

**DETERMINATION OF DYNAMIC BUCKLING LOAD OF A  
CLAMPED FINITE COLUMN RESTING ON QUADRATIC-CUBIC  
ELASTIC FOUNDATION**

**BY**

**BASSEY JULIUS EFFIONG, B.Sc-MOUAU, M.Sc - FUTO  
(20204256598)**

**THESIS SUBMITTED TO  
DEPARTMENT OF MATHEMATICS, FEDERAL UNIVERSITY OF  
TECHNOLOGY, OWERRI**

**FOR PARTIAL FULFILMENT OF THE REQUIREMENT FOR  
AWARD OF DEGREE OF DOCTOR OF PHILOSOPHY (PhD) IN  
INDUSTRIAL AND APPLIED MATHEMATICS**

**DECEMBER, 2024**

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STATE**

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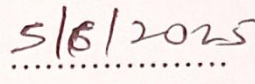
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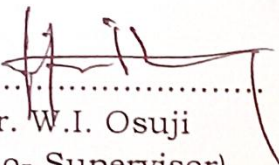
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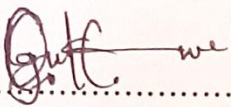
  
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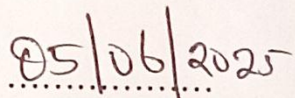
  
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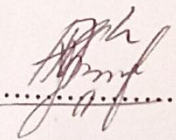
  
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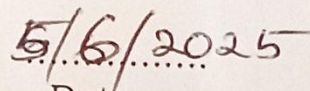
  
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## **DEDICATION**

This Thesis is dedicated to Almighty God, the Author of knowledge.

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Several scholarly materials and resources were consulted in the course of carrying out this research work, and I would like to extend my sincere gratitude to all the authors whose work has been cited.

I am deeply indebted to my supervisor, Dr. (Mrs) J.U. Chukwuchekwa, who provided expert advice, demonstrated a genuine interest in my work, offered constructive criticism feedback, and demonstrated outstanding academic excellence. In the same light, my gratitude also goes to my assistant supervisors, Prof. E.N. Erumaka and Dr. W. I. Osuji for their tremendous assistance.

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## TABLE OF CONTENTS

Title	i
Certification	ii
Dedication	iii
Acknowledgements	iv-v
Table of Contents	vi-vii
Abstract	viii
<b>CHAPTER ONE: INTRODUCTION</b>	
1.1 Background Information	1-2
1.2 Problem Statement	3
1.3 Aim and Objectives of the Study	3
1.4 Significance of the Study	3
1.5 Scope of the Study	4
<b>CHAPTER TWO: LITERATURE REVIEW</b>	
2.1 Conceptual Literature	5-7
2.2 Specific Literature	7-8
<b>CHAPTER THREE: METHODOLOGY</b>	
3.1 Background Theorem	9
3.2 Formulation of the Problem	10-11
3.3 Non-Dimensionalization of the Problem	11-12
3.4 Analytic Method by Perturbation Procedure	12-15
3.5 Numerical Approach by Block Unification Method	15-16
3.6 Derivation of the Block Unification Method	17-21
<b>CHAPTER FOUR: APPLICATION, ANALYSIS AND RESULTS</b>	
4.1 Application by Perturbation	22-65
4.2 Values of Independent Variables at Maximum Displacement	65-76
4.3 Maximum Displacement	76-89
4.4 The Dynamic Buckling Load of the Column	86-91

4.5 Results from Analytic Procedures	91-92
4.6 Computational Procedure by Block Unification Method (BUM)	93-94
4.7 Numerical Results	94-95
4.8 Discussion of Results	96
4.9 Analysis of the Result	96

## **CHAPTER FIVE: SUMMARY, CONCLUSION AND RECOMMENDATION**

5.1 Summary	97
5.2 Conclusion	97
5.3 Recommendation	97
5.4 Contribution to Knowledge	97-98
REFERENCES	99-104

## **ABSTRACT**

This research work centred on the determination of the dynamic buckling load of a clamped finite column structure that is lying on a quadratic-cubic foundation by the analytic and numerical procedures. The formulation of the governing equation contains two small but mathematically independent parameters ( $\delta$  and  $\epsilon$ ) which are used for asymptotic expansion of the variables. For the solutions, two techniques were applied to investigate namely, two-timing regular perturbation procedure and Block Unification Method (BUM). BUM was adopted to solve the governing equation through the method of lines and finite difference methods. The results obtained indicate that the dynamic buckling load decreases with increased imperfection, and decreases with increased in damping. More so, it was shown that the results obtained are strictly asymptotic and valid in the limit as the small parameters become increasingly small relative to unity.

Keywords: Dynamic, Buckling, Load, Block unification method, Stability

# CHAPTER ONE

## INTRODUCTION

### 1.1 BACKGROUND INFORMATION

Buckling is associated with failure of column-like structures in buildings, bridges, shells and plates which are subjected to loading. Buckling of column is a phenomenon that occurs when a slender structural element such as a column or a beam, fails under compressive loads. It is characterized by sudden, lateral deflection or bending of column due to loss of stability. Buckling can also be seen as a sudden change in shape (deformation) of a structural component under load, such as the bowing of a column under compressive or the wrinkling of a plate under shear.

Globally, the collapse of buildings, bridges and other material structures are issues of concern. Structural failures are forms of material failures which are dangerous and should be prevented by all means. Several researches and studies have been carried out by Engineers and Applied Mathematicians to determine the maximum load structures can carry prior to buckling. The dynamic buckling load of a viscously damped but clamped elastic structure that is trapped by a step load is a typical engineering real-life problem and the governing equation or equation of displacement is the mathematical generalization of some of the physical phenomena encountered in engineering practice.

Structural elastic materials normally display certain tendencies of failure and instability when loaded either statically or dynamically, and one of the responsibilities of the Structural Engineer and Applied Mathematician is to help in the determination of the load which a given elastic material can support prior to buckling. A vast quantum of insights on dynamic stability of elastic structure has been achieved by subjecting elastic materials to diverse dynamic loading conditions. From findings, it has become firmly established that initial imperfections and to some extent, the loading duration, are some of the main factors that have been seriously implicated as causative agents of reduction of the elastic strength of materials.

This research work is aimed at determining the dynamic buckling load of an imperfect elastic structure that is viscously damped but clamped column trapped by a step load using analytic and numerical procedures. The investigation is anchored within the framework of the already established research work proposed by Ette & Osuji (2019).

Most of the studies in the dynamic buckling of structures, the equations of motion (displacement) which are often nonlinear are usually solved by numerical procedure or experimental approach. This research work is distinguished from most previous investigations by the fact that its equations are solved by analytical method by the use of a two-timing perturbation technique and asymptotic expansions of the variables, and numerical approach using Block Unification Method (BUM) by converting the governing equation into system of ordinary ODEs through method of line.

The characteristic problems to be solved are strictly nonlinear and no simple method exists. The methods used in this work are made possible by the presence of two small but mathematically independent parameters upon which the two-timing perturbation and iteration schemes are formulated.

The available literature has shown that buckling analysis can be investigated through analytical, numerical and experimental methods. However, many such investigations are described by equations in which the solutions cannot be obtained easily, in other words, exact solutions to such problems are impossible because of the high level of nonlinearity that are inherent in such formulations. Consequently, there is a great need to seek for analytic methods to address such uncommon problems.

The dynamic buckling load  $\lambda_D$  can be defined as the maximum load parameter for which the displacement or solution of the governing equation remains bounded for all time.

According to Ette & Udo-Akpan (2016), the condition of obtaining dynamic buckling load is the maximization of the equation;

$$\frac{d\lambda}{dU_a} = 0 \tag{1.1}$$

where  $\lambda$  is the load parameter and  $U_a$  is the maximum value of displacement.

## **1.2 PROBLEM STATEMENT**

Despite significant advancement in dynamic buckling and stability analysis, structural failure remains a cause for concern. This study addresses the critical issue of determining the dynamic buckling load of a viscously damped but clamped finite column lying on a quadratic-cubic nonlinear elastic foundation using analytical method (asymptotic and perturbation techniques) and numerical method (Block Unification Method). The effects of light viscous damping as well as imperfection on the dynamic stability of a finite column structure will be considered.

## **1.3 OBJECTIVES OF THE STUDY**

The main objective is a determination of the dynamic buckling load of a clamped finite column resting on quadratic-cubic elastic foundation using Perturbation technique and Block Unification Method (BUM) for analytic and numerical solutions respectively.

In this work, the specific objectives include to:

- a. Adopt the relevant governing equations (displacement equation).
- b. Normalize the governing equations and solve the equations using perturbation and asymptotic technique for exact solutions, and Block Unification Method for numerical solutions.
- c. Determine the maximum displacement which the solutions remain bounded even after dynamic buckling in the case of perturbation method.
- d. Develop a mathematical model that will relate the dynamic buckling load to the imperfection parameter in the case of analytic approach and results simulated to draw general conclusion.

## **1.4 JUSTIFICATION OF THE STUDY**

The use of both analytic and numerical methods for the investigation and analysis of dynamic buckling load of a clamped finite column lying on the quadratic-cubic elastic foundation offers a balanced approach that combines theoretical insights, practical applicability and methodological robustness. This comprehensive approach enhanced the reliability of results, broadens the scope of possible applications and innovation in structural engineering.

## **1.5 SCOPE OF THE STUDY**

Investigation in this study is limited to a viscously damped imperfect finite column resting on quadratic-cubic elastic foundation with clamped boundary conditions,

hence other types of boundary conditions such as the simply supported boundary condition and mixed boundary condition are not discussed. The study is restricted to dynamic buckling; hence the equivalent static buckling load is not to be considered. Also, the analysis assumes only homogeneous boundary conditions as opposed to the inhomogeneous boundary conditions. In the case of numerical method, the convergence properties are not discussed.

## CHAPTER TWO

### LITERATURE REVIEW

#### 2.1 CONCEPTUAL LITERATURE

Eglitis (2011) studied the dynamic response of columns under impulsive axial compression. The results have shown that initial geometrical imperfections, duration of impulse and effective slenderness have a major influence on the buckling loads, whereas the effect of the material is secondary.

Recent investigations on dynamic buckling have been directed principally on columns, beams, plates, spherical and cylindrical shells, and, extensive literature (most often numerical approach), have since come to limelight. In this regard, mention must be made of Kowal-Michalska (2010), studied some important parameters in dynamic buckling analysis of plated structures subjected to pulse loading. The author established that the strain-rate is dependence of material properties buckling loads in dynamic buckling analysis. Ferri, Antinucci, He, Zok & Evans (2006) equally investigated the buckling of impulsively loaded prismatic cores. The simulations revealed that the stresses induced differ on the front and back faces, and duration of stress pulse remains the same. In the same vein, Kubiak (2005) studied the dynamic buckling of thin-walled composite plates with varying width-wise material properties, while Kubiak (2007) also investigated interactive dynamic buckling of thin-walled columns. Mania (2010), studied the dynamic buckling of thin-walled viscoplastic columns, while Kowal-Michalska & Mania (2008) similarly investigated some aspects of dynamic buckling of plates under in-plane pulse compression. Zaczynska & Mania (2021) investigated the dynamic buckling phenomenon in thin-walled fiber metal laminate short columns subjected to axial compression loading. The work revealed that the value of the amplitude of initial geometric imperfection and the loading shape of pulse load influence the value of critical dynamic load factor strongly. Aghdam & Schroeder (2021), analyzed the structural behaviour of Crash Box under an impact load, and a new criterion called sector points criterion was introduced in order to detect the dynamic buckling point.

Vo, Lee & Lee (2010) investigated triply coupled vibrations of axially loaded thin-walled composite beams. The authors analyzed the effects of axial force, fiber orientation and modulus ratio on the natural frequencies, load-frequency interaction curves and corresponding vibration mode shapes. Soltani & Soltani (2022) investigated the discrepancies between the endurable transverse buckling load of multi-layer fibrous composite and fiber-metal laminate I-section beams. The results

show that the transverse buckling load of the selected I-beam is significantly affected by some parameters. An investigation on computational nonlinear stochastic dynamics was undertaken by Capiet-Leinout, Soize & Mignolet (2003). Similarly, Chitra & Priyardarsini (2013) as well as Priyardarsini, Kalyangraman & Srinvasam (2012), and Mcshane, Pingle, Deshpande & Fleck (2012) made excellent contributions to the dynamics of dynamic buckling. The dynamic effect of lateral buckling of high temperature/high-pressure offshore pipeline was carried out by Reda & Forbes (2012). In the same token, Gladden, Hamdy, Belmonte & Villermaux (2005) investigated the dynamic buckling and fragmentation in brittle rods, while a study on the vibration of nonlocal Kelvin-Voigt viscoelastic damped Timoshenko beams was undertaken by Lei et al. (2013). The study by Salehi and Safi-Djahanshahi (2010) on non-linear analysis of viscoelastic rectangular plates subjected to in-plane compression was insightful. Capiet-Leinout, Soize and Mignolet (2013) discussed nonlinear stochastic dynamical post-buckling analysis of uncertain cylindrical shells.

Batra & Geng (2001), studied numerically the transient three-dimensional elastic deformations of a plate with piezoceramic elements perfectly bonded to its top and bottom surfaces and analyzed the effect of the shape and size of the piezoceramic actuators on the increasing the buckling load of the plate. Kolakowski and Teller (2016) investigated the static and dynamic coupled buckling and post-buckling behaviour of thin-walled structures through applications of the Finite Element Method (FEM) and analytical-numerical method to solve interaction buckling problems. Bisagni (2004) investigated the elastic dynamic buckling of carbon fiber reinforced plastic cylindrical shells subjected to pulse axial compression. Wu (2017) presented basic equation of motion of a beam and plate in an oscillating magnetic field and touches on the dynamic instability behaviour of beam and plate system which is the Isotropic material and composites made of piezoelectric materials. Kuzkin (2015) derived analytic expression for critical buckling force as a function of compression velocity and established that the time required for buckling is inversely proportional to the cubic root of the compression velocity and logarithmically depends on the initial disturbance. Yanze & Jiawei (2023), investigated the dynamic buckling of thin-walled cylindrical shells under radial impact pressures randomly distributed in the circumferential direction. The numerical results show that nonlinear terms from Green's strain tensors and the change of curvature are important for shell large deformation. It was observed that the pressure characteristics, materials and thickness of the cylindrical shell affect its buckling behaviour remarkably.

Kubiak & Kowal-Michalska (2012) estimated the dynamic buckling load relatively to static buckling load for imperfect plates and established that dynamic local factor is a ratio of pulse load amplitude and the static buckling load for imperfect plates. Sinir, Ozhan and Reddy (2014), presented exact solutions of buckling configurations and vibration response of post-buckling configurations of beams with non-classical boundary conditions using the Euler-Bernoulli theory. Soltani & Soltani (2018), investigated the stability and buckling of a load of columns with variable flexural rigidity, different boundary conditions and subjected to variable axial loads using the Finite Difference Method (FDM). The critical buckling loads were finally determined by solving the eigenvalue problems of the obtained algebraic system resulting from FDM expansion. Kalathur et al (2013), investigated theoretically and experimentally the buckling of compressed flat-end columns loaded by unattached flat platens.

Onuoha (2023), worked on analytical analysis of the dynamic buckling load of a geometrical imperfect column lying on a nonlinear elastic foundation trapped by a time dependent load. The result of the investigation shows that increase in geometric imperfection decreases the dynamic buckling load and that damping alters the vibration amplitude of the dynamic buckling load. The author postulated that increase in the mode reduces the dynamic buckling load. Jurczynska & Szmidla (2015), studied the influence of the Winkler elastic foundation of a geometrically nonlinear column loaded by force directed towards the positive pole on a type of system. The authors determine the criterion that allows for classification of an analyzed structure to divergent or divergent pseudo-flustering type of system. On the basis of obtained formulas, the ranges of parameters describing the Winker elastic foundation for which the considered system were determined.

Kounadis, Mallis & Sbarounis (2006), employed an approximate analytic technique on the post buckling columns resting on elastic foundation leading to very reliable results in the vicinity of the critical state. It was found that the critical state of perfect columns is a stable symmetric bifurcation point and consequently there is no sensitivity to initial geometrical imperfections. A simple but readily analyzed mechanical model is proposed to simulate the salient features of buckling mechanism of the column on elastic foundation with the obtained models. Zhang & Ma (2020), worked on chaotic analysis of a buckled curved beam resting on nonlinear elastic foundations. The authors adopted Melnikov's method to find the necessary conditions for chaotic behaviours of the beam, and obtained the maximum Lyapunov exponent which is greater than zero and notice that the system is sensitive to small differences in the initial values. The chaotic behaviours can occur and numerical analysis verifies the theoretical and analytical predictions.

## 2.2 SPECIFIC LITERATURE

The following investigations are specific related to column-like structures. Kalathur, Hoang, Lakes & Drugan (2013) studied buckling of compressed flat-end columns loaded by unattached flat platens. The Authors noted that the theoretical critical load for secondary or end tilt buckling for a column is slightly higher than the critical load for primary buckling. Artem & Aydin (2010), investigated on exact solution and dynamic buckling analysis of beam-column system having the elliptic type loading. The solution to the governing is obtained in the form of Fourier Sine Series, and the effects of the static load and the frequency ratio on the critical buckling load were investigated. Aristitabal-Ochoa (2007), studied static and dynamic stability of uniform shear beam-columns under generalized boundary conditions; Pavlovic, Kosic, Rajkovic & Pavlovic (2007), investigated the dynamic stability of a thin-walled beam subjected to axial loads and moments. The uniform stochastic stability regions were shown in intensity of stochastic loadings and constant parts of axial loads and end moments.

Ette, Chukwuchekwa, Osuji, Udo-Akpan & Ozoigbo (2018), investigated static buckling of infinitely column lying on quadratic-cubic elastic foundations. Static buckling load of some harmonically imperfect infinitely long columns lying on a quadratic-cubic nonlinear elastic foundation was determined using two slightly different perturbation schemes. The static buckling load and displacement are asymptotically obtained and the mathematical relationship between the static buckling and imperfection amplitude was deduced. Ette, Chukwuchekwa & Udo-Akpan (2016), analyzed the dynamic stability of an imperfect viscously damped but clamped finite column that rest on elastic nonlinear (cubic) foundation subjected to a step load. Multi-timing perturbation procedure was used to obtain results which are strictly asymptotic. The result show that the column buckles at higher buckling loads than if it was subjected to simply-supported end conditions. It was observed that in general, damping increases the dynamic buckling load, and dynamic load is higher than the corresponding static buckling load if damping is present, otherwise without damping, the static buckling load is always higher than the dynamic load for a step loading consideration. Ette & Osuji (2019), investigated the dynamic buckling of a viscously damped modified quadratic model elastic structure struck by an impulse. An asymptotic expansion was applied on the perturbed system of nonlinear coupled nonhomogeneous ordinary differential equations, and impulse loading was imposed on the viscously damped system with light damping coefficient.

The result revealed an increasing dynamic buckling load with increase in damping coefficient.

Ette (2008, investigated the initial post dynamic buckling of a quadratic-cubic column pressurized by a sinusoidal slowly varying dynamic load. The imperfection is assumed in the shape of the  $m$ th term in the Fourier sine series expansion with small Fourier coefficients. The results obtained are strictly asymptotic and valid for small absolute values of the perturbation parameters. Possidente, Freddi & Tondini (2024), studied the effects of steel structures subjected to progressive collapse when buckling of column is relevant. A numerical procedure is introduced to evaluate the dynamic increase factors considering two different Engineering Demand Parameters (EDPs), suited for describing beam and column type mechanisms. The authors derived dynamic increase factors and failure mechanisms.

# CHAPTER THREE

## METHODOLOGY

### 3.1 BACKGROUND THEOREM

The methods to be adopted in this work are based on a two-timing perturbation and asymptotic analysis, and Block Unification Method. The nature of the problems to be solved naturally calls for the adoption of these methods. The problems encountered in the work are nonlinear and so, their exact analytical solutions cannot be obtained in a closed-form.

In the case of Perturbation Method, the formulations contain two small but mathematical independent parameters upon which asymptotic series expansions are initiated. To obtain a uniformly valid asymptotic solution in each case, we used a two-timing regular perturbation technique in which, all expansions are asymptotic and are valid in the limit as the two small parameters become increasingly small relative to unity.

Here, the aim is to first obtain the displacement and thereafter, obtain the maximum displacement  $U_a$ .

Thereafter, the dynamic buckling load can be obtained by using the maximization

$$\frac{d\lambda}{dU_a} = 0 \tag{3.1.1}$$

Equation (3.1.1) is lastly evaluated to obtain the dynamic buckling load,  $\lambda_D$ .

When an elastic structure is suddenly struck by a load that is maintained in a constant magnitude, it begins a back-and-forth pre-buckling motion (deflection). Subsequently, the pre-buckling motion may excite a buckling motion. The presence of imperfections in a structure may further the excitation of the buckling motion. Enough of the energy in the pre-buckling motion may be transmitted to the buckling motion to result in the collapse of the structure.

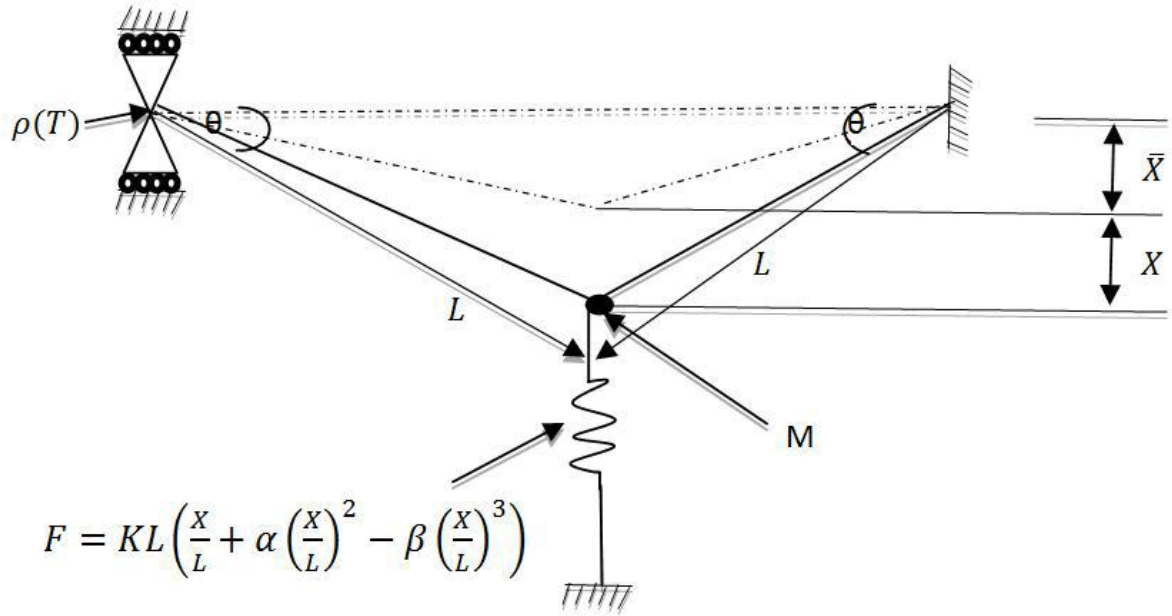
### 3.2 FORMULATION OF THE PROBLEM

The elastic quadratic-cubic model structure under investigation was recently studied by Ette & Osuji (2019) and has continued to remain a source of reference because it is a mathematical generalization of some of the physical structures encountered in real-life engineering practice.

The Fig 3.1 below is a two-arm elastic quadratic-cubic model structure, each arm of length  $L$ , and whose meeting point carries a mass  $M$ , which is attached to a nonlinear spring with restoring force per unit length given by:

$$kL \left( \frac{X}{L} + \alpha \left( \frac{X}{L} \right)^2 - \beta \left( \frac{X}{L} \right)^3 \right) \quad (3.1.2)$$

Where  $k$ ,  $L$ ,  $\alpha$  and  $\beta$  are positive constants, and  $\rho(T)$  is a longitudinal force while  $T$  is the time and  $\bar{X}$  is the initial imperfection with  $X$  as additional displacement from equilibrium position.



**Figure 3.1 A simple Quadratic-cubic Elastic Model Structure.**

The usual dimensional differential equation satisfied by the deflection  $W(X,T)$  of the column under consideration, as in Amazigo and Frank (1973) and Amazigo and Ette (1987) is:

$$m_0 W_{,TT} + c_0 W_{,T} + EI W_{,XXXX} + 2\rho(T) W_{,XX} + W k_1 - W^2 k_2 - W^3 k_3 = -2\rho(T) \frac{d^2 \bar{W}}{dX^2}, T > \quad (3.2.1)$$

$$0 < X < \pi \quad (3.2.2)$$

$$W(X, 0) = 0 = W_{,T}(X, 0), 0 < X < \pi \quad (3.3)$$

$$W = W_{,X} = 0 \text{ at } X = 0, \pi \quad (3.4)$$

Where  $m_0$  is the mass per unit length,  $c_0$  is the damping coefficient,  $EI$  is the bending stiffness, where  $E$  and  $I$  are the Young's modulus and the moment of inertia respectively. Here the nonlinear elastic foundation exerts a force per unit length given by  $Wk_1 - W^2k_2 - W^3k_3$  on the column, where  $k_1$ ,  $k_2$  and  $k_3$  are constants such that  $k_1 > 0, k_2 > 0, k_3 > 0$ . In this formulation, all nonlinearities higher than cubic are excluded, while all nonlinear derivatives of  $W(X, T)$  are also excluded. Here,  $\bar{W}$  is the stress-free twice-differentiable imperfection displacement and all aspects of axial inertia are neglected.

### 3.3 NON-DIMENSIONALIZATION OF THE PROBLEM

To reduce equation (3.2) -(3.4) to non-dimensional form, we adopt the following quantities:

$$x = \left(\frac{k_1}{EI}\right)^{\frac{1}{4}}X, \quad \omega = \left(\frac{k_2}{k_1}\right)^{\frac{1}{2}}W, \quad \lambda f(t) = \frac{P(T)}{2(EIk_1)^{\frac{1}{2}}}, \quad t = \left(\frac{k_1}{m_0}\right)^{\frac{1}{2}}T, \quad \epsilon \bar{\omega} = \left(\frac{k_3}{k_1}\right)^{\frac{1}{2}}\bar{W},$$

$$2\delta = \frac{c_0}{(m_0k_1)^{\frac{1}{2}}}, \quad \alpha = \frac{k_2}{\sqrt{k_1k_2}}, \quad \beta = \left(\frac{k_3}{k_1}\right)^{\frac{3}{2}} \quad (3.5.1)$$

Here, we shall assume the following inequalities

$$0 < \delta \ll 1, \quad 0 < \epsilon \ll 1. \quad (3.5.2)$$

On substituting (3.5a) in (3.2a) and simplifying, the following is obtained:

$$\omega_{,tt} + 2\delta\omega_{,t} + \omega_{,xxxx} + 2\lambda f(t)\omega_{,xx} + \omega - \alpha\omega^2 - \beta\omega^3 = -2\epsilon\lambda f(t)\frac{d^2\bar{\omega}}{dx^2}, \quad t > 0 \quad (3.6)$$

$$0 < X < \pi \quad (3.7.1)$$

$$\omega(x, 0) = 0 = \omega_{,t}(x, 0) = 0, \quad 0 < x < \pi \quad (3.7.2)$$

$$\omega = \omega_{,x} = 0 \text{ at } x = 0, \pi \quad (3.7.3)$$

Where  $\omega$  is the displacement,  $t$  is the time variable,  $\delta$  is the damping coefficient,  $\alpha$  and  $\beta$  are the imperfection – sensitivity parameters,  $\epsilon$  is the amplitude of the imperfection,  $\bar{\omega}$  is a stress-free twice-differentiable imperfection and  $f(t)$  is a time dependent loading function while  $\lambda$  is the non-dimensional amplitude (or magnitude) of the loading.

Here, a subscript following a comma indicates partial differentiation and  $f(t)$  is a step load such that;

$$f(t) = \begin{cases} 1, & t > 0 \\ 0, & t < 0 \end{cases} \quad (3.8)$$

It is assumed  $\delta$  and  $\epsilon$  are two small but mathematically unrelated parameters that satisfy the inequalities as in (3.5.2). The ultimate aim is to determine the dynamic buckling load  $\lambda_D$  which is obtained by using the maximization (3.1) as in Ette & Osuji (2019).

### 3.4 ANALYTIC METHOD BY PERTURBATION PROCEDURE

From equation (3.6), let:

$$\tau = \delta t \quad (3.9.1)$$

$$\hat{t} = t + \frac{1}{\delta} [\omega_1(\tau)\epsilon + \omega_2(\tau)\epsilon^2 + \omega_3(\tau)\epsilon^3 + \omega_4(\tau)\epsilon^4 + \dots] \quad (3.9.2)$$

Where,

$$\omega_i(0) = 0, \quad i = 1, 2, 3, \dots, \quad (3.10.1)$$

Let

$$\omega(\mathbf{x}, t) = U(\mathbf{x}, \hat{t}, \tau, \epsilon, \delta) \quad (3.10.2)$$

From equation (3.10b) we have,

$$\omega_{,t} = \left( \frac{\partial U}{\partial \hat{t}} \cdot \frac{\partial \hat{t}}{\partial t} \right) + \left( \frac{\partial U}{\partial \hat{t}} \cdot \frac{\partial \hat{t}}{\partial \tau} \cdot \frac{d\tau}{dt} \right) + \left( \frac{\partial U}{\partial \tau} \cdot \frac{d\tau}{dt} \right) \quad (3.11)$$

i.e

$$\omega_{,t} = U_{,\hat{t}} + (\omega'_1\epsilon + \omega'_2\epsilon^2 + \omega'_3\epsilon^3 + \dots)U_{,\tau} + \delta U_{,\tau} \quad (3.12)$$

The following also follows:

$$\begin{aligned} \omega_{,tt} = & U_{,\hat{t}\hat{t}} + (\omega'_1\epsilon + \omega'_2\epsilon^2 + \omega'_3\epsilon^3 + \dots)^2 U_{,\hat{t}\hat{t}} + \delta^2 U_{,\tau\tau} + 2(\omega'_1\epsilon + \omega'_2\epsilon^2 + \omega'_3\epsilon^3 + \\ & \dots)U_{,\hat{t}\hat{t}} + 2(\delta U_{,\hat{t}\tau} + 2\delta(\omega'_1\epsilon + \omega'_2\epsilon^2 + \omega'_3\epsilon^3 + \dots)U_{,\hat{t}\tau} + \delta(\omega'_1\epsilon + \omega'_2\epsilon^2 + \omega'_3\epsilon^3 + \dots)U_{,\hat{t}} \end{aligned} \quad (3.13)$$

Substituting (3.12) and (3.13) into equation (3.6) results to;

$$\begin{aligned} & U_{,\hat{t}\hat{t}} + (\omega'_1\epsilon + \omega'_2\epsilon^2 + \omega'_3\epsilon^3 + \dots)^2 U_{,\hat{t}\hat{t}} + \delta^2 U_{,\tau\tau} + 2(\omega'_1\epsilon + \omega'_2\epsilon^2 + \omega'_3\epsilon^3 + \dots)U_{,\hat{t}\hat{t}} \\ & + 2\delta U_{,\hat{t}\tau} + 2\delta(\omega'_1\epsilon + \omega'_2\epsilon^2 + \omega'_3\epsilon^3 + \dots)U_{,\hat{t}\tau} \\ & + \delta(\omega'_1\epsilon + \omega'_2\epsilon^2 + \omega'_3\epsilon^3 + \dots)U_{,\hat{t}} \\ & + 2\delta[U_{,\hat{t}} + (\omega'_1\epsilon + \omega'_2\epsilon^2 + \omega'_3\epsilon^3 + \dots)U_{,\hat{t}} + \delta U_{,\tau}] + U_{,xxxx} + 2\lambda U_{,xx} + U \\ & + \alpha U^2 - \beta U^3 \end{aligned}$$

$$= -2\lambda\epsilon\frac{d^2\bar{\omega}}{dx^2} \quad (3.14)$$

Let

$$U(x, \epsilon, \tau) = \sum_{i=1}^{\infty} \sum_{j=0}^{\infty} U_n^{(i,j)}(x, t, \tau) \epsilon^i \delta^j \quad (3.15)$$

$$= \epsilon(U^{(10)} + \delta U^{(11)} + \delta^2 U^{(12)} + \dots) + \epsilon^2(U^{(20)} + \delta U^{(21)} + \delta^2 U^{(22)} + \dots) + \epsilon^3(U^{(30)} + \delta U^{(31)} + \delta^2 U^{(32)} + \dots) + \dots \quad (3.16)$$

Here, the  $ij$  in  $U^{(ij)}$  are not powers but superscripts. Therefore, the following orders of equations are obtained

$$O(\epsilon) : U_{,\hat{t}\hat{t}}^{(10)} + U_{,xxxx}^{(10)} + 2\lambda U_{,xx}^{(10)} + U^{(10)} = -2\lambda\frac{d^2\bar{\omega}}{dx^2}, \quad (3.17)$$

$$O(\epsilon\delta) : U_{,\hat{t}\hat{t}}^{(11)} + U_{,xxxx}^{(11)} + 2\lambda U_{,xx}^{(11)} + U^{(11)} = -2U_{,\hat{t}\tau}^{(10)} - 2U_{,\hat{t}}^{(10)} \quad (3.18)$$

$$O(\epsilon\delta^2) : U_{,\hat{t}\hat{t}}^{(12)} + U_{,xxxx}^{(12)} + 2\lambda U_{,xx}^{(12)} + U^{(12)} = -2U_{,\hat{t}\tau}^{(11)} - 2U_{,\hat{t}}^{(11)} - U_{,\tau\tau}^{(10)} \quad (3.19)$$

$$O(\epsilon^2) : U_{,\hat{t}\hat{t}}^{(20)} + U_{,xxxx}^{(20)} + 2\lambda U_{,xx}^{(20)} + U^{(20)} = -\alpha(U^{(10)})^2 - 2\omega_1' U_{,\hat{t}\hat{t}}^{(10)} \quad (3.20)$$

$$O(\epsilon^2\delta) : U_{,\hat{t}\hat{t}}^{(21)} + U_{,xxxx}^{(21)} + 2\lambda U_{,xx}^{(21)} + U^{(21)} = -2\alpha U^{(10)} U^{(11)} - 2U_{,\hat{t}\tau}^{(20)} - 2U_{,\hat{t}}^{(20)} - 2\omega_1' U_{,\hat{t}\hat{t}}^{(11)} - \omega_1' U_{,\hat{t}}^{(10)} - 2\omega_1' U_{,\hat{t}}^{(10)} \quad (3.21)$$

$$O(\epsilon^2\delta^2) : U_{,\hat{t}\hat{t}}^{(22)} + U_{,xxxx}^{(22)} + 2\lambda U_{,xx}^{(22)} + U^{(22)} = -U_{,\tau\tau}^{(20)} - 2\omega_1' U_{,\hat{t}\hat{t}}^{(12)} - 2U_{,\hat{t}\tau}^{(21)} - 2\omega_1' U_{,\hat{t}\hat{t}}^{(12)} - 2\omega_1' U_{,\hat{t}}^{(11)} - 2U_{,\hat{t}}^{(21)} - 2\omega_1' U_{,\hat{t}}^{(11)} - \alpha\{(U^{(11)})^2 + U^{(10)}U^{(12)}\} \quad (3.22)$$

$$O(\epsilon^3) : U_{,\hat{t}\hat{t}}^{(30)} + U_{,xxxx}^{(30)} + 2\lambda U_{,xx}^{(30)} + U^{(30)} = -(\omega_1')^2 U_{,\hat{t}\hat{t}}^{(10)} - 2(\omega_1' U_{,\hat{t}\hat{t}}^{(20)} + \omega_2' U_{,\hat{t}\hat{t}}^{(10)}) - 2\alpha U^{(20)} U^{(12)} + \beta(U^{(10)})^3 \quad (3.23)$$

$$O(\epsilon^3\delta) : U_{,\hat{t}\hat{t}}^{(31)} + U_{,xxxx}^{(31)} + 2\lambda U_{,xx}^{(31)} + U^{(31)} = -(\omega_1')^2 U_{,\hat{t}\hat{t}}^{(10)} - 2(\omega_1' U_{,\hat{t}\tau}^{(21)} + \omega_2' U_{,\hat{t}\tau}^{(11)}) - 2U_{,\hat{t}\tau}^{(30)} + 2(\omega_1' U_{,\hat{t}\hat{t}}^{(20)} + \omega_2' U_{,\hat{t}\hat{t}}^{(10)}) - (\omega_1' U_{,\hat{t}}^{(20)} + \omega_2' U_{,\hat{t}}^{(10)}) - 2\{\omega_1' U_{,\hat{t}}^{(30)} + (\omega_1' U_{,\hat{t}}^{(20)} + \omega_2' U_{,\hat{t}}^{(10)})\} - \alpha(U^{(10)}U^{(21)} + U^{(11)}U^{(20)}) + 3\beta(U^{(10)})^2(U^{(11)}) \quad (3.24)$$

$$O(\epsilon^3\delta^2) : U_{,\hat{t}\hat{t}}^{(32)} + U_{,xxxx}^{(32)} + 2\lambda U_{,xx}^{(32)} + U^{(32)} = -(\omega_1')^2 U_{,\hat{t}\hat{t}}^{(12)} - U_{,\tau\tau}^{(30)} - 2(\omega_1' U_{,\hat{t}\hat{t}}^{(22)} + \omega_2' U_{,\hat{t}\hat{t}}^{(12)}) - 2U_{,\hat{t}\tau}^{(31)} - 2(\omega_1' U_{,\hat{t}\tau}^{(21)} + \omega_2' U_{,\hat{t}\tau}^{(11)}) - (\omega_1'' U_{,\hat{t}}^{(21)} + \omega_2'' U_{,\hat{t}}^{(11)}) - 2(U_{,\hat{t}}^{(31)} + \omega_1' U_{,\hat{t}}^{(21)} + \omega_2' U_{,\hat{t}}^{(11)})$$

$$\omega_2' U_{,\hat{t}}^{(11)} - 2U_{,\tau}^{(30)} - 2\alpha(U^{(10)}U^{(32)} + U^{(11)}U^{(21)} + U^{(12)}U^{(20)}) + \beta[(U^{(10)})^2U^{(12)} + 3U^{(10)}(U^{(10)})^2] \quad (3.25)$$

The associated initial conditions are as follows:

$$O(\epsilon): U^{(ij)}(x, 0, 0) = 0; i = 1, 2, 3 \dots, j = 1, 2, 3 \dots \quad (3.26)$$

$$O(\epsilon\delta): U_{,\hat{t}}^{(11)}(x, 0, 0) + U_{,\tau}^{(10)}(x, 0, 0) = 0 \quad (3.27)$$

$$O(\epsilon\delta^2): U_{,\hat{t}}^{(12)}(x, 0, 0) + U_{,\tau}^{(11)}(x, 0, 0) = 0 \quad (3.28)$$

$$O(\epsilon^2): U_{,\hat{t}}^{(20)}(x, 0, 0) + \omega_1'(0)U_{,\hat{t}}^{(10)}(x, 0, 0) = 0 \quad (3.29)$$

$$O(\epsilon^2\delta): U_{,\hat{t}}^{(21)}(x, 0, 0) + \omega_1'(0)U_{,\hat{t}}^{(11)}(x, 0, 0) + U_{,\tau}^{(20)}(x, 0, 0) = 0 \quad (3.30)$$

$$O(\epsilon^2\delta^2): U_{,\hat{t}}^{(22)}(x, 0, 0) + \omega_1'(0)U_{,\hat{t}}^{(12)}(x, 0, 0) + U_{,\tau}^{(21)}(x, 0, 0) = 0 \quad (3.31)$$

$$O(\epsilon^3): U_{,\hat{t}}^{(30)}(x, 0, 0) + \omega_1'(0)U_{,\hat{t}}^{(20)}(x, 0, 0) + \omega_2'(0)U_{,\tau}^{(10)}(x, 0, 0) = 0 \quad (3.32)$$

$$O(\epsilon^3\delta): U_{,\hat{t}}^{(31)}(x, 0, 0) + \omega_1'(0)U_{,\hat{t}}^{(21)}(x, 0, 0) + \omega_2'(0)U_{,\hat{t}}^{(11)}(x, 0, 0) + U_{,\tau}^{(30)}(x, 0, 0) = 0 \quad (3.33)$$

$$O(\epsilon^3\delta^2): U_{,\hat{t}}^{(32)}(x, 0, 0) + \omega_1'(0)U_{,\hat{t}}^{(22)}(x, 0, 0) + \omega_2'(0)U_{,\hat{t}}^{(12)}(x, 0, 0) + U_{,\tau}^{(31)}(x, 0, 0) = 0 \quad (3.34)$$

The associated boundary conditions are:

$$U^{(ij)} = U_{,x}^{(ij)} = 0; x = 0, \pi \quad (3.35)$$

### 3.5 NUMERICAL APPROACH BY BLOCK UNIFICATION METHOD

The equation (3.6) can be resolved by the use of Block Unification Method.

