_w = [w,z,yx][wz,y,x][w,z,y]^{-1}[w,zy,x]^{-1}$

   $\implies [x^\lambda,y,x]R_{w^\rho}L_w = [w,x^\lambda,yx][wx^\lambda,y,x][w,x^\lambda,y]^{-1}[w,x^\lambda y,x]^{-1}$

3. Set $z = x^\lambda$ in Theorem 4.8.5(3),

   $[x,z,y]R_{w^\rho}L_w = [w,x,zy][wx,z,y][w,x,z]^{-1}[w,xz,y]^{-1}$

   $\implies [x,x^\lambda,y]R_{w^\rho}L_w = [w,x,x^\lambda y][wx,x^\lambda,y][w,x,x^\lambda]^{-1}[w,xx^\lambda,y]^{-1}$

□

□

Theorem 4.8.27. Let $(Q,\cdot)$ be a Basarab loop, and $A(Q)$ its associator subloop. Then for every $x,y,w \in Q$:

1. $(y,x,x^\lambda)L_{w^\lambda}R_w = (y,xx^\lambda,w)^{-1}(x,x^\lambda,w)^{-1}(y,x,x^\lambda w)(yx,x^\lambda,w)$

2. $(x^\lambda,y,x)L_{w^\lambda}R_w = (x^\lambda,yx,w)^{-1}(y,x,w)^{-1}(x^\lambda,y,xw)(x^\lambda y,x,w)$

3. $(x,x^\lambda,y)L_{w^\lambda}R_w = (x,x^\lambda y,w)^{-1}(x^\lambda,y,w)^{-1}(x,x^\lambda,yw)(xx^\lambda,y,w)$

Proof. Let $(Q,\cdot)$ be a Basarab loop, then every associator of three elements contains in $N(Q)$.

1. Set $z = x^\lambda$ in Theorem 4.8.6(1),

   $(y,x,z)L_{w^\lambda}R_w = (y,xz,w)^{-1}(x,z,w)^{-1}(y,xzw)(yx,z,w)$

   $\implies (y,x,x^\lambda)L_{w^\lambda}R_w = (y,xx^\lambda,w)^{-1}(x,x^\lambda,w)^{-1}(y,x,x^\lambda w)(yx,x^\lambda,w)$

2. Set $z = x^\lambda$ in Theorem 4.8.6(2),

   $(z,y,x)L_{w^\lambda}R_w = (z,yx,w)^{-1}(y,x,w)^{-1}(z,y,xw)(zy,x,w)$

   $\implies (x^\lambda,y,x)L_{w^\lambda}R_w = (x^\lambda,yx,w)^{-1}(y,x,w)^{-1}(x^\lambda,y,xw)(x^\lambda y,x,w)$

3. Set $z = x^\lambda$ in Theorem 4.8.6(3),

$(x, z, y)L_{w^\lambda}R_w = (x, zy, w)^{-1}(z, y, w)^{-1}(x, z, yw)(xz, y, w)$

$\implies (x, x^\lambda, y)L_{w^\lambda}R_w = (x, x^\lambda y, w)^{-1}(x^\lambda, y, w)^{-1}(x, x^\lambda, yw)(xx^\lambda, y, w)$

□

Theorem 4.8.28. Let $(Q, \cdot)$ be a Basarab loop, and $A(Q)$ its associator subloop. Then for every $x, z, w \in Q$:

1. $[x, z^\lambda, z]R_{w^\rho}L_w = [w, x, z^\lambda z][wx, z^\lambda, z][w, x, z^\lambda]^{-1}[w, xz^\lambda, z]^{-1}$

2. $(x, z^\lambda, z)L_{w^\lambda}R_w = (x, z^\lambda z, w)^{-1}(z^\lambda, z, w)^{-1}(x, z^\lambda, zw)(xz^\lambda, z, w)$

Proof. Let $(Q, \cdot)$ be a Basarab loop, then every associator of three elements contains in $N(Q)$.