From equation (3.6) –(3.7), we have

$$\omega_{,tt} + 2\delta\omega_{,t} + \omega_{,xxxx} + 2\lambda f(t)\omega_{,xx} + \omega - \alpha\omega^2 - \beta\omega^3 = -2\epsilon\lambda f(t)\frac{d^2\bar{\omega}}{dx^2}, t > 0, 0 < x < \pi$$

With the initial and boundary conditions,

$$\omega(x, 0) = 0 = \omega_t(x, 0) = 0, \quad 0 < x < \pi$$

$$\omega = \omega_t = 0 \quad \text{at } x = 0, \pi$$

A Continuous Linear Multistep Method (CLMM) is derived and used to formulate Block Unification Method, which is applied to solve equation (3.6) with appropriate initial and boundary conditions by first changing the PDEs into system of fourth order ODEs through the method of lines. The changing is done by replacing one of the variables with a finite difference method.

Let consider equation (3.6) with associated initials and boundary conditions,

$$\omega(x, t) = \emptyset(x), \quad \omega_t(x, 0) = \varphi(x)$$

$$\omega(\eta_0, t) = g_0(t), \quad \omega(\eta_1, t) = g_1(t)$$

$$\omega_{xx}(\eta_0, t) = p_0(t) = \omega_{xx}(\eta_1, t) = p_1(t)$$

$$\omega(x, 0) = \varphi(x), \quad \omega_t(x, 0) = \varphi(x)$$

$$\omega(\eta_0, t) = g_0(t), \quad \omega(\eta_1, t) = g_1(t)$$

$$\omega_x(\eta_0, t) = p_0(t), \quad \omega_x(\eta_1, t) = p_1(t) \quad (3.36)$$

Where  $\omega(x, t)$  is the dependent variable,  $x$  and  $t$  are variables such that  $\eta_0 \leq x \leq \eta_1$ ,  $\eta_0$  and  $\eta_1$  are finite real numbers,  $t \geq 0$ ,  $\emptyset(x)$ ,  $\varphi(x)$ ,  $g_i(t)$ ,  $p_i(t)$ ,  $i = 0, 1$  are continuous functions.

Applying BUM to equation (3.6), the problem is first converted into system of ODEs through method of lines and Central Difference Methods.

Thus, for real numbers,  $L_1$ ,  $L_2$ ,  $L_3$  and  $L_4$ , and solution  $\omega(x, t)$  of equation (3.6), where  $(x, t)$  is in the rectangle  $[L_1, L_2] \times [L_3, L_4]$ . The  $t$  variable is discretized with mesh sparing.

$$\Delta t = \frac{L_4 - L_3}{M}, \quad t_m = L_3 + m\Delta t, m = 0, 1, \dots, M, \quad (3.37)$$

And noting that;

$$\Delta x = \frac{L_2 - L_1}{N}, \quad x_n = L_1 + n\Delta x, \quad n = 0, 1, \dots, N \quad (3.38)$$

With the vector

$$\omega = [\omega_{1,1}, \omega_{1,2}, \dots, \omega_{n-i, m-1}]^T \quad (3.40)$$

And

$$f = [f_{1,1}, f_{1,2}, \dots, f_{n-i,m-1}]^T \quad (3.41)$$

Where  $\omega_m \cong \omega(x, t_m)$  and  $f_m \cong f(x, t_m)$ ; using the central difference method yields;

$$\omega(x, t_m) \cong \frac{\omega(x, t_{m+1}) - 2\omega(x, t_m) + \omega(x, t_{m-1}))}{(\Delta t)^2} \quad (3.42)$$

In semi-discretized form,

$$\frac{d\omega_m^4}{dx^4} = - \left( \frac{\omega_{m+1} - 2\omega_m + \omega_{m-1}}{(\Delta t)^2} \right) + f_m \quad (3.43)$$

Which can be written in the form;

$$\omega^{(iv)} = f(x, \omega, \omega', \omega'', \omega''') = A\omega + g \quad (3.44)$$

Subject to the appropriate initial and boundary conditions,

Where,  $\omega^j = (\omega_{1,1}^{(j)}, \omega_{1,2}^{(j)}, \omega_{2,1}^{(j)}, \dots, \omega_{n-1,m-1}^{(j)})^T$ ,  $j = 1, 2, 3$ ,  $A$  is an  $(M-1) \times (M-1)$  matrix arising from the semi-discretized system (3.43), and solved by the BUM.

A Continuous Linear Multistep Method (CLMM) is derived through a block technique, which is used to formulate Continuous Block Finite Difference Methods (CBFDMs).

Thus, using multistep interpolation and collocation, a continuous FDM is derived and additional  $k-1$  methods which are assembled and solved simultaneously to obtain approximation  $\omega_{n,m}$  for  $n = 1, \dots, N-1$ , which is applied to solve forth order PDEs using method of line.

### 3.6 DERIVATION OF THE BLOCK UNIFICATION METHOD

According to Modebei, Adeniyi & Jator (2020), the exact solution  $U_{n,m}$  of equation (3.6) can be approximated by seeking the continuous solution of the form:

$$p(x) = \sum_{i=0}^{k=r+s-1} \rho_i S_i^*(x) \cong u(x) \quad (3.45)$$

With the 4<sup>th</sup> derivative given as

$$p^{(iv)}(x) = \sum_{i=4}^{r+s-1} \rho_i S_i^{*(iv)}(x) \cong u^{(iv)}(x) \quad (3.46)$$

Where  $x \in [a, b]$ ,  $\rho_i$ 's are constants to be determined.  $S_i^*(x)$  in the interval  $[x_n, x_{n+k}]$ ,  $i = 0(1)r + s + 1$ , are shifted Chebyshev polynomials in the interval  $[0, k]$ . The parameters  $r$  and  $s$  are respectively the number of interpolation points that satisfies

$4 \leq r \leq \mu$  and the number of collocation points satisfying  $0 < s \leq \mu + 1$ ,  $\mu$  is the order of the differential equation.

Interpolating equation (3.45) at the points  $x_{n+i}$ ;  $i = 0, 1, 2, \dots, r-1$  and collocating equation (3.46) at the points  $x_{n+s}$ ;  $s = 0, 1, 2, \dots, s-1$ .

The method has the following specifications  $k = 5, r = 4, s = 6, S_n^*(x_{n+j})$  for  $x_{n+j} \in [x_n, x_{n+5}]$  yield the system noting that

$$U_{n+i,m} \cong U(x_{n+1}, t_m)$$

And,

$$U_{n+i,m}^{(iv)} \cong f_{n+i,m} = f(x_{n+i}, t_m, U_{n+i,m}, \dots, U_{n+i,m}''')$$

The interpolation of equation (3.45) at the points  $x_{n+j}$ ,  $j = 0(1)(3)$  and collocation of equation (3.46) at the points  $x_{n+j}$ ,  $j = 0(1)(5)$  yields a system which after solving, the values of the coefficients  $\rho_i$ ,  $i = 0, 1, \dots, 9$  are obtained.

These values are substituted into equation (3.45) and after some simplification; the approximate polynomial in equation (3.46) adopts the continuous form:

$$p(x) = \sum_{i=0}^3 \alpha_i(x) u_{n+i,m} + h^4 \sum_{i=0}^5 \beta_i(x) f_{n+i,m} \quad (3.47)$$

Where the  $\alpha_i$ 's and  $\beta_i$ 's are continuous coefficients expressed as functions of  $\xi$  and given as:

$$\alpha_0 = \frac{1}{48} (3 + 5\xi - 75\xi^2 - 125\xi^3)$$

$$\alpha_1 = \frac{5}{16} (-1 - \xi + 25\xi^2 + 25\xi^3)$$

$$\alpha_2 = -\frac{5}{16} (-3 + 7\xi + 35\xi^2 + 25\xi^3)$$

$$\alpha_3 = \frac{5}{48} (3 + 23\xi + 45\xi^2 + 25\xi^3)$$

$$\beta_0 = \frac{1}{185794560} (36729 + 28779\xi - 1049700\xi^2 - 695500\xi^3 + 3543750\xi^4)$$

$$\beta_1 = \frac{1}{37158912} (-430479 - 676299\xi + 10981980\xi^2 + 16840300\xi^3 - 5906250\xi^4 + 1968750\xi^5 + 10237500\xi^6 - 7312500\xi^7 - 2109375\xi^8 + 1953125\xi^9)$$

$$\beta_2 = \frac{1}{18579456} (-533547 - 1478277\xi + 12637020\xi^2 + 37650500\xi^3 + 17718750\xi^4 - 17718750\xi^5 - 4462500\xi^6 + 9562500\xi^7 + 703125\xi^8 - 1953125\xi^9)$$

$$\begin{aligned}
\beta_3 &= \frac{1}{18579456} (30933 - 129147\xi - 1474980\xi^2 + 2535100\xi^3 + 17718750\xi^4 \\
&\quad + 17718750\xi^5 - 4462500\xi^6 - 9562500\xi^7 + 703125\xi^8 - 1953125\xi^9) \\
\beta_3 &= \frac{1}{18579456} (30933 - 129147\xi - 1474980\xi^2 + 2535100\xi^3 + 17718750\xi^4 \\
&\quad + 17718750\xi^5 - 4462500\xi^6 - 9562500\xi^7 + 703125\xi^8 - 1953125\xi^9) \\
\beta_4 &= \frac{1}{37158912} (-27279 + 15051\xi + 901080\xi^2 - 309100\xi^3 - 5906250\xi^4 \\
&\quad - 1968750\xi^5 + 10237500\xi^6 + 7312500\xi^7 - 2109375\xi^8 - 1953125\xi^9) \\
\beta_5 &= \frac{1}{185794560} (20601 - 1899\xi - 646500\xi^2 + 23500\xi^3 + 3543750\xi^4 + \\
&\quad 708750\xi^5 - 6562500\xi^6 - 2812500\xi^7 + 3515625\xi^8 + 1953125\xi^9)
\end{aligned} \tag{3.48}$$

Where,

$$\xi = \frac{1}{h} \left( \frac{2x}{5} - 1 \right) \tag{3.49}$$

Evaluating  $p(x)$  in equation (3.47) at the points  $x = x_{n+4,m}$ ,  $x_{n+5,m}$ , which implies that  $\xi = \frac{3}{5}, 1$ , the following 5- step discrete LMMs are obtained:

$$\begin{aligned}
u_{n+4,m} &= 4u_{n+3,m} - 6u_{n+2,m} + 4u_{n+1,m} - u_{n,m} + h^4 \left( \frac{-1}{720} f_{n,m} + \frac{31}{180} f_{n+1,m} + \frac{79}{120} f_{n+2,m} + \right. \\
&\quad \left. \frac{31}{180} f_{n+3,m} - \frac{1}{720} f_{n+4,m} \right)
\end{aligned} \tag{3.50}$$

$$\begin{aligned}
u_{n+5,m} &= -4u_{n,m} + 15u_{n+1,m} - 20u_{n+2,m} + 10u_{n+3,m} + h^4 \left( \frac{-1}{180} f_{n,m} + \frac{11}{16} f_{n+1,m} + \right. \\
&\quad \left. \frac{101}{36} f_{n+2,m} + \frac{97}{72} f_{n+3,m} + \frac{1}{6} f_{n+4,m} - \frac{1}{720} f_{n+5,m} \right)
\end{aligned} \tag{3.51}$$

From the transformation  $x = \frac{(5h+5)}{2}, \frac{2}{5} dx = h d\xi$ , equation (3.49) is differentiated three times respectively and on each differentiation,  $p'(x)$ ,  $p''(x)$  and  $p'''(x)$  in equation (3.48) are evaluated at the point  $x = x_{n+i,m}$ ,  $i = 0(1)5$ , which implies  $\xi = -1, -\frac{3}{5}, -\frac{1}{5}, \frac{1}{5}, \frac{3}{5}, 1$ , the following 5-step discrete LMMs are obtained:

$$\begin{aligned}
hu'_{n,m} &= -\frac{11u_{n,m}}{6} + 3u_{n+1,m} - \frac{3u_{n+2,m}}{2} + \frac{u_{n+3,m}}{3} - h^4 \left( -\frac{937}{100800} f_{n,m} - \frac{19}{105} f_{n+1,m} - \right. \\
&\quad \left. \frac{599}{10080} f_{n+2,m} - \frac{1}{720} f_{n+3,m} + \frac{3}{2240} f_{n+4,m} - \frac{1}{3600} f_{n+5,m} \right)
\end{aligned}$$

$$\begin{aligned}
hu'_{n+1,m} &= -u_{n,m} - \frac{1}{2}u_{n+1,m} + u_{n+2,m} - \frac{1}{6}u_{n+3,m} + h^4 \left( -\frac{1}{5400} f_{n,m} + \frac{2809}{60480} f_{n+1,m} + \right. \\
&\quad \left. \frac{43}{1008} f_{n+2,m} - \frac{229}{302400} f_{n+3,m} + \frac{1}{432} f_{n+4,m} - \frac{11}{33600} f_{n+5,m} \right)
\end{aligned}$$

$$\begin{aligned}
hu'_{n+2,m} &= \frac{1}{6}u_{n,m} - u_{n+1,m} + \frac{1}{2}u_{n+2,m} + \frac{1}{3}u_{n+3,m} + h^4 \left( \frac{169}{302400}f_{n,m} - \frac{311}{10080}f_{n+1,m} - \right. \\
&\left. \frac{353}{6048}f_{n+2,m} + \frac{1}{135}f_{n+3,m} - \frac{7}{2880}f_{n+4,m} + \frac{53}{151200}f_{n+5,m} \right) \\
hu'_{n+3,m} &= -\frac{1}{3}u_{n,m} + \frac{3}{2}u_{n+1,m} - 3u_{n+2,m} + \frac{11}{6}u_{n+3,m} + h^4 \left( -\frac{41}{50400}f_{n,m} + \frac{173}{2880}f_{n+1,m} + \right. \\
&\left. \frac{11}{60}f_{n+2,m} + \frac{61}{10080}f_{n+3,m} + \frac{17}{1080}f_{n+4,m} - \frac{11}{33600}f_{n+5,m} \right) \\
hu'_{n+4,m} &= -\frac{11}{6}u_{n,m} + 7u_{n+1,m} - \frac{19}{2}u_{n+2,m} + \frac{13}{3}u_{n+3,m} + h^4 \left( -\frac{671}{302400}f_{n,m} + \right. \\
&\left. \frac{169}{540}f_{n+1,m} + \frac{12821}{10080}f_{n+2,m} + \frac{7447}{15120}f_{n+3,m} + \frac{509}{60480}f_{n+4,m} - \frac{1}{3600}f_{n+5,m} \right) \\
hu'_{n+5,m} &= -\frac{13}{3}u_{n,m} + \frac{31}{2}u_{n+1,m} - 19u_{n+2,m} + \frac{47}{6}u_{n+3,m} + h^4 \left( -\frac{31}{5400}f_{n,m} + \right. \\
&\left. \frac{1663}{2240}f_{n+1,m} + \frac{6847}{2160}f_{n+2,m} + \frac{60863}{30240}f_{n+3,m} + \frac{2473}{5040}f_{n+4,m} + \frac{2041}{30240}f_{n+5,m} \right) \tag{3.52}
\end{aligned}$$

$$\begin{aligned}
h^2u''_{n,m} &= 2u_{n,m} - 5u_{n+1,m} + 4u_{n+2,m} - u_{n+3,m} \\
&+ h^2 \left( \frac{1411}{20160}f_{n,m} + \frac{3091}{4320}f_{n+1,m} + \frac{2831}{30240}f_{n+2,m} + \frac{143}{2520}f_{n+3,m} - \frac{1391}{60480}f_{n+4,m} \right. \\
&\left. + \frac{23}{6048}f_{n+5,m} \right)
\end{aligned}$$

$$\begin{aligned}
h^2u''_{n+1,m} &= u_{n,m} - 2u_{n+1,m} + u_{n+2,m} + h^4 \left( -\frac{73}{30240}f_{n,m} - \frac{1601}{20160}f_{n+1,m} + \frac{1}{7560}f_{n+2,m} - \right. \\
&\left. \frac{11}{4320}f_{n+3,m} + \frac{11}{10080}f_{n+4,m} - \frac{11}{60480}f_{n+5,m} \right)
\end{aligned}$$

$$\begin{aligned}
h^2u''_{n+2,m} &= u_{n+1,m} - 2u_{n+2,m} + u_{n+3,m} + h^4 \left( \frac{11}{60480}f_{n,m} - \frac{53}{15120}f_{n+1,m} - \frac{773}{10080}f_{n+2,m} - \right. \\
&\left. \frac{53}{15120}f_{n+3,m} + \frac{11}{604800}f_{n+4,m} \right)
\end{aligned}$$

$$\begin{aligned}
h^2u''_{n+3,m} &= -u_{n,m} + 4u_{n+1,m} - 5u_{n+2,m} + 2u_{n+3,m} + h^4 \left( -\frac{1}{720}f_{n,m} + \frac{10427}{60480}f_{n+1,m} + \right. \\
&\left. \frac{9901}{15120}f_{n+2,m} + \frac{107}{1120}f_{n+3,m} - \frac{37}{75600}f_{n+4,m} + \frac{11}{60480}f_{n+5,m} \right)
\end{aligned}$$

$$\begin{aligned}
h^2u''_{n+4,m} &= -2u_{n,m} + 7u_{n+1,m} - 8u_{n+2,m} + 3u_{n+3,m} + h^4 \left( -\frac{179}{60480}f_{n,m} + \right. \\
&\left. \frac{3469}{10080}f_{n+1,m} + \frac{6421}{4320}f_{n+2,m} + \frac{3791}{3780}f_{n+3,m} + \frac{121}{1344}f_{n+4,m} - \frac{23}{6048}f_{n+5,m} \right)
\end{aligned}$$

$$\begin{aligned}
h^2u''_{n+5,m} &= -3u_{n,m} + 10u_{n+1,m} - 11u_{n+2,m} + 4u_{n+3,m} + h^4 \left( -\frac{11}{30240}f_{n,m} + \right. \\
&\left. \frac{29689}{604800}f_{n+1,m} + \frac{499}{120}f_{n+2,m} + \frac{58271}{30240}f_{n+3,m} + \frac{4561}{4320}f_{n+4,m} + \frac{271}{4032}f_{n+5,m} \right) \tag{3.53}
\end{aligned}$$

$$\begin{aligned}
h^3u'''_{n,m} &= -u_{n,m} + 3u_{n+1,m} - 3u_{n+2,m} + u_{n+3,m} - h^3 \left( -\frac{19151}{60480}f_{n,m} - \frac{73967}{60480}f_{n+1,m} + \right. \\
&\left. \frac{1261}{6048}f_{n+2,m} - \frac{7439}{30240}f_{n+3,m} + \frac{5549}{60480}f_{n+4,m} - \frac{883}{60480}f_{n+5,m} \right)
\end{aligned}$$

$$\begin{aligned}
h^3 u'''_{n+1,m} &= -u_{n,m} + 3u_{n+1,m} - 3u_{n+2,m} + u_{n+3,m} + h^4 \left( \frac{799}{60480} f_{n,m} - \frac{14033}{60480} f_{n+1,m} - \right. \\
&\left. \frac{10453}{30240} f_{n+2,m} + \frac{2683}{30240} f_{n+3,m} - \frac{1717}{60480} f_{n+4,m} + \frac{251}{60480} f_{n+5,m} \right) \\
h^3 u'''_{n+2,m} &= -u_{n,m} + 3u_{n+1,m} - 3u_{n+2,m} + u_{n+3,m} + h^4 \left( -\frac{67}{12096} f_{n,m} + \frac{12721}{60480} f_{n+1,m} + \right. \\
&\left. \frac{11009}{30240} f_{n+2,m} - \frac{547}{6048} f_{n+3,m} + \frac{1517}{60480} f_{n+4,m} - \frac{211}{60480} f_{n+5,m} \right) \\
h^3 u'''_{n+3,m} &= -u_{n,m} + 3u_{n+1,m} - 3u_{n+2,m} + u_{n+3,m} + h^4 \left( \frac{127}{60480} f_{n,m} + \frac{1763}{12096} f_{n+1,m} + \right. \\
&\left. \frac{27851}{30240} f_{n+2,m} + \frac{14107}{30240} f_{n+3,m} - \frac{2389}{60480} f_{n+4,m} + \frac{251}{60480} f_{n+5,m} \right) \\
h^3 u'''_{n+4,m} &= -u_{n,m} + 3u_{n+1,m} - 3u_{n+2,m} + u_{n+3,m} + h^4 \left( -\frac{67}{120960} f_{n,m} + \frac{12049}{60480} f_{n+1,m} + \right. \\
&\left. \frac{22433}{30240} f_{n+2,m} + \frac{35569}{30240} f_{n+3,m} + \frac{4873}{12096} f_{n+4,m} - \frac{883}{60480} f_{n+5,m} \right) \\
h^3 u'''_{n+5,m} &= -u_{n,m} + 3u_{n+1,m} - 3u_{n+2,m} + u_{n+3,m} + h^4 \left( \frac{799}{60480} f_{n,m} + \frac{4783}{60480} f_{n+1,m} + \right. \\
&\left. \frac{6511}{60480} f_{n+2,m} + \frac{18811}{30240} f_{n+3,m} + \frac{84299}{60480} f_{n+4,m} + \frac{19067}{60480} f_{n+5,m} \right) \tag{3.54}
\end{aligned}$$

The equation (3.52), (3.53) and (3.54) are the additional methods and considered for  $n = 0, 5, \dots, N - 5$ .

The equation (3.51), (3.52), (3.53) and (3.54) together form the Block Unification Method (BUM) which is used to solve equation (3.6) numerically with the appropriate boundary conditions.

## CHAPTER FOUR

### APPLICATION, ANALYSIS AND RESULTS

#### 4.1 APPLICATION BY PERTURBATION PROCEDURES

This Chapter solves the problem earlier posed in Chapter Three using a two-timing perturbation and asymptotic methods.

Let the imperfection  $\bar{\omega}$  be

$$\bar{\omega} = \bar{a}_m(1 - \cos 2mx) \quad (4.0)$$

where  $\bar{a}_m$  is a constant. Based on the clamped boundary condition which the problem assumes, it is necessary to take the displacement  $U^{(ij)}$  as:

$$U^{(ij)}(t, \tau, x) = \sum_{n=1}^{\infty} U_n^{(ij)}(\hat{t}\tau)(1 - \cos 2nx) \quad (4.1)$$

#### Solution of equation of order $\epsilon\delta^j$ , $j=0,1,2$

From equation (3.17), substituting (4.1) into (3.17) gives

$$\begin{aligned} U_{,\hat{t}\hat{t}}^{(10)} + U_{,xxxx}^{(10)} + 2\lambda U_{,xx}^{(10)} + U^{(10)} &= -2\lambda \frac{d^2\bar{\omega}}{dx^2}, \\ \sum_{n=1}^{\infty} \left[ U_{n,\hat{t}\hat{t}}^{(10)}(1 - \cos 2nx) + \{-16n^4 + 8\lambda n^2 + (1 - \cos 2nx)\}U_n^{(10)} \right] \\ &= -8\lambda m^2 \bar{a}_m \cos 2mx \end{aligned} \quad (4.2)$$

i.e

$$\begin{aligned} \sum_{n=1}^{\infty} (1 - \cos 2nx)U_{n,\hat{t}\hat{t}}^{(10)} + \{-16n^4 + 8\lambda n^2 + (1 - \cos 2nx)\}U_n^{(10)} \\ = -8\lambda m^2 \bar{a}_m \cos 2mx \end{aligned} \quad (4.3)$$

Multiplying (4.3) through by  $\cos 2mx$  and integrating from 0 to  $\pi$  and for  $n = m$ , the result is,

$$\int_0^\pi \sum_{n=1}^{\infty} \left[ \{(1 - \cos 2nx) \cos 2mx\} U_{n,\hat{t}\hat{t}}^{(10)} + U_n^{(10)} \{(-16n^4 + 8\lambda n^2) \cos 2nx \cos 2mx + (1 - \cos 2nx)\} \cos 2mx \right] dx = - \int_0^\pi 8\lambda m^2 \bar{a}_m \cos 2mxdx = -8\lambda m^2 \bar{a}_m \int_0^\pi \frac{(1 + \cos 4mx)}{2} dx = \frac{-8\lambda m^2 \bar{a}_m \pi}{2} = -4\lambda m^2 \bar{a}_m \pi \quad (4.4)$$

The left hand side of (4.4) vanishes for all  $n$  except where  $n = m$ . Thus,

for  $n = m$ , it easily follows that

$$\int_0^\pi \sum_{n=1}^{\infty} \left[ \{(1 - \cos 2nx) \cos 2mx\} U_{n,\hat{t}\hat{t}}^{(10)} + \left\{ U_n^{(10)} (-16n^4 + 8\lambda n^2) \cos 2nx + (1 - \cos 2nx) U_n^{(10)} \right\} \cos 2mx \right] dx \quad (4.5)$$

It is to be noted that, when  $n = m$ , then

$$\int_0^\pi U_n^{(10)} (-16n^4 + 8\lambda n^2) \cos 2nx \cos 2mxdx = U_m^{(10)} (-16m^4 + 8\lambda m^2) \int_0^\pi \cos^2 2mxdx = \frac{\pi}{2} U_m^{(10)} (-16m^4 + 8\lambda m^2) \quad (4.6)$$

Thus, substituting (4.6) into (4.4), gives;

$$-\frac{\pi}{2} U_{m,\hat{t}\hat{t}}^{(10)} + \frac{\pi}{2} (-16m^4 + 8\lambda m^2) U_m^{(10)} - \frac{\pi}{2} U_m^{(10)} = -8\lambda m^2 \bar{a}_m \left( \frac{\pi}{2} \right) \quad (4.7.1)$$

and this yields,

$$U_{m,\hat{t}\hat{t}}^{(10)} + (16m^4 - 8\lambda m^2 + 1) U_m^{(10)} + U_m^{(10)} = 8\lambda m^2 \bar{a}_m \quad (4.7.2)$$

Let

$$16m^4 - 8\lambda m^2 + 1 = \theta^2 > 0 \quad \forall m \quad (4.7.3)$$

Then (4.7.2) yields

$$U_{m,\hat{t}\hat{t}}^{(10)} + \theta^2 U_m^{(10)} + U_m^{(10)} = 8\lambda m^2 \bar{a}_m \quad (4.7.4)$$

With initial conditions,

$$U_m^{(10)}(0,0) = 0; \quad U_{m,\hat{t}}^{(10)}(0,0) = 0$$

Therefore, the solutions of (4.7.4) is

$$U_m^{(10)} = \alpha_1(\tau)\cos\theta\hat{t} + \beta_1(\tau)\sin\theta\hat{t} + B \quad (4.7.5)$$

Where ,

$$B = \frac{8\lambda m^2 \bar{a}_m}{\theta^2} \quad (4.7.6)$$

The use of initial conditions gives

$$\alpha_1(0) = -\frac{8\lambda m^2 \bar{a}_m}{\theta^2}, \beta_1 = 0 \quad (4.7.7)$$

Thus

$$U^{(10)} = U_m^{(10)}(1 - \cos 2mx) \quad (4.8)$$

From (3.18), we have

$$O(\epsilon\delta) : U_{,\hat{t}\hat{t}}^{(11)} + U_{,xxxx}^{(11)} + 2\lambda U_{,xx}^{(11)} + U^{(11)} = -2U_{,\hat{t}\tau}^{(10)} - 2U_{,\hat{t}}^{(10)}$$

Let  $U^{(11)} = \sum_{n=1}^{\infty} U_n^{(11)}(\hat{t}, \tau)(1 - \cos 2nx)$

Therefore, (3.18) becomes

$$\begin{aligned} & \sum_{n=1}^{\infty} \left[ \left[ U_{n,\hat{t}\hat{t}}^{(11)}(1 - \cos 2nx) + (-16n^4 + 8\lambda n^2)U_n^{(11)}\cos 2nx + (1 - \cos nx)U_n^{(11)} \right] U_n^{(10)} \right] \\ & = -2 \left[ U_{m,\hat{t}\tau}^{(10)} + U_{m,\hat{t}}^{(10)} \right] (1 - \cos 2mx) \end{aligned} \quad (4.9.1)$$

Multiplying both sides of (4.9.1) through by  $\cos 2mx$  and integrating from 0 to  $\pi$  and for  $n = m$ , gives

$$\begin{aligned} & \int_0^{\pi} \sum_{n=1}^{\infty} \left[ \{(1 - \cos 2nx)\cos 2mx\} U_{n,\hat{t}\hat{t}}^{(11)} \right. \\ & \quad \left. + U_n^{(11)} \{(-16n^4 + 8\lambda n^2)\cos 2nx\cos 2mx + (1 - \cos 2nx)\}\cos 2mx \right] dx \\ & = -2 \left[ U_{m,\hat{t}\tau}^{(10)} + U_{m,\hat{t}}^{(10)} \right] \int_0^{\pi} (1 - \cos 2mx)\cos 2mx dx \\ & = -\frac{\pi}{2} U_{m,\hat{t}\hat{t}}^{(11)} + \frac{\pi}{2} (-16m^4 + 8\lambda m^2) U_m^{(11)} - \frac{\pi}{2} U_m^{(11)} \end{aligned}$$

$$= -2\left(U_{m,\hat{t}\tau}^{(10)} + U_{m,\hat{t}}^{(10)}\right)\left(-\frac{\pi}{2}\right) \quad (4.9.2)$$

Further simplification gives

$$U_{m,\hat{t}\hat{t}}^{(11)} + (16m^4 - 8\lambda m^2 + 1)U_m^{(11)} = -2\left(U_{m,\hat{t}\tau}^{(10)} + U_{m,\hat{t}}^{(10)}\right) \quad (4.9.3)$$

i.e.

$$U_{m,\hat{t}\hat{t}}^{(11)} + \theta^2 U_m^{(11)} = -2\left(U_{m,\hat{t}\tau}^{(10)} + U_{m,\hat{t}}^{(10)}\right) \quad (4.10)$$

The initial conditions are

$$U_m^{(11)}(0,0) = 0; U_{m,\hat{t}}^{(11)}(0,0) + U_{m,\tau}^{(10)}$$

Substituting for  $U_m^{(10)}$  on the right-hand side of (4.10), from (4.7.5) gives

$$\begin{aligned} U_{m,\hat{t}\hat{t}}^{(11)} + \theta^2 U_m^{(11)} &= -2[-\theta\alpha'_1 \sin\theta\hat{t} + \theta\beta'_1 \cos\theta\hat{t} + (-\theta\alpha_1 \sin\theta\hat{t} + \theta\beta_1 \cos\theta\hat{t})] \\ &= -2\theta[-(\alpha'_1 + \alpha_1)\sin\theta\hat{t} + (\beta'_1 + \beta)\cos\theta\hat{t}] \end{aligned} \quad (4.11.1)$$

To ensure a uniformly valid solution in  $\hat{t}$ , implies equating to zero the coefficients of  $\cos\theta\hat{t}$  and  $\sin\theta\hat{t}$  on the RHS of (4.11.1). Therefore, the coefficient of  $\cos\theta\hat{t}$  gives;

$$\beta'_1 + \beta = 0 \quad (4.11.2)$$

The integrating factor is  $e^\tau$ , then,

$$\frac{d(e^\tau\beta_1)}{d\tau} = 0 \quad (4.11.3)$$

This gives,

$$\beta_1(\tau) = Ae^{-\tau} \text{ and } \beta_1(\tau) = 0 \quad (4.11.4)$$

Similarly, the coefficient of  $\sin\theta\hat{t}$  gives,

$$\alpha'_1 + \alpha_1 = 0 \quad (4.11.5)$$

This gives,

$$\alpha'_1(0) = -\alpha_1(0) = B \text{ and } \alpha_1(\tau) = -Be^{-\tau} \quad (4.11.6)$$

Therefore,

$$U_m^{(10)} = \alpha_1(\tau)\cos\theta\hat{t} + B \quad (4.11.7)$$

The remaining equation in (4.11.1) is

$$U_{m,\hat{t}\hat{t}}^{(11)} + \theta^2 U_m^{(11)} = 0 \quad (4.11.8)$$

Therefore,

$$U_m^{(11)} = \alpha_2(\tau)\cos\theta\hat{t} + \beta_2(\tau)\sin\theta\hat{t} \quad (4.12.1)$$

$$U_m^{(11)}(0,0) = 0 \text{ implies } \alpha_2(0) = 0 \quad (4.12.2)$$

$$U_{m,\hat{t}}^{(11)}(0,0) + U_{m,\tau}^{(10)} = 0$$

Implies that

$$\beta_2(0)\theta + \alpha_1'(0) = 0 \text{ and } \beta_2(0) = -\frac{\alpha_1'(0)}{\theta} = \frac{-B}{\theta} \quad (4.12.3)$$

Therefore,

$$U_m^{(11)} = U_m^{(11)}(1 - \cos 2m\hat{t}) \quad (4.12.4)$$

From (3.19); the next equation is

$$O(\epsilon\delta^2): U_{,\hat{t}\hat{t}}^{(12)} + U_{,xxxx}^{(12)} + 2\lambda U_{,xx}^{(12)} + U^{(12)} = -2U_{,\hat{t}\tau}^{(11)} - 2U_{,\hat{t}}^{(11)} - U_{,\tau\tau}^{(10)}$$

Substituting for  $U_m^{(11)}$  and  $U_m^{(10)}$  from (4.11.7) and (4.12.1) respectively on the right-hand side of (3.19), gives

$$\begin{aligned} U_{m,\hat{t}\hat{t}}^{(12)} + \theta^2 U_m^{(12)} &= -2[-\theta\alpha_2'(\tau)\sin\theta\hat{t} + \theta\beta_2'(\tau)\cos\theta\hat{t} + (-\theta\alpha_2(\tau)\sin\theta\hat{t} + \\ &\theta\beta_2(\tau)\cos\theta\hat{t})] - \alpha_1''(\tau)\cos\theta\hat{t} = 2\theta\alpha_2'(\tau)\sin\theta\hat{t} - 2\theta\beta_2'(\tau)\cos\theta\hat{t} - \theta\alpha_2(\tau)\sin\theta\hat{t} + \\ &\theta\beta_2(\tau)\cos\theta\hat{t} - \alpha_1''(\tau)\cos\theta\hat{t} \\ &= (2\theta\alpha_2'(\tau) - 2\theta\alpha_2(\tau)\sin\theta\hat{t} + (2\theta\beta_2(\tau) - 2\theta\beta_2'(\tau) - \alpha_1''(\tau))\cos\theta\hat{t}) \end{aligned} \quad (4.13.1)$$

To remove secular terms in the solution of  $U_m^{(12)}$ , i.e, to ensure a uniformly valid solution in  $\hat{t}$  implies equating to zero the coefficients of  $\cos\theta\hat{t}$  and  $\sin\theta\hat{t}$  on the right-hand side. These respectively give:

$$\cos\theta\hat{t}: -2(\theta\beta_2' + \theta\beta_2) - \alpha_1'' = 0$$

And

$$\sin\theta\hat{t}: -2(-\theta\alpha'_2 - \theta\alpha_2) = 0$$

Therefore

$$\beta'_2 + \beta_2 = \frac{-\alpha''_1}{2\theta} \text{ and } [\beta'_2(0) = \frac{3B}{2\theta}] \quad (4.13.2)$$

$$\alpha'_2 + \alpha_2 = 0 \quad (4.13.3)$$

Therefore from (4.13),

$$\alpha_2(\tau) = 0, \quad (4.13.4)$$

And from (4.13.3),

$$\beta_2(\tau) = e^{-\tau} \left[ -\int_0^\tau \frac{e^s \alpha''_1}{2\theta} ds + \beta_2(0) \right] \quad (4.13.5)$$

That is,

$$\beta_2(\tau) = e^{-\tau} \left[ -\int_0^\tau \frac{e^s \alpha''_1}{2\theta} ds - \frac{B}{\theta} \right] \quad (4.13.6)$$

Therefore,

$$U_m^{(11)} = \beta_2(\tau) \sin\theta\hat{t} \quad (4.13.7)$$

Equating the left hand side of (4.13.1) to zero,

i.e,

$$U_{m,\hat{t}\hat{t}}^{(12)} + \theta^2 U_m^{(12)} = 0$$

The Initial conditions are

$$U_m^{(12)}(0,0) = 0, \quad U_{m,\hat{t}}^{(12)} + U_{m,\tau}^{(11)}(0,0) = 0$$

Then

$$U_m^{(12)}(\hat{t}, \tau) = \alpha_3(\tau) \cos\theta\hat{t} + \beta_3(\tau) \sin\theta\hat{t} \quad (4.13.8)$$

Applying the initial conditions yield,

$$\alpha_3(0) = 0, \quad \beta_3(0) = 0 \quad (4.13.9)$$

$$U_{m,\hat{t}}^{(12)}(0,0) = \theta\beta_3(0) = 0$$

Therefore,

$$\beta_3(0) = 0 \quad (4.13.10)$$

It follows that

$$U^{12} = U_m^{(12)}(1 - \cos 2mx) \quad (4.13.11)$$

### Solution of equation of order $\epsilon^2 \delta^j$ , $j=0, 1, 2$

From (3.20),

$$O(\epsilon^2) : U_{,\hat{t}\hat{t}}^{(20)} + U_{,xxxx}^{(20)} + 2\lambda U_{,xx}^{(20)} + U^{(20)} = -\alpha(U^{(10)})^2 - 2\omega'_1 U_{,\hat{t}\hat{t}}^{(10)}$$

Let,

$$U^{(20)} = \sum_{n=1}^{\infty} U_n^{(20)}(\hat{t}\tau)(1 - \cos 2nx) \quad (4.14.1)$$

Substituting (4.14.1) into (3.20) gives

$$\begin{aligned} & \sum_{n=1}^{\infty} U_{n,\hat{t}\hat{t}}^{(20)}(1 - \cos 2nx) + (-16n^4 + 8\lambda n^2)U_n^{(20)} \cos 2nx + (1 - \cos 2nx)U_n^{(20)} \\ & = \alpha \left[ \left( U_m^{(10)} \right)^2 (1 - \cos 2mx)^2 \right] - 2\omega'_1 U_{m,\hat{t}\hat{t}}^{(10)}(1 - \cos 2mx) \end{aligned} \quad (4.14.2)$$

This further gives,

$$\begin{aligned} & \sum_{n=1} \left[ U_{n,\hat{t}\hat{t}}^{(20)}(1 - \cos 2nx) + (-16n^4 + 8\lambda n^2)U_n^{(20)} \cos 2nx \right] \\ & \quad + (1 - \cos 2nx)U_n^{(20)} \\ & = -\alpha \left( U_m^{(10)} \right)^2 \left[ \frac{3}{2} - 2\cos 2mx + \frac{1}{2} \cos 4mx \right] - 2\omega'_1 U_{m,\hat{t}\hat{t}}^{(10)}(1 - \cos 2mx) \end{aligned} \quad (4.15.1)$$

Multiplying both sides of (4.15.1) through by  $\cos 2mx$  and integrating from 0 to  $\pi$  and for  $n = m$ , the result gives:

$$\begin{aligned} & \left[ -\frac{\pi}{2} U_{m,\hat{t}\hat{t}}^{(20)} + (-16m^4 + 8\lambda m^2)U_m^{(20)} \left( \frac{\pi}{2} \right) + \left( \frac{\pi}{2} U_m^{(20)} \right) \right] \\ & = -\alpha \left( U_m^{(10)} \right)^2 \left[ -2 \left( \frac{\pi}{2} \right) - 2\omega'_1 U_{m,\hat{t}\hat{t}}^{(10)} \left( \frac{-\pi}{2} \right) \right] \end{aligned} \quad (4.15.2)$$

i.e

$$\frac{\pi}{2} \left[ -U_{m,\hat{t}\hat{t}}^{(20)} + (-16m^4 + 8\lambda m^2)U_m^{(20)} - U_m^{(20)} \right] = \frac{\pi}{2} \left[ 2\alpha \left( U_m^{(10)} \right)^2 + 2\omega'_1 U_{m,\hat{t}\hat{t}}^{(10)} \right] \quad (4.15.3)$$

Simplification of (4.15.2) gives;

$$U_{m,\hat{t}\hat{t}}^{(20)} + (16m^4 - 8\lambda m^2 + 1)U_m^{(20)} = - \left[ 2\alpha \left( U_m^{(10)} \right)^2 + 2\omega'_1 U_{m,\hat{t}\hat{t}}^{(10)} \right] \quad (4.15.4)$$

and this further gives

$$U_{m,\hat{t}\hat{t}}^{(20)} + \theta^2 U_m^{(20)} = - \left[ 2\alpha \left( U_m^{(10)} \right)^2 + 2\omega' U_{m,\hat{t}\hat{t}}^{(10)} \right] \quad (4.16.1)$$

The initial conditions are;

$$U_m^{(20)}(0,0) = 0, U_{m,\hat{t}}^{(20)}(0,0) + \omega'(0)U_{m,\hat{t}}^{(10)}(0,0) = 0$$

Next multiplying equation (4.15.1) by  $\cos 4mx$  and integrating from 0 to  $\pi$  and for  $n=m$ , the result gives;

$$-\frac{\pi}{2} U_{2m,\hat{t}\hat{t}}^{(20)} + (-256m^4 + 32\lambda m^2) \left( \frac{\pi}{2} \right) U_{2m}^{(20)} - \left( \frac{\pi}{2} \right) U_{2m}^{(20)} = -\alpha \left( U_m^{(10)} \right)^2 \left( \frac{1}{2} \cdot \frac{\pi}{2} \right) \quad (4.16.2)$$

Simplifying (4.16b) gives;

$$U_{2m,\hat{t}\hat{t}}^{(20)} + (256m^4 - 32\lambda m^2 + 1)U_{2m}^{(20)} = \frac{\alpha}{2} \left( U_m^{(10)} \right)^2 \quad (4.16.3)$$

Let

$$\varphi^2 = (256m^4 + 32\lambda m^2 + 1) > 0 \quad (4.16.4)$$

Then, the final equation is

$$U_{2m,\hat{t}\hat{t}}^{(20)} + \varphi^2 U_{2m}^{(20)} = \frac{\alpha}{2} \left( U_m^{(10)} \right)^2 \quad (4.17)$$

The initial conditions are;

$$U_{2m}^{(20)}(0,0) = 0; U_{2m,\hat{t}}^{(20)}(0,0) + \omega'_1 U_{2m,\hat{t}}^{(10)}(0,0) = 0$$

On substituting for  $U_m^{(10)}$  on the RHS of (4.16.1), the simplification is

$$\begin{aligned}
U_{2m,\hat{t}\hat{t}}^{(20)} + \theta^2 U_m^{(20)} &= -[\{2\alpha(\alpha_1 \cos\theta\hat{t} + B)^2\} + \{2\omega'_1(-\alpha_1 \cos\theta\hat{t})\}] \\
&= -\left[2\alpha\left\{\left(\frac{\alpha_1^2}{2} + B^2\right) + 2B\alpha_1 \cos\theta\hat{t} + \frac{\alpha_1^2}{2}\alpha_1 \cos 2\theta\hat{t}\right\} + 2\omega'_1(-\alpha_1 \theta^2 \cos\theta\hat{t})\right]
\end{aligned} \tag{4.18.1}$$

To ensure a uniformly valid solution in  $\hat{t}$ , we equate to zero, the coefficients of  $\cos\theta\hat{t}$  on the right-hand side of (4.18.1)

That is

$$-[2B\alpha_1 - 2\omega'_1\theta^2\alpha_1] = 0 \tag{4.18.2}$$

Therefore,

$$\omega'_1 = \frac{B}{\theta^2}; \quad \omega_1 = \int \frac{B}{\theta^2} d\tau \tag{4.18.3}$$

The remaining part of equation (4.18.1) for  $U_m^{(20)}$  is

$$U_{m,\hat{t}\hat{t}}^{(20)} + \theta^2 U_m^{(20)} = r_0 + r_1 \cos 2\theta\hat{t} \tag{4.19.1}$$

Where,

$$r_0 = -2\alpha\left(\frac{\alpha_1^2}{2} + B^2\right), \quad r_0(0) = -3\alpha B^2$$

$$r_1 = -\alpha\alpha_1^2, \quad r_1(0) = -\alpha B^2, \quad r'_0(0) = 2\alpha B^2, \quad r'_1(0) = 2\alpha B^2$$

Therefore, it follows that

$$U_m^{(20)}(\hat{t}, \tau) = \alpha_4(\tau) \cos\theta\hat{t} + \beta_4(\tau) \sin\theta\hat{t} + \frac{r_0}{\theta^2} - \frac{r_1 \cos 2\theta\hat{t}}{3\theta^2} \tag{4.19.2}$$

From the initial condition;

$$U_m^{(20)}(0,0) = 0;$$

That is,

$$\alpha_4(0) + \frac{r_0(0)}{\theta^2} - \frac{r_1}{3\theta^2} = 0$$

Therefore,

$$\alpha_4(0) = \frac{r_1}{3\theta^2} - \frac{r_0(0)}{\theta^2} = \frac{8\alpha B^2}{3\theta^2} \tag{4.19.3}$$

Applying the initial condition  $U_{m,\hat{t}}^{(20)}(0,0) + \omega'(0) + U_{m,\hat{t}}^{(10)}(0,0)$  yields

$$\beta_4(0) = 0 \quad (4.19.4)$$

Simplification of (4.17) yields,

$$U_{2m,\hat{t}\hat{t}}^{(20)} + \varphi^2 U_m^{(20)} = \frac{\alpha}{2} \left[ \left( \frac{\alpha_1^2}{2} + B^2 \right) + 2B\alpha_1 \cos\theta\hat{t} + \frac{\alpha_1^2}{2} \cos 2\theta\hat{t} \right] \quad (4.20.1)$$

Therefore,

$$U_{2m}^{(20)}(\hat{t}, \tau) = \alpha_5(\tau) \cos\varphi\hat{t} + \beta_5(\tau) \sin\varphi\hat{t} + \frac{\alpha}{2} \left[ \frac{\left(\frac{\alpha_1^2}{2} + B^2\right)}{\varphi^2} + \frac{2B\alpha_1 \cos\theta\hat{t}}{(\varphi^2 - \theta^2)} + \frac{\alpha_1^2 \cos 2\theta\hat{t}}{2(\varphi^2 - \theta^2)} \right] \quad (4.20.2)$$

From the initial conditions;

$$U_{2m}^{(20)}(0,0) = 0; \quad U_{2m,\hat{t}}^{(20)}(0,0) + \omega_1' U_{2m,\hat{t}}^{(10)}(0,0) = 0$$

It follows that,

$$\alpha_5(0) + \frac{\alpha}{2} \left[ \frac{\left(\frac{\alpha_1^2}{2} + B^2\right)}{\varphi^2} + \frac{2B\alpha_1}{(\varphi^2 - \theta^2)} + \frac{\alpha_1^2}{2(\varphi^2 - 4\theta^2)} \right] = 0$$

Therefore,

$$\alpha_5(0) = -\frac{\alpha}{2} \left[ \frac{\left(\frac{\alpha_1^2}{2} + B^2\right)}{\varphi^2} + \frac{2B\alpha_1}{(\varphi^2 - \theta^2)} + \frac{\alpha_1^2}{2(\varphi^2 - 4\theta^2)} = 0 \right] \text{ at } \tau=0$$

That is,

$$\alpha_5(0) = -\frac{\alpha}{2} \left[ \frac{3B^2}{2\varphi^2} - \frac{2B^2}{(\varphi^2 - \theta^2)} + \frac{B^2}{2(\varphi^2 - 4\theta^2)} \right] = B^2 \alpha S_0$$

And

$$\beta_5(0) = 0 \quad (4.20.3)$$

Where,

$$S_0 = \left( -\frac{3\alpha}{2\varphi^2} + \frac{\alpha}{(\varphi^2 - \theta^2)} - \frac{\alpha}{4(\varphi^2 - 4\theta^2)} \right)$$

Therefore,

$$U^{(20)} = U_m^{(20)}(1 - \cos 2mx) + U_{2m}^{(20)}(1 - \cos 4mx) \quad (4.20.4)$$

From (3.21),

$$\begin{aligned} O(\epsilon^2 \delta) : & U_{,\hat{t}\hat{t}}^{(21)} + U_{,xxxx}^{(21)} + 2\lambda U_{,xx}^{(21)} + U^{(21)} \\ & = -2\alpha U^{(10)}U^{(11)} - 2U_{,\hat{t}\tau}^{(20)} - 2U_{,\hat{t}}^{(20)} - 2\omega_1' U_{,\hat{t}\hat{t}}^{(11)} - \omega_1'' U_{,\hat{t}}^{(10)} - 2\omega_1' U_{,\hat{t}}^{(10)} \end{aligned}$$

That is,

$$\begin{aligned} & U_{,\hat{t}\hat{t}}^{(21)} + U_{,xxxx}^{(21)} + 2\lambda U_{,xx}^{(21)} + U^{(21)} \\ & = -2\alpha U_m^{(10)}(1 - \cos 2mx)U_m^{(11)}(1 - \cos 2mx) - 2U_{m,\hat{t}\tau}^{(20)}(1 - \cos 2mx) \\ & - 2U_{m,\hat{t}}^{(20)}(1 - \cos 2mx) - 2\omega_1' U_{m,\hat{t}\hat{t}}^{(11)}(1 - \cos 2mx) - \omega_1'' U_{m,\hat{t}}^{(10)}(1 - \cos 2mx) \\ & - 2\omega_1' U_{m,\hat{t}}^{(10)}(1 - \cos 2mx) \end{aligned} \quad (4.21)$$

Let,

$$U^{(21)} = \sum_{n=1}^{\infty} U_n^{(21)}(\hat{t}\tau)(1 - \cos 2nx)$$

Substituting into (4.21), gives

$$\begin{aligned} & \sum_{n=1} \left[ U_{n,\hat{t}\hat{t}}^{(21)}(1 - \cos 2nx) + (-16n^4 + 8\lambda n^2)U_n^{(21)} \cos 2nx + (1 - \cos 2nx)U_n^{(21)} \right] \\ & = -2\alpha U_m^{(10)}U_m^{(11)} \left[ \frac{3}{2} - 2\cos 2mx + \frac{1}{2}\cos 4mx \right] - 2U_{m,\hat{t}\tau}^{(20)}(1 - \cos 2mx) - 2U_{m,\hat{t}}^{(20)}(1 - \cos 2mx) \\ & - 2\omega_1' U_{m,\hat{t}\hat{t}}^{(11)}(1 - \cos 2mx) - \omega_1'' U_{m,\hat{t}}^{(10)}(1 - \cos 2mx) - 2\omega_1' U_{m,\hat{t}}^{(10)}(1 - \cos 2mx) \end{aligned} \quad (4.22.1)$$

Multiplying both sides of (4.22.1) through by  $\cos 2mx$  and integrating from 0 to  $\pi$  and for  $n = m$  gives

$$\begin{aligned} & \left[ -\frac{\pi}{2} U_{m,\hat{t}\hat{t}}^{(21)} + (-16m^4 + 8\lambda m^2)U_m^{(21)} \left( \frac{\pi}{2} \right) + \left( -\frac{\pi}{2} U_m^{(21)} \right) \right] \\ & = -2\alpha U_m^{(10)}U_m^{(11)} \left( -\frac{\pi}{2} \right) - 2U_{m,\hat{t}\tau}^{(20)} \left( -\frac{\pi}{2} \right) - 2U_{m,\hat{t}}^{(20)} \left( -\frac{\pi}{2} \right) - 2\omega_1' U_{m,\hat{t}\hat{t}}^{(11)} \left( -\frac{\pi}{2} \right) \\ & - \omega_1'' U_{m,\hat{t}}^{(10)} \left( -\frac{\pi}{2} \right) - 2\omega_1' U_{m,\hat{t}}^{(10)} \left( -\frac{\pi}{2} \right) \end{aligned} \quad (4.22.2)$$

Further simplification yields,

$$\begin{aligned}
& U_{m,\hat{t}\hat{t}}^{(21)} + (16m^4 - 8\lambda m^2 + 1)U_m^{(21)} \\
& = -2\alpha U_m^{(10)}U_m^{(11)} - 2U_{m,\hat{t}\tau}^{(20)} - 2U_{m,\hat{t}}^{(20)} - 2\omega_1' U_{m,\hat{t}\hat{t}}^{(11)} - \omega_1'' U_{m,\hat{t}}^{(10)} - 2\omega_1' U_{m,\hat{t}}^{(10)}
\end{aligned} \tag{4.22.3}$$

The above finally yields

$$\begin{aligned}
& U_{m,\hat{t}\hat{t}}^{(21)} + \theta^2 U_m^{(21)} \\
& = -2\alpha U_m^{(10)}U_m^{(11)} - 2U_{m,\hat{t}\tau}^{(20)} - 2U_{m,\hat{t}}^{(20)} - 2\omega_1' U_{m,\hat{t}\hat{t}}^{(11)} - \omega_1'' U_{m,\hat{t}}^{(10)} - 2\omega_1' U_{m,\hat{t}}^{(10)}
\end{aligned} \tag{4.23.1}$$

The initial conditions are

$$U_m^{(21)}(0,0) = 0; U_{m,\hat{t}}^{(21)}(0,0) + \omega_1'(0)U_{m,\hat{t}}^{(11)}(x,0,0) + U_{m,\tau}^{(20)}(0,0) = 0$$

Next, multiplying (4.22.1) by  $\cos 4mx$  and integrating from 0 to  $\pi$  for  $n = m$ , gives

$$\begin{aligned}
& \left[ -\frac{\pi}{2} U_{2m,\hat{t}\hat{t}}^{(21)} + (-256m^4 + 32\lambda m^2)U_{2m}^{(21)}\left(\frac{\pi}{2}\right) - \frac{\pi}{2} U_{2m}^{(21)} \right] \\
& = -2\alpha U_m^{(10)}U_m^{(11)}\left(\frac{\pi}{2}\right)\left(\frac{1}{2}\right) - 2\left(U_{2m,\hat{t}\tau}^{(20)} + U_{m,\hat{t}}^{(20)}\right)
\end{aligned} \tag{4.23.2}$$

$$U_{2m,\hat{t}\hat{t}}^{(21)} + \varphi^2 U_{2m}^{(21)} = \alpha U_m^{(10)}U_m^{(11)} + 2\left(U_{2m,\hat{t}\tau}^{(20)} + U_{2m,\hat{t}}^{(20)}\right) \tag{4.24}$$

The initial conditions are

$$U_{2m}^{(21)}(0,0) = 0; U_{2m,\hat{t}}^{(21)}(0,0) = 0$$

Substituting for  $U_m^{(10)}$ ,  $U_m^{(11)}$  and  $U_m^{(20)}$  in (4.24) yields

$$\begin{aligned}
& U_{m,\hat{t}\hat{t}}^{(21)} + \theta^2 U_m^{(21)} - 2\alpha U_m^{(10)}U_m^{(11)} - 2U_{m,\hat{t}\tau}^{(20)} - 2U_{m,\hat{t}}^{(20)} - 2\omega_1' U_{m,\hat{t}\hat{t}}^{(11)} - \omega_1'' U_{m,\hat{t}}^{(10)} \\
& - 2\omega_1' U_{m,\hat{t}}^{(10)}
\end{aligned} \tag{4.25.1}$$

That is,

$$\begin{aligned}
U_{m,\hat{t}\hat{t}}^{(21)} + \theta^2 U_m^{(21)} &= -2\alpha \left( \frac{\alpha_1 \beta_2}{2} \sin\theta\hat{t} + B\beta_2 \sin\theta\hat{t} \right) \\
&- 2 \left( -\theta\alpha'_4 \sin\theta\hat{t} + \theta\beta'_4 \cos\theta\hat{t} + \frac{2\theta r'_1 \sin 2\theta\hat{t}}{3\theta^2} \right) \\
&- 2 \left( -\theta\alpha_4 \sin\theta\hat{t} + \theta\beta_4 \cos\theta\hat{t} + \frac{2\theta r_1 \sin 2\theta\hat{t}}{3\theta^2} \right) - 2\omega'_1(-\theta^2)\beta_2 \sin\theta\hat{t} \\
&- \omega''_1(-\theta\alpha_1 \sin\theta\hat{t}) - 2\omega'_1(-\alpha_1\theta \sin\theta\hat{t})
\end{aligned} \tag{4.25.2}$$

To ensure a uniformly valid solution in  $\hat{t}$ , equating to zero the coefficients of  $\cos\theta\hat{t}$  and  $\sin\theta\hat{t}$ .

This yields,

$$-2\theta\beta'_4 - 2\theta\beta_4 = 0 \tag{4.25.3}$$

$$\alpha B\beta_2 + 2\theta\alpha'_4 + 2\theta\alpha_4 + 2\theta^2\omega'_1\beta_2 + \omega''_1\theta\alpha_1 + 2\omega'_1\alpha_1\theta \tag{4.26.4}$$

The (4.25.3) gives,

$$\beta'_4 + \beta_4 = 0 \tag{4.26.5}$$

Solving (4.26.5) yields,

$$\beta_4(\tau) = \beta_4(0)e^{-\tau} = 0 \text{ since } \beta_4(0) = 0 \tag{4.26.6}$$

Solving (4.25.4) yields,

$$\alpha'_4 + \alpha_4 = \rho_1(\tau) = \frac{1}{2\theta}(\alpha B\beta_2 - 2\theta^2\omega'_1\beta_2 - \omega''_1\theta\alpha_1 - 2\omega'_1\alpha_1\theta) \tag{4.26.7}$$

Therefore,

$$\alpha_4(\tau) = e^{-\tau} \left[ \int_0^\tau e^s \rho_1(s) ds + \alpha_4(0) \right]$$

Then,

$$\alpha'_4(0) = \rho_1(0) - \alpha_4(0) \tag{4.26.8}$$

Where,

$$\alpha_4(0) = \frac{8\alpha B^2}{3\theta^2}, \quad \alpha_4'(0) = \frac{-13\alpha B^2}{3\theta^2} + \frac{4B^2}{\theta} \quad (4.26.9)$$

Therefore,

$$\rho_1(0) = \frac{-5\alpha B^2}{3\theta^2} + \frac{4B^2}{\theta} = B^2 \left( \frac{-5\alpha}{3\theta^2} + \frac{4}{\theta} \right) = B^2 V \quad (4.26.10)$$

Where,

$$V = \left( \frac{-5\alpha}{3\theta^2} + \frac{4}{\theta} \right)$$

The remaining equation in (4.25.1) is

$$U_{m,\hat{t}\hat{t}}^{(21)} + \theta^2 U_m^{(21)} = r_2 + r_3 \cos 2\theta \hat{t} + r_4 \sin 2\theta \hat{t} \quad (4.26.11)$$

With the initial conditions;

$$U_m^{(21)}(0,0) = 0; \quad U_{m,\hat{t}}^{(21)}(0,0) + \omega_1'(0)U_{m,\hat{t}}^{(11)} + U_{m,\hat{t}}^{(20)}(0,0) = 0$$

Where,

$$r_2 = \alpha_1 \alpha_2; \quad r_2(0) = \alpha_1(0) \alpha_2(0) = 0$$

$$r_3 = \alpha \alpha_1 \alpha_2; \quad r_3(0) = \alpha \alpha_1(0) \alpha_2(0) = 0, r_3'(0) = 0$$

$$r_4 = \left[ \alpha \alpha_1 \beta_2 + \frac{8\alpha}{3\theta} (\alpha_1 \alpha_1' + \alpha_1^2) \right]$$

$$r_4(0) = \left[ \alpha(-B) \left( \frac{-B}{\theta} \right) + \frac{8\alpha}{3\theta} (-B \cdot B + B^2) \right] = \frac{\alpha B^2}{\theta}, r_4'(0) = \frac{-5\alpha B^2}{2\theta}$$

The solution of (4.26.11) is;

$$U_m^{(21)}(\hat{t}, \tau) = \alpha_6 \cos \theta \hat{t} + \beta_6 \sin \theta \hat{t} + \frac{r_2}{\theta^2} - \left( \frac{r_3 \cos 2\theta \hat{t} + r_4 \sin 2\theta \hat{t}}{3\theta^2} \right) \quad (4.27.1)$$

With the initial conditions,

$$\alpha_6(0) + \frac{r_2}{\theta^2} - \frac{r_3}{3\theta^2} = 0$$

Therefore,

$$\alpha_6(0) = \frac{r_3 - 3r_2}{3\theta^2} = 0, \quad \beta_6(0) = 0 \quad (4.27.2)$$

From (4.24),

$$\begin{aligned} U_{2m, \hat{t}\hat{t}}^{(21)} + \varphi^2 U_{2m}^{(21)} &= \alpha \left[ \frac{\alpha_1 \beta_2}{2} \sin 2\theta \hat{t} + B \beta_2 \sin \theta \hat{t} \right] + 2 \left( U_{2m, \hat{t}\tau}^{(20)} + U_{2m, \hat{t}}^{(20)} \right) \\ &= \alpha \left[ \frac{\alpha_1 \beta_2}{2} \sin 2\theta \hat{t} + B \beta_2 \sin \theta \hat{t} \right] + 2 \left[ -\varphi \alpha_5' \sin \varphi \hat{t} + \beta_5' \varphi \cos \varphi \hat{t} + \frac{\alpha}{2} \left\{ \frac{-2\theta \alpha_1' B \sin \theta \hat{t}}{\varphi^2 - \theta^2} - \right. \right. \\ &\left. \left. \frac{2\theta (\alpha_1^2)' \sin 2\theta \hat{t}}{2(\varphi^2 - 4\theta^2)} \right\} + \left\{ -\varphi \alpha_5 \sin \varphi \hat{t} + \beta_5 \cos \varphi \hat{t} + \frac{\alpha}{2} \left\{ \frac{-2\theta \alpha_1 B \sin \theta \hat{t}}{\varphi^2 - \theta^2} - \frac{2\theta \alpha_1^2 \sin 2\theta \hat{t}}{2(\varphi^2 - 4\theta^2)} \right\} \right\} \right] \end{aligned} \quad (4.28.1)$$

To ensure a uniformly valid solution in  $\hat{t}$ , we equate to zero the coefficient of  $\cos \varphi \hat{t}$  and  $\sin \varphi \hat{t}$

$$2\varphi \beta_5' + 2\varphi \beta_5 = 0 \Rightarrow \beta_5' + \beta_5 = 0 \Rightarrow \beta_5'(0) = -\beta_5(0) \quad (4.28.2)$$

$$-2\varphi \alpha_5' - 2\varphi \alpha_5 = 0 \Rightarrow \alpha_5' + \alpha_5 = 0 \Rightarrow \alpha_5'(0) = -\alpha_5(0) \quad (4.28.3)$$

$$\beta_5 = \beta_5(0) e^{-\tau} = 0,$$

$$\alpha_5 = \alpha_5(0) e^{-\tau} = 0 \quad (4.28.4)$$

The remaining equation of (4.27.1) is

$$\begin{aligned}
& U_{2m,\hat{t}\hat{t}}^{(21)} + \varphi^2 U_{2m}^{(21)} \\
&= \left[ \alpha B \beta_2 + \frac{\alpha}{2} \left( \frac{-2\theta B}{\varphi^2 - \theta^2} \right) (\alpha'_1 + \alpha_1) \right] \sin\theta\hat{t} + \left[ \frac{\alpha\alpha_1\beta_2}{2} + \frac{\alpha}{2} \left\{ \frac{-\theta(\alpha_1^2)' + \alpha_1^2}{2(\varphi^2 - 4\theta^2)} \right\} \right] \sin 2\theta\hat{t} \\
&= r_5 \sin\theta\hat{t} + r_6 \sin 2\theta\hat{t} \tag{4.29.1}
\end{aligned}$$

Where,

$$\begin{aligned}
r_5 &= \left[ \alpha B \beta_2 - \frac{-2\theta B}{(\varphi^2 - \theta^2)} (\alpha'_1 + \alpha) \right] = \alpha B \beta_2, \text{ since } \alpha'_1 + \alpha = 0, \\
\alpha'_1 &= B; \quad r_5(0) = \frac{-\alpha B^2}{\theta}, \quad r_6 = \left[ \frac{\alpha\alpha_1\beta_2}{2} + \frac{\alpha}{2} \left\{ \frac{-\theta(\alpha_1^2)' + \alpha_1^2}{2(\varphi^2 - 4\theta^2)} \right\} \right]
\end{aligned}$$

Therefore,

$$r_6(0) = \left[ \frac{\alpha\alpha_1(0)\beta_2(0)}{2} + \frac{\alpha}{2} \left\{ \frac{-\theta(\alpha_1^2(0))' + \alpha_1^2(0)}{2(\varphi^2 - 4\theta^2)} \right\} \right] = \frac{B^2\alpha}{2\theta} + \frac{B^2\theta\alpha}{4(\varphi^2 - 4\theta^2)} = B^2 S_1 \tag{4.29.2}$$

Therefore,

$$S_1 = \left( \frac{\alpha}{2\theta} + \frac{\alpha\theta}{4(\varphi^2 - 4\theta^2)} \right)$$

Therefore, the result is;

$$U_{2m}^{(21)} = \alpha_7(\tau) \cos\varphi\hat{t} + \beta_7(\tau) \sin\varphi\hat{t} + \frac{r_5 \cos\theta\hat{t}}{\varphi^2 - \theta^2} + \frac{r_6 \cos 2\theta\hat{t}}{\varphi^2 - 4\theta^2} \tag{4.30}$$

The initial conditions are

$$U_{2m}^{(21)}(0,0) = 0; \quad U_{2m,\hat{t}}^{(21)}(0,0) + U_{2m,\hat{t}}^{(20)}(0,0) = 0;$$

This implies;

$$\begin{aligned}
& -\varphi\alpha_7(0) \sin\varphi\hat{t} + \varphi\beta_7(0) \cos\varphi\hat{t} + \frac{\theta r_5(0) \cos\theta\hat{t}}{\varphi^2 - \theta^2} + \frac{2\theta r_6(0) \cos\theta\hat{t}}{\varphi^2 - 4\theta^2} + \alpha'_5(0) \cos\varphi\hat{t} \\
&+ \frac{\alpha}{2} \left[ \frac{\alpha'_1\alpha_1}{\varphi^2} + \frac{2B\theta\alpha'_1 \cos\theta\hat{t}}{\varphi^2 - \theta^2} + \frac{2\alpha'_1\alpha_1 \cos 2\theta\hat{t}}{2(\varphi^2 - 4\theta^2)} \right] = 0 \tag{4.31.1}
\end{aligned}$$

Therefore,

$$\alpha_7(0) = 0 \tag{4.31.2}$$

Similarly, the following is obtained

$$\varphi\beta_7(0) + \frac{\theta r_5(0)}{\varphi^2 - \theta^2} + \frac{2\theta r_6(0)}{\varphi^2 - 4\theta^2} + \alpha'_5(0) + \frac{\alpha}{2} \left[ \frac{\alpha'_1(0)\alpha_1(0)}{\varphi^2} + \frac{2B\theta\alpha'_1(0)}{\varphi^2 - \theta^2} + \frac{\alpha'_1(0)\alpha_1(0)}{(\varphi^2 - 4\theta^2)} \right] = 0 \quad (4.32.1)$$

$$\beta_7(0) = -\frac{1}{\varphi} \left[ \frac{\theta r_5(0)}{\varphi^2 - \theta^2} + \frac{2\theta r_6(0)}{\varphi^2 - 4\theta^2} + \alpha'_5(0) + \frac{\alpha}{2} \left( \frac{\alpha'_1(0)\alpha_1(0)}{\varphi^2} + \frac{2B\theta\alpha'_1(0)}{\varphi^2 - \theta^2} + \frac{\alpha'_1(0)\alpha_1(0)}{(\varphi^2 - 4\theta^2)} \right) \right] \quad (4.32.2)$$

That is,

$$\beta_7(0) = B^2 \left( \frac{\alpha S_0}{\varphi} + \frac{\alpha}{2\varphi^3} + \frac{\alpha}{2\alpha(\varphi^2 - 4\theta^2)} - \frac{\alpha}{\alpha(\varphi^2 - \theta^2)} - \frac{2\theta\alpha S_1}{\varphi(\varphi^2 - 4\theta^2)} \right) \quad (4.32.3)$$

So far, it follows that,

$$U^{(21)} = U_m^{(21)}(1 - \cos 2mx) + U_{2m}^{(21)}(1 - \cos 4mx) \quad (4.33)$$

From (3.23),

$$\begin{aligned} O(\epsilon^2 \delta^2) : U_{,\hat{t}\hat{t}}^{(22)} + U_{,xxxx}^{(22)} + 2\lambda U_{,xx}^{(22)} \\ = -U_{,\tau\tau}^{(20)} - 2\omega'_1 U_{,\hat{t}\hat{t}}^{(12)} - 2U_{,\hat{t}\tau}^{(21)} - 2\omega'_1 U_{,\hat{t}\hat{t}}^{(12)} - 2\omega''_1 U_{,\hat{t}}^{(11)} - 2U_{,\hat{t}}^{(21)} - \\ 2\omega'_1 U_{,\hat{t}}^{(11)} - \alpha \left\{ (U^{(11)})^2 + U^{(10)} \right\} \end{aligned}$$

Substituting on the right-hand side gives,

$$\begin{aligned} U_{,\hat{t}\hat{t}}^{(22)} + U_{,xxxx}^{(22)} + 2\lambda U_{,xx}^{(22)} \\ = - \left[ U_{m,\tau\tau}^{(20)}(1 - \cos 2mx) + U_{2m,\tau\tau}^{(20)}(1 - \cos 4mx) + 2\omega'_1 U_{m,\hat{t}\hat{t}}^{(12)}(1 - \cos 2mx) + 2 \left\{ U_{m,\hat{t}\tau}^{(21)}(1 - \cos 2mx) + U_{2m,\hat{t}\tau}^{(21)}(1 - \cos 2mx) \right\} + 2\omega''_1 U_{m,\hat{t}}^{(11)}(1 - \cos 2mx) + 2 \left\{ U_{m,\hat{t}}^{(21)}(1 - \cos 2mx) + U_{2m,\hat{t}}^{(21)}(1 - \cos 4mx) \right\} + 2\omega'_1 U_{m,\hat{t}\hat{t}}^{(11)}(1 - \cos 2mx) + \alpha \left( U_m^{(11)} \right)^2 \left\{ \frac{3}{2} - 2\cos 2mx + \frac{1}{2}\cos 4mx \right\} + 2 \left\{ U_m^{(10)} U_m^{(12)} \right\} \left\{ \frac{3}{2} - 2\cos 2mx + \frac{1}{2}\cos 4mx \right\} \right] \quad (4.34) \end{aligned}$$

Let,

$$U^{(22)} = \sum_{n=1}^{\infty} U_n^{(22)}(\hat{t}\tau)(1 - \cos 2nx)$$

The left-hand side of (4.34) simplifies to

$$\sum_{n=1} \left[ U_{n,\hat{t}\hat{t}}^{(22)} (1 - \cos 2nx) + (-16n^4 + 8\lambda n^2) U_n^{(22)} + U_n^{(22)} (1 - \cos 2nx) \right]$$