1. Set $y = z^\lambda$ in Theorem 4.8.4(1),

$[x, y, z]R_{w^\rho}L_w = [w, x, yz][wx, y, z][w, x, y]^{-1}[w, xy, z]^{-1}$

$\implies [x, z^\lambda, z]R_{w^\rho}L_w = [w, x, z^\lambda z][wx, z^\lambda, z][w, x, z^\lambda]^{-1}[w, xz^\lambda, z]^{-1}$

2. Set $y = z^\lambda$ in Theorem 4.8.4(2),

$(x, y, z)L_{w^\lambda}R_w = (x, yz, w)^{-1}(y, z, w)^{-1}(x, y, zw)(xy, z, w)$

$\implies (x, z^\lambda, z)L_{w^\lambda}R_w = (x, z^\lambda z, w)^{-1}(z^\lambda, z, w)^{-1}(x, z^\lambda, zw)(xz^\lambda, z, w)$

□

Theorem 4.8.29. Let $(Q, \cdot)$ be a Basarab loop, and $A(Q)$ its associator subloop. Then for every $x, z, w \in Q$:

1. $[z^\lambda, x, z]R_{w^\rho}L_w = [w, z^\lambda, xz][wz^\lambda, x, z][w, z^\lambda, x]^{-1}[w, z^\lambda x, z]^{-1}$

2. $[z, z^\lambda, x]R_{w^\rho}L_w = [w, z, z^\lambda x][wz, z^\lambda, x][w, z, z^\lambda]^{-1}[w, zz^\lambda, x]^{-1}$

3. $[x, z, z^\lambda]R_{w^\rho}L_w = [w, x, zz^\lambda][wx, z, z^\lambda][w, x, z]^{-1}[w, xz, z^\lambda]^{-1}$

Proof. Let $(Q, \cdot)$ be a Basarab loop, then every associator of three elements contains in $N(Q)$.

1. Set $y = z^\lambda$ in Theorem 4.8.5(1),

   $[y, x, z]R_{w^\rho}L_w = [w, y, xz][wy, x, z][w, y, x]^{-1}[w, yx, z]^{-1}$

   $\Longrightarrow [z^\lambda, x, z]R_{w^\rho}L_w = [w, z^\lambda, xz][wz^\lambda, x, z][w, z^\lambda, x]^{-1}[w, z^\lambda x, z]^{-1}$

2. Set $y = z^\lambda$ in Theorem 4.8.5(2),

   $[z, y, x]R_{w^\rho}L_w = [w, z, yx][wz, y, x][w, z, y]^{-1}[w, zy, x]^{-1}$

   $\Longrightarrow [z, z^\lambda, x]R_{w^\rho}L_w = [w, z, z^\lambda x][wz, z^\lambda, x][w, z, z^\lambda]^{-1}[w, zz^\lambda, x]^{-1}$

3. Set $y = z^\lambda$ in Theorem 4.8.5(3),

   $[x, z, y]R_{w^\rho}L_w = [w, x, zy][wx, z, y][w, x, z]^{-1}[w, xz, y]^{-1}$

   $\Longrightarrow [x, z, z^\lambda]R_{w^\rho}L_w = [w, x, zz^\lambda][wx, z, z^\lambda][w, x, z]^{-1}[w, xz, z^\lambda]^{-1}$

□

Theorem 4.8.30. Let $(Q, \cdot)$ be a Basarab loop, and $A(Q)$ its associator subloop. Then for every $x, z, w \in Q$:

1. $(z^\lambda, x, z)L_{w^\lambda}R_w = (z^\lambda, xz, w)^{-1}(x, z, w)^{-1}(z^\lambda, x, zw)(z^\lambda x, z, w)$

2. $(z, z^\lambda, x)L_{w^\lambda}R_w = (z, z^\lambda x, w)^{-1}(z^\lambda, x, w)^{-1}(z, z^\lambda, xw)(zz^\lambda, x, w)$

3. $(x, z, z^\lambda)L_{w^\lambda}R_w = (x, zz^\lambda, w)^{-1}(z, z^\lambda, w)^{-1}(x, z, z^\lambda w)(xz, z^\lambda, w)$

Proof. Let $(Q, \cdot)$ be a Basarab loop, then every associator of three elements contains in $N(Q)$.

1. Set $y = z^\lambda$ in Theorem 4.8.6(1),

   $(y, x, z)L_{w^\lambda}R_w = (y, xz, w)^{-1}(x, z, w)^{-1}(y, x, zw)(yx, z, w)$

   $\Longrightarrow (z^\lambda, x, z)L_{w^\lambda}R_w = (z^\lambda, xz, w)^{-1}(x, z, w)^{-1}(z^\lambda, x, zw)(z^\lambda x, z, w)$

2. Set $y = z^\lambda$ in Theorem 4.8.6(2),

   $(z, y, x)L_{w^\lambda}R_w = (z, yx, w)^{-1}(y, x, w)^{-1}(z, y, xw)(zy, x, w)$

   $\Longrightarrow (z, z^\lambda, x)L_{w^\lambda}R_w = (z, z^\lambda x, w)^{-1}(z^\lambda, x, w)^{-1}(z, z^\lambda, xw)(zz^\lambda, x, w)$