Multiplying (4.30) through by  $\cos 2mx$  and integrating from 0 to  $\pi$  and for  $n=m$ , gives;

$$\begin{aligned} & -\frac{\pi}{2} U_{m,\hat{t}\hat{t}}^{(22)} + (-16m^4 + 8\lambda m^2) U_m^{(22)} \left( \frac{\pi}{2} \right) + \left( -\frac{\pi}{2} U_m^{(22)} \right) = - \left[ \left( -\frac{\pi}{2} \right) U_{m,\tau\tau}^{(20)} + \right. \\ & 2\omega_1' U_{m,\hat{t}\hat{t}}^{(12)} \left( -\frac{\pi}{2} \right) + 2U_{m,\hat{t}\tau}^{(12)} \left( -\frac{\pi}{2} \right) + 2\omega_1'' U_{m,\hat{t}}^{(11)} \left( -\frac{\pi}{2} \right) + 2U_{m,\hat{t}}^{(21)} \left( -\frac{\pi}{2} \right) + \\ & \left. + 2\omega_1' U_{m,\hat{t}}^{(11)} \left( -\frac{\pi}{2} \right) + \alpha \left( U_m^{(11)} \right)^2 \left( -2 \cdot \frac{\pi}{2} \right) + \alpha U_m^{(10)} U_m^{(12)} \left( -2 \cdot \frac{\pi}{2} \right) \right] \end{aligned} \quad (4.35.1)$$

Further simplification of (4.35.1) gives

$$\begin{aligned} & -\frac{\pi}{2} U_{m,\hat{t}\hat{t}}^{(22)} + (16m^4 - 8\lambda m^2 + 1) U_m^{(22)} = -\frac{\pi}{2} \left[ -U_{m,\tau\tau}^{(20)} - 2\omega_1' U_{m,\hat{t}\hat{t}}^{(12)} - 2U_{m,\hat{t}\tau}^{(21)} - \right. \\ & \left. 2\omega_1'' U_{m,\hat{t}}^{(11)} - 2U_{m,\hat{t}}^{(21)} - 2\omega_1' U_{m,\hat{t}}^{(11)} + 2\alpha \left( U_m^{(11)} \right)^2 + 2\alpha U_m^{(10)} U_m^{(12)} \right] \end{aligned} \quad (4.35.2)$$

Further simplification yields

$$\begin{aligned} & U_{m,\hat{t}\hat{t}}^{(22)} + \theta^2 U_m^{(22)} = - \left[ U_{m,\tau\tau}^{(20)} + 2\omega_1' U_{m,\hat{t}\hat{t}}^{(12)} + 2U_{m,\hat{t}\tau}^{(21)} + 2\omega_1'' U_{m,\hat{t}}^{(11)} + 2U_{m,\hat{t}}^{(21)} + \right. \\ & \left. 2\omega_1' U_{m,\hat{t}}^{(11)} - 2 \left\{ \left( U_m^{(11)} \right)^2 + U_m^{(10)} U_m^{(12)} \right\} \right] \end{aligned} \quad (4.35.3)$$

The initial conditions are as follows:

$$U_m^{(22)}(0,0) = 0; U_{m,\hat{t}}^{(22)}(0,0) + \omega_1'(0) U_{m,\hat{t}}^{(12)} + U_{m,\tau}^{(21)}(0,0) = 0$$

Next from equation (4.34) for  $n=2m$ ,

Let,

$$U^{(22)} = \sum_{n=1}^{\infty} U_n^{(22)} (1 - \cos 4mx)$$

Multiplying (4.34) through by  $\cos 4mx$  and integrating from 0 to  $\pi$  and for  $n = 2m$ , gives

$$\begin{aligned} & -\frac{\pi}{2} U_{2m,\hat{t}\hat{t}}^{(22)} + \frac{\pi}{2} (-256m^4 + 32\lambda m^2) U_{2m}^{(22)} - \frac{\pi}{2} U_{2m}^{(22)} = \frac{\alpha}{2} \left\{ \left( U_m^{(11)} \right)^2 + \right. \\ & \left. \left( U_m^{(10)} U_m^{(12)} \right) \left( \frac{\pi}{2} \right) \right\} \end{aligned} \quad (4.36.1)$$

This further gives

$$U_{2m,\hat{t}\hat{t}}^{(22)} + \varphi^2 U_{2m}^{(22)} = -\frac{\alpha}{2} \left\{ \left( U_m^{(11)} \right)^2 + \left( U_m^{(10)} U_m^{(12)} \right) \right\} \quad (4.36.2)$$

The initial conditions are

$$U_m^{(22)}(0,0) = 0; U_{2m,\hat{t}}^{(22)}(0,0) + \omega_1'(0)U_{2m,\hat{t}}^{(12)} + U_{m,\tau}^{(21)}(0,0) = 0$$

On substituting for terms in (4.35.3) and simplifying, the result is

$$\begin{aligned} U_{2m,\hat{t}\hat{t}}^{(22)} + \theta^2 U_m^{(22)} = & - \left[ \left\{ \alpha_4'' \cos\theta\hat{t} + \frac{r_0''}{\theta^2} - \frac{r_1'' \cos 2\theta\hat{t}}{3\theta^2} \right\} + 2\omega_1' \{ -\theta^2 \alpha_3 \sin\theta\hat{t} + \right. \\ & \left. \beta_3 \theta^2 \cos\theta\hat{t} \} + 2 \left\{ -\theta \alpha_6' \sin\theta\hat{t} + \theta \beta_6' \cos\theta\hat{t} - \frac{2\theta r_3' \sin 2\theta\hat{t} + 2\theta r_4' \cos 2\theta\hat{t}}{3\theta^2} \right\} + \right. \\ & \left. 2\omega_1' (\theta \beta_2 \cos\theta\hat{t}) + 2 \left\{ -\alpha_6 \theta \sin\theta\hat{t} + \beta_6 \theta \cos\theta\hat{t} + \left( \frac{2\theta r_3 \sin\theta\hat{t} - 2\theta r_4 \cos\theta\hat{t}}{3\theta^2} \right) \right\} + \right. \\ & \left. 2\omega_1' \{ \theta \beta_2 \cos\theta\hat{t} \} + 2\alpha \left\{ \frac{\beta_2}{2} (1 - \cos 2\theta\hat{t}) + \left( \frac{\alpha_1 \beta_2}{2} \sin 2\theta\hat{t} + B \beta_2 \sin 2\theta\hat{t} \right) \right\} \right] \quad (4.37.1) \end{aligned}$$

To ensure a uniformly valid solution in  $\hat{t}$ ; equate to zero the coefficients of  $\cos\theta\hat{t}$  and  $\sin\theta\hat{t}$  of (4.37.1) and this yields respectively

$$-\alpha_4'' + 2\omega_1' \theta \beta_3 - 2\theta \beta_6' - 2\omega_1'' \theta \beta_2 - 2\beta_6 \theta - 2\omega_1' \theta \beta_2 = 0 \quad (4.37.2)$$

And,

$$2\omega_1' \theta^2 \alpha_3 + 2\theta \alpha_6' + 2\alpha_6 \theta - 2\alpha B \beta_2 = 0 \quad (4.37.3)$$

Simplification of (4.37.2) gives

$$\beta_6' + \beta_6 = \frac{1}{2\theta} [\alpha_4'' - 2\omega_1' \theta^2 \beta_3 + 2\omega_1'' \beta_2 + 2\omega_1' \theta \beta_2] = \rho_2(\tau) \quad (4.37.4)$$

This gives

$$\beta_6(\tau) = e^{-\tau} [\int e^\tau \rho_2(\tau) d\tau + \beta_6(0)] = e^{-\tau} [\int e^\tau \rho_2(\tau) d\tau] \quad (4.37.5)$$

Similarly, simplification of (4.37.3) yields

$$\alpha_6' + \alpha_6 = \frac{1}{2\theta} [-2\omega_1' \theta^2 \alpha_3 + 2\alpha B \beta_2] = \rho_3(\tau) \quad (4.37.6)$$

Therefore

$$\alpha_6 = e^{-\tau} [\int e^\tau \rho_3(\tau) d\tau + \alpha_6(0)] \quad (4.37.7)$$

The remaining part of equation (4.37.1) is;

$$U_{m,\hat{t}\hat{t}}^{(22)} + \theta^2 U_m^{(22)} = r_7 + r_8 \cos 2\theta\hat{t} + r_9 \sin 2\theta\hat{t} \quad (4.38)$$

$$r_7 = -\left[\frac{r_0''}{\theta^2} - \frac{2r_4'}{3\theta} + \alpha\beta_2\right], r_8 = \left[\frac{r_1''}{3\theta^2} - \frac{4r_4}{3\theta} - \alpha\beta_2\right], r_9 = -\left[\frac{2r_3'}{3\theta^2} - \frac{4r_3}{3\theta} - \alpha\alpha_1\beta_2\right]$$

It is to be recalled that

$$r_0 = -2\alpha\left(\frac{\alpha_1^2}{2} + B^2\right)$$

Therefore,

$$r_0' = -2\alpha\alpha_1\alpha_1', r_0'' = -2\alpha(\alpha_1'^2 + \alpha_1\alpha_1''), r_0'(0) = 2\alpha B^2, r_0''(0) = -4\alpha B^2$$

$$r_4 = \alpha\alpha_1\beta_2 + \frac{8\alpha}{3\theta}(\alpha_1\alpha_1' + \alpha_1'^2)$$

$$r_4' = \alpha(\alpha_1'\beta_2 + \alpha_1'\beta_2') + \frac{8\alpha}{3\theta}(\alpha_1'\alpha_1' + \alpha_1\alpha_1'' + 2\alpha_1\alpha_1')$$

$$r_4'(0) = \frac{-5\alpha B^2}{2\theta}, r_7(0) = \frac{17\alpha B^2}{3\theta^2} - \frac{\alpha B}{\theta}$$

$$r_1 = -\alpha_1\alpha_1^2; r_1' = 2\alpha\alpha_1\alpha_1'; r_1'' = -2\alpha(\alpha_1'^2 + \alpha_1\alpha_1'')$$

$$r_1'(0) = -2\alpha\alpha_1(0)\alpha_1'(0) = 2\alpha B^2; r_1''(0) = -4\alpha B^2$$

$$r_8(0) = \left(\frac{-4\alpha B^2}{3\theta^2} + \frac{4\alpha B^2}{3\theta^2} + \frac{\alpha B}{3\theta}\right) = \frac{\alpha B}{\theta}, r_3 = \alpha\alpha_1\alpha_2; r_3(0) = 0$$

$$r_3' = \alpha(\alpha_1'\alpha_1 + \alpha_1\alpha_2'); r_3'(0) = 0, \beta_2' = \frac{3B}{2\theta}, r_9(0) = \frac{3\theta B^2}{2\theta}$$

It therefore follows that

$$U_m^{(22)} = \alpha_8 \cos\theta + \beta_8 \sin\theta + \frac{r_7}{3\theta^2} + \frac{r_8 \cos 2\theta}{\theta^2} + \frac{r_9 \sin 2\theta \hat{t}}{\theta^2} \quad (4.39.1)$$

The initial conditions are

$$U_m^{(22)}(0,0) = 0; U_m^{(22)}(0,0) + \omega_1'(0)U_{m,\hat{t}}^{(12)}(0,0) + U_{m,\tau}^{(21)}(0,0) = 0$$

It follows that;

$$\alpha_8(0) = \left(\frac{r_8(0)}{3\theta^2} - \frac{r_7(0)}{\theta^2}\right) = \frac{4\alpha B}{3\theta^3} - \frac{17\alpha B^2}{3\theta^4} \quad (4.39.2)$$

$$\theta\beta_8(0) + \frac{2\theta r_9(0)}{\theta^2} = 0$$

Therefore,

$$\beta_8(0) = \frac{-2\theta r_9(0)}{\theta^2} = \frac{-3\alpha B^2}{\theta^3} \quad (4.39.3)$$

### Solution for equations of order $\epsilon^3 \delta^j$ for $j=0, 1, 2$

From equation (3.23);

$$O(\epsilon^3) : U_{,\hat{t}\hat{t}}^{(30)} + U_{,xxxx}^{(30)} + 2\lambda U_{,xx}^{(30)} + U^{(30)} = -(\omega_1')^2 U_{,\hat{t}\hat{t}}^{(10)} - 2(\omega_1' U_{,\hat{t}\hat{t}}^{(20)} + \omega_2' U_{,\hat{t}\hat{t}}^{(20)}) - 2\alpha U^{(20)} U^{(10)} + \beta (U^{(10)})^3$$

Let,

$$U^{(30)} = U_m^{(30)} (1 - \cos 2mx)$$

Then, substituting on the right-hand side of (3.23) yields

$$\begin{aligned} U_{,\hat{t}\hat{t}}^{(30)} + U_{,xxxx}^{(30)} + 2\lambda U_{,xx}^{(30)} + U^{(30)} &= -(\omega_1')^2 U_{,\hat{t}\hat{t}}^{(10)} (1 - \cos 2mx) - \\ 2 \left[ \omega_1' U_{,\hat{t}\hat{t}}^{(20)} + (1 - \cos 2mx) + \omega_1' U_{2m,\hat{t}\hat{t}}^{(20)} (1 - \cos 2mx) \right] &- 2\omega_2' U_{,\hat{t}\hat{t}}^{(10)} (1 - \cos 2mx) - \\ 2\alpha \left[ U_m^{(10)} (1 - \cos 2mx) \left\{ U_m^{(20)} (1 - \cos 2mx) + U_{2m}^{(20)} (1 - \cos 4mx) \right\} \right] &+ \beta (U_m^{(10)})^3 (1 - \\ \cos 2mx)^3 & \end{aligned} \quad (4.40)$$

The following simplifications are necessary as they appear in equation (4.40);

$$\begin{aligned} &U_m^{(10)} U_m^{(20)} (1 - \cos 2mx) (1 - \cos 2mx) \\ &= U_m^{(10)} U_m^{(20)} (1 - \cos 2mx)^2 = U_m^{(10)} U_m^{(20)} [1 - \cos 2mx + \cos^2 2mx] \\ &= U_m^{(10)} U_m^{(20)} \left[ 1 - 2\cos 2mx + \left( \frac{1 + \cos 4mx}{2} \right) \right] \\ &= U_m^{(10)} U_m^{(20)} \left( \frac{3}{2} - 2\cos 2mx + \frac{1}{2} \cos 4mx \right) \end{aligned}$$

Also,

$$\begin{aligned} &U_m^{(10)} U_{2m}^{(20)} (1 - \cos 2mx) (1 - \cos 4mx) \\ &= U_m^{(10)} U_{2m}^{(20)} \left( 1 - \frac{1}{2} \cos 2mx - \cos 4mx + \frac{1}{2} \cos 6m \right) \end{aligned}$$

Similarly,

$$\begin{aligned}
& \beta(U_m^{(10)})^3 (1 - \cos 2mx)^3 = \beta(U_m^{(10)})^3 [1 - 3\cos 2mx + 3\cos^2 2mx - \\
& \cos 2^3 mx] \\
& = \beta(U_m^{(10)})^3 \left[ 1 - 3\cos 2mx + \frac{3}{2}(1 + \cos 4mx) - \frac{1}{4}(3\cos 2mx + \cos 6mx) \right] \\
& = \beta(U_m^{(10)})^3 \left[ \frac{5}{2} - \frac{15}{4}\cos 2mx + \frac{3}{2}\cos 4mx - \frac{1}{4}\cos 6mx \right]
\end{aligned}$$

Therefore; substituting all these on the right-hand side of (4.40) yields

$$\begin{aligned}
U_{,\hat{t}\hat{t}}^{(30)} + U_{,xxxx}^{(30)} + 2\lambda U_{,xx}^{(30)} + U^{(30)} = & -(\omega'_1)^2 U_{m,\hat{t}\hat{t}}^{(10)}(1 - \cos 2mx) \\
& - 2 \left[ \omega'_1 U_{m,\hat{t}\hat{t}}^{(20)}(1 - \cos 2mx) + \omega'_1 U_{2m,\hat{t}\hat{t}}^{(20)}(1 - \cos 4mx) + \omega'_2 U_m^{(10)}(1 - \right. \\
& \left. \cos 2mx) \right] - 2\alpha \left[ U_m^{(10)} U_m^{(20)} \left\{ \frac{3}{2} - 2\cos 2mx + \frac{1}{2}\cos 4mx \right\} + U_m^{(10)} U_m^{(20)} \left\{ 1 - \frac{1}{2}\cos 2mx - \right. \right. \\
& \left. \left. \cos 4mx + \frac{1}{2}\cos 6mx \right\} \right] + \beta(U_m^{(10)})^3 \left[ \frac{5}{2} - \frac{15}{4}\cos 2mx + \frac{3}{2}\cos 4mx + \frac{1}{4}\cos 6mx \right] \quad (4.41)
\end{aligned}$$

Let,

$$U^{(30)} = \sum_{n=1}^{\infty} U_n^{(30)} (1 - \cos 2nx)$$

Left hand side of (4.41) becomes

$$\sum_{n=1}^{\infty} \left[ U_{n,\hat{t}\hat{t}}^{(30)} (1 - \cos 2nx) + (-16n^4 + 8\lambda n^2 + 1) U_n^{(30)} \cos 2nx \right]$$

Multiplying (4.41) through by  $\cos 2mx$  and integrating from 0 to  $\pi$  and for  $n=m$ , the result is,

$$\begin{aligned}
& -\frac{\pi}{2} U_{m,\hat{t}\hat{t}}^{(30)} + (-16m^4 + 8\lambda m^2 + 1) U_m^{(30)} \left( -\frac{\pi}{2} \right) = - \left[ (\omega'_1)^2 U_{m,\hat{t}\hat{t}}^{(10)} \left( -\frac{\pi}{2} \right) + \right. \\
& 2\omega'_1 U_{m,\hat{t}\hat{t}}^{(20)} \left( -\frac{\pi}{2} \right) + 2\omega'_2 U_{m,\hat{t}\hat{t}}^{(10)} \left( -\frac{\pi}{2} \right) + 2\alpha U_m^{(10)} U_m^{(20)} \left( -2, -\frac{\pi}{2} \right) - \alpha U_m^{(10)} U_{2m}^{(20)} \left( -\frac{\pi}{2} \right) - \\
& \left. \frac{15}{4} (U_m^{(10)})^3 \left( -\frac{\pi}{2} \right) \right] \quad (4.42.1)
\end{aligned}$$

Further simplification of (4.42.1) yields;

$$-\frac{\pi}{2} \left[ U_{m,\hat{t}\hat{t}}^{(30)} + (16m^4 - 8\lambda m^2 + 1)U_m^{(30)} \right] = -\frac{\pi}{2} \left[ -(\omega'_1)^2 U_{m,\hat{t}\hat{t}}^{(10)} - 2\omega'_1 U_{m,\hat{t}\hat{t}}^{(20)} - 2\omega'_2 U_{m,\hat{t}\hat{t}}^{(10)} - 2\alpha \left[ 2U_m^{(10)} U_m^{(20)} + U_m^{(10)} U_{2m}^{(20)} \right] - \frac{15}{4} \beta (U_m^{(10)})^3 \right] \quad (4.42.2)$$

Yet, further simplification yields,

$$U_{m,\hat{t}\hat{t}}^{(30)} + \theta^2 U_m^{(30)} = -(\omega'_1)^2 U_{m,\hat{t}\hat{t}}^{(10)} - 2\omega'_1 U_{m,\hat{t}\hat{t}}^{(20)} - 2\omega'_2 U_{m,\hat{t}\hat{t}}^{(10)} - 2\alpha \left[ 2U_m^{(10)} U_m^{(20)} + U_m^{(10)} U_{2m}^{(20)} \right] - \frac{15}{4} \beta (U_m^{(10)})^3 \quad (4.43)$$

The initial conditions are

$$U_m^{(30)}(0,0) = 0;$$

$$U_{m,\hat{t}}^{(30)}(0,0) + \omega'(0)U_{m,\hat{t}}^{(20)}(0,0) + \omega'_2(0)U_{\tau}^{(10)}(0,0) = 0$$

Multiplying (4.41) through by  $\cos 4mx$  and integrating from 0 to  $\pi$  and for  $n=2m$ , the result gives

$$\begin{aligned} & -\frac{\pi}{2} \left[ U_{2m,\hat{t}\hat{t}}^{(30)} + (256m^4 - 32\lambda m^2 + 1)U_{2m}^{(30)} \right] \\ & = 2 \left[ \omega'_1 U_{2m,\hat{t}\hat{t}}^{(20)} \left( -\frac{\pi}{2} \right) \right] + 2\alpha \left[ U_m^{(10)} U_m^{(20)} \cdot \frac{1}{2} \left( -\frac{\pi}{2} \right) + U_m^{(10)} U_{2m}^{(20)} \left( -\frac{\pi}{2} \right) \right] \\ & \quad - \beta (U_m^{(10)})^3 \cdot \frac{3}{2} \left( -\frac{\pi}{2} \right) \\ & U_{2m,\hat{t}\hat{t}}^{(30)} + \varphi^2 U_{2m}^{(30)} = -2 \left[ -\omega'_1 U_{2m,\hat{t}\hat{t}}^{(20)} \right] + 2\alpha \left[ U_m^{(10)} U_m^{(20)} - U_m^{(10)} U_{2m}^{(20)} \right] \\ & - \frac{3}{2} \beta (U_m^{(10)})^3 \end{aligned} \quad (4.44)$$

The initial conditions are

$$U_{2m}^{(30)}(0,0) = 0; U_{2m,\hat{t}}^{(30)}(0,0) + \omega'_1(0)U_{2m,\hat{t}}^{(20)}(0,0) = 0$$

Multiplying (4.41) through by  $\cos 6mx$  and integrating from 0 to  $\pi$  and for  $n=3m$  and get,

$$U_{3m,\hat{t}\hat{t}}^{(30)} + (1296m^4 - 72\lambda m^2 + 1)U_{3m}^{(30)} = \alpha U_m^{(10)} U_{2m}^{(20)} - \frac{1}{4} \beta (U_m^{(10)})^3 \quad (4.45.1)$$

Let,

$$\Omega^2 = 1296m^4 - 72\lambda m^2 + 1 > 0 \text{ for all } m$$

Therefore,

$$U_{3m,\hat{t}\hat{t}}^{(30)} + \Omega^2 U_{3m}^{(30)} = \alpha U_m^{(10)} U_{2m}^{(20)} - \frac{1}{4} \beta (U_m^{(10)})^3 \quad (4.45.2)$$

The initial conditions for (4.45.2) are

$$U_{3m}^{(30)}(0,0) = 0; U_{3m,\hat{t}}^{(30)}(0,0) = 0$$

The following term is now simplified before substitution

$$\begin{aligned} (U_m^{(10)} U_{2m}^{(20)}) &= (\alpha_1 \cos \theta \hat{t} + B) \left[ \alpha_4 \cos \theta \hat{t} + \frac{r_0}{\theta^2} - \frac{r_1 \cos 2\theta \hat{t}}{3\theta^2} \right] \\ &= \frac{\alpha_1 \alpha_4}{2} \{ \cos^2 \theta \hat{t} \} + \frac{\alpha_1 r_0}{2} \cos \theta \hat{t} - \frac{\alpha_1 r_1}{6\theta^2} (\cos 3\theta \hat{t} + \cos \theta \hat{t}) + B \alpha_4 \cos \theta \hat{t} + \frac{B r_0}{\theta^2} - \frac{B r_1 \cos 2\theta \hat{t}}{3\theta^2} \\ &= \frac{\alpha_1 \alpha_4}{4} \{ 1 + \cos 2\theta \hat{t} \} + \frac{\alpha_1 r_0}{2} \cos \theta \hat{t} - \frac{\alpha_1 r_1}{6\theta^2} (\cos 3\theta \hat{t} + \cos \theta \hat{t}) + B \alpha_4 \cos \theta \hat{t} + \frac{B r_0}{\theta^2} \\ &\quad - \frac{B r_1 \cos 2\theta \hat{t}}{3\theta^2} \\ &= \left( \frac{\alpha_1 \alpha_4}{4} + \frac{B r_0}{\theta^2} \right) + \left( \frac{\alpha_1 r_0}{\theta^2} - \frac{\alpha_1 r_1}{6\theta^2} + B \alpha_4 \right) \cos \theta \hat{t} + \left( \frac{\alpha_1 \alpha_4}{4} - \frac{B r_0}{3\theta^2} \right) \cos 2\theta \hat{t} - \\ &\quad \frac{\alpha_1 r_1}{6\theta^2} \cos 3\theta \hat{t} \end{aligned} \quad (4.45.3)$$

The following is further simplified;

$$\begin{aligned} (U_m^{(10)} U_{2m}^{(20)}) &= (\alpha_1 \cos \theta \hat{t} + B) \left[ \alpha_5 \cos \varphi \hat{t} + \beta_5 \sin \varphi \hat{t} + \frac{\alpha}{2} \left\{ \frac{\left( \frac{\alpha_1^2}{2} + B^2 \right)}{\varphi^2} + \frac{2B\alpha_1 \cos \theta \hat{t}}{\varphi^2 - \theta^2} \right\} \right. \\ &\quad \left. + \frac{\alpha_1^2 \cos \theta \hat{t}}{2(\varphi^2 - \theta^2)} \right] \\ &= \alpha_1 \alpha_5 \left\{ \frac{\cos(\varphi + \theta) \hat{t} + \cos(\varphi - \theta) \hat{t}}{2} \right\} + \frac{\alpha_1 \beta_5}{2} \left\{ \frac{\sin(\varphi + \theta) \hat{t} + \sin(\varphi - \theta) \hat{t}}{2} \right\} \\ &\quad + \frac{\alpha_1 \alpha}{2} \frac{\left( \frac{\alpha_1^2}{2} + B^2 \right)}{\varphi^2} \cos \theta \hat{t} + \frac{\alpha \alpha_1^2 B}{2(\varphi^2 - \theta^2)} (1 + \cos 2\theta \hat{t}) \\ &\quad + \frac{\alpha_1^3 \alpha}{8(\varphi^2 - \theta^2)} \{ \cos 3\theta \hat{t} + \cos \theta \hat{t} \} \\ &= \frac{\alpha \alpha_1^2 B}{2(\varphi^2 - \theta^2)} + \left\{ \frac{\alpha \alpha_1 \left( \frac{\alpha_1^2}{2} + B^2 \right)}{2\varphi^2} + \frac{\alpha_1^3 \alpha}{8(\varphi^2 - 4\theta^2)} \right\} \cos \theta \hat{t} + \frac{\alpha_1 \alpha_5}{2} \cos(\varphi + \theta) \hat{t} + \\ &\quad \frac{\alpha \alpha_1^2 B \cos 2\theta \hat{t}}{2(\varphi^2 - \theta^2)} + \frac{\alpha_1 \beta_5}{2} \sin(\varphi + \theta) \hat{t} + \frac{\alpha_1 \alpha_5}{2} \cos(\varphi - \theta) \hat{t} + \frac{\alpha_1 \beta_5}{2} \sin(\varphi - \theta) \hat{t} + \frac{\alpha_1^3 \alpha \cos 3\theta \hat{t}}{8(\varphi^2 - 4\theta^2)} \end{aligned} \quad (4.45.4)$$

Similarly,

$$(U_m^{(10)})^3 = \left[ \left( B^3 + \frac{3\alpha_1^3 B}{2} \right) + 3 \left( \frac{\alpha_1^3}{4} + \alpha_1 B^2 \right) \cos\theta \hat{t} + \frac{3\alpha_1^2 B}{2} \cos 2\theta \hat{t} + \frac{\alpha_1^3}{4} \cos 3\theta \hat{t} \right] \quad (4.45.5)$$

Substituting in (4.45.3) to (4.45.5) into (4.43) gives

$$\begin{aligned} U_{m,\hat{t}\hat{t}}^{(30)} + \theta^2 U_m^{(30)} = & -(\omega'_1)^2 (-\theta^2 \alpha_1 \cos\theta \hat{t}) - 2\omega'_1 \left( -\alpha_4 \theta^2 \cos\theta \hat{t} + \right. \\ & \left. \frac{4r_1 \cos 2\theta \hat{t}}{3} \right) - 2\omega'_2 (-\theta^2 \alpha_1 \cos\theta \hat{t}) - 2\alpha \left[ \left( \frac{\alpha_1 \alpha_4}{4} + \frac{Br_0}{\theta^2} \right) + \left( \frac{\alpha_1 r_0}{\theta^2} - \frac{\alpha_1 r_1}{6\theta^2} + B\alpha_4 \right) \cos\theta \hat{t} + \left( \frac{\alpha_1 \alpha_4}{4} - \right. \right. \\ & \left. \left. \frac{Br_1}{3\theta^2} \right) \cos 2\theta \hat{t} - \frac{\alpha_1 r_1}{6\theta^2} \cos 3\theta \hat{t} \right] - 2\alpha \left[ \frac{\alpha \alpha_1^2 B}{2(\varphi^2 - \theta^2)} + \left\{ \frac{\alpha \alpha_1^2 \left( \frac{\alpha_1^2}{2} + B^2 \right)}{2\varphi^2} + \frac{\alpha \alpha_1^3}{8(\varphi^2 - 4\theta^2)} \right\} \cos\theta \hat{t} + \right. \\ & \left. \frac{\alpha_1 \alpha_5}{2} \cos(\varphi + \theta) \hat{t} + \frac{\alpha \alpha_1^2 B \cos 2\theta \hat{t}}{2(\varphi^2 - \theta^2)} + \frac{\alpha_1 \beta_5}{2} \sin(\varphi + \theta) \hat{t} + \frac{\alpha_1 \alpha_5}{2} \cos(\varphi - \theta) \hat{t} + \frac{\alpha_1 \beta_5}{2} \sin(\varphi - \right. \\ & \left. \theta) \hat{t} + \frac{\alpha \alpha_1^3 \cos 3\theta \hat{t}}{8(\varphi^2 - 4\theta^2)} \right] - \frac{15\beta}{4} \left[ \left( B^3 + \frac{3\alpha_1^2 B}{2} \right) \right] + 3 \left( \frac{\alpha_1^3}{4} + \alpha_1 B^2 \right) \cos\theta \hat{t} + \frac{3\alpha_1^2}{2} B \cos 2\theta \hat{t} + \frac{\alpha_1^3}{4} \cos 3\theta \hat{t} \end{aligned} \quad (4.45.6)$$

To ensure a uniformly valid solution in  $\hat{t}$ , equate to zero the coefficients of  $\cos\theta \hat{t}$  and this yields

$$\begin{aligned} & (\omega'_1)^2 \theta^2 + 2\omega'_1 \alpha_4 \theta^2 + 2\omega'_2 \theta^2 \alpha_1 - 2\alpha \left( \frac{\alpha_1 r_0}{\theta^2} - \frac{\alpha_1 r_1}{6\theta^2} + B\alpha_4 \right) - \left\{ \frac{\alpha^2 \alpha_1 \left( \frac{\alpha_1^2}{2} + B^2 \right)}{\varphi^2} + \right. \\ & \left. \frac{\alpha_1^3 \alpha_1}{4(\varphi^2 - 4\theta^2)} \right\} - \frac{45\beta}{4} \left( \frac{\alpha_1^3}{4} + \alpha_1 B^2 \right) = 0 \end{aligned}$$

Therefore,

$$\begin{aligned} \omega'_2 = & -\frac{1}{2\theta^2 \alpha_1} \left[ (\omega'_1)^2 \theta^2 + 2\omega'_1 \alpha_4 \theta^2 - 2\alpha \left( \frac{\alpha_1 r_0}{\theta^2} - \frac{\alpha_1 r_1}{6\theta^2} + B\alpha_4 \right) - \left\{ \frac{\alpha^2 \alpha_1 \left( \frac{\alpha_1^2}{2} + B^2 \right)}{\varphi^2} + \right. \right. \\ & \left. \left. \frac{\alpha_1^3 \alpha_1}{4(\varphi^2 - 4\theta^2)} \right\} - \frac{45\beta}{4} \left( \frac{\alpha_1^3}{4} + \alpha_1 B^2 \right) \right] \end{aligned} \quad (4.46)$$

The remaining equation in (4.45.6) is

$$\begin{aligned} U_{m,\hat{t}\hat{t}}^{(30)} + \theta^2 U_m^{(30)} = & r_{10} + r_{11} \cos 2\theta \hat{t} + r_{12} \cos 3\theta \hat{t} + r_{13} \cos(\varphi + \theta) \hat{t} + r_{14} \sin(\varphi + \\ & \theta) \hat{t} + r_{15} \cos(\varphi - \theta) \hat{t} + r_{16} \sin(\varphi - \theta) \hat{t} \end{aligned} \quad (4.47)$$

Solving (4.47) gives,

$$U_m^{(30)}(\hat{t}, \tau) = \alpha_9(\tau) \cos \theta \hat{t} + \beta_9(\tau) \sin \theta \hat{t} + \frac{r_{10}}{\theta^2} - \frac{r_{11} \cos \theta \hat{t}}{3\theta^2} - \frac{r_{12} \cos 3\theta \hat{t}}{8\theta^2} - \frac{1}{\varphi(2\theta+\varphi)} [r_{13} \cos(\varphi + \theta) \hat{t} + r_{14} \sin(\varphi + \theta) \hat{t}] + \frac{1}{\varphi(2\theta-\varphi)} [r_{15} \cos(\varphi - \theta) \hat{t} + r_{16} \sin(\varphi - \theta) \hat{t}] \quad (4.48)$$

The initial conditions are

$$U_m^{(30)}(0,0) = 0$$

$$U_{m,\hat{t}}^{(30)}(0,0) + \omega_1'(0) U_{m,\hat{t}}^{(20)}(0,0) + \omega_2'(0) U_{m,\tau}^{(10)}(0,0) = 0$$

Where,

$$\alpha_9(0) = \left[ -\frac{r_{10}}{\theta^2} + \frac{r_{11}}{3\theta^2} + \frac{r_{12}}{8\theta^2} + \frac{r_{13}}{\varphi(2\theta+\varphi)} - \frac{r_{15}}{\varphi(2\theta-\varphi)} \right] \text{ at } \tau = 0$$

And

$$\beta_9(0) = \frac{1}{\theta} \left[ \frac{r_{14}(\varphi+\theta)}{\varphi(2\theta+\varphi)} - \frac{r_{16}(\varphi-\theta)}{\varphi(2\theta-\varphi)} \right] \text{ at } \tau = 0$$

Where,

$$r_{10} = - \left[ 2\alpha \left( \frac{\alpha_1 \alpha_4}{4} + \frac{Br_0}{\theta^2} \right) + \frac{\alpha^2 \alpha_1^2 B}{\varphi^2 - \theta^2} - \frac{15\beta}{4} \left( B^3 + \frac{3\alpha_1^2 B}{2} \right) \right]$$

$$r_{10}(0) = B^3 \left( \frac{10\alpha^2}{3\theta^2} - \frac{\alpha^2}{\varphi^2 - \theta^2} + \frac{75\beta}{8} \right)$$

$$r_{10}'(0) = B^3 \left[ \frac{\alpha S_5}{2} - \frac{8\alpha^2}{3\theta^2} - \frac{4\alpha^2}{\theta} - \frac{2\alpha^2}{(\varphi^2 - \theta^2)} - \frac{45\beta}{4} \right]$$

$$r_{11} = - \left( \frac{8r_1 \omega_1'}{3} + 2\alpha \left( \frac{\alpha_1 \alpha_4}{4} + \frac{Br_0}{3\theta^2} \right) + \frac{\alpha^2 \alpha_1^2 B}{\varphi^2 - \theta^2} + \frac{45\beta \alpha_1^2 B}{2} \right)$$

$$r_{11}(0) = B^3 \left( \frac{8\alpha}{3\theta^2} + \frac{2\alpha^2}{3\theta^2} - \frac{45\beta}{8} + \frac{\alpha^2}{\varphi^2 - \theta^2} \right)$$

$$r_{11}'(0) = B^3 \left[ \frac{4\alpha^2}{3\theta^2} - \frac{16\alpha}{3\theta^2} - \frac{\alpha S_5}{2} - \frac{2\alpha}{(\varphi^2 - \theta^2)} - \frac{45\beta}{4} \right]$$

$$r_{12} = \frac{\alpha \alpha_1 r_1}{3\theta^2} - \frac{\alpha_1^3 \alpha^2}{4(\varphi^2 - \theta^2)} - \frac{15\beta \alpha_1^3}{16},$$

$$r_{12}(0) = B^3 \left[ \frac{\alpha^2}{3\theta^2} + \frac{\alpha^2}{4(\varphi^2 - \theta^2)} + \frac{15\beta}{16} \right]$$

$$r'_{12}(0) = B^3 \left[ \frac{3\alpha^2}{4(\varphi^2 - \theta^2)} - \frac{\alpha^2}{3\theta^2} + \frac{45\beta}{4} \right]$$

$$r_{13} = -\alpha\alpha_1\alpha_5 = r_{15}$$

$$r_{13}(0) = r_{15}(0) = B^3\alpha^2S_0, \quad r'_{13}(0) = r'_{15}(0) = -2\alpha S_0 B^3$$

$$r_{14} = -\alpha\alpha_1\beta_5 = r_{16}, \quad r_{14}(0) = r_{16}(0) = 0 \text{ since } \beta_5(0) = 0,$$

$$r'_{16}(0) = r'_{14}(0) = 0$$

Substituting in (4.44) gives

$$\begin{aligned} U_{2m,\hat{t}\hat{t}}^{(30)} + \varphi^2 U_{2m}^{(30)} &= 2\omega'_1 \left[ -\varphi^2 \alpha_5 \cos\varphi\hat{t} - \varphi^2 \beta_5 \sin\varphi\hat{t} + \frac{\alpha}{2} \left\{ \frac{-2\theta^2 B \alpha_1 \cos\theta\hat{t}}{\varphi^2 - \theta^2} - \right. \right. \\ &\left. \left. \frac{2\alpha_1^2 \theta^2 \cos 2\theta\hat{t}}{\varphi^2 - 4\theta^2} \right\} \right] + \\ &2\alpha \left[ \left( \frac{\alpha_1 \alpha_4}{4} + \frac{B r_0}{\theta^2} \right) + \left( \frac{\alpha_1 r_0}{\theta^2} - \frac{\alpha_1 r_1}{6\theta^2} + B \alpha_4 \right) \cos\theta\hat{t} + \left( \frac{\alpha_1 \alpha_4}{4} + \frac{B r_1}{3\theta^2} \right) \cos 2\theta\hat{t} \right] - \\ &\left[ -\frac{\alpha_1 r_1}{6\theta^2} \cos 3\theta\hat{t} \right] - \\ &2\alpha \left[ \frac{\alpha \alpha_1^2 B}{2(\varphi^2 - \theta^2)} + \left\{ \frac{\alpha \alpha_1 \left( \frac{\alpha_1^2}{2} + B^2 \right)}{2\varphi^2} + \frac{\alpha_1^3 \alpha}{8(\varphi^2 - 4\theta^2)} \right\} \cos\theta\hat{t} + \frac{\alpha_1 \alpha_5}{2} \cos(\varphi + \theta)\hat{t} + \frac{\alpha \alpha_1^2 B \cos 2\theta\hat{t}}{2(\varphi^2 - \theta^2)} + \right. \\ &\left. \frac{\alpha_1 \beta_5}{2} \sin(\varphi + \theta)\hat{t} + \frac{\alpha_1 \alpha_5}{2} \cos(\varphi - \theta)\hat{t} + \frac{\alpha_1 \beta_5}{2} \sin(\varphi - \theta)\hat{t} + \frac{\alpha_1^3 \alpha \cos 3\theta\hat{t}}{8(\varphi^2 - 4\theta^2)} \right] - \frac{3\beta}{2} \left[ \left( B^3 + \right. \right. \\ &\left. \left. \frac{3\alpha_1^2 B}{2} \right) + 3 \left( \frac{\alpha_1^3}{4} + \alpha_1 B^2 \right) \cos\theta\hat{t} + \frac{3\alpha_1^2 B}{2} \cos 2\theta\hat{t} + \frac{\alpha_1^3}{4} \cos 3\theta\hat{t} \right] \quad (4.49) \end{aligned}$$

To ensure a uniformly valid solution in  $\hat{t}$ , needs equating to zero the coefficients of  $\cos\varphi\hat{t}$  and  $\sin\varphi\hat{t}$ .