3. Set $y = z^\lambda$ in Theorem 4.8.6(3),

$$(x, z, y)L_{w^\lambda}R_w = (x, zy, w)^{-1}(z, y, w)^{-1}(x, z, yw)(xz, y, w)$$

$$\implies (x, z, z^\lambda)L_{w^\lambda}R_w = (x, zz^\lambda, w)^{-1}(z, z^\lambda, w)^{-1}(x, z, z^\lambda w)(xz, z^\lambda, w)$$

□

CHAPTER FIVE

SUMMARY, CONCLUSION AND RECOMMENDATIONS

5.1 SUMMARY

This study examined Basarab loop which was introduced by Basarab (1992), and some new properties have been determined. These new properties together with the existing ones give more pertinent tools for further exploration of Basarab loops. The results on this study began on constructions of Basarab loops, and two abelian groups were considered. A mapping f which takes a cross product of one of the abelian groups into the other was considered with a condition. A multiplication $\circ$ satisfying some laws with respect to the elements of these abelian groups and the mapping f was defined, on the cross product of these abelian groups with an identity $(0, 0)$. This algebraic structure with multiplication $\circ$ is shown to be a loop. In particular, the necessary and sufficient conditions for such an algebraic structure with multiplication $\circ$ to be a left (right) Basarab loop, and Basarab loop were established.

Basarab loop was considered as a type of an Osborn loop, and it is proved that Basarab loop and CC-loop are Osborn loops. It is also proved that any Osborn loop is a Basarab loop if and only if it is a left (right) Basarab loop; and a Basarab loop is flexible if and only if it is an extra loop. Also, an extra loop is proved to be both a Buchsteiner loop and a Basarab loop. Necessary and sufficient conditions for a Basarab loop to be a CC-loop, an RCC-loop, and an LCC-loop have been established. It is shown that the center and centrum of a Basarab loop coincide, and both are contained in the Nucleus of the Basarab loop. Necessary and sufficient conditions for a Basarab loop to be power alternative have been established. The following result have been obtained: the left (right) inner mapping of a Basarab loop is a nuclear automorphism; a Basarab loop with the RIP or LIP is an extra loop; a Basarab loop with the IP (AAIP) is an extra loop; a left (right) Basarab loop with the RIP (LIP) is an extra loop; and the middle inner mapping generates the inner mapping group of a Basarab loop. It is also proved that, the left and right inner mappings of a Basarab loop are nuclear.

The study investigated the isotopes of a Basarab loop. It is shown that every right isotope of a right Basarab loop is a right conjugacy closed loop and every left isotope of a left Basarab loop is a left conjugacy closed loop. The left (right) isotope of a right (left) Basarab loop is shown to be a right (left) Basarab loop. It is established that every principal isotope of a left Basarab loop is a left Basarab loop and every principal isotope of a right Basarab loop is a right Basarab loop. Hence, every principal isotope of a Basarab loop is a Basarab loop. Necessary and sufficient conditions for isotopes and principal isotopes of a Basarab loop have been determined.

Investigations were carried out on the holomorphy of a Basarab loop by considering a group $A(Q)$ of automorphisms of a loop. Some necessary and sufficient conditions for an $A(Q)$-holomorph of a loop $(Q,\cdot)$ to be left (right) Basarab loop, and Basarab loop were established, respectively. These necessary and sufficient conditions are also expressed in terms of autotopisms of a loop. Some left (right) translation mapping of the holomorph of a left (right) Basarab loop is shown to be left (right) regular. It is proved that $A(Q)$ -holomorph of a loop $(Q,\cdot)$ which satisfies the inverse property is a Basarab loop if and only if $(Q,\cdot)$ is a Basarab loop and every automorphism of Q is nuclear.

Properties of some functions defined on a Basarab loop have been studied. Some subloops of a Basarab loop which are characterized by permutations were obtained, by application of these functions. The middle inner mapping is expressed in terms of a left (right) translation mapping and the middle translation mapping. Using some functions, necessary and sufficient conditions for a loop to be a left (right) Basarab loop, and Basarab loop were fine-tuned in the cases of a left (right) CC-loop, and CC-loop, respectively. Necessary and sufficient condition for a Basarab loop to be associative is given in terms of the middle inner mapping. Also, necessary and sufficient condition for the left and right inner mappings of a Basarab loop to coincide is established. With consideration to these functions defined on a Basarab loop, it is proved that a Basarab loop $(Q, \cdot)$ is a cross inverse property loop if and only if Q is commutative or an abelian group.