A further simplification of (4.49) gives

$$U_{2m,\hat{t}\hat{t}}^{(30)} + \varphi^2 U_{2m}^{(30)} = r_{17} + r_{18} \cos\theta\hat{t} + r_{19} \cos 2\theta\hat{t} + r_{20} \cos 3\theta\hat{t} \quad (4.50)$$

Where,

$$r_{17} = \left[ 2\alpha \left( \frac{\alpha_1 \alpha_4}{4} + \frac{B r_0}{\theta^2} \right) - \frac{\alpha^2 \alpha_1^2 B}{\varphi^2 - \theta^2} - \frac{3\beta}{2} \left( B^3 + \frac{3\alpha_1^2 B}{2} \right) \right]$$

$$r_{17}(0) = B^3 \left( -\frac{22\alpha^2}{3\theta^2} + \frac{\alpha^2}{\varphi^2 - \theta^2} + \frac{15\beta}{4} \right)$$

$$r'_{17}(0) = B^3 \left[ \frac{-S_5\alpha}{2} + \frac{20\alpha^2}{3\theta^2} + \frac{2\alpha^2}{2(\varphi^2-\theta^2)} + \frac{9\beta}{2} \right]$$

$$r_{18} = \left[ \frac{-2\theta^2\omega'_1 B\alpha_1\alpha}{\varphi^2-\theta^2} + 2\alpha \left( \frac{\alpha_1 r_0}{\theta^2} - \frac{\alpha_1 r_1}{6\theta^2} + B\alpha_4 \right) - \frac{\alpha^2\alpha_1 \left( \frac{\alpha_1^2}{2} + B^2 \right)}{\varphi^2} - \frac{\alpha_2\alpha_1^3}{4(\varphi^2-4\theta^2)} - \frac{9}{2} \left( \frac{\alpha_1^3}{4} + \alpha_1 B^2 \right) \right]$$

$$r_{18}(0) = B^3 \left( \frac{2\alpha}{\varphi^2-\theta^2} + \frac{18\alpha^2}{6\theta^2} + \frac{3\alpha^2}{\varphi^2} + \frac{\alpha^2}{4(\varphi^2-4\theta^2)} + \frac{45}{2} \right)$$

$$r'_{18}(0) = B^3 \left[ 2\alpha S_5 - 2 - \frac{43\alpha^2}{\theta^2} - \frac{5\alpha^2}{2\varphi^2} - \frac{3\alpha^2}{4(\varphi^2-\theta^2)} - \frac{63}{8} \right]$$

$$r_{19} = \left[ \frac{-2\omega'_1\alpha_1^2\alpha\theta^2}{\varphi^2-\theta^2} + 2\alpha \left( \frac{\alpha_1\alpha_4}{4} - \frac{Br_1}{3\theta^2} \right) + \frac{\alpha^2\alpha_1^2 B}{2(\varphi^2-\theta^2)} - \frac{9\alpha_1^2 B\beta}{4} \right]$$

$$r'_{19}(0) = B^3 \left[ \frac{4\alpha}{B(\varphi^2-4\theta^2)} - \frac{S_5}{2} + \frac{4\alpha^2}{3\theta^2} - \frac{\alpha^2}{(\varphi^2-4\theta^2)} + \frac{9\beta}{2} \right]$$

$$r_{20} = \left[ -\frac{\alpha\alpha_1 r_1}{3\theta^2} - \frac{\alpha_1^3\alpha^2}{4(\varphi^2-4\theta^2)} - \frac{3\beta\alpha_1^3}{8} \right] r_{20}(0) = B^3 \left( -\frac{\alpha^2}{3\theta^2} + \frac{\alpha^2}{4(\varphi^2-\theta^2)} + \frac{3\beta}{8} \right)$$

$$r'_{20}(0) = B^3 \left[ \frac{\alpha^2}{\theta^2} + \frac{3\alpha^2}{4(\varphi^2-4\theta^2)} - \frac{9\beta}{8} \right]$$

The solution of (4.50) is,

$$U_{2m}^{(30)} = \alpha_{10}\cos\varphi\hat{t} + \beta_{10}\sin\varphi\hat{t} + \frac{r_{17}}{\varphi^2} + \frac{r_{18}\cos\theta\hat{t}}{(\varphi^2-\theta^2)} + \frac{r_{19}\cos2\theta\hat{t}}{(\varphi^2-4\theta^2)} + \frac{r_{20}\cos3\theta\hat{t}}{(\varphi^2-9\theta^2)} \quad (4.51.1)$$

The initial conditions for (4.51.1) are,

$$U_{2m}^{(30)}(0,0) = 0; U_{2m,\hat{t}}^{(30)}(0,0) + \omega'_1(0)U_{2m,\hat{t}}^{(20)}(0,0) = 0$$

Therefore,

$$\alpha_{10}(0) = - \left[ \frac{r_{18}}{(\varphi^2-\theta^2)} + \frac{r_{19}}{(\varphi^2-4\theta^2)} + \frac{r_{20}}{(\varphi^2-9\theta^2)} \right] \text{ at } \tau = 0 \quad (4.51.2)$$

$$\beta_{10}(0) = 0 \quad (4.51.3)$$

Substituting in (4.45.2) to obtained

$$\begin{aligned}
U_{3m,\hat{t}\hat{t}}^{(30)} + \Omega^2 U_{3m}^{(30)} &= \alpha \left[ \frac{\alpha \alpha_1^2 B}{2(\varphi^2 - \theta^2)} + \left\{ \frac{\alpha \alpha_1 \left( \frac{\alpha_1^2}{2} + B^2 \right)}{2\varphi^2} + \frac{\alpha_1^3 \alpha}{8(\varphi^2 - 4\theta^2)} \right\} \cos\theta \hat{t} + \right. \\
&\frac{\alpha_1 \alpha_5}{2} \cos(\varphi + \theta) \hat{t} + \frac{\alpha \alpha_1^2 B \cos 2\theta \hat{t}}{2(\varphi^2 - \theta^2)} + \frac{\alpha_1 \beta_5}{2} \sin(\varphi + \theta) \hat{t} + \frac{\alpha_1 \alpha_5}{2} \cos(\varphi - \theta) \hat{t} + \frac{\alpha_1 \beta_5}{2} \sin(\varphi - \\
&\left. \theta) \hat{t} + \frac{\alpha_1^3 \alpha \cos 3\theta \hat{t}}{8(\varphi^2 - 4\theta^2)} \right] - \frac{\beta}{4} \left[ \left( B^3 + \frac{3\alpha_1^2 B}{2} \right) + 3 \left( \frac{\alpha_1^3}{4} + \alpha_1 B^2 \right) \cos\theta \hat{t} + \frac{3\alpha_1^2}{2} B \cos 2\theta \hat{t} + \frac{\alpha_1^3}{4} \cos 3\theta \hat{t} \right]
\end{aligned} \tag{4.52.1}$$

Rewriting (4.52.1) gives;

$$U_{3m,\hat{t}\hat{t}}^{(30)} + \Omega^2 U_{3m}^{(30)} = r_{21} + r_{22} \cos\theta \hat{t} + r_{23} \cos 2\theta \hat{t} + r_{24} \cos 3\theta \hat{t} + r_{25} \cos(\varphi + \theta) \hat{t} + r_{26} \sin(\varphi + \theta) \hat{t} + r_{27} \cos(\varphi - \theta) \hat{t} + r_{28} \sin(\varphi - \theta) \hat{t} \tag{4.52.2}$$

The initial conditions are;

$$U_{3m}^{(30)}(0,0) = 0; \quad U_{3m,\hat{t}}^{(30)}(0,0) = 0$$

Where,

$$\begin{aligned}
r_{21} &= \left\{ \frac{\alpha^2 \alpha_1^2 B}{2(\varphi^2 - \theta^2)} - \frac{\beta}{4} \left( B^3 + \frac{3\alpha_1^2 B}{2} \right) \right\}, \quad r_{21}(0) = B^3 \left( \frac{\alpha^2 \alpha_1^2}{2(\varphi^2 - \theta^2)} - \frac{5\beta}{8} \right) \\
r_{22} &= \left\{ \frac{\alpha^2 \alpha_1 \left( \frac{\alpha_1^2}{2} + B^2 \right)}{2\varphi^2} + \frac{\alpha_1^3 \alpha^2}{8(\varphi^2 - 4\theta^2)} - \frac{3\beta}{4} \left( \frac{\alpha_1^3}{4} + \alpha_1 B^2 \right) \right\} \\
r_{21}(0) &= B^3 \left( \frac{15\beta}{16} - \frac{3\alpha^2}{4\varphi^2} - \frac{\alpha^2}{8(\varphi^2 - 4\theta^2)} \right) \\
r_{22}'(0) &= B^3 \left( \frac{5\alpha^2}{4\varphi} + \frac{3\alpha^2}{8(\varphi^2 - 4\theta^2)} - \frac{21\beta}{16} \right) \\
r_{23} &= \left\{ \frac{\alpha^2 \alpha_1^2 B}{2(\varphi^2 - \theta^2)} - \frac{3\alpha_1^2 B}{8} \right\}, \quad r_{23}(0) = B^3 \left( \frac{\alpha^2}{2(\varphi^2 - \theta^2)} - \frac{3\beta}{8} \right) \\
r_{23}'(0) &= B^3 \left( \frac{3\beta}{4} - \frac{\alpha^2}{(\varphi^2 - \theta^2)} \right), \quad r_{24} = \left( \frac{\alpha^2 \alpha_1^3 B}{8(\varphi^2 - \theta^2)} - \frac{3\alpha_1^3 \beta}{16} \right) \\
r_{24}(0) &= B^3 \left( \frac{\beta}{16} - \frac{\alpha^2}{8(\varphi^2 - 4\theta^2)} \right), \quad r_{24}'(0) = B^3 \left( \frac{3\alpha^2}{8(\varphi^2 - \theta^2)} - \frac{3\beta}{16} \right) \\
r_{25} &= \frac{\alpha \alpha_1 \alpha_5}{2} = r_{27}, \quad r_{25}(0) = r_{27}(0) = B^2 \alpha^2 S_0 \\
r_{25}'(0) &= r_{27}'(0) = \alpha S_0 B^3
\end{aligned}$$

$$r_{26} = \frac{\alpha\alpha_1\beta_5}{2} = r_{28}; \quad r_{26}(0) = r_{28}(0) = 0, \quad r'_{26}(0) = r'_{28}(0) = 0$$

Therefore,

$$U_{3m}^{(30)}(\hat{t}, \tau) = \alpha_{11}(\tau)\cos\Omega\hat{t} + \beta_{11}(\tau)\sin\Omega\hat{t} + \frac{r_{22}\cos\theta\hat{t}}{\Omega^2 - \theta^2} + \frac{r_{23}\cos 2\theta\hat{t}}{\Omega^2 - 4\theta^2} + \frac{r_{24}\cos 3\theta\hat{t}}{\Omega^2 - 9\theta^2} + \left\{ \frac{r_{25}\cos(\varphi+\theta)\hat{t} + r_{26}\sin(\varphi+\theta)\hat{t}}{\Omega^2 - (\varphi+\theta)^2} \right\} + \left\{ \frac{r_{27}\cos(\varphi-\theta)\hat{t} + r_{28}\sin(\varphi-\theta)\hat{t}}{\Omega^2 - (\varphi-\theta)^2} \right\} \quad (4.52.2)$$

Therefore,

$$\alpha_{11}(0) = - \left[ \frac{r_{22}}{\Omega^2 - \theta^2} + \frac{r_{23}}{\Omega^2 - 4\theta^2} + \frac{r_{24}}{\Omega^2 - 9\theta^2} + \frac{r_{25}}{\Omega^2 - (\varphi+\theta)^2} + \frac{r_{27}}{\Omega^2 - (\varphi+\theta)^2} \right] \tau = 0 \quad (4.52.3)$$

$$\beta_{11}(0) = \frac{-1}{\Omega} \left[ \frac{r_{26}(\varphi+\theta)}{\Omega^2 - (\varphi+\theta)^2} + \frac{r_{28}(\varphi-\theta)}{\Omega^2 - (\varphi-\theta)^2} \right] \tau = 0 \quad (4.52.4)$$

So far, it follows that,

$$U^{(30)} = U_m^{(30)}(1 - \cos 2mx) + U_{2m}^{(30)}(1 - \cos 4mx) + U_{3m}^{(30)}(1 - \cos 6mx)$$

From (3.24),

$$\begin{aligned} O(\epsilon^3 \delta) : & U_{,\hat{t}\hat{t}}^{(31)} + U_{,xxxx}^{(31)} + 2\lambda U_{,xx}^{(31)} + U^{(31)} \\ & = -(\omega'_1)^2 U_{,\hat{t}\hat{t}}^{(11)} - 2 \left( \omega'_1 U_{,\hat{t}\tau}^{(21)} + \omega'_2 U_{,\hat{t}\tau}^{(11)} \right) - 2U_{,\hat{t}\tau}^{(30)} \\ & + 2 \left( \omega'_1 U_{,\hat{t}\hat{t}}^{(20)} + \omega'_2 U_{,\hat{t}\hat{t}}^{(10)} \right) - \left( \omega''_1 U_{,\hat{t}}^{(20)} + \omega''_2 U_{,\hat{t}}^{(10)} \right) \\ & - 2 \left\{ U_{,\hat{t}}^{(30)} + \left( \omega'_1 U_{,\hat{t}}^{(20)} + \omega'_2 U_{,\hat{t}}^{(10)} \right) \right\} - \alpha (U^{(10)} U^{(21)} + U^{(11)} U^{(20)}) \\ & + 3\beta (U^{(10)})^2 (U^{(11)}) \end{aligned}$$

Substituting on the right-hand side of (3.24) gives;

$$\begin{aligned} & U_{,\hat{t}\hat{t}}^{(31)} + U_{,xxxx}^{(31)} + 2\lambda U_{,xx}^{(31)} + U^{(31)} = - \left[ (\omega'_1)^2 U_{m,\hat{t}}^{(11)} (1 - \cos 2mx) + \right. \\ & 2 \left\{ \omega'_1 \left( U_{m,\hat{t}\tau}^{(21)} (1 - \cos 2mx) + U_{2m,\hat{t}\tau}^{(21)} (1 - \cos 4mx) \right) + \omega'_2 U_{2m,\hat{t}\tau}^{(11)} (1 - \cos 2mx) \right\} + \\ & 2 \left\{ U_{m,\hat{t}\tau}^{(30)} (1 - \cos 2mx) + U_{2m,\hat{t}\tau}^{(30)} (1 - \cos 4mx) + U_{3m,\hat{t}\tau}^{(30)} (1 - \cos 6mx) \right\} - \\ & 2 \left\{ \omega'_1 \left( U_{m,\hat{t}\hat{t}}^{(20)} (1 - \cos 2mx) + U_{2m,\hat{t}\hat{t}}^{(20)} (1 - \cos 4mx) \right) + \omega'_2 U_{m,\hat{t}\hat{t}}^{(10)} (1 - \cos 2mx) \right\} + \\ & \left. \left\{ \omega''_1 U_{m,\hat{t}}^{(20)} (1 - \cos 2mx) + U_{2m,\hat{t}}^{(20)} (1 - \cos 4mx) + \omega''_2 U_{m,\hat{t}}^{(10)} (1 - \cos 2mx) \right\} + 2 \left\{ U_{m,\hat{t}}^{(30)} (1 - \right. \\ & \left. \cos 2mx) + U_{2m,\hat{t}}^{(30)} (1 - \cos 4mx) + U_{3m,\hat{t}}^{(30)} (1 - \cos 6mx) + \omega'_1 \left( U_{m,\hat{t}}^{(20)} (1 - \cos 2mx) + \right. \right. \end{aligned}$$

$$\begin{aligned}
& U_{2m\hat{t}}^{(20)}(1 - \cos 4mx) + \omega'_2 U_{m,\hat{t}}^{(10)}(1 - \cos 2mx) \Big\} + \alpha \Big\{ U_m^{(10)}(1 - \cos 2mx) \left( U_m^{(21)}(1 - \right. \\
& \left. \cos 2mx) + U_{2m}^{(21)}(1 - \cos 4mx) \right) + U_m^{(11)}(1 - \cos 2mx) \left( U_m^{(20)}(1 - \cos 2mx) + U_{2m}^{(20)}(1 - \right. \\
& \left. \cos 4mx) \right) \Big\} - 3\beta \left\{ \left( U_m^{(10)} \right)^2 U_m^{(11)}(1 - \cos 2mx)^3 \right\} \\
(4.53)
\end{aligned}$$

The following simplifications are necessary

$$\begin{aligned}
& U_m^{(10)} U_m^{(21)}(1 - \cos 2mx)(1 - \cos 2mx) \\
& = U_m^{(10)} U_m^{(21)} \left( \frac{3}{2} - 2\cos mx + \frac{1}{2}\cos 4mx \right) \tag{4.54.1}
\end{aligned}$$

$$\begin{aligned}
& U_m^{(10)} U_{2m}^{(21)}(1 - \cos 2mx)(1 - \cos 4mx) = U_m^{(10)} U_{2m}^{(21)} \left( 1 - \frac{1}{2}\cos 2mx - \right. \\
& \left. \cos 4mx + \frac{1}{2}\cos 6mx \right) \tag{4.54.2}
\end{aligned}$$

$$\begin{aligned}
& U_m^{(11)} U_{2m}^{(20)}(1 - \cos 2mx)(1 - \cos 2mx) \\
& = U_m^{(11)} U_m^{(21)} \left( \frac{3}{2} - 2\cos 2mx + \frac{1}{2}\cos 4mx \right) \tag{4.54.3}
\end{aligned}$$

$$\begin{aligned}
& U_m^{(11)} U_{2m}^{(20)}(1 - \cos 2mx)(1 - \cos 4mx) = U_m^{(11)} U_{2m}^{(20)} \left[ 1 - \frac{1}{2}\cos 2mx - \right. \\
& \left. \cos 4mx + \frac{1}{2}\cos 6mx \right] \tag{4.54.4}
\end{aligned}$$

$$\begin{aligned}
& 3\beta \left( U_m^{(10)} \right)^2 U_m^{(11)}(1 - \cos 2mx)^3 = 3\beta \left( U_m^{(10)} \right)^2 U_m^{(11)} \left[ \frac{5}{2} - \frac{15}{4}\cos 2mx + \right. \\
& \left. \frac{3}{2}\cos 4mx - \frac{1}{4}\cos 6mx \right] \tag{4.54.5}
\end{aligned}$$

Therefore, substituting (4.54.1) - (4.54.5) in (4.53) yields,

$$\begin{aligned}
& U_{,\hat{t}\hat{t}}^{(31)} + U_{,xxxx}^{(31)} + 2\lambda U^{(31)} + U^{(31)} = - \left[ (\omega'_1)^2 U_{m,\hat{t}\hat{t}}^{(21)}(1 - \cos 2mx) + \right. \\
& 2 \left\{ \omega'_1 \left( U_{m,\hat{t}\tau}^{(21)}(1 - \cos 2mx) + U_{2m,\hat{t}\tau}^{(21)}(1 - \cos 4mx) \right) \omega'_2 U_{m,\hat{t}\tau}^{(11)}(1 - \cos 2mx) \right\} + \\
& 2 \left\{ U_{m,\hat{t}\tau}^{(30)}(1 - \cos 2mx) + U_{2m,\hat{t}\tau}^{(30)}(1 - \cos 4mx) + U_{3m,\hat{t}\tau}^{(30)}(1 - \cos 6mx) \right\} - 2 \left\{ \omega'_1 U_{m,\hat{t}\hat{t}}^{(20)}(1 - \right. \\
& \left. \cos 2mx) + U_{2m,\hat{t}\hat{t}}^{(20)}(1 - \cos 4mx) + \omega'_2 U_{m,\hat{t}\hat{t}}^{(10)}(1 - \cos 2mx) \right\} + \left\{ \omega''_1 U_{m,\hat{t}}^{(20)}(1 - \cos 2mx) + \right. \\
& \left. U_{2m,\hat{t}}^{(20)}(1 - \cos 4mx) + \omega''_2 U_{m,\hat{t}}^{(10)}(1 - \cos 2mx) \right\} + 2 \left\{ U_{m,\hat{t}}^{(30)}(1 - \cos 2mx) + U_{2m,\hat{t}}^{(30)}(1 - \right. \\
& \left. \cos 4mx) + U_{3m,\hat{t}}^{(30)}(1 - \cos 6mx) + \omega'_1 \left( U_{m,\hat{t}}^{(20)}(1 - \cos 2mx) + U_{2m,\hat{t}}^{(20)}(1 - \cos 4mx) \right) + \right. \\
& \left. \omega'_2 \left( U_{m,\hat{t}}^{(10)}(1 - \cos 2mx) \right) \right\} + \alpha \left\{ U_m^{(10)} U_m^{(21)} \left( \frac{3}{2} - 2\cos 2mx + \frac{1}{2}\cos 4mx \right) + U_m^{(10)} U_{2m}^{(21)} \left( 1 - \right. \right.
\end{aligned}$$

$$\begin{aligned} & \left. \frac{1}{2} \cos 2mx - \cos 4mx + \frac{1}{2} \cos 6mx \right\} + \alpha \left\{ U_m^{(11)} U_m^{(20)} \left( \frac{3}{2} - 2 \cos 2mx + \frac{1}{2} \cos 4mx \right) + \right. \\ & U_m^{(11)} U_{2m}^{(20)} \left( 1 - \frac{1}{2} \cos 2mx - \cos 4mx + \frac{1}{2} \cos 6mx \right) \left. \right\} - 3\beta \left( U_m^{(10)} \right)^2 U_m^{(11)} \left( \frac{5}{2} - \frac{15}{4} \cos 2mx - \right. \\ & \left. \frac{3}{2} \cos 4mx - \frac{1}{4} \cos 6mx \right) \left. \right] \end{aligned} \quad (4.55)$$

Let,

$$U^{(31)} \sum_{n=1}^{\infty} U^{(31)} (1 - \cos 2nx)$$

The left-hand side of (4.55) yields,

$$\sum_{n=1}^{\infty} \left[ U_{n,\hat{t}\hat{t}}^{(31)} (1 - \cos 2nx) + (-16n^4 + 8\lambda n^2 + 1) U_n^{(31)} \cos 2nx \right]$$

Multiplying (4.55) through  $\cos 2mx$  and integrating from 0 to  $\pi$  and from  $n=m$ , gives,

$$\begin{aligned} -\frac{\pi}{2} \left[ U_{m,\hat{t}\hat{t}}^{(31)} + (16m^4 - 8\lambda m^2 + 1) U_m^{(31)} \right] = & - \left[ (\omega_1')^2 U_{m,\hat{t}\hat{t}}^{(11)} \left( -\frac{\pi}{2} \right) + 2 \left\{ \omega_1' U_{m,\hat{t}\tau}^{(21)} \left( -\frac{\pi}{2} \right) + \right. \right. \\ & \omega_2' U_{m,\hat{t}\tau}^{(11)} \left( -\frac{\pi}{2} \right) \left. \right\} + 2 \left\{ U_{m,\hat{t}\tau}^{(30)} \left( -\frac{\pi}{2} \right) \right\} - 2 \left\{ \begin{array}{l} \omega_1' U_{m,\hat{t}\hat{t}}^{(20)} \\ + \omega_2' U_{m,\hat{t}\hat{t}}^{(10)} \left( -\frac{\pi}{2} \right) \end{array} \right\} + \left\{ \omega_1' U_{m,\hat{t}\tau}^{(20)} \left( -\frac{\pi}{2} \right) + \right. \\ & \omega_2'' U_{m,\hat{t}}^{(10)} \left( -\frac{\pi}{2} \right) \left. \right\} + 2 \left\{ U_{m,\hat{t}}^{(30)} \left( -\frac{\pi}{2} \right) + \omega_1' U_{m,\hat{t}\tau}^{(20)} \left( -\frac{\pi}{2} \right) + \omega_2' U_{m,\hat{t}}^{(10)} \left( -\frac{\pi}{2} \right) \right\} + \\ & \alpha \left\{ -2 U_m^{(10)} U_m^{(21)} \left( -\frac{\pi}{2} \right) - U_m^{(10)} U_m^{(21)} \left( -\frac{\pi}{2} \right) - 2 U_m^{(11)} U_m^{(21)} \left( -\frac{\pi}{2} \right) - 2 U_m^{(11)} U_m^{(20)} \left( -\frac{\pi}{2} \right) - \right. \\ & \left. U_m^{(11)} U_{2m}^{(20)} \left( -\frac{\pi}{2} \right) \right\} + 3\beta \left( U_m^{(10)} \right)^2 U_m^{(11)} \left( -\frac{15}{4} \right) \left. \right] \end{aligned} \quad (4.56)$$

A further simplification of the above yields;

$$\begin{aligned} U_{m,\hat{t}\hat{t}}^{(31)} + \theta^2 U_m^{(31)} = & (\omega_1')^2 U_{m,\hat{t}\hat{t}}^{(11)} - 2 \left\{ \omega_1' U_{m,\hat{t}\tau}^{(21)} + \omega_2' U_{m,\hat{t}\tau}^{(11)} \right\} - 2 \left\{ U_{m,\hat{t}\tau}^{(30)} \right\} + 2 \left\{ \omega_1' U_{m,\hat{t}\hat{t}}^{(20)} + \right. \\ & \omega_2' U_{m,\hat{t}\hat{t}}^{(10)} \left. \right\} - \left\{ \omega_1' U_{m,\hat{t}}^{(20)} + \omega_2'' U_{m,\hat{t}}^{(10)} \right\} - 2 \left\{ U_{m,\hat{t}}^{(30)} + \omega_1' U_{m,\hat{t}}^{(20)} + \omega_2' U_{m,\hat{t}}^{(10)} \right\} + \alpha \left\{ 2 U_m^{(10)} U_m^{(21)} + \right. \\ & \left. U_m^{(10)} U_m^{(21)} + 2 U_m^{(11)} U_m^{(21)} + U_m^{(11)} U_{2m}^{(20)} \right\} - \frac{45}{4} \beta \left( U_m^{(10)} \right)^2 U_m^{(11)} \end{aligned} \quad (4.57)$$

The initial conditions for (4.57) are;

$$U_m^{(31)}(0,0) = 0;$$

$$U_{m,\hat{t}}^{(31)}(0,0) + \omega'_1(0)U_{m,\hat{t}}^{(21)}(0,0) + \omega'_2(0)U_{m,\hat{t}}^{(11)}(0,0) + U_{m,\tau}^{(30)}(0,0) = 0$$

Multiplying (4.55) through  $\cos 4mx$  and integrating from 0 to  $\pi$  and for  $n=2m$  gives,

$$\begin{aligned} & -\frac{\pi}{2} \left[ U_{m,\hat{t}\hat{t}}^{(31)} + (256m^4 - 32\lambda m^2 + 1)U_{2m}^{(30)} \right] = - \left[ 2\omega'_1 U_{2m,\hat{t}\tau}^{(21)} \left( -\frac{\pi}{2} \right) + \right. \\ & 2U_{2m,\hat{t}\tau}^{(30)} \left( -\frac{\pi}{2} \right) - 2\omega'_1 U_{2m,\hat{t}\hat{t}}^{(20)} \left( -\frac{\pi}{2} \right) + \omega''_1 U_{2m,\hat{t}}^{(20)} \left( -\frac{\pi}{2} \right) + 2U_{2m,\hat{t}}^{(30)} \left( -\frac{\pi}{2} \right) + \\ & 2\omega'_1 U_{2m,\hat{t}}^{(20)} \left( -\frac{\pi}{2} \right) + \alpha \left\{ \frac{1}{2} U_m^{(10)} U_m^{(21)} \left( -\frac{\pi}{2} \right) - U_m^{(10)} U_{2m}^{(21)} \left( -\frac{\pi}{2} \right) + \frac{1}{2} U_m^{(11)} U_m^{(20)} \left( -\frac{\pi}{2} \right) - \right. \\ & \left. \left. U_m^{(11)} U_{2m}^{(20)} \left( -\frac{\pi}{2} \right) \right\} - 3\beta \left( U_m^{(10)} \right)^2 U_m^{(11)} \left( \frac{3}{2} \right) \right] \end{aligned} \quad (4.58)$$

This yield,

$$\begin{aligned} & U_{2m,\hat{t}\hat{t}}^{(31)} + \varphi^2 U_{2m}^{(30)} = - \left[ 2\omega'_1 U_{2m,\hat{t}\tau}^{(21)} + 2U_{2m,\hat{t}\tau}^{(30)} - 2\omega'_1 U_{2m,\hat{t}\hat{t}}^{(20)} + \omega''_1 U_{2m,\hat{t}}^{(20)} + \right. \\ & 2U_{2m,\hat{t}}^{(30)} + 2\omega'_1 U_{2m,\hat{t}}^{(20)} + \alpha \left\{ \frac{1}{2} U_m^{(10)} U_m^{(21)} - U_m^{(10)} U_{2m}^{(21)} + \frac{1}{2} U_m^{(11)} U_m^{(20)} - U_m^{(11)} U_{2m}^{(20)} \right\} - \\ & \left. \frac{9}{4} \beta \left( U_m^{(10)} \right)^2 U_m^{(11)} \right] \end{aligned} \quad (4.59)$$

The initial conditions are,

$$U_{2m}^{(31)}(0,0) = 0; \quad U_{2m}^{(31)}(0,0) + \omega'_1(0)U_{2m,\hat{t}}^{(20)}(0,0) = 0$$

Multiplying through  $\cos 6mx$  and integrating from 0 to  $\pi$  and for  $n=3m$ , the result is;

$$\begin{aligned} & -\frac{\pi}{2} \left[ U_{m,\hat{t}\hat{t}}^{(31)} + (1296m^4 - 72\lambda m^2 + 1)U_{2m}^{(31)} \right] = - \left[ 2U_{3m,\hat{t}\tau}^{(30)} \left( -\frac{\pi}{2} \right) + \right. \\ & 2U_{3m,\hat{t}}^{(30)} \left( -\frac{\pi}{2} \right) + \alpha \left\{ \frac{1}{2} U_m^{(10)} U_{2m}^{(21)} \left( -\frac{\pi}{2} \right) + \frac{1}{2} U_m^{(11)} U_{2m}^{(20)} \left( -\frac{\pi}{2} \right) \right\} - 3\beta \left( -\frac{1}{4} \right) \left( U_m^{(10)} \right)^2 U_m^{(11)} \right] \end{aligned} \quad (4.60)$$

A further simplification of above yields,

$$\begin{aligned} & U_{3m,\hat{t}\hat{t}}^{(31)} + \Omega^2 U_{3m}^{(30)} = - \left[ 2U_{3m,\hat{t}\tau}^{(30)} + 2U_{3m,\hat{t}}^{(30)} + \alpha \left\{ \frac{1}{2} U_m^{(10)} U_{2m}^{(21)} + \frac{1}{2} U_m^{(11)} U_{2m}^{(20)} \right\} + \right. \\ & \left. \frac{3}{4} \beta \left( U_m^{(10)} \right)^2 U_m^{(11)} \right] \end{aligned} \quad (4.61)$$

The initial conditions for (4.61) are,

$$U_{3m}^{(31)}(0,0) = 0; \quad U_{3m,\hat{t}}^{(31)}(0,0) = 0$$

Certain terms in (4.61) are now simplified as

$$U_m^{(10)}U_m^{(21)} = \left[ \alpha_6 \cos \theta \hat{t} + \beta_6 \sin \theta \hat{t} + \frac{r_2}{\theta^2} - \left( \frac{r_3 \cos 2\theta \hat{t} - r_4 \sin 2\theta \hat{t}}{3\theta^2} \right) \right] + \left[ \frac{\alpha_1 \alpha_6}{2} (1 + \cos 2\theta \hat{t}) + \frac{\alpha_1 \beta_6}{2} \sin 2\theta \hat{t} + \frac{\alpha_1 r_2}{2} \cos 2\theta \hat{t} - \frac{1}{3\theta^2} \left[ \frac{\alpha_1 r_3}{2} (\cos 3\theta \hat{t} + \cos \theta \hat{t}) + \frac{\alpha_1 r_4}{2} (\sin 3\theta \hat{t} + \sin \theta \hat{t}) \right] + B \left[ \alpha_6 \cos \theta \hat{t} + \beta_6 \sin \theta \hat{t} + \frac{r_2}{\theta^2} - \left( \frac{r_3 \cos 2\theta \hat{t} + r_4 \sin 2\theta \hat{t}}{3\theta^2} \right) \right] \right] \quad (4.62)$$

The following simplification is necessary

$$\begin{aligned} U_m^{(10)}U_{2m}^{(21)} &= (\alpha_6 \cos \theta \hat{t}) \left[ \alpha_7 \cos \varphi \hat{t} + \beta_7 \sin \varphi \hat{t} + \frac{r_5 \sin \theta \hat{t}}{\varphi^2 - \theta^2} + \frac{r_6 \sin 2\theta \hat{t}}{\varphi^2 - 4\theta^2} \right] \\ &= \frac{\alpha_1 \alpha_7}{2} \{ \cos(\varphi + \theta) \hat{t} + \cos(\varphi - \theta) \hat{t} \} + \frac{\alpha_1 \beta_7}{2} \{ \sin(\varphi + \theta) \hat{t} + \sin(\varphi - \theta) \hat{t} \} + \frac{r_5 \alpha_1 \sin 2\theta \hat{t}}{2(\varphi^2 - \theta^2)} + \frac{\alpha_1 \alpha_6}{2(\varphi^2 - 4\theta^2)} (\sin 3\theta \hat{t} + \sin \theta \hat{t}) + B \left[ \alpha_7 \cos \varphi \hat{t} + \beta_7 \sin \varphi \hat{t} + \frac{r_5 \sin \theta \hat{t}}{\varphi^2 - \theta^2} + \frac{r_6 \sin 2\theta \hat{t}}{\varphi^2 - 4\theta^2} \right] \end{aligned} \quad (4.63)$$

Certain terms in (4.63) are now simplified as

$$U_m^{(11)}U_m^{(20)} = \beta_2 \sin \theta \hat{t} \left[ \alpha_4 \cos \theta \hat{t} + \frac{r_0}{\theta^2} - \frac{r_1 \cos 2\theta \hat{t}}{3\theta^2} \right] = \frac{\beta_2 \alpha_4 \sin 2\theta \hat{t}}{2} + \frac{\beta_2 r_0 \sin \theta \hat{t}}{\theta^2} - \frac{\beta_2 r_1}{6\theta^2} (\sin 3\theta \hat{t} - \sin \theta \hat{t}) \quad (4.64.1)$$

$$\begin{aligned} U_m^{(11)}U_{2m}^{(20)} &= \beta_2 \sin \theta \hat{t} \left[ \alpha_5 \cos \varphi \hat{t} + \beta_5 \sin \varphi \hat{t} + \frac{\alpha}{2} \left\{ \frac{\left( \frac{\alpha_1^2}{2} + B \right)}{\varphi^2} + \frac{2B\alpha_1 \cos \theta \hat{t}}{(\varphi^2 - \theta^2)} + \frac{\alpha_1^2 \cos 2\theta \hat{t}}{2(\varphi^2 - 4\theta^2)} \right\} \right] \\ &= \frac{\beta_2 \alpha_5}{2} (\sin(\varphi + \theta) \hat{t} + \sin(\varphi - \theta) \hat{t}) + \frac{\beta_2 \beta_5}{2} (\cos(\varphi - \theta) \hat{t} - \cos(\varphi + \theta) \hat{t}) + \frac{\alpha}{2} \left\{ \beta_2 \frac{\left( \frac{\alpha_1^2}{2} + B \right)}{\varphi^2} \sin \theta \hat{t} + \frac{\beta_2 B \alpha_1 \sin 2\theta \hat{t}}{(\varphi^2 - \theta^2)} + \frac{\beta_2 \alpha_1^2 \sin 3\theta \hat{t} - \sin \theta \hat{t}}{4(\varphi^2 - 4\theta^2)} \right\} \end{aligned} \quad (4.46.2)$$

Lastly, the following is further simplified,

$$\left( U_m^{(10)} \right)^2 U_m^{(11)} = (\alpha_1 \cos \theta \hat{t} + B)^2 (\beta_2 \sin \theta \hat{t}) = (B^2 + \alpha_1^2 \cos^2 \theta \hat{t} + 2B\alpha_1 \cos 2\theta \hat{t}) (\beta_2 \sin \theta \hat{t})$$

$$\begin{aligned}
&= \left[ \left( B^2 + \frac{\alpha_1^2}{2} \right) + 2B\alpha_1 \cos\theta \hat{t} + \frac{\alpha_1^2}{2} \cos 2\theta \hat{t} \right] (\beta_2 \sin\theta \hat{t}) = \beta_2 \left( B^2 + \frac{\alpha_1^2}{2} \right) \sin\theta \hat{t} + \\
&\beta_2 B \alpha_1 \sin 2\theta \hat{t} + \frac{\beta_2 \alpha_1^2}{4} (\sin 3\theta \hat{t} - \sin\theta \hat{t})
\end{aligned} \tag{4.65}$$

Substituting in (4.57) the following equation results;

$$\begin{aligned}
U_{m,\hat{t}\hat{t}}^{(31)} + \theta^2 U_m^{(31)} &= (\omega'_1)^2 \theta^2 \beta_2 \sin\theta \hat{t} - 2 \left[ \omega'_1 (-\theta \alpha'_6 \sin\theta \hat{t} + \theta \beta'_6 \cos\theta \hat{t}) - \right. \\
&\left. \left( \frac{-2\theta r'_3 \sin 2\theta \hat{t} + 2\theta r'_4 \cos 2\theta \hat{t}}{3\theta^2} \right) \right] - 2\omega'_2 \beta'_2 \theta \cos\theta \hat{t} - 2 \left[ -\alpha'_9 \theta \sin\theta \hat{t} + \beta'_9 \theta \cos\theta \hat{t} + \right. \\
&\frac{2r'_{11} \sin 2\theta \hat{t}}{3\theta} + \frac{3r'_{12} \sin 3\theta \hat{t}}{8\theta} - \frac{1}{\varphi(2\theta+\varphi)} \{ -r'_{13}(\varphi+\theta) \sin(\varphi+\theta) \hat{t} + r'_{14}(\varphi+\theta) \cos(\varphi+\theta) \\
&\left. \theta \hat{t} \} + \frac{1}{\varphi(2\theta-\varphi)} \{ -r'_{15}(\varphi-\theta) \sin(\varphi-\theta) \hat{t} + r'_{16}(\varphi-\theta) \cos(\varphi-\theta) \hat{t} \} \right] + \\
&2\omega'_1 \left\{ -\theta^2 \alpha_4 \cos\theta \hat{t} + \frac{4r_1 \cos 2\theta \hat{t}}{3} \right\} + 2\omega'_2 (-\alpha_1 \theta^2 \cos\theta \hat{t}) - \omega'_1 \left\{ -\theta \alpha_4 \sin\theta \hat{t} + \right. \\
&\left. \frac{2r_1 \sin 2\theta \hat{t}}{3\theta} \right\} - \omega''_2 (-\alpha_1 \theta \sin\theta \hat{t}) - 2 \left\{ -\alpha_9 \sin\theta \hat{t} + \beta_9 \cos\theta \hat{t} + \frac{2r_{11} \sin 2\theta \hat{t}}{3\theta} + \frac{3r_{12} \sin 3\theta \hat{t}}{8\theta} - \right. \\
&\frac{1}{\varphi(2\theta+\varphi)} \{ -(\varphi+\theta) r_{13} \sin(\varphi+\theta) \hat{t} + r_{14}(\varphi+\theta) \cos(\varphi+\theta) \hat{t} \} + \frac{1}{\varphi(2\theta-\varphi)} \{ -(\varphi+\theta) \\
&\left. \theta r_{15} \sin(\varphi-\theta) \hat{t} + (\varphi-\theta) r_{16} \cos(\varphi-\theta) \hat{t} \} \right\} - 2\omega'_1 \left\{ -\theta \alpha_4 \sin\theta \hat{t} + \frac{2r_1 \sin 2\theta \hat{t}}{3\theta} \right\} - \\
&2\omega'_2 (-\theta \alpha_1 \sin\theta \hat{t}) + 2\alpha \left\{ \left( \frac{\alpha_1 \alpha_6}{2} - \frac{Br_2}{\theta^2} \right) + \left( \frac{\alpha_1 r_2}{2} - \frac{Br_3}{6\theta^2} + B\alpha_6 \right) \cos\theta \hat{t} + \right. \\
&\left. \left( \frac{B\beta_6 - \alpha_1 \alpha_4}{6\theta^2} \right) \sin\theta \hat{t} + \left( \frac{\alpha_1 \alpha_6}{2} - \frac{Br_3}{3\theta^2} \right) \cos 2\theta \hat{t} + \left( \frac{\alpha_1 \beta_6}{2} - \frac{Br_4}{3\theta^2} \right) \sin\theta \hat{t} - \frac{\alpha_1 r_3}{6\theta^2} \cos 3\theta \hat{t} - \right. \\
&\left. \frac{\alpha_1 r_4}{6\theta^2} \sin 3\theta \hat{t} \right\} + \alpha \left\{ \left( \frac{\alpha_1 r_6}{2(\varphi^2 - 4\theta^2)} + \frac{Br_5}{\varphi^2 - \theta^2} \right) \sin\theta \hat{t} + \left( \frac{\alpha_1 r_5}{2(\varphi^2 - 4\theta^2)} - \frac{Br_6}{\varphi^2 - 4\theta^2} \right) \sin 2\theta \hat{t} + \right. \\
&\frac{\alpha_1 r_6 \sin 3\theta \hat{t}}{2(\varphi^2 - 4\theta^2)} + B\alpha_7 \cos\varphi \hat{t} + B\beta_7 \sin\varphi \hat{t} + \frac{\alpha_1 \alpha_7}{2} \cos(\varphi+\theta) \hat{t} + \frac{\alpha_1 \beta_7}{2} \sin(\varphi+\theta) \hat{t} + \\
&\frac{\alpha_1 \alpha_7}{2} \cos(\varphi-\theta) \hat{t} + \frac{\alpha_1 \beta_7}{2} \sin(\varphi-\theta) \hat{t} \left. \right\} + 2\alpha \left\{ \frac{\beta_2 \alpha_4}{2} \sin 2\theta \hat{t} + \frac{\beta_2 r_0}{\theta^2} \sin\theta \hat{t} - \right. \\
&\left. \frac{\beta_2 r_1}{6\theta^2} (\sin 3\theta \hat{t} - \sin\theta \hat{t}) \right\}
\end{aligned} \tag{4.66}$$

To ensure a uniformly valid solution in  $\hat{t}$ , demands equating to zero the coefficients of  $\sin\theta \hat{t}$  and  $\cos\theta \hat{t}$  in (4.66) as further expanded.

The coefficient of  $\sin\theta \hat{t}$  leads to;

$$\alpha'_9 + \alpha_9 = h_1(\tau) \tag{4.67}$$

$$h_1(\tau) = -\frac{1}{2\theta} \left[ 2\omega'_1 \theta \alpha'_6 + \omega''_2 \alpha_1 \theta + \omega'_1 \theta \alpha_4 + 2\omega'_1 \theta \alpha_4 + 2\omega'_2 \theta \alpha_1 + 2\alpha \left( \frac{B\beta_6 - \alpha_1 r_4}{6\theta^2} \right) \right] \tag{4.68.1}$$

Therefore,

$$\alpha_9 = e^{-\tau} \left[ \int e^s h_1(s) ds + \alpha_9(0) \right] \quad (4.68.2)$$

Equating the coefficient of  $\cos\theta \hat{t}$  yields

$$\beta'_9 + \beta_9 = h_2(\tau) \quad (4.68.3)$$

$$h_2(\tau) = -\frac{1}{2\theta} \left[ 2\omega'_1 \theta \beta'_6 + 2\omega'_2 \beta'_2 \theta + 2\omega'_1 \theta^2 \alpha_4 + 2\omega'_2 \alpha_1 \theta^2 - 2\alpha \left( \frac{\alpha_1 r_2}{\theta^2} - \frac{\alpha_1 r_3}{6\theta^2} + B\alpha_6 \right) \right] \quad (4.68.4)$$

Therefore,

$$\beta_9 = e^{-\tau} \left[ \int e^s h_2(s) ds + \beta_9(0) \right] \quad (4.68.5)$$

The remaining equation in (4.66);

$$\begin{aligned} U_{m,\hat{t}\hat{t}}^{(31)} + \theta^2 U_m^{(31)} &= r_{29} + r_{30} \sin 2\theta \hat{t} + r_{31} \cos 2\theta \hat{t} + r_{32} \cos 3\theta \hat{t} + r_{33} \cos 3\theta \hat{t} + \\ &r_{34} \cos \varphi \hat{t} + r_{35} \sin \varphi \hat{t} + r_{36} \cos(\varphi + \theta) \hat{t} + r_{37} \sin(\varphi + \theta) \hat{t} + r_{38} \cos(\varphi - \theta) \hat{t} + \\ &r_{39} \sin(\varphi - \theta) \hat{t} \end{aligned} \quad (4.69)$$

The initial conditions are

$$U_m^{(31)}(0,0) = 0; U_{m,\hat{t}}^{(31)}(0,0) + \omega'_1(0) U_{m,\hat{t}}^{(21)}(0,0) + \omega'_2(0) U_{m,\hat{t}}^{(11)}(0,0) + U_{m,\tau}^{(30)}(0,0) = 0$$

Where,

$$r_{29} = 2\alpha \left( \frac{\alpha_1 \alpha_6}{2} + \frac{r_2 B}{\theta^2} \right); \quad r_{29}(0) = 0$$

$$\begin{aligned} r_{30} &= \left[ \frac{-4r'_3}{3\theta} - \frac{4r'_1}{3\theta} - \frac{4r'_{11}}{3\theta} - \frac{2\omega'_1 r_1}{3\theta} - \frac{4r_{11}}{3\theta} - \frac{4\omega'_1 r_1}{3\theta} + 2\alpha \left( \frac{\alpha_1 \beta_6}{2} - \frac{r_4 B}{3\theta^2} \right) + \alpha \left( \frac{\alpha_1 r_5}{2(\varphi^2 - \theta^2)} + \right. \right. \\ &\left. \left. \frac{Br_6}{\varphi^2 - 4\theta^2} \right) + \alpha \beta_2 \alpha_4 + \frac{\alpha^2 \alpha_1 B \beta_2}{2(\varphi^2 - \theta^2)} - \frac{45}{4} \beta B \alpha_1 \beta_2 \right] \end{aligned}$$

$$R_{30}(0) = B^3 \left( \frac{-8\alpha}{3\theta B} - \frac{4S_{21}}{3\theta} + \frac{2\alpha}{3\theta^3} - \frac{4S_4}{3\theta} + \frac{4\alpha}{3\theta^3} - \frac{2\alpha^2}{3\theta^3} + \frac{\alpha^2}{2\theta(\varphi^2 - \theta^2)} + \frac{\alpha^2}{2\theta B(\varphi^2 - \theta^2)} - \frac{45}{4\theta} \right)$$

$$= B^3 S_{25}$$

$$S_{25} = \left( -\frac{8\alpha}{3B\theta} - \frac{4S_{21}}{2\theta} + \frac{2\alpha}{\theta^3} - \frac{4S_4}{3\theta} - \frac{2\alpha^2}{\theta^3} + \frac{\alpha^2}{2\theta(\varphi^2 - \theta^2)} + \frac{S_1}{(\varphi^2 - \theta^2)} + \frac{\alpha^2}{2\theta B(\varphi^2 - \theta^2)} - \frac{45\beta}{4\theta} \right)$$

$$r_{31} = \left[ \frac{8\omega'_1 r_1}{3} + 2\alpha \left( \frac{\alpha_1 \alpha_6}{2} - \frac{Br_3}{3\theta^2} \right) \right], \quad r_{31}(0) = \frac{-8\alpha B^3}{3\theta^2}$$

$$r_{32} = \left[ -\frac{\alpha \alpha_1 r_3}{3\theta^2} \right], \quad r_{32}(0) = 0$$

$$r_{33} = \left[ \begin{array}{c} \frac{-3r'_{12}}{4\theta} - \frac{3r_{12}}{4\theta} - \frac{\alpha \alpha_1 r_4}{3\theta^2} + \frac{\alpha \alpha_1 r_6}{2(\varphi^2 - 4\theta^2)} - \frac{\alpha \beta_2 r_1}{3\theta^2} + \frac{\alpha^2 \alpha_1^2 \beta_2}{8(\varphi^2 - 4\theta^2)} \\ -\frac{45}{16} \beta B \alpha_1 \beta_2 \end{array} \right]$$

$$r_{33}(0) = B^3 \left( -\frac{3S_{61}}{4\theta} - \frac{3S_5}{3\theta} + \frac{\alpha^2}{3\theta^3} - \frac{\alpha S_1}{2\theta(\varphi^2 - \theta^2)} \right) = B^3 S_{26}$$

Where,

$$S_{26} = \left( -\frac{3S_{16}}{3\theta} - \frac{3S_5}{3\theta} + \frac{\alpha^2}{3\theta^3} - \frac{\alpha S_1}{2(\varphi^2 - 4\theta^2)} \right)$$

$$r_{34} = [\alpha B \alpha_7], \quad r_{34}(0) = 0,$$

$$r_{35} = [\alpha B \beta_7], \quad r_{35}(0) = \frac{\alpha^2}{3\varphi^3} + \frac{\alpha^2 S_0}{\varphi} + \frac{\alpha^2}{2\varphi(\varphi^2 - 4\theta^2)} - \frac{\alpha^2}{\varphi(\varphi^2 - 4\theta^2)}$$

$$r_{36} = \left[ \frac{2r'_{14}(\varphi + \theta)}{\varphi(2\theta + \varphi)} - \frac{2r_{14}(\varphi + \theta)}{\varphi(2\theta + \varphi)} + \frac{\alpha \alpha_1 \alpha_7}{2} - \frac{\alpha \beta_2 \beta_5}{2} \right], \quad r_{36}(0) = 0$$

$$r_{37} = \left[ -\frac{2r'_{13}(\varphi + \theta)}{\varphi(2\theta + \varphi)} + \frac{2r_{13}(\varphi + \theta)}{\varphi(2\theta + \varphi)} + \frac{\alpha \alpha_1 \beta_7}{2} + \alpha \left( \frac{\beta_2 \alpha_5}{2} \right) \right]$$

$$r_{37}(0) = B^3 \left( \frac{6\alpha(\varphi + \theta)S_0}{\varphi(2\theta + \varphi)} + \frac{\alpha S_{23}}{2} - \frac{S_0 \alpha}{2\theta} \right) = B^3 S_{27}$$

Where,

$$S_{27} = \left( \frac{6\alpha(\varphi + \theta)S_0}{\varphi(2\theta + \varphi)} - \frac{\alpha S_{43}}{2} - \frac{\alpha S_0}{2\theta} \right)$$

$$r_{38} = \left[ \frac{-2r'_{16}(\varphi - \theta)}{\varphi(2\theta - \varphi)} + \frac{2r_{16}(\varphi - \theta)}{\varphi(2\theta - \varphi)} + \frac{\alpha \alpha_1 \alpha_7}{2} + \frac{\alpha \beta_2 \beta_5}{2} \right],$$

$$r_{38}(0) = 0 \quad r_{39} = \left[ \frac{2r'_{15}(\varphi - \theta)}{\varphi(2\theta - \varphi)} + \frac{2r_{15}(\varphi - \theta)}{\varphi(2\theta - \varphi)} + \frac{\alpha \alpha_1 \beta_7}{2} - \alpha \left( \frac{\beta_2 \alpha_5}{2} \right) \right]$$

$$r_{39}(0) = B^3 \left( \frac{4\alpha(\varphi - \theta)S_0}{\varphi(2\theta + \varphi)} - \frac{2\alpha(\varphi - \theta)S_0}{\varphi(2\theta - \varphi)} - \frac{\alpha^2 S_3}{2\varphi} + \frac{\alpha S_0}{2\theta} \right) = B^3 S_{29}$$

Where,

$$S_{29} = \left( -\frac{3\alpha}{\theta^2(\varphi^2 - \theta^2)} - \frac{2\alpha}{(\varphi^2 - \theta^2)} \right)$$

Therefore, the following is obtained

$$U_m^{(31)} = \alpha_{12} \cos \theta \hat{t} + \beta_{12} \sin \theta \hat{t} + \frac{r_{29}}{\theta^2} + \frac{r_{30} \sin 2\theta \hat{t} + r_{31} \cos 2\theta \hat{t}}{\theta^2 - 4\theta^2} + \frac{r_{32} \cos 3\theta \hat{t} + r_{33} \sin 3\theta \hat{t}}{\theta^2 - 9\theta^2} + \frac{r_{34} \cos \varphi \hat{t} + r_{35} \sin \varphi \hat{t}}{\theta^2 - \varphi^2} + \frac{r_{36} \cos(\varphi + \theta) \hat{t} + r_{37} \sin(\varphi + \theta) \hat{t}}{\varphi(2\theta - \varphi)} + \frac{r_{38} \cos(\varphi - \theta) \hat{t} + r_{39} \sin(\varphi - \theta) \hat{t}}{\varphi(2\theta - \varphi)} \quad (4.70.1)$$

$$\begin{aligned} \alpha_{12}(0) = & - \left[ \frac{r_{29}}{\theta^2} + \frac{r_{31}}{\theta^2 - 4\theta^2} + \frac{r_{32}}{\theta^2 - 9\theta^2} + \frac{r_{34}}{\theta^2 - \varphi^2} - \frac{r_{36}}{\varphi(2\theta - \varphi)} + \frac{r_{38}}{\varphi(2\theta - \varphi)} \right. \\ & - \frac{2\alpha B^3}{3\theta^4} \\ & + \frac{1}{2\theta^2} \left( \frac{B^2}{\theta^4} + \frac{16\alpha B^3}{3\theta^2} - \frac{10\alpha^2 B^3}{\theta^2} + \frac{3\alpha^2 B^3}{2\varphi^2} + \frac{\alpha^2 B^3}{4(\varphi^2 - 4\theta^2)} + \frac{225\beta B^3}{16} \right) \\ & \left. + \left( \alpha'_9 + \frac{r'_{10}}{\theta^2} - \frac{r'_{11}}{3\theta^2} - \frac{r'_{12}}{8\theta^2} - \frac{r'_{13}}{\varphi(2\theta + \varphi)} + \frac{r'_{15}}{\varphi(2\theta - \varphi)} \right) \right] \Big|_{\tau=0} \end{aligned} \quad (4.70.2)$$

$$\beta_{12}(0) = \frac{-1}{\theta} \left[ \frac{2\theta r_{30}}{\theta^2 - 4\theta^2} + \frac{3\theta r_{33}}{\theta^2 - 9\theta^2} + \frac{\varphi r_{35}}{\theta^2 - \varphi^2} + \frac{(\varphi + \theta)r_{37}}{\varphi(2\theta - \varphi)} + \frac{(\varphi - \theta)r_{39}}{\varphi(2\theta - \varphi)} \right] \text{ at } \tau = 0 \quad (4.70.3)$$

Substituting in (4.59) gives

$$\begin{aligned} U_{2m, \hat{t}\hat{t}}^{(31)} + \varphi^2 U_{2m}^{(31)} = & - \left[ 2\omega'_1 \left\{ -\varphi \alpha'_7 \sin \varphi \hat{t} + \varphi \beta'_7 \cos \varphi \hat{t} + \frac{\theta r'_5 \cos \theta \hat{t}}{\varphi^2 - \theta^2} + \frac{2\theta r'_6 \cos 2\theta \hat{t}}{\varphi^2 - 4\theta^2} \right\} + \right. \\ & 2 \left\{ -\varphi \alpha'_{10} \sin \varphi \hat{t} + \varphi \beta'_{10} \cos \varphi \hat{t} - \frac{\theta r'_{18} \sin \theta \hat{t}}{\varphi^2 - \theta^2} - \frac{2\theta r'_{19} \sin 2\theta \hat{t}}{\varphi^2 - 4\theta^2} - \frac{3\theta r'_{20} \sin 3\theta \hat{t}}{\varphi^2 - 9\theta^2} \right\} - \alpha \omega'_1 \left\{ \frac{2B\alpha_1 \theta^2 \cos \theta \hat{t}}{\varphi^2 - \theta^2} - \right. \\ & \left. \frac{2\theta \alpha_1^2 \cos 2\theta \hat{t}}{\varphi^2 - 4\theta^2} \right\} + \alpha (\omega''_1 + 2\omega'_1) \left\{ \frac{B\alpha_1 \theta \sin \theta \hat{t}}{\varphi^2 - \theta^2} - \frac{\theta \alpha_1^2 \sin 2\theta \hat{t}}{\varphi^2 - 4\theta^2} \right\} + 2 \left\{ -\varphi \alpha_{10} \sin \varphi \hat{t} + \varphi \beta_{10} \cos \varphi \hat{t} - \right. \\ & \left. \frac{\theta r_{18} \sin \theta \hat{t}}{\varphi^2 - \theta^2} - \frac{2\theta r_{19} \sin 2\theta \hat{t}}{\varphi^2 - 4\theta^2} - \frac{3\theta r_{20} \sin 3\theta \hat{t}}{\varphi^2 - 9\theta^2} \right\} + \frac{\alpha}{2} \left\{ \left( \frac{\alpha_1 \alpha_6}{2} + \frac{r_2 B}{\theta^2} \right) + \left( \frac{\alpha_1 r_2}{\theta^2} - \frac{\alpha_1 r_3}{6\theta^2} + B\alpha_6 \right) \cos \theta \hat{t} + \left( B\beta_6 - \right. \right. \\ & \left. \left. \frac{\alpha_1 r_4}{6\theta^2} \right) \sin \theta \hat{t} + \left( \frac{\alpha_1 \beta_6}{2} - \frac{r_4 B}{3\theta^2} \right) \sin 2\theta \hat{t} - \frac{\alpha_1 r_3}{6\theta^2} \cos 3\theta \hat{t} - \frac{\alpha_1 r_4}{6\theta^2} \sin 3\theta \hat{t} \right\} - \alpha \left\{ \left( \frac{\alpha_1 r_6}{2(\varphi^2 - 4\theta^2)} + \right. \right. \\ & \left. \left. \frac{B r_5}{\varphi^2 - \theta^2} \right) \sin \theta \hat{t} + \frac{\alpha_1 \alpha_7}{2} \cos(\varphi + \theta) \hat{t} + \frac{\alpha_1 \beta_7}{2} \sin(\varphi + \theta) \hat{t} + \frac{\alpha_1 \alpha_7}{2} \cos(\varphi - \theta) \hat{t} + \frac{\alpha_1 \beta_7}{2} \sin(\varphi - \right. \\ & \left. \theta) \hat{t} + \left( \frac{\alpha_1 r_5}{2(\varphi^2 - \theta^2)} + \frac{B r_6}{\varphi^2 - 4\theta^2} \right) \sin 2\theta \hat{t} + \frac{\alpha_1 r_6}{2(\varphi^2 - 4\theta^2)} \sin 3\theta \hat{t} + B\alpha_7 \cos \varphi \hat{t} + B\beta_7 \sin \varphi \hat{t} \right\} + \\ & \frac{\alpha}{2} \left\{ \frac{\beta_2 \alpha_4 \sin 2\theta \hat{t}}{2} + \frac{\beta_2 r_0 \sin \theta \hat{t}}{\theta^2} - \frac{\beta_2 r_1}{6\theta^2} (\sin 3\theta \hat{t} - \sin \theta \hat{t}) \right\} - \alpha \left\{ \frac{\alpha \beta_2}{2} \left( \frac{\alpha_1^2 + B^2}{\varphi^2} \right) - \frac{\alpha \alpha_1^2 \beta_2}{8(\varphi^2 - 4\theta^2)} \sin \theta \hat{t} + \right. \\ & \left. \frac{\alpha \alpha_1 B \beta_2}{2(\varphi^2 - \theta^2)} \sin 2\theta \hat{t} + \frac{\alpha \alpha_1^2 \beta_2}{8(\varphi^2 - 4\theta^2)} \sin 3\theta \hat{t} \right\} + \frac{9\beta}{4} \left[ \left\{ \beta_2 \left( B^2 + \frac{\alpha_1^2}{2} \right) - \frac{\beta_2 \alpha_1^2}{4} \right\} \sin \varphi \hat{t} + \beta_2 B \alpha_1 \sin 2\theta \hat{t} + \right. \end{aligned}$$

$$\left. \frac{\beta_2 \alpha_1^2}{4} \sin 3\theta \hat{t} \right] \quad (4.71)$$

To ensure uniformly valid solution in  $\hat{t}$  needs equating the coefficients of  $\cos\varphi\hat{t}$  and  $\sin\varphi\hat{t}$  to zero.

Equating the coefficient of  $\cos\varphi\hat{t}$  yields

$$-2\omega'_1 \varphi \beta'_7 - 2\varphi \beta'_{10} - 2\beta_{10} \varphi + B\alpha_1 \alpha_7 = 0$$

Therefore,

$$\beta'_{10} + \beta_{10} = \frac{1}{2\varphi} [-2\omega'_1 \varphi \beta'_7 + B\alpha_1 \alpha_7] \quad (4.72.1)$$

$$\beta'_{10} + \beta_{10} = h_3(\tau) \quad (4.72.2)$$

Where,

$$h_3(\tau) = \frac{1}{2\varphi} [-2\omega'_1 \varphi \beta'_7 + B\alpha_1 \alpha_7] \quad (4.72.3)$$

It therefore follows that:

$$\beta_{10} = e^{-\tau} [\int h_3(s) e^s ds + \beta_{10}(0)] \quad (4.72.4)$$

The coefficient of  $\sin\varphi\hat{t}$  leads to

$$2\omega'_1 \varphi \alpha'_7 + 2\varphi \alpha'_{10} + 2\alpha_{10} \varphi + B\beta_7 \alpha = 0$$

$$\alpha'_{10} + \alpha_{10} = h_4(\tau) \quad (4.72.5)$$

$$h_4(\tau) = -\frac{1}{2\varphi} [2\omega'_1 \varphi \alpha'_7 + B\beta_7 \alpha] \quad (4.72.6)$$

Therefore:

$$\alpha_{10} = e^{-\tau} [\int h_4(s) e^s ds + \alpha_{10}(0)] \quad (4.72.7)$$

The remaining equation in (4.71) is

$$\begin{aligned} U_{2m,\hat{t}\hat{t}}^{(31)} + \varphi^2 U_{2m}^{(31)} = & r_{40} \cos\theta \hat{t} + r_{41} \sin\theta \hat{t} + r_{42} \cos 2\theta \hat{t} + r_{43} \sin 2\theta \hat{t} + \\ & r_{44} \cos 3\theta \hat{t} + r_{45} \sin 3\theta \hat{t} + r_{46} \cos(\varphi + \theta) \hat{t} + r_{47} \sin(\varphi + \theta) \hat{t} + r_{48} \cos(\varphi - \theta) \hat{t} + \\ & r_{49} \sin(\varphi - \theta) \hat{t} \end{aligned} \quad (4.73)$$

The initial conditions are;

$$U_{2m}^{(31)}(0,0) = 0; \quad U_{2m,\hat{t}}^{(31)}(0,0) + \omega_1'(0)U_{2m,\hat{t}}^{(21)}(0,0) + U_{2m,\tau}^{(30)}(0,0) = 0$$

Where,

$$r_{40} = \frac{-2\theta r_{15}'\omega_1'}{\varphi^2 - \theta^2} + \frac{2\alpha\omega_1'B\alpha_1\theta^2}{\varphi^2 - \theta^2} - \frac{\alpha}{2} \left( \left( \frac{\alpha_1\alpha_6}{2} + \frac{r_2B}{\theta^2} \right) + \left( \frac{\alpha_1r_2}{\theta^2} - \frac{\alpha_1r_3}{6\theta^2} \right) + B\alpha_6 \right)$$

$$r_{40}(0) = B^3 \left( \frac{-3\alpha}{\theta^2(\varphi^2 - \theta^2)} - \frac{2\alpha}{(\varphi^2 - \theta^2)} \right)$$

$$r_{41} = \left[ \frac{2\theta r_{18}^1}{\varphi^2 - \theta^2} + \frac{(\omega_1'' + 2\omega_1')\alpha B\alpha_1\theta}{\varphi^2 - \theta^2} + \frac{2\theta r_{18}}{\varphi^2 - \theta^2} - \frac{\alpha}{2} \left( B\beta_6 - \frac{\alpha_1r_4}{6\theta^2} \right) + \alpha \left( \frac{\alpha_1r_6}{2(\varphi^2 - 4\theta^2)} + \frac{Br_5}{\varphi^2 - \theta^2} \right) - \frac{\alpha\beta_2r_0}{2\theta^2} - \frac{\alpha\beta_2r_1}{12\theta^2} + \left\{ \frac{\alpha^2\beta_2}{2\varphi^2} \left( \frac{\alpha_1^2}{2} + B^2 \right) - \frac{\alpha^2\alpha_1B\beta_2}{8(\varphi^2 - 4\theta^2)} + \frac{9\beta}{4} \left( \beta_2 \left( B^2 + \frac{\alpha_1^2}{2} \right) \right) \right\} \right]$$

$$r_{41}(0) = B^3 \left( \frac{2\theta S_{24}}{(\varphi^2 - \theta^2)} - \frac{2\alpha}{\theta(\varphi^2 - \theta^2)} + \frac{2\theta S_7}{\varphi^2 - \theta^2} - \frac{2\alpha^2}{9\theta^4} - \frac{\alpha S_1}{2(\varphi^2 - 4\theta^2)} + \frac{\alpha^2}{\theta(\varphi^2 - \theta^2)} - \frac{17\alpha^2}{12\theta^3} - \frac{3\alpha^2}{(4\theta - \varphi^2)} - \frac{\alpha^2}{8\theta(\varphi^2 - 4\theta^2)} - \frac{45\beta}{16\theta} \right)$$

$$r_{42} = \left[ \frac{4\theta\omega_1'r_6}{(\varphi^2 - 4\theta^2)} - \frac{2\theta\alpha\omega_1'\alpha_1^2}{(\varphi^2 - 4\theta^2)} \right]$$

$$r_{42}(0) = B^3 \left( \frac{2\alpha}{\theta(\varphi^2 - 4\theta^2)} - \frac{4S_{14}}{\theta(\varphi^2 - 4\theta^2)} \right)$$

$$r_{43} = \left[ \frac{4\theta\omega_1'r_{19}}{(\varphi^2 - 4\theta^2)} + \frac{(\omega_1'' + 2\omega_1')\alpha\theta\alpha_1^2}{\varphi^2 - 4\theta^2} + \frac{4\theta r_{19}}{(\varphi^2 - 4\theta^2)} - \frac{\alpha}{2} \left( \frac{\alpha_1\beta_6}{2} - \frac{Br_4}{3\theta^2} \right) + \alpha \left( \frac{\alpha_1r_5}{2(\varphi^2 - \theta^2)} + \frac{Br_6}{\varphi^2 - 4\theta^2} \right) + \frac{\alpha^2\alpha_1B\beta_2}{2(\varphi^2 - \theta^2)} + \frac{9}{4}\beta\beta_2B\alpha_1 \right]$$

$$r_{43}(0) = B^3 \left( \frac{4\theta S_{10}}{(\varphi^2 - 4\theta^2)} + \frac{2\alpha}{\theta(\varphi^2 - \theta^2)} + \frac{4\theta S_8}{\varphi^2 - 4\theta^2} + \frac{\alpha^2}{6\theta^3} + \frac{2\alpha^2}{2\theta(\varphi^2 - \theta^2)} + \frac{\alpha S_1}{(\varphi^2 - \theta^2)} + \frac{9\beta}{4\theta} \right)$$

$$r_{44} = \left[ \frac{\alpha}{2} \left( \frac{\alpha_2r_3}{6\theta^2} \right) \right], \quad r_{44}(0) = 0$$

$$r_{45} = \left[ \frac{6\theta r_{20}'}{(\varphi^2 - 9\theta^2)} + \frac{6\theta r_{20}}{(\varphi^2 - 9\theta^2)} + \frac{\alpha}{2} \left( \frac{\alpha_1r_4}{6\theta^2} \right) + \frac{\alpha\alpha_1r_6}{2(\varphi^2 - 4\theta^2)} + \frac{\alpha\beta_2r_1}{12} + \frac{\alpha^2\alpha_1^2\beta_2}{8(\varphi^2 - 4\theta^2)} + \frac{9}{16}\beta\beta_2\alpha_1^2B \right]$$

$$r_{45}(0) = B^3 \left( \frac{6\theta S_{34}}{(\varphi^2 - 4\theta^2)} + \frac{6\theta S_9}{(\varphi^2 - \theta^2)} - \frac{\alpha^2}{12\theta^3} - \frac{\alpha S_1}{2(\varphi^2 - \theta^2)} - \frac{\alpha^2}{12\theta} - \frac{9\beta}{16\theta} \right)$$

$$r_{46} = \left[ \frac{\alpha_1\alpha_7}{2} \right], \quad r_{46}(0) = 0, \quad r_{47} = \left[ \frac{\alpha\alpha_1\beta_7}{2} \right], \quad r_{47}(0) = -\alpha B^3 S_{43}$$

$$r_{48} = \left[ \frac{\alpha\alpha_1\alpha_7}{2} \right], \quad r_{48}(0) = 0, \quad r_{49} = \left[ \frac{\alpha\alpha_1\beta_7}{2} \right], \quad r_{49}(0) = -\alpha B^3 S_{43}$$

$$S_{43} = \frac{\alpha S_0}{\varphi} + \frac{\alpha}{2\varphi^3} + \frac{\alpha}{2\alpha(\varphi^2-4\theta^2)} - \frac{\alpha}{\alpha(\varphi^2-\theta^2)} - \frac{2\theta\alpha S_1}{\varphi(\varphi^2-4\theta^2)}$$

Therefore;

$$\begin{aligned} U_{2m}^{(31)} = & \alpha_{13} \cos\varphi\hat{t} + \beta_{13} \sin\varphi\hat{t} + \frac{r_{42}\cos\theta\hat{t}+r_{43}\sin\theta\hat{t}}{\varphi^2-\theta^2} + \frac{r_{44}\cos2\theta\hat{t}+r_{45}\sin2\theta\hat{t}}{\varphi^2-4\theta^2} + \\ & \frac{r_{46}\cos3\theta\hat{t}+r_{47}\sin3\theta\hat{t}}{\varphi^2-9\theta^2} - \frac{r_{48}\cos(\varphi+\theta)\hat{t}+r_{49}\sin(\varphi+\theta)\hat{t}}{\theta(2\varphi+\theta)} + \frac{r_{50}\cos(\varphi-\theta)\hat{t}+r_{39}\sin(\varphi-\theta)\hat{t}}{\theta(2\varphi-\theta)} \end{aligned} \quad (4.74)$$

From the first initial condition,

$$\alpha_{13}(0) = - \left[ \frac{r_{42}}{\varphi^2-\theta^2} + \frac{r_{44}}{\varphi^2-4\theta^2} + \frac{r_{46}}{\varphi^2-9\theta^2} - \frac{r_{48}}{\theta(2\varphi+\theta)} + \frac{r_{50}}{\theta(2\varphi-\theta)} \right] \text{ at } \tau = 0 \quad (4.75.1)$$

And from the second initial condition, it follows that,

$$\begin{aligned} \left[ \beta_{13}(0)\varphi + \frac{\theta r_{43}}{\varphi^2-\theta^2} + \frac{2\theta r_{45}}{\varphi^2-4\theta^2} + \frac{3\theta r_{47}}{\varphi^2-9\theta^2} - \frac{(\theta+\varphi)r_{49}}{\theta(2\varphi+\theta)} + \frac{(\theta-\varphi)r_{51}}{\theta(2\varphi-\theta)} + \alpha'_{10}(0) \right. \\ \left. + \frac{r'_{17}}{\varphi^2} + \frac{r'_{18}}{\varphi^2-\theta^2} + \frac{r'_{19}}{\varphi^2-4\theta^2} + \frac{r'_{20}}{\varphi^2-9\theta^2} \right] = 0 \end{aligned}$$

Therefore,

$$\begin{aligned} \beta_{13}(0) = - \frac{1}{\varphi} \left[ \frac{\theta r_{43}}{\varphi^2-\theta^2} + \frac{2\theta r_{45}}{\varphi^2-4\theta^2} + \frac{3\theta r_{47}}{\varphi^2-9\theta^2} - \frac{(\theta+\varphi)r_{49}}{\theta(2\varphi+\theta)} + \frac{(\theta-\varphi)r_{51}}{\theta(2\varphi-\theta)} + \alpha'_{10}(0) + \frac{r'_{17}}{\varphi^2} + \right. \\ \left. \frac{r'_{18}}{\varphi^2-\theta^2} + \frac{r'_{19}}{\varphi^2-4\theta^2} + \frac{r'_{20}}{\varphi^2-9\theta^2} \right] \end{aligned} \quad (4.75.2)$$