It is proved that a Basarab loop $(Q, \cdot)$ is a centrum abelian inner mappings loop. In Basarab loop $(Q, \cdot)$, the following are shown to be true: the center $Z(Q, \cdot)$ of a Basarab loop is normal; the quotient $Q/Z(Q, \cdot)$ is an abelian group; the centrum $C(Q, \cdot)$ is normal; and the quotient $Q/C(Q, \cdot)$ is an abelian group.

Finally, some associators in the nucleus and center of a Basarab loop were examined. Relationship between associators and inner mappings of a Basarab loop is defined. Some special cases like, associator with a right inverse component, and associator with a left inverse component were considered. It is shown that the associator of any three elements of a Basarab loop is contained in the center and centrum of a Basarab loop. Some expressions for an associator of a loop have been obtained for when: one component of the associator is a product of the loop and its nucleus; one component of the associator is a product of the loop and its nucleus which is a normal subloop of the loop; one component of the associator is a product of the loop and its center; two components of the associator are products of the loop and its nucleus; two components of the associator are products of the loop and its nucleus which is a normal subloop of the loop; two components of the associator are products of the loop and its center;

three components of the associator are products of the loop and its nucleus; three components of the associator are products of the loop and its nucleus which is a normal subloop of the loop; and three components of the associator are products of the loop and its center.

5.2 CONCLUSION

This investigation has determined some algebraic properties of Basarab loops using some loop notions and some basic features of Basarab loops. The multiplication group and total multiplication group of a loop also generated vital results about the properties of Basarab loops. The study was limited to the centrum, isotopes, holomorphs and associators of a Basarab loop, construction of a Basarab loop, the relationship between a Basarab loop and centrum abelian inner mappings loop, and to obtain some subloops of a Basarab loop that are characterized by permutations and the relationship among them.

The study has determined the relationship between Basarab loop and Osborn loop. It was proved that an Obsorn loop is a Basarab loop if and only if it is both left and right Basarab loop. Investigation was carried out on the isotopes of a Basarab loop and it was proved that every principal isotope of a Basarab loop is a Basarab loop. It was shown that the centrum of a Basarab loop is a subloop and it is equal to the center of a Basarab loop. An investigation was carried out on the holomorphs of Basarab loops, and some necessary and sufficient conditions for the holomorphs of a loop to be a Basarab loop were determined. Also, some subloops of a basarab loop were obtained and characterized. These subloops were obtained by some permutations of the loop defined using the middle translation mapping. The algebraic properties of associators of a Basarab loop were examined and it was found that the associator of any three elements of a Basarab loop is contained in the center and centrum of a Basarab loop. It was also proved that a Basarab loop is a centrum abelian inner mapping loop.

All these properties before now were not considered in the literature for Basarab loops. With this study, sufficient theoretical investigations have been carried out on the aforementioned properties. Thus, some important properties of Basarab loops are now made available to both loop theorist and cryptographers.

5.3 CONTRIBUTIONS TO KNOWLEDGE

(i) The study has determined the relationship between Basarab loop and other loops like Osborn loop, CC-loop, and AIM- loop.

(ii) It has been proved that the centrum of a Basarab loop is a subloop and it is equal to the center of a Basarab loop and that an Obsorn loop is a Basarab loop if and only if it is both left and right Basarab loop. Every principal isotope of a Basarab loop has been proved to be a Basarab loop.

(iii) Some necessary and sufficient conditions for the holomorphs of a loop to be a Basarab loop have been determined. Some subloops of a Basarab loop have been obtained and characterized.

(iv) The algebraic properties of associators of a Basarab loop have been examined and it has been shown that the associator of any three elements of a Basarab loop is contained in the center and centrum of a Basarab loop. The study has also proved that a Basarab loop is a centrum-abelian inner mapping loop.

5.4 RECOMMENDATIONS

This study has presented additional properties of Basarab loops which are for the first time considered in the literature. These properties have not been applied by other researchers nor cryptographers since these properties were yet to be developed and made available for possible applications. Therefore, it is recommended that researchers and cryptographers should use the properties of Basarab loops determined by this study, for further research and application. It is known that every Basarab loop is an Osborn loop, and it could be justified whether or not every holomorph of a left (right) Basarab loop is a holomorph of an Osborn loop. Similarly, in order to prevent an authorized user of a certain information under transmission, the cryptographer could think of encrypting such information by isotopes of a Basarab loop. More studies are yet to be carried out on the weak inverse property, automorphic inverse property, anti-automorphic inverse property, and semi-automorphic inverse property of Basarab loops. Thus, researchers are encouraged to undertake these investigations.

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