Substituting in (4.61);

$$\begin{aligned} U_{3m,\hat{t}\hat{t}}^{(31)} + \Omega^2 U_{3m}^{(31)} = - \left[ 2U_{3m,\hat{t}\hat{t}}^{(30)} + 2U_{3m,\hat{t}}^{(30)} + \alpha \left\{ \frac{1}{2} U_m^{(10)} U_{2m}^{(21)} + \frac{1}{2} U_m^{(11)} U_{2m}^{(20)} \right\} + \right. \\ \left. \frac{3}{4} \beta \left( U_m^{(10)} \right)^2 U_m^{(11)} \right] \end{aligned}$$

Simplification yields,

$$\begin{aligned} U_{3m,\hat{t}\hat{t}}^{(31)} + \Omega^2 U_{3m}^{(31)} = - \left[ 2 \left\{ -\Omega\alpha'_{11} \sin\Omega\hat{t} + \Omega\beta'_{11} \cos\Omega\hat{t} - \frac{\theta r'_{22} \sin\theta\hat{t}}{\Omega^2-\theta^2} - \right. \right. \\ \left. \frac{2\theta r'_{23} \sin2\theta\hat{t}}{\Omega^2-4\theta^2} - \frac{3\theta r'_{24} \sin3\theta\hat{t}}{\Omega^2-9\theta^2} - \frac{(\varphi+\theta)r'_{25} \sin(\varphi+\theta)\hat{t}}{\Omega^2-(\varphi+\theta)^2} + \frac{(\varphi+\theta)r'_{26} \cos(\varphi+\theta)\hat{t}}{\Omega^2-(\varphi+\theta)^2} - \frac{(\varphi-\theta)r'_{27} \sin(\varphi-\theta)\hat{t}}{\Omega^2-(\varphi-\theta)^2} \right. \end{aligned}$$

$$\begin{aligned}
& \left. \frac{(\varphi-\theta)r'_{28}\cos(\varphi-\theta)\hat{t}}{\Omega^2-(\varphi-\theta)^2} \right\} + 2 \left\{ -\Omega\alpha_{11}\sin\Omega\hat{t} + \Omega\beta_{11}\cos\Omega\hat{t} - \frac{\theta r_{22}\sin\theta\hat{t}}{\Omega^2-\theta^2} - \frac{2\theta r_{23}\sin 2\theta\hat{t}}{\Omega^2-4\theta^2} - \frac{3\theta r_{24}\sin 3\theta\hat{t}}{\Omega^2-9\theta^2} - \right. \\
& \left. \frac{(\varphi+\theta)r_{25}\sin(\varphi+\theta)\hat{t}}{\Omega^2-(\varphi+\theta)^2} + \frac{(\varphi+\theta)r_{26}\cos(\varphi+\theta)\hat{t}}{\Omega^2-(\varphi+\theta)^2} - \frac{(\varphi-\theta)r_{27}\sin(\varphi-\theta)\hat{t}}{\Omega^2-(\varphi-\theta)^2} + \frac{(\varphi-\theta)r_{28}\cos(\varphi-\theta)\hat{t}}{\Omega^2-(\varphi-\theta)^2} \right\} + \\
& \frac{\alpha}{2} \left\{ \left( \frac{\alpha_1 r_6}{2(\varphi^2-4\theta^2)} + \frac{Br_5}{\varphi^2-\theta^2} \right) \sin\theta\hat{t} + \frac{\alpha_1\alpha_7}{2} \cos(\varphi+\theta)\hat{t} + \frac{\alpha_1\beta_7}{2} \sin(\varphi+\theta)\hat{t} + \frac{\alpha_1\alpha_7}{2} \cos(\varphi-\theta)\hat{t} + \right. \\
& \left. \frac{\alpha_1\beta_7}{2} \sin(\varphi-\theta)\hat{t} + \left( \frac{\alpha_1 r_5}{2(\varphi^2-\theta^2)} + \frac{Br_6}{\varphi^2-4\theta^2} \right) \sin 2\theta\hat{t} + \frac{\alpha_1 r_6}{2(\varphi^2-4\theta^2)} \sin 3\theta\hat{t} + B\alpha_7 \sin\varphi\hat{t} + \right. \\
& \left. B\beta_7 \sin\varphi\hat{t} \right\} + \frac{\alpha}{2} \left\{ \frac{\beta_2\alpha}{2} \left( \frac{B^2+\frac{\alpha_1^2}{2}}{\varphi^2} \right) - \frac{\alpha\alpha_1^2\beta_2}{8(\varphi^2-4\theta^2)} \sin\theta\hat{t} + \frac{\alpha\alpha_1 B\beta_2}{2(\varphi^2-\theta^2)} \sin 2\theta\hat{t} + \frac{\alpha\alpha_1^2\beta_2}{8(\varphi^2-4\theta^2)} \sin 3\theta\hat{t} - \right. \\
& \left. \frac{\beta_2\beta_5}{2} \cos(\varphi+\theta)\hat{t} \right\} + \frac{3\beta}{4} \left\{ \left( \beta_2 \left( B^2 + \frac{\alpha_1^2}{2} \right) - \frac{\beta_2\alpha_1^2}{4} \right) \sin\theta\hat{t} + \beta_2 B\alpha_1 \sin 2\theta\hat{t} + \frac{\beta_2\alpha_1^2}{4} \sin 3\theta\hat{t} \right\} \Bigg\} \quad (4.76)
\end{aligned}$$

To ensure uniformly valid solution in  $\hat{t}$ , needs equating the coefficients of  $\cos\Omega\hat{t}$  and  $\sin\Omega\hat{t}$  to zero. The coefficients of  $\cos\Omega\hat{t}$  yields

$$-2\Omega\beta'_{11} - 2\Omega\beta_{11} - \frac{\alpha B\beta_7}{2} = 0 \quad (4.77.1)$$

Therefore,

$$\beta'_{11} + \beta_{11} = -\frac{\alpha B\alpha_7}{2\Omega} = h_5(\tau) \quad (4.77.2)$$

Where,

$$h_5(\tau) = -\frac{\alpha B\alpha_7}{2\Omega} \quad (4.77.3)$$

Therefore,

$$\beta_{11} = e^{-\tau} \left[ \int h_5(\tau) e^s ds + \beta_{11}(0) \right] \quad (4.77.4)$$

The coefficients  $\sin\Omega\hat{t}$  of yields

$$-2\Omega\alpha'_{11} - 2\Omega\alpha_{11} - \frac{\alpha B\beta_7}{2} = 0 \quad (4.77.5)$$

Where,

$$h_6(\tau) = \frac{\alpha B\beta_7}{4\Omega} \quad (4.77.6)$$

Therefore;

$$\alpha_{11} = e^{-\tau} \left[ \int h_6(\tau) e^s ds + \alpha_{11}(0) \right] \quad (4.77.7)$$

The remaining equation (4.76) is:

$$U_{3m,\hat{t}\hat{t}}^{(31)} + \Omega^2 U_{2m}^{(31)} = r_{50} \sin\theta \hat{t} + r_{51} \sin 2\theta \hat{t} + r_{52} \sin 3\theta \hat{t} + r_{53} \cos(\varphi + \theta) \hat{t} + r_{54} \sin(\varphi + \theta) \hat{t} + r_{55} \cos(\varphi - \theta) \hat{t} + r_{56} \sin(\varphi - \theta) \hat{t} \quad (4.78)$$

The initial conditions are

$$U_{3m}^{(31)}(0,0) = 0; U_{3m,\hat{t}}^{(31)}(0,0) + U_{3m,\tau}^{(30)}(0,0) = 0$$

$$r_{50} = \frac{2\theta r_{22}^1}{\Omega^2 - \theta^2} + \frac{2\theta r_{22}}{\Omega^2 - \theta^2} - \frac{\alpha\alpha_1 r_6}{4(\varphi^2 - 4\theta^2)} - \frac{B\alpha r_5}{2(\varphi^2 - 4\theta^2)} - \frac{\alpha^2 \beta_2}{4} \left( \frac{\alpha_1^2 + B^2}{\varphi^2} \right) + \frac{\alpha^2 \alpha_1 \beta_2}{16(\varphi^2 - 4\theta^2)} - \frac{3\alpha\beta\beta_2}{8} \left( B^2 + \frac{\alpha_1^2}{2} \right) + \frac{3\alpha\beta\alpha_1^2}{32}$$

$$r_{50}(0) = B^3 \left( \frac{2\theta S_{17}}{(\Omega^2 - \theta^2)} + \frac{2\theta S_{11}}{(\Omega^2 - \theta^2)} + \frac{\alpha S_1}{4(\varphi^2 - 4\theta^2)} + \frac{\alpha^2}{2\theta(\varphi^2 - 4\theta^2)} + \frac{3\alpha^2}{8\theta\varphi^2} + \frac{\alpha^2}{16B(\varphi^2 - 4\theta^2)} + \frac{9\alpha\beta}{16\theta} + \frac{3\alpha\beta}{32B} \right)$$

$$r_{51} = \frac{4\theta r_{23}^1}{\Omega^2 - 4\theta^2} + \frac{4\theta r_{23}}{\Omega^2 - 4\theta^2} - \frac{\alpha\alpha_1 r_5}{4(\varphi^2 - \theta^2)} - \frac{B\alpha r_6}{2(\varphi^2 - 4\theta^2)} - \frac{\alpha^2 \alpha_1 B \beta_2}{4(\varphi^2 - 4\theta^2)} - \frac{\alpha\alpha_1 \beta_2 B}{2}$$

$$r_{51}(0) = B^3 \left( \frac{4\theta S_{18}}{(\Omega^2 - 4\theta^2)} + \frac{4\theta S_{12}}{(\Omega^2 - 4\theta^2)} + \frac{\alpha^2}{2\theta(\varphi^2 - \theta^2)} - \frac{\alpha S_1}{2(\varphi^2 - 4\theta^2)} - \frac{\alpha}{2\theta} \right)$$

$$r_{52} = \frac{6\theta r_{24}^1}{\Omega^2 - 9\theta^2} + \frac{6\theta r_{24}}{\varphi^2 - 9\theta^2} - \frac{\alpha\alpha_1 r_6}{4(\varphi^2 - 4\theta^2)} - \frac{\alpha^2 \beta_2 \alpha_1^2}{16(\varphi^2 - 4\theta^2)} + \frac{\alpha\alpha_1^2 \beta_2}{8}$$

$$r_{52}(0) = B^3 \left( \frac{6\theta S_{19}}{(\Omega^2 - 9\theta^2)} + \frac{6\theta S_{13}}{(\Omega^2 - 9\theta^2)} + \frac{\alpha S_1}{4(\varphi^2 - 4\theta^2)} - \frac{\alpha^2}{16\theta(\varphi^2 - 4\theta^2)} + \frac{\alpha}{8\theta} \right)$$

$$r_{53} = -\frac{2r'_{26}(\varphi+\theta)}{\Omega^2 - (\varphi+\theta)^2} - \frac{2r_{26}(\varphi+\theta)}{\Omega^2 - (\varphi+\theta)^2} - \frac{\alpha\alpha_1\alpha_7}{4}, \quad r_{53}(0) = 0$$

$$r_{54} = \frac{2r'_{25}(\varphi+\theta)}{\Omega^2 - (\varphi+\theta)^2} + \frac{2r_{25}(\varphi+\theta)}{\Omega^2 - (\varphi+\theta)^2} - \frac{\alpha\alpha_1\beta_7}{4}, \quad r_{54}(0) = B^3 \left( \frac{6\alpha S_0(\varphi+\theta)}{\Omega^2 - (\varphi+\theta)^2} + \frac{\alpha S_{43}}{4} \right)$$

$$r_{55} = \frac{-2r'_{28}(\varphi-\theta)}{\Omega^2 - (\varphi-\theta)^2} - \frac{2r_{28}(\varphi-\theta)}{\Omega^2 - (\varphi-\theta)^2} - \frac{\alpha\alpha_1\alpha_7}{4}, \quad r_{55}(0) = \frac{-4\alpha S_0 B^3}{\Omega^2 - (\varphi-\theta)^2}$$

$$r_{56} = \frac{2r'_{27}(\varphi-\theta)}{\Omega^2 - (\varphi-\theta)^2} + \frac{2r_{27}(\varphi-\theta)}{\Omega^2 - (\varphi-\theta)^2} + \frac{\alpha\alpha_1\beta_7}{4}, \quad r_{56}(0) = B^3 \left( \frac{6\alpha S_0(\varphi-\theta)}{\Omega^2 - (\varphi-\theta)^2} + \frac{\alpha S_{43}}{4} \right)$$

Therefore;

$$U_{3m}^{(31)} = \alpha_{14} \cos \Omega \hat{t} + \beta_{14} \sin \Omega \hat{t} + \frac{r_{50} \sin \theta \hat{t}}{\Omega^2 - \theta^2} + \frac{r_{51} \sin 2\theta \hat{t}}{\Omega^2 - 4\theta^2} + \frac{r_{52} \sin 3\theta \hat{t}}{\Omega^2 - 9\theta^2} +$$

$$\left( \frac{r_{53} \cos(\varphi + \theta) \hat{t} + r_{54} \sin(\varphi + \theta) \hat{t}}{\Omega^2 - (\varphi + \theta)^2} \right) + \left( \frac{r_{55} \cos(\varphi - \theta) \hat{t} + r_{56} \sin(\varphi - \theta) \hat{t}}{\Omega^2 - (\varphi - \theta)^2} \right)$$

(4.79)

$$\alpha_{14}(0) = - \left[ \frac{r_{53}}{\Omega^2 - (\varphi + \theta)^2} + \frac{r_{55}}{\Omega^2 - (\varphi - \theta)^2} \right] \Big|_{\tau = 0} \quad (4.80.1)$$

$$\Omega \beta_{14}(0) = - \frac{\theta r_{50}}{\Omega^2 - \theta^2} - \frac{2\theta r_{51}}{\Omega^2 - 4\theta^2} - \frac{3\theta r_{52}}{\Omega^2 - 9\theta^2} - \frac{(\varphi + \theta) r_{54}}{\Omega^2 - (\varphi + \theta)^2} - \frac{(\varphi - \theta) r_{56}}{\Omega^2 - (\varphi - \theta)^2} - \alpha'_{11} - \frac{r'_{22}}{\Omega^2 - \theta^2} -$$

$$\frac{r'_{23}}{\Omega^2 - 4\theta^2} - \frac{r'_{24}}{\Omega^2 - 9\theta^2} - \frac{r'_{25}}{\Omega^2 - (\varphi + \theta)^2} - \frac{r'_{27}}{\Omega^2 - (\varphi - \theta)^2}$$

Therefore,

$$\beta_{14}(0) = \frac{-1}{\Omega} \left[ \frac{(\theta r_{50} + r'_{22})}{\varphi^2 - \theta^2} + \frac{(2\theta r_{51} + r'_{23})}{\varphi^2 - 4\theta^2} + \frac{(3\theta r_{52} + r'_{24})}{\varphi^2 - 9\theta^2} - \frac{((\theta + \varphi) r_{54} + r'_{24})}{\Omega^2 - (\varphi + \theta)^2} + \frac{(\theta - \varphi) r_{56} + r'_{27}}{\Omega^2 - (\varphi - \theta)^2} + \right.$$

$$\left. \alpha'_{11}(0) \right] \quad (4.80.2)$$

So far, it follows that,

$$U^{(31)} = U_m^{(31)}(1 - \cos 2mx) + U_{2m}^{(31)}(1 - \cos 4mx) + U_{3m}^{(31)}(1 - \cos 6mx) \quad (4.81)$$

The summary of the solution so far is,

$$U(x, t, \tau) = \epsilon(U^{(10)} + \delta U^{(11)} + \delta^2 U^{(12)} + \dots) + \epsilon^2(U^{(20)} + \delta U^{(21)} + \delta^2 U^{(22)} +$$

$$\dots) + \epsilon^3(U^{(30)} + \delta U^{(31)} + \delta^2 U^{(32)} + \dots) + \dots \quad (4.82)$$

## 4.2 VALUES OF INDEPENDENT VARIABLES AT MAXIMUM DISPLACEMENT

The dynamic buckling load is obtained from the maximization  $\frac{d\lambda}{dU_a} = 0$ ; where  $U_a$  is the maximum displacement and  $\lambda$  is the load parameter.

The conditions for maximum displacement are

$$\frac{\partial U}{\partial x} = 0, \quad \frac{\partial w}{\partial t} = 0 \quad (4.83.1)$$

But from (3.12), it follows that

$$\frac{\partial w}{\partial t} = U_{,\hat{t}} + (\omega'_1 \epsilon + \omega'_2 \epsilon^2 + \dots) U_{,\hat{t}} + \delta U_{,\tau} = 0 \quad (4.83.2)$$

The aim is to determine the maximum displacement;

$$U_a = U(x_a, \hat{t}_a, t_a)$$

Where  $x_a, t_a, \tau_a$  and  $\hat{t}_a$  are the values of  $x, t, \tau$ , and  $\hat{t}$  respectively at maximum displacement and are to be next determined before finally determining the maximum displacement.

Recall:

$$U^{(10)} = U_m^{(10)}(1 - \cos 2mx)$$

$$U^{(11)} = U_m^{(11)}(1 - \cos 2mx)$$

$$U^{(20)} = U_m^{(20)}(1 - \cos 2mx) + U_{2m}^{(20)}(1 - \cos 4mx)$$

$$U^{(21)} = U_m^{(21)}(1 - \cos 2mx) + U_{2m}^{(21)}(1 - \cos 4mx)$$

$$U^{(30)} = U_m^{(30)}(1 - \cos 2mx) + U_{2m}^{(30)}(1 - \cos 4mx) + U_{3m}^{(30)}(1 - \cos 6mx)$$

$$U^{(31)} = U_m^{(31)}(1 - \cos 2mx) + U_{2m}^{(31)}(1 - \cos 4mx) + U_{3m}^{(31)}(1 - \cos 6mx)$$

From the first condition of maximization  $\frac{\partial U}{\partial x} = 0$

This means,

$$\epsilon \left[ \frac{\partial U^{(10)}}{\partial x} + \delta \frac{\partial U^{(11)}}{\partial x} + \dots \right] + \epsilon^2 \left[ \frac{\partial U^{(20)}}{\partial x} + \delta \frac{\partial U^{(21)}}{\partial x} + \dots \right] + \epsilon^3 \left[ \frac{\partial U^{(30)}}{\partial x} + \delta \frac{\partial U^{(31)}}{\partial x} + \dots \right] = 0 \quad (4.84)$$

That is,

$$\begin{aligned} & 2m\epsilon \left[ U_m^{(10)} \sin 2mx + \delta U_m^{(11)} \sin 2mx + \dots \right] + \epsilon^2 \left[ 2m U_m^{(20)} \sin 2mx + 4m U_{2m}^{(20)} \sin 4mx + \dots \right. \\ & \left. \delta \left\{ 2m U_m^{(21)} \sin 2mx + 4m U_{2m}^{(21)} \sin 4mx + \dots \right\} \right] + \epsilon^3 \left[ 2m U_m^{(30)} \sin 2mx + \dots \right. \\ & \left. 4m U_{2m}^{(30)} \sin 4mx + 6m U_{3m}^{(30)} \sin 6mx + \dots \right. \\ & \left. \delta \left\{ 2m U_m^{(31)} \sin 2mx + 4m U_{2m}^{(31)} \sin 4mx + 6m U_{3m}^{(31)} \sin 6mx + \dots \right\} + \dots \right] = 0 \end{aligned} \quad (4.85)$$

The equation (4.85) is satisfied if  $\sin 2mx_a = 0$

Where  $x_a$  is the value of  $x$  at maximum displacement.

This means,

$$2mx_a = \pi n, \quad n = 0, 1, 2, 3 \dots$$

$$\text{Set } n = 1$$

Therefore,

$$x_a = \frac{\pi}{2m}$$

Substituting  $x_a = \frac{\pi}{2m}$  in  $U(x, \hat{t}, \tau)$ , gives

$$\begin{aligned} U(x_a, \hat{t}, \tau) = & 2\epsilon \left[ U_m^{(10)} + \delta U_m^{(11)} + \dots \right] + 2\epsilon^2 \left[ U_m^{(20)} + \delta U_{2m}^{(21)} + \dots \right] + \\ & 2\epsilon^3 \left[ \left( U_m^{(30)} + U_{3m}^{(30)} \right) + \delta \left( U_m^{(31)} + U_{3m}^{(31)} \right) + \dots \right] \end{aligned} \quad (4.86)$$

Let  $\hat{t}_a$ ,  $t_a$  and  $\tau_a$  be the values of  $\hat{t}$ ,  $t$  and  $\tau$  respectively at maximum displacement and let them be expanded asymptotically as

$$\hat{t}_a = \hat{t}_0 + \delta \hat{t}_{01} + \delta^2 \hat{t}_{02} + \epsilon (\hat{t}_{10} + \delta \hat{t}_{11} + \delta^2 \hat{t}_{12} + \dots) + \epsilon^2 (\hat{t}_{20} + \delta \hat{t}_{21} + \delta^2 \hat{t}_{22} + \dots) \quad (4.87.1)$$

$$t_a = t_0 + \delta t_{01} + \delta^2 t_{02} + \dots + \epsilon (t_{10} + \delta t_{11} + \delta^2 t_{12} + \dots) + \epsilon^2 (t_{20} + \delta t_{21} + \delta^2 t_{22} + \dots) \quad (4.87.2)$$

$$\tau_a = \delta [t_0 + \delta t_{01} + \delta^2 t_{02} + \dots + \epsilon (t_{10} + \delta t_{11} + \delta^2 t_{12} + \dots) + \epsilon^2 (t_{20} + \delta t_{21} + \delta^2 t_{22} + \dots)] \quad (4.87.3)$$

The individual terms in (4.86) are expanded as:

$$\begin{aligned} \epsilon U_{m, \hat{t}}^{(10)} = & \epsilon \left[ U_{m, \hat{t}}^{(10)} + \{(\delta \hat{t}_{01} + \delta^2 \hat{t}_{02} + \dots) + \epsilon (\hat{t}_{10} + \delta \hat{t}_{11} + \delta^2 \hat{t}_{12} + \dots) + \right. \\ & \left. \epsilon^2 (\hat{t}_{20} + \delta \hat{t}_{21} + \delta^2 \hat{t}_{22} + \dots) \right] U_{m, \hat{t}\hat{t}}^{(10)} + \delta \{ (t_0 + \delta t_{01} + \delta^2 t_{02} + \dots) + \epsilon (t_{10} + \\ & \delta t_{11} + \delta^2 t_{12} + \dots) + \epsilon^2 (t_{20} + \delta t_{21} + \delta^2 t_{22} + \dots) \} U_{m, \hat{t}\tau}^{(10)} + \frac{1}{2} \{ \{ (\delta \hat{t}_{01} + \delta^2 \hat{t}_{02} + \\ & \dots) + \epsilon (\hat{t}_{10} + \delta \hat{t}_{11} + \delta^2 \hat{t}_{12} + \dots) + \epsilon^2 (\hat{t}_{20} + \delta \hat{t}_{21} + \delta^2 \hat{t}_{22} + \dots) + \dots \}^2 U_{m, \hat{t}\hat{t}\hat{t}}^{(10)} + \\ & 2\delta \{ (\delta \hat{t}_{01} + \delta^2 \hat{t}_{02} + \dots) + \epsilon (\hat{t}_{10} + \delta \hat{t}_{11} + \delta^2 \hat{t}_{12} + \dots) + \epsilon^2 (\hat{t}_{20} + \delta \hat{t}_{21} + \delta^2 \hat{t}_{22} + \\ & \dots) \} \{ \delta t_{01} + \delta^2 t_{02} + \dots + \epsilon (t_{10} + \delta t_{11} + \delta^2 t_{12} + \dots) + \epsilon^2 (t_{20} + \delta t_{21} + \delta^2 t_{22} + \dots) \} \end{aligned}$$

$$\dots\}U_{m,\hat{t}\hat{\tau}}^{(10)} + \delta^2\{\delta t_{01} + \delta^2 t_{02} + \dots + \epsilon(t_{10} + \delta t_{11} + \delta^2 t_{12} + \dots) + \epsilon^2(t_{20} + \delta t_{21} + \delta^2 t_{22} + \dots)\}^2 \times U_{m,\hat{t}\hat{\tau}\hat{\tau}}^{(10)} \Big|_{(\hat{t}_{0,0})} \quad (4.88)$$

$$\begin{aligned} \epsilon \delta U_{m,\hat{t}}^{(11)} = & \epsilon \delta \left[ U_{m,\hat{t}}^{(11)} + \{(\delta \hat{t}_{01} + \delta^2 \hat{t}_{02} + \dots) + \epsilon(\hat{t}_{10} + \delta \hat{t}_{11} + \delta^2 \hat{t}_{12} + \dots) + \epsilon^2(\hat{t}_{20} + \delta \hat{t}_{21} + \delta^2 \hat{t}_{22} + \dots)\} U_{m,\hat{t}\hat{t}}^{(11)} + \delta\{t_0 + \delta t_{01} + \delta^2 t_{02} + \dots + \epsilon(t_{10} + \delta t_{11} + \dots) + \epsilon^2(t_{20} + \delta t_{21} + \delta^2 t_{22} + \dots)\} U_{m,\hat{t}\hat{\tau}}^{(11)} + \{\delta \hat{t}_{01} + \epsilon(\hat{t}_{10} + \delta \hat{t}_{11} + \delta^2 \hat{t}_{12} + \dots) + \epsilon^2(\hat{t}_{20} + \delta \hat{t}_{21} + \delta^2 \hat{t}_{22} + \dots) + \dots\}^2 U_{m,\hat{t}\hat{t}\hat{t}}^{(11)} + \delta\{\delta \hat{t}_{01} + \dots + \epsilon(\hat{t}_{10} + \delta \hat{t}_{11} + \delta^2 \hat{t}_{12} + \dots) + \epsilon^2(\hat{t}_{20} + \delta \hat{t}_{21} + \delta^2 \hat{t}_{22} + \dots) + \dots\} \times \{t_0 + \delta t_{01} + \delta^2 t_{02} + \dots + \epsilon(t_{10} + \delta t_{11} + \delta^2 t_{12} + \dots) + \epsilon^2(t_{20} + \delta t_{21} + \delta^2 t_{22} + \dots)\} \right] U_{m,\hat{t}\hat{\tau}}^{(11)} \Big|_{(\hat{t}_{0,0})} \end{aligned} \quad (4.89)$$

$$\begin{aligned} \epsilon^2 U_{m,\hat{t}}^{(20)} = & \epsilon^2 \left[ U_{m,\hat{t}}^{(20)} + \{(\hat{t}_{01} \delta + \delta^2 \hat{t}_{02} + \dots) + \epsilon^2(\hat{t}_{10} + \delta \hat{t}_{11} + \delta^2 \hat{t}_{12} + \dots)\} U_{m,\hat{t}\hat{t}}^{(20)} + \{(t_0 + \delta t_{01}) + \epsilon(t_{10} + \delta t_{11} + \delta^2 t_{12} + \dots) + \epsilon^2(t_{20} + \delta t_{21} + \dots) \dots\} U_{m,\hat{t}\hat{\tau}}^{(20)} + \frac{1}{2} \left\{ \{(\delta \hat{t}_{01} + \delta^2 \hat{t}_{02} + \dots) + \epsilon(\hat{t}_{10} + \delta \hat{t}_{11} + \delta^2 \hat{t}_{12} + \dots) + \dots\}^2 U_{m,\hat{t}\hat{t}\hat{t}}^{(20)} + 2\delta\{\delta t_{01} + \dots + \epsilon(t_{10} + \delta t_{11} + \dots) \dots\} \{t_0 + \delta t_{01} \dots + \epsilon(t_{10} + \delta t_{11} + \delta t_{11} + \dots) + \dots\} U_{m,\hat{t}\hat{t}\hat{\tau}}^{(20)} \right\} + \delta^2 \{(t_0 + \delta t_{01}) \dots + \epsilon(t_{10} + \delta t_{11} + \dots)\} U_{m,\hat{t}\hat{\tau}\hat{\tau}}^{(20)} + \dots \right] \Big|_{(\hat{t}_{0,0})} \end{aligned} \quad (4.90)$$

$$\begin{aligned} \delta \epsilon^2 U_{m,\hat{t}}^{(21)} = & \delta \epsilon^2 \left[ U_{m,\hat{t}}^{(21)} + \{\hat{t}_{01} \delta + \dots + \epsilon^2(\hat{t}_{10} + \delta \hat{t}_{11} + \dots)\} U_{m,\hat{t}\hat{t}}^{(21)} + \delta\{t_0 + \delta t_{01} + \dots + \epsilon(t_{10} + \delta t_{11} + \delta^2 t_{12} + \dots)\} U_{m,\hat{t}\hat{\tau}}^{(21)} + \{\delta \hat{t}_{01} + \dots + \epsilon(\hat{t}_{10} + \delta \hat{t}_{11} + \dots) + \dots\}^2 U_{m,\hat{t}\hat{t}\hat{t}}^{(21)} + \delta\{t_0 + t_{11} + \dots + \epsilon(t_{10} + \delta t_{11} + \dots) \dots\} \times \{t_{01} \delta \dots + \epsilon(t_{10} + \delta t_{11} + \dots) + \dots\} \times U_{m,\hat{t}\hat{t}\hat{\tau}}^{(21)} \right] \Big|_{(\hat{t}_{0,0})} \end{aligned} \quad (4.91)$$

$$\begin{aligned} \epsilon^3 \left( U_m^{(30)} + U_{3m}^{(30)} \right)_{,\hat{t}} = & \epsilon^3 \left[ \left( U_m^{(30)} + U_{3m}^{(30)} \right)_{,\hat{t}} + \{(\delta \hat{t}_{01} + \delta^2 \hat{t}_{02} + \dots)\} \left( U_m^{(30)} + U_{3m}^{(30)} \right)_{,\hat{t}\hat{t}} + \delta\{t_0 + \delta t_{01} + \dots\} \left( U_m^{(30)} + U_{3m}^{(30)} \right)_{,\hat{t}\hat{\tau}} + \frac{1}{2} \{(\delta \hat{t}_{01} + \delta^2 \hat{t}_{02} + \dots)\}^2 \left( U_m^{(30)} + U_{3m}^{(30)} \right)_{,\hat{t}\hat{t}\hat{t}} + \delta \left\{ (\delta \hat{t}_{01} + \delta^2 \hat{t}_{02} + \dots)(t_0 + \delta t_{01} + \dots) \left( U_m^{(30)} + U_{3m}^{(30)} \right)_{,\hat{t}\hat{t}\hat{\tau}} \right\} + \frac{1}{2} \delta^2 (t_0 + \dots)^2 \left( U_m^{(30)} + U_{3m}^{(30)} \right)_{,\hat{t}\hat{\tau}\hat{\tau}} \right] \Big|_{(\hat{t}_{0,0})} \end{aligned} \quad (4.92)$$

$$\delta \epsilon^3 \left( U_m^{(31)} + U_{3m}^{(31)} \right)_{,\hat{t}} = \epsilon^3 \delta \left[ \left( U_m^{(31)} + U_{3m}^{(31)} \right)_{,\hat{t}} + \{(\delta \hat{t}_{01} + \delta^2 \hat{t}_{02} + \dots)\} \left( U_m^{(31)} + U_{3m}^{(31)} \right)_{,\hat{t}\hat{t}} \right] \Big|_{(\hat{t}_{0,0})} \quad (4.93)$$

$$\begin{aligned} \omega'_1 \epsilon U_{m,\hat{t}}^{(10)} &= \epsilon^2 \left[ \omega'_1 U_{m,\hat{t}}^{(10)} + \omega'_1 \{ \delta \hat{t}_{01} + \delta^2 \hat{t}_{02} + \dots \} \epsilon (\delta \hat{t}_{10} + \delta \hat{t}_{11} + \delta \hat{t}_{12} + \dots) + \right. \\ &\epsilon^2 (\hat{t}_{20} + \delta \hat{t}_{21} + \delta^2 \hat{t}_{22} + \dots) \left. \right] U_{m,\hat{t}\hat{t}}^{(10)} + \delta \{ (t_0 + \delta t_{01} + \dots) + \epsilon (t_{10} + \delta t_{11} + \dots) + \epsilon^2 (t_{20} + \delta t_{21} + \dots) \} \\ &\left\{ \left( \omega'_1 U_{m,\hat{t}}^{(10)} \right)_{,\tau} \right\} + \frac{\omega'_1}{2} \{ \delta \hat{t}_{01} + \delta^2 \hat{t}_{02} + \dots \} \epsilon (\delta \hat{t}_{10} + \delta \hat{t}_{11} + \delta^2 \hat{t}_{12}) + \epsilon^2 (\hat{t}_{20} + \delta \hat{t}_{21} + \delta^2 \hat{t}_{22} + \dots) \}^2 \\ &U_{m,\hat{t}\hat{t}\hat{t}}^{(10)} + \delta \{ \delta \hat{t}_{01} + \dots \} \epsilon (\hat{t}_{10} + \delta \hat{t}_{11} + \dots) + \epsilon^2 (\hat{t}_{20} + \delta \hat{t}_{21} + \dots) \} \times \\ &\{ t_0 + \delta t_{01} + \dots + \epsilon (t_{10} + \delta t_{11} + \delta^2 t_{12} \dots) + \epsilon^2 (t_{20} + \delta t_{21} + \dots) \} \left( \omega'_1 U_{m,\hat{t},\hat{t}}^{(20)} \right)_{,\tau} + \frac{1}{2} \delta^2 \{ t_0 + \delta t_{01} + \dots + \epsilon (t_{10} + \delta t_{11} + \dots) + \epsilon^2 (t_{20} + \delta t_{21} + \dots) \}^2 \\ &\left( U_{m,\hat{t}}^{(20)} \right)_{,\tau\tau} \Big|_{(\hat{t}_{0,0})} \end{aligned} \quad (4.94)$$

$$\begin{aligned} \omega'_1 \epsilon \delta U_{m,\hat{t}}^{(11)} &= \epsilon^2 \delta \left[ \omega'_1 U_{m,\hat{t}}^{(11)} + \{ \delta \hat{t}_{01} + \dots \} \epsilon (\delta \hat{t}_{10} + \delta \hat{t}_{11} + \delta \hat{t}_{12} + \dots) + \epsilon^2 (\hat{t}_{20} + \delta \hat{t}_{21} + \delta^2 \hat{t}_{22} + \dots) \right] \\ &\omega'_1 U_{m,\hat{t}\hat{t}}^{(11)} + \delta \{ t_0 + \delta t_{01} + \dots + \epsilon (t_{10} + \delta t_{11} + \dots) + \epsilon^2 (t_{20} + \delta t_{21} + \dots) \} \\ &\left\{ \left( \omega'_1 U_{m,\hat{t}}^{(20)} \right)_{,\tau} + \frac{1}{2} \{ \delta \hat{t}_{01} + \delta^2 \hat{t}_{02} + \dots \} \epsilon (\hat{t}_{10} + \delta \hat{t}_{11} + \dots) + \epsilon^2 (\hat{t}_{20} + \delta \hat{t}_{21} + \delta^2 \hat{t}_{22} + \dots) \right\}^2 \\ &\omega'_1 U_{m,\hat{t}\hat{t}\hat{t}}^{(21)} + \delta \{ \delta \hat{t}_{01} + \dots \} \epsilon (\hat{t}_{10} + \delta \hat{t}_{11} + \dots) + \epsilon^2 (\hat{t}_{20} + \delta \hat{t}_{21} + \dots) \} \times \{ t_0 + \delta t_{01} + \dots + \epsilon (t_{10} + \delta t_{11} + \dots) + \epsilon^2 (t_{20} + \delta t_{21} + \dots) \} \\ &\left( \omega'_1 U_{m,\hat{t},\hat{t}}^{(20)} \right)_{,\tau} + \frac{1}{2} \{ t_0 + \delta t_{01} + \dots + \epsilon (t_{10} + \delta t_{11} + \dots) + \epsilon^2 (t_{20} + \delta t_{21} + \dots) \}^2 \times \\ &\left( \omega'_1 U_{m,\hat{t}}^{(20)} \right)_{,\tau\tau} \Big|_{(\hat{t}_{0,0})} \end{aligned} \quad (4.95)$$

$$\begin{aligned} \epsilon^3 \omega'_1 \delta U_{m,\hat{t}}^{(20)} &= \epsilon^3 \left[ \omega'_1 U_{m,\hat{t}}^{(20)} + \{ \hat{t}_{01} \delta + \hat{t}_{01} \delta^2 \} \omega'_1 U_{m,\hat{t}\hat{t}}^{(20)} + \delta \{ t_0 + \delta t_{01} + \dots + \epsilon (t_{10} + \delta t_{11} + \dots) + \dots \} \right. \\ &\left( \omega'_1 U_{m,\hat{t}}^{(20)} \right)_{,\tau} + \frac{1}{2} \{ \delta \hat{t}_{01} + \delta^2 \hat{t}_{02} + \dots \}^2 \omega'_1 U_{m,\hat{t}\hat{t}}^{(20)} + \delta \{ \delta \hat{t}_{01} + \delta \hat{t}_{02} + \dots \} \times \\ &\{ t_0 + \delta t_{01} + \delta^2 t_{02} + \dots \} \left( \omega'_1 U_{m,\hat{t},\hat{t}}^{(20)} \right)_{,\hat{t}\tau} + \frac{1}{2} \delta^2 \{ t_0 + \delta t_{01} + \dots \}^2 \times \\ &\left. \left( \omega'_1 U_{m,\hat{t}}^{(20)} \right)_{,\tau\tau} \right] \Big|_{(\hat{t}_{0,0})} \end{aligned} \quad (4.96)$$

$$\begin{aligned} \epsilon^3 \delta \omega'_1 \delta U_{m,\hat{t}}^{(21)} &= \epsilon^3 \delta \left[ \omega'_1 U_{m,\hat{t}}^{(21)} + \{ \delta \hat{t}_{01} + \delta^2 \hat{t}_{01} \} \omega'_1 U_{m,\hat{t}\hat{t}}^{(21)} + \delta \{ t_0 + \delta t_{01} + \dots + \epsilon (t_{10} + \delta t_{11} + \dots) + \dots \} \right. \\ &\left. \left( \omega'_1 U_{m,\hat{t}}^{(21)} \right)_{,\tau} \right] \Big|_{(\hat{t}_{0,0})} \end{aligned} \quad (4.97)$$

$$\begin{aligned} \epsilon^3 \delta \omega'_2 U_{m,\hat{t}}^{(10)} &= \epsilon^3 \left[ \omega'_2 U_{m,\hat{t}}^{(10)} + \{\delta \hat{t}_{01} + \delta^2 \hat{t}_{01}\} \omega'_2 U_{m,\hat{t}\hat{t}}^{(10)} + \delta \{t_0 + \delta t_{01} + \dots + \epsilon(t_{10} + \right. \\ &\delta t_{11} + \dots) + \dots \} \left( \omega'_2 U_{m,\hat{t}}^{(10)} \right)_{,\tau} + (\delta \hat{t}_{01} + \delta^2 \hat{t}_{02} + \dots)^2 \omega'_2 U_{m,\hat{t}\hat{t}\hat{t}}^{(10)} + \delta(\delta \hat{t}_{01} + \delta \hat{t}_{02} + \dots)(t_0 + \\ &\delta t_{01} + \delta^2 t_{02} + \dots) \left( \omega'_2 U_{m,\hat{t},\hat{t}}^{(10)} \right)_{,\tau} + \frac{1}{2} \delta^2 (t_0^2) \times \left( \omega'_2 U_{m,\hat{t}}^{(10)} \right)_{,\tau\tau} \Big]_{(\hat{t}_{0,0})} \end{aligned} \quad (4.98)$$

$$\begin{aligned} \epsilon \delta U_{m,\tau}^{(10)} &= \epsilon \delta \left[ U_{m,\tau}^{(10)} + \{(\delta \hat{t}_{01} + \delta^2 \hat{t}_{02} + \dots) + \epsilon(\hat{t}_{10} + \delta \hat{t}_{11} + \delta^2 \hat{t}_{12} + \dots) + \right. \\ &\epsilon^2(t_{20} + \delta t_{21} + \delta^2 t_{22} + \dots) \} U_{m,\hat{t}\tau}^{(10)} + \delta \{t_0 + \delta t_{01} + \delta^2 t_{02} + \dots + \epsilon(t_{10} + \delta t_{11} + \dots) + \\ &\epsilon^2(t_{20} + \delta t_{21} + \delta^2 t_{22} + \dots) \} U_{m,\tau\tau}^{(10)} + \frac{1}{2} \{ \delta \hat{t}_{01} + \epsilon(\hat{t}_{10} + \delta \hat{t}_{11} + \delta^2 \hat{t}_{12} + \dots) + \epsilon^2(t_{20} + \\ &\delta t_{21} + \delta^2 t_{22} + \dots) + \dots \}^2 U_{m,\tau\hat{t}\hat{t}}^{(10)} + \delta \{ \delta \hat{t}_{01} + \dots + \epsilon(\hat{t}_{10} + \delta \hat{t}_{11} + \delta^2 \hat{t}_{12} + \dots) + \epsilon^2(t_{20} + \\ &\delta t_{21} + \delta^2 t_{22} + \dots) \} \times \{ t_0 + \delta t_{01} + \delta^2 t_{02} + \dots + \epsilon(t_{10} + \delta t_{11} + \delta^2 t_{12} + \dots) + \epsilon^2(t_{20} + \\ &\delta t_{21} + \delta^2 t_{22} + \dots) \} U_{m,\tau\hat{t}\hat{t}}^{(10)} \Big]_{(\hat{t}_{0,0})} \end{aligned} \quad (4.99)$$

$$\begin{aligned} \epsilon \delta^2 U_{m,\tau}^{(11)} &= \epsilon \delta^2 \left[ U_{m,\tau}^{(11)} + \{ \hat{t}_{01} + \epsilon \hat{t}_{10} + \epsilon^2 \hat{t}_{20} + \dots \} U_{m,\hat{t}\tau}^{(11)} + \delta \{ (t_0 + \epsilon t_{10} + t_{11} \dots + \right. \\ &\left. \epsilon^2 t_{20}) \} U_{m,\tau\tau}^{(11)} + \frac{1}{2} \{ \epsilon \hat{t}_{10} + \epsilon^2 \hat{t}_{20} + \dots \}^2 U_{m,\tau\hat{t}\hat{t}}^{(11)} + \dots \Big]_{(\hat{t}_{0,0})} \end{aligned} \quad (4.100)$$

$$\begin{aligned} \epsilon \delta^2 U_{m,\tau}^{(20)} &= \epsilon^2 \delta \left[ U_{m,\tau}^{(20)} + \{ \hat{t}_{01} \delta + \epsilon(\hat{t}_{10} + \delta \hat{t}_{11}) \} U_{m,\tau\hat{t}}^{(20)} + \delta \{ t_0 + \epsilon(t_{10} + \right. \\ &\dots + \epsilon) \} U_{m,\tau\tau}^{(20)} + \frac{1}{2} \{ \hat{t}_{01} \delta + \epsilon(\hat{t}_{10} + \delta \hat{t}_{11} + \dots) \}^2 U_{m,\tau\hat{t}\hat{t}}^{(20)} + \delta \{ \delta \hat{t}_{01} + \epsilon(\hat{t}_{10} + \delta \hat{t}_{11} + \\ &\dots) + \dots \} \{ t_0 + \epsilon(t_{10} + \delta t_{11} + \delta^2 t_{12} + \dots) + \epsilon^2 t_{20} \} \times U_{m,\tau\hat{t}\tau}^{(20)} \Big]_{(\hat{t}_{0,0})} \end{aligned} \quad (4.101)$$

$$\epsilon^3 \delta \left( U_m^{(30)} + U_{3m}^{(30)} \right)_{,\tau} = \epsilon^3 \delta \left[ \left( U_m^{(30)} + U_{3m}^{(30)} \right)_{,\tau} \right]_{(\hat{t}_{0,0})} \quad (4.102)$$

Therefore, the following orders of equations are obtained

$$O(\epsilon): U_{m,\hat{t}}^{(10)}(t_0, 0) = 0 \quad (4.103.1)$$

$$O(\epsilon\delta): \hat{t}_{01} U_{m,\hat{t}\hat{t}}^{(10)} + t_0 U_{m,\hat{t}\tau}^{(10)} + U_{m,\tau}^{(10)} + U_{m,\hat{t}}^{(11)} = 0 \quad (4.103.2)$$

$$O(\epsilon\delta^2): \hat{t}_{02} U_{m,\hat{t}\hat{t}}^{(10)} + t_{01} U_{m,\hat{t}\tau}^{(10)} + \hat{t}_{01} U_{m,\hat{t}\hat{t}}^{(11)} + t_0 U_{m,\hat{t}\tau}^{(11)} + \dots = 0 \quad (4.103.3)$$

$$O(\epsilon^2): \hat{t}_{10} U_{m,\hat{t}\hat{t}}^{(10)} + U_{m,\hat{t}}^{(20)} + \omega'_1 U_{m,\hat{t}}^{(10)} = 0 \quad (4.103.4)$$

$$\begin{aligned} O(\epsilon^2\delta): \hat{t}_{11} U_{m,\hat{t}\hat{t}}^{(10)} + t_{10} U_{m,\hat{t}\tau}^{(10)} + \hat{t}_{01} \hat{t}_{10} U_{m,\hat{t}\hat{t}\hat{t}}^{(10)} + \hat{t}_{10} U_{m,\hat{t}\hat{t}}^{(11)} + \hat{t}_{01} U_{m,\hat{t}\hat{t}}^{(20)} + t_0 U_{m,\hat{t}\tau}^{(20)} + \\ U_{m,\hat{t}}^{(21)} + \omega'_1 \hat{t}_{01} U_{m,\hat{t}\hat{t}}^{(10)} + t_0 \left( \omega'_1 U_{m,\hat{t}}^{(10)} \right)_{,\tau} + \omega'_1 U_{m,\hat{t}}^{(11)} + \hat{t}_{10} U_{m,\hat{t}\tau}^{(10)} = 0 \end{aligned} \quad (4.103.5)$$

$$\begin{aligned}
O(\epsilon^3): \hat{t}_{20}U_{m,\hat{t}\hat{t}}^{(10)} + \frac{1}{2}(\hat{t}_{10})^2U_{m,\hat{t}\hat{t}\hat{t}}^{(10)} + \hat{t}_{10}U_{m,\hat{t}\hat{t}}^{(20)} + \left(U_{m,\hat{t}}^{(30)} + U_{3m,\hat{t}}^{(30)}\right) + \omega'_1\hat{t}_{10}U_{m,\hat{t}\hat{t}}^{(10)} + \\
\omega'_1U_{m,\hat{t}}^{(20)} + \omega'_2U_{m,\hat{t}}^{(10)} = 0 \tag{4.103.6} \\
O(\epsilon) : U_{m,\hat{t}}^{(10)}(\hat{t}_0, 0) = 0 \Rightarrow \theta\alpha_1(0) \sin \theta \hat{t}_0 \\
\sin \theta \hat{t}_0 = 0 ; \theta\hat{t}_0 = \pi r, r = 0,1,2 \dots
\end{aligned}$$

Therefore,

$$\hat{t}_0 = \frac{\theta}{\pi} \tag{4.104.1}$$

$$O(\epsilon\delta) : \hat{t}_{01}U_{m,\hat{t}\hat{t}}^{(10)} + t_0U_{m,\hat{t}\tau}^{(10)} + U_{m,\tau}^{(10)} + U_{m,\hat{t}}^{(11)} = 0$$

This implies that,

$$\hat{t}_{01}(-\omega'_1(0)\theta^2 \cos \hat{t}_0) + t_0(-\omega'_1(0)\theta \sin \theta \hat{t}_0) + \omega'_1(0)\theta \cos \theta \hat{t}_0 + (-\beta_2(0)\cos \theta \hat{t}_0) = 0$$

And,

$$\hat{t}_{01}(-\alpha'_1(0)\theta^2 \cos \hat{t}_0) + t_0(-\alpha'_1(0)\theta \sin \theta \hat{t}_0) + \alpha'_1(0)\theta \cos \theta \hat{t}_0 + (-\beta_2(0)\cos \theta \hat{t}_0) = 0$$

This yields

$$\hat{t}_{01} \left( -B\theta^2 \cos \left( \frac{\pi}{\theta} \right) \right) + B \cos \theta \left( \frac{\pi}{\theta} \right) + \left( \theta \left( \frac{-B}{\theta} \right) \cos \theta \left( \frac{\pi}{\theta} \right) \right) = 0$$

This implies,

$$-B\theta^2\hat{t}_{01} - B + B = 0$$

Therefore,

$$\hat{t}_{01} = 0 \tag{4.104.2}$$

$$O(\epsilon\delta^2): \hat{t}_{02}U_{m,\hat{t}\hat{t}}^{(10)} + t_{01}U_{m,\hat{t}\tau}^{(10)} + \hat{t}_{01}U_{m,\hat{t}\hat{t}}^{(11)} + t_0U_{m,\hat{t}\tau}^{(11)} + \dots = 0$$

This implies

$$\hat{t}_{02}(-\alpha_1(0)\theta^2 \cos \hat{t}_0) + \hat{t}_{01}(-\theta^2\beta_2(0)\sin \theta \hat{t}_0) + t_0(-\beta'_2(0)\cos \theta \hat{t}_0) = 0$$

This implies,

$$\hat{t}_{02}(B\theta^2 \cos \pi) + \hat{t}_{01} \left( -\theta^2 \left( \frac{B}{\theta} \right) \sin \pi \right) + t_0 \left( \theta \left( \frac{3B}{2\theta} \right) \cos \pi \right) = 0$$

This implies,

$$-B\theta^2\hat{t}_{02} - \frac{3Bt_0}{2} = 0$$

Therefore,

$$\hat{t}_{02} = \frac{-3t_0}{2\theta^2} \quad (4.104.3)$$

$$O(\epsilon^2): \hat{t}_{10}U_{m,\hat{t}\hat{t}}^{(10)} + U_{m,\hat{t}}^{(20)} + \omega'_1 U_{m,\hat{t}}^{(10)} = 0$$

This implies,

$$\hat{t}_{10}(\theta^2\alpha_1(0)\cos\theta\hat{t}_0) + \left\{-\theta\alpha_4(0)\sin\theta\hat{t}_0 + \theta\beta_4(0)\cos\theta\hat{t}_0 + \frac{2\theta r_1(0)\sin 2\theta\hat{t}_0}{3\theta^2}\right\} + \omega'_1(0)\{-\theta\alpha_1(0)\sin\theta\hat{t}_0\} - \theta^2 B\hat{t}_{10} = 0$$

Since  $\theta^2 \neq 0$  and  $B \neq 0$

$$\hat{t}_{10} = 0 \quad (4.104.4)$$

$$O(\epsilon^2\delta): \hat{t}_{11}U_{m,\hat{t}\hat{t}}^{(10)} + t_{10}U_{m,\hat{t}\tau}^{(10)} + \hat{t}_{01}\hat{t}_{10}U_{m,\hat{t}\hat{t}}^{(10)} + \hat{t}_{10}U_{m,\hat{t}\hat{t}}^{(11)} + \hat{t}_{01}U_{m,\hat{t}\hat{t}}^{(20)} + t_0U_{m,\hat{t}\tau}^{(20)} + U_{m,\hat{t}}^{(21)} + \omega'_1\hat{t}_{01}U_{m,\hat{t}\hat{t}}^{(10)} + t_0\left(\omega'_1U_{m,\hat{t}}^{(10)}\right)_\tau + \omega'_1U_{m,\hat{t}}^{(11)} + \hat{t}_{10}U_{m,\hat{t}\tau}^{(10)} = 0$$

This implies,

$$\hat{t}_{11}\alpha_1(0)\theta^2\cos\theta\hat{t}_0 - \frac{2r_4(0)}{3\theta} + \omega'_1(0)(\beta_2(0)\theta\cos\theta\hat{t}_0) = 0$$

That is,

$$-B\theta\hat{t}_{11}\theta^2 + \left(\frac{2\alpha B^2}{3\theta^2}\right) - \frac{B}{\theta^2}\left(\frac{-B}{\theta}\right)\theta = 0$$

This implies,

$$-B\theta\hat{t}_{11} - \frac{2\alpha B^2}{3\theta^2} + \frac{B}{\theta^2} = 0$$

Therefore,

$$\hat{t}_{11} = \frac{1}{\theta^3}\left(1 - \frac{2\alpha B}{3}\right) \quad (4.104.5)$$

$$O(\epsilon^3): \hat{t}_{20}U_{m,\hat{t}\hat{t}}^{(10)} + \frac{1}{2}(\hat{t}_{10})^2U_{m,\hat{t}\hat{t}}^{(10)} + \hat{t}_{10}U_{m,\hat{t}\hat{t}}^{(20)} + \left(U_{m,\hat{t}}^{(30)} + U_{3m,\hat{t}}^{(30)}\right) + \omega'_1\hat{t}_{10}U_{m,\hat{t}\hat{t}}^{(10)} + \omega'_1U_{m,\hat{t}}^{(20)} + \omega'_2U_{m,\hat{t}}^{(10)} = 0$$

This implies,

$$\hat{t}_{20}U_{m,\hat{t}\hat{t}}^{(10)} + U_{m,\hat{t}}^{(30)} + U_{3m,\hat{t}}^{(30)} = 0 \quad (4.104.6)$$

Certain terms in (4.104.1) - (4.104.6) are now simplified as

From,  $U_{m,\hat{t}}^{(30)}$ ,

$$\begin{aligned} & \frac{-1}{\varphi(2\theta + \varphi)} [-(\varphi + \theta)r_{13}(0)\sin(\varphi + \theta)\hat{t}_0 + (\varphi + \theta)r_{14}(0)\cos(\varphi + \theta)\hat{t}_0] \\ &= \frac{-1}{\varphi(2\theta + \varphi)} [(\varphi + \theta)r_{13}(0)\sin\varphi\hat{t}_0] \end{aligned} \quad (4.104.7)$$

But

$$\sin(\varphi + \theta)\hat{t}_0 = \sin\varphi\hat{t}_0\cos\theta\hat{t}_0 + \cos\varphi\hat{t}_0\sin\theta\hat{t}_0 = -\sin\varphi\hat{t}_0$$

Similarly,

$$\begin{aligned} & \frac{1}{\varphi(2\theta - \varphi)} [-(\varphi - \theta)r_{15}(0)\sin(\varphi - \theta)\hat{t}_0 + (\varphi - \theta)r_{16}(0)\cos(\varphi - \theta)\hat{t}_0] \\ &= \frac{1}{\varphi(2\theta - \varphi)} [(\varphi - \theta)r_{15}(0)\sin\varphi\hat{t}_0] \end{aligned} \quad (4.104.8)$$

But,

$$\sin(\varphi - \theta)\hat{t}_0 = \sin\varphi\hat{t}_0\cos\theta\hat{t}_0 - \cos\varphi\hat{t}_0\sin\theta\hat{t}_0 = -\sin\varphi\hat{t}_0$$

The non-vanishing terms in  $U_{m,\hat{t}}^{(30)}(\hat{t}_0, 0)$  are;

$$\frac{-(\varphi + \theta)r_{13}(0)\sin\varphi\hat{t}_0}{\varphi(2\theta + \varphi)} \quad (4.104.9)$$

And

$$\frac{[(\varphi - \theta)r_{15}(0)\sin\varphi\hat{t}_0]}{\varphi(2\theta - \varphi)} \quad (4.104.10)$$

Similarly;

The non – vanishing terms in  $U_{3m,\hat{t}}^{(30)}(\hat{t}_0, 0)$  are ;

$$\frac{1}{\Omega^2 - (\varphi + \theta)^2} [-(\varphi + \theta)r_{25}(0)\sin\varphi\hat{t}_0] \quad (4.104.11)$$

And,

$$\frac{1}{\Omega^2 - (\varphi - \theta)^2} [(\varphi - \theta)r_{27}(0)\sin\varphi\hat{t}_0] \quad (4.104.12)$$

From (4.104.6),

$$\hat{t}_{20}U_{m,\hat{t}}^{(10)} + U_{m,\hat{t}}^{(30)} + U_{3m,\hat{t}}^{(30)} = 0$$

Substituting into the above to obtained

$$\begin{aligned} & -\hat{t}_{20}\theta^2B - \frac{(\varphi+\theta)r_{13}(0)\sin\varphi\hat{t}_0}{\varphi(2\theta+\varphi)} + \frac{[(\varphi-\theta)r_{15}(0)\sin\varphi\hat{t}_0]}{\varphi(2\theta-\varphi)} - \frac{(\varphi+\theta)r_{25}(0)\sin\varphi\hat{t}_0}{\Omega^2 - (\varphi+\theta)^2} + \\ & \frac{[(\varphi-\theta)r_{27}(0)\sin\varphi\hat{t}_0]}{\Omega^2 - (\varphi-\theta)^2} = 0 \end{aligned}$$

This implies

$$\begin{aligned} \hat{t}_{20} &= \frac{\sin\varphi\hat{t}_0}{\theta^2B} \left[ \frac{r_{27}(0)(\varphi-\theta)}{\Omega^2 - (\varphi-\theta)^2} - \frac{(\varphi+\theta)r_{25}(0)}{\Omega^2 - (\varphi+\theta)^2} - \frac{(\varphi+\theta)r_{13}(0)}{\varphi(2\theta+\varphi)} + \frac{r_{15}(0)(\varphi-\theta)}{\varphi(2\theta-\varphi)} \right] \\ &= \frac{\sin\varphi\hat{t}_0}{\theta^2B} \left[ \frac{(\varphi-\theta)B^3\alpha S_0}{\Omega^2 - (\varphi-\theta)^2} - \frac{(\varphi+\theta)B^3\alpha S_0}{\Omega^2 - (\varphi+\theta)^2} - \frac{(\varphi+\theta)B^3\alpha S_0}{\varphi(2\theta+\varphi)} + \frac{B^3\alpha S_0(\varphi-\theta)}{\varphi(2\theta-\varphi)} \right] \\ &= \frac{B^3\alpha S_0\sin\varphi\hat{t}_0}{\theta^2B} \left[ \frac{(\varphi-\theta)}{\Omega^2 - (\varphi-\theta)^2} - \frac{(\varphi+\theta)}{\Omega^2 - (\varphi+\theta)^2} - \frac{(\varphi+\theta)}{\varphi(2\theta+\varphi)} + \frac{(\varphi-\theta)}{\varphi(2\theta-\varphi)} \right] \\ &= \frac{B^2\alpha S_0\sin\varphi\hat{t}_0}{\theta^2B} \left[ \frac{(\varphi-\theta)}{\Omega^2 - (\varphi-\theta)^2} - \frac{(\varphi+\theta)}{\Omega^2 - (\varphi+\theta)^2} - \frac{(\varphi+\theta)}{\varphi(2\theta+\varphi)} + \frac{(\varphi-\theta)}{\varphi(2\theta-\varphi)} \right] \end{aligned}$$

Therefore,

$$\hat{t}_{20} = \frac{B^2\alpha S_0Q}{\theta^2} \quad (4.105)$$

Where,

$$Q = \left[ \frac{(\varphi-\theta)}{\Omega^2 - (\varphi-\theta)^2} - \frac{(\varphi+\theta)}{\Omega^2 - (\varphi+\theta)^2} - \frac{(\varphi+\theta)}{\varphi(2\theta+\varphi)} + \frac{(\varphi-\theta)}{\varphi(2\theta-\varphi)} \right]$$

At maximum displacement  $t$  becomes  $t_a$  and  $\hat{t} = \hat{t}_a, \tau = \tau_a$

$$\hat{t}_a = t_a + \frac{1}{8}(\omega_1(\tau_a)t + \omega_2(\tau_a)t^2 + \dots) \quad (4.106.1)$$

Expansion of both sides of (4.106.1) gives,

$$\begin{aligned}
& \hat{t}_0 + \delta \hat{t}_{01} + \delta^2 \hat{t}_{02} + \epsilon (\hat{t}_{10} + \delta \hat{t}_{11} + \delta^2 \hat{t}_{12} + \dots) + \epsilon^2 (\hat{t}_{20} + \delta \hat{t}_{21} + \delta^2 \hat{t}_{22} + \dots) \\
& = t_0 + \delta t_{01} + \delta^2 t_{02} + \dots + \epsilon (t_{10} + \delta t_{11} + \delta^2 t_{12} + \dots) + \epsilon^2 (t_{20} + \delta t_{21} + \delta^2 t_{22} + \\
& \dots) + \dots + \frac{\tau_a}{\delta} \left[ \epsilon \left\{ \omega'_1(0) + \frac{\omega''_1(0)\tau_a}{2} + \frac{\omega'''_1(0)\tau_a^2}{6} + \dots \right\} + \epsilon^2 \left\{ \omega'_2(0) + \frac{\omega''_2(0)\tau_a}{2} + \frac{\omega'''_2(0)\tau_a^2}{6} + \dots \right\} + \dots \right]
\end{aligned} \tag{4.106.2}$$

$$\begin{aligned}
& = t_0 + \delta t_{01} + \delta^2 t_{02} + \dots + \epsilon (t_{10} + \delta t_{11} + \delta^2 t_{12} + \dots) + \epsilon^2 (t_{20} + \delta t_{21} + \delta^2 t_{22} + \\
& \dots) + t_a \left[ \epsilon \left\{ \omega'_1(0) + \frac{\omega''_1(0)\tau_a}{2} + \frac{\omega'''_1(0)\tau_a^2}{6} + \dots \right\} + \epsilon^2 \left\{ \omega'_2(0) + \frac{\omega''_2(0)\tau_a}{2} + \frac{\omega'''_2(0)\tau_a^2}{6} + \dots \right\} + \dots \right]
\end{aligned} \tag{4.106.3}$$

### Equating the coefficients of $\epsilon, \delta, \epsilon\delta$

$$O(1): \hat{t}_0 = \frac{\pi}{\theta} = t_0$$

Therefore,

$$t_0 = \frac{\pi}{\theta} \tag{4.107.1}$$

$$O(\epsilon): \hat{t}_{10} = 0 = t_{10} + t_0 \omega'_1(0)$$

$$= t_{10} + t_0 \omega'_1(0) = 0$$

Therefore,

$$t_{10} = -t_0 \omega'_1(0) = -t_0 \left( \frac{B}{\theta^2} \right) = \frac{-t_0 B}{\theta^2} \tag{4.107.2}$$

$$O(\epsilon^2): \hat{t}_{20} = t_{20} + t_0 \omega'_1(0) + t_0 \omega'_2(0)$$

Therefore,

$$t_{20} = \hat{t}_{20} - t_{10} \omega'_1(0) - t_0 \omega'_2(0) \tag{4.107.3}$$

Following from equation (4.87.1) – (4.87.3) and (4.106.1), at maximum displacement (3.9.2) takes the form,

$$\begin{aligned}
\frac{1}{\delta} \omega_1(\tau_a) &= \frac{1}{\delta} \left[ \omega_1(0) + \tau_a \omega_1'(0) + \frac{\tau_a^2 \omega_1''(0)}{2} + \frac{\tau_a^3 \omega_1'''(0)}{6} + \dots \right] \\
&= \frac{1}{\delta} \left[ \tau_a \omega_1'(0) + \frac{\tau_a^2 \omega_1''(0)}{2} + \frac{\tau_a^3 \omega_1'''(0)}{6} + \dots \right] \\
\frac{1}{\delta} \omega_1(\tau_a) &= t_a \omega_1'(0) + \frac{\delta t_a^2 \omega_1''(0)}{2} + \frac{\delta^2 t_a^3 \omega_1'''(0)}{6} + \dots
\end{aligned} \tag{4.108.1}$$

Therefore,

$$\begin{aligned}
&\frac{1}{\delta} [\omega_1(\tau_a) \in t + \omega_2(\tau_a) \in^2 t^2 + \omega_3(\tau_a) \in^3 t^3] = \left\{ t_0 \omega_1'(0) + \frac{\delta t_a^2 \omega_1''(0)}{2} + \right. \\
&\left. \frac{\delta^2 t_a^3 \omega_1'''(0)}{6} + \dots \right\} \epsilon + \left\{ t_a \omega_2'(0) + \frac{\delta t_a^2 \omega_2''(0)}{2} + \frac{\delta^2 t_a^3 \omega_2'''(0)}{6} + \dots \right\} \epsilon^2 + \left\{ t_a \omega_3'(0) + \frac{\delta t_a^2 \omega_3''(0)}{2} + \right. \\
&\left. \frac{\delta^2 t_a^3 \omega_3'''(0)}{6} + \dots \right\} \epsilon^3
\end{aligned} \tag{4.108.2}$$

From equation (4.106.1), we obtained,

$$\begin{aligned}
&\hat{t}_0 + \delta \hat{t}_{01} + \delta^2 \hat{t}_{02} + \dots + \epsilon (\hat{t}_{10} + \delta \hat{t}_{11} + \delta^2 \hat{t}_{12} + \dots) + \epsilon^2 (\hat{t}_{20} + \delta \hat{t}_{21} + \delta^2 \hat{t}_{22} + \dots) \\
&= t_0 + \delta t_{01} + \delta^2 t_{02} + \dots + \epsilon (t_{10} + \delta t_{11} + \delta^2 t_{12} + \dots) \\
&\quad + \epsilon^2 (t_{20} + \delta t_{21} + \delta^2 t_{22} + \dots) \\
&\quad + \{t_0 + \delta t_{01} + \delta^2 t_{02} + \dots + \epsilon (t_{10} + \delta t_{11} + \delta^2 t_{12} + \dots) \\
&\quad + \epsilon^2 (t_{20} + \delta t_{21} + \delta^2 t_{22} + \dots)\} \omega_1'(0) \epsilon \\
&\quad + \frac{\delta}{2} \{t_0 + \delta t_{01} + \delta^2 t_{02} + \dots + \epsilon (t_{10} + \delta t_{11} + \delta^2 t_{12} + \dots) \\
&\quad + \epsilon^2 (t_{20} + \delta t_{21} + \delta^2 t_{22} + \dots)\}^2 \omega_1''(0) \epsilon \\
&\quad + \frac{1}{6} \delta^2 \omega_1'''(0) \epsilon \{t_0 + \delta t_{01} + \delta^2 t_{02} + \dots + \epsilon (t_{10} + \delta t_{11} + \delta^2 t_{12} + \dots)\}^2 \\
&\quad + \omega_2'(0) \epsilon^2 \{t_0 + \delta t_{01} + \delta^2 t_{02} + \dots + \epsilon (t_{10} + \delta t_{11} + \delta^2 t_{12} + \dots)\} \\
&\quad + \frac{\epsilon^2 \delta \omega_2''(0)}{2} \{t_0 + \delta t_{01} + \delta^2 t_{02} + \dots + \epsilon (t_{10} + \delta t_{11} + \delta^2 t_{12} + \dots)\}^2
\end{aligned} \tag{4.108.3}$$

**Equating the coefficients of orders 1,  $\delta$ ,  $\epsilon$ , and  $\epsilon^2$**

$$O(1): \hat{t}_0 = t_0 = \frac{\pi}{\theta} \tag{4.109.1}$$

$$O(\delta): \hat{t}_{01} = 0 = t_{01}$$

$$t_{01} = 0 \quad (4.109.2)$$

$$O(\epsilon): \hat{t}_{10} = t_{10} + t_0 \omega'_1(0)$$

$$t_{10} = -t_0 \omega'_1(0) = -\frac{\pi \omega'_1(0)}{\theta} \quad (4.109.3)$$

$$O(\epsilon^2): \hat{t}_{20} = t_{20} = t_{10} \omega'_1(0) + t_0 \omega'_2(0)$$

$$t_{20} = -\frac{\pi(\omega'_1(0))^2}{\theta} + t_0 \omega'_2(0) \quad (4.109.4)$$

### 4.3 Maximum Displacement

Let  $U_a$  be the maximum displacement. We now substitute for  $x_a$  in (4.86);

$$U\left(\frac{\pi}{2m}, \hat{t}, \tau\right) = \epsilon \left[ 2U_m^{(10)} + 2\delta U_m^{(11)} \dots \right] + \epsilon^2 \left[ 2U_m^{(20)} + 2\delta U_{2m}^{(21)} \dots \right] + \epsilon^3 \left[ \left( 2U_m^{(30)} + 2U_{3m}^{(30)} \dots \right) + \delta \left( 2U_m^{(31)} + 2\delta U_{3m}^{(31)} \dots \right) \right] \quad (4.110)$$

Where (4.110) is evaluated at maximum values and each of the terms in (4.110) is expanded as follows:

$$\begin{aligned} \epsilon U_m^{10}(\hat{t}_a, \tau_a) = & \epsilon \left[ U_{m,\hat{t}}^{(10)} \right. \\ & + \{(\delta \hat{t}_{01} + \delta^2 \hat{t}_{02} + \dots) + \epsilon(\hat{t}_{10} + \delta \hat{t}_{11} + \delta^2 \hat{t}_{12} + \dots)\} \\ & + \epsilon^2(t_{20} + \delta t_{21} + \delta^2 t_{22} + \dots) \} U_{m,\hat{t}}^{(10)} \\ & + \delta \{ (t_0 + \delta t_{01} + \delta^2 t_{02} + \dots) + \epsilon(t_{10} + \delta t_{11} + \delta^2 t_{12} + \dots) \\ & + \epsilon^2(t_{20} + \delta t_{21} + \delta^2 t_{22} + \dots) \} U_{m,\tau}^{(10)} \\ & + \frac{1}{2} \{ \{ (\delta \hat{t}_{01} + \delta^2 \hat{t}_{02} + \dots) + \epsilon(\hat{t}_{10} + \delta \hat{t}_{11} + \delta^2 \hat{t}_{12} + \dots) \\ & + \epsilon^2(t_{20} + \delta t_{21} + \delta^2 t_{22} + \dots) + \dots \}^2 U_{m,\hat{t}\hat{t}}^{(10)} \\ & + 2\delta \{ (\delta \hat{t}_{01} + \delta^2 \hat{t}_{02} + \dots) + \epsilon(\hat{t}_{10} + \delta \hat{t}_{11} + \delta^2 \hat{t}_{12} + \dots) \\ & + \epsilon^2(t_{20} + \delta t_{21} + \delta^2 t_{22} + \dots) + \dots \} x \{ t_0 + \delta t_{01} + \delta^2 t_{02} + \dots \\ & + \epsilon(t_{10} + \delta t_{11} + \delta^2 t_{12} + \dots) \\ & + \epsilon^2(t_{20} + \delta t_{21} + \delta^2 t_{22} + \dots) \} U_{m,\hat{t}\tau}^{(10)} \\ & + \delta^2 \{ \delta t_{01} + \delta^2 t_{02} + \dots \\ & + \epsilon(t_{10} + \delta t_{11} + \delta^2 t_{12} + \dots) \} U_{m,\tau\tau}^{(10)} \} \Big] \text{ at } \tau = 0 \end{aligned} \quad (4.111)$$

$$\begin{aligned}
\delta \epsilon U_m^{(11)} = & \delta \epsilon \left[ U_m^{(11)} + \{(\delta \hat{t}_{01} + \delta^2 \hat{t}_{02} + \dots) + \epsilon(\hat{t}_{10} + \delta \hat{t}_{11} + \delta^2 \hat{t}_{12} + \dots) + \right. \\
& \left. \epsilon^2(t_{20} + \delta t_{21} + \delta^2 t_{22} + \dots)\} U_{m,\hat{t}}^{(11)} + \delta\{(t_0 + \delta t_{01} + \delta^2 t_{02} + \dots) + \epsilon(t_{10} + \delta t_{11} + \right. \\
& \left. \delta^2 t_{12} + \dots) + \epsilon^2(t_{20} + \delta t_{21} + \delta^2 t_{22} + \dots)\} U_{m,\tau}^{(11)} + \frac{1}{2} \{ \{(\delta \hat{t}_{01} + \delta^2 \hat{t}_{02} + \dots) + \right. \\
& \left. \epsilon(\hat{t}_{10} + \delta \hat{t}_{11} + \delta^2 \hat{t}_{12} + \dots) + \epsilon^2(t_{20} + \delta t_{21} + \delta^2 t_{22} + \dots) + \dots \}^2 U_{m,\hat{t}\hat{t}}^{(11)} + \right. \\
& \left. 2\delta\{(\delta \hat{t}_{01} + \delta^2 \hat{t}_{02} + \dots) + \epsilon(\hat{t}_{10} + \delta \hat{t}_{11} + \delta^2 \hat{t}_{12} + \dots) + \epsilon^2(t_{20} + \delta t_{21} + \delta^2 t_{22} + \right. \\
& \left. \dots) + \dots\} \times \{t_0 + \delta t_{01} + \delta^2 t_{02} + \dots + \epsilon(t_{10} + \delta t_{11} + \delta^2 t_{12} + \dots) + \epsilon^2(t_{20} + \right. \\
& \left. \delta t_{21} + \delta^2 t_{22} + \dots)\} \} U_{m,\hat{t}\tau}^{(11)} + \dots \right] \text{ at } \tau = 0 \tag{4.112}
\end{aligned}$$

$$\begin{aligned}
\epsilon^2 U_m^{(20)} = & \epsilon^2 \left[ U_m^{(20)} + \{(\delta \hat{t}_{01} + \delta^2 \hat{t}_{02} + \dots) + \epsilon(\hat{t}_{10} + \delta \hat{t}_{11} + \delta^2 \hat{t}_{12} + \dots)\} U_{m,\tau}^{(20)} + \right. \\
& \left. \frac{1}{2} \{ \{(\delta \hat{t}_{01} + \delta^2 \hat{t}_{02} + \dots) + \epsilon(\hat{t}_{10} + \delta \hat{t}_{11} + \delta^2 \hat{t}_{12} + \dots)\}^2 U_{m,\hat{t}\hat{t}}^{(20)} + \delta\{(\delta \hat{t}_{01} + \delta^2 \hat{t}_{02} + \right. \\
& \left. \dots) + \epsilon(\hat{t}_{10} + \delta \hat{t}_{11} + \delta^2 \hat{t}_{12} + \dots) + \dots\} \times \{t_0 + \delta t_{01} + \delta^2 t_{02} + \dots + \epsilon(t_{10} + \delta t_{11} + \right. \\
& \left. \delta^2 t_{12} + \dots) + \epsilon^2(t_{20} + \delta t_{21} + \delta^2 t_{22} + \dots)\} U_{m,\hat{t}\tau}^{(20)} + \delta^2\{\delta t_{01} + \delta^2 t_{02} + \dots + \epsilon(t_{10} + \right. \\
& \left. \delta t_{11} + \delta^2 t_{12} + \dots)\} U_{m,\tau\tau}^{(10)} \} \right] \text{ at } \tau = 0 \tag{4.113}
\end{aligned}$$

$$\epsilon^2 \delta U_m^{(21)} = \epsilon^2 \delta \left[ U_m^{(21)} + \{\hat{t}_{01} \delta + \dots + \epsilon^2(\hat{t}_{10} + \delta \hat{t}_{11} + \dots)\} U_{m,\hat{t}}^{(21)} + \dots \right] \text{ at } \tau = 0 \tag{4.114}$$

$$\begin{aligned}
\epsilon^3 (U_m^{(30)} + U_{3m}^{(30)}) = & \epsilon^3 \left[ (U_m^{(30)} + U_{3m}^{(30)}) + \{\delta \hat{t}_{01} + \delta^2 \hat{t}_{02} + \right. \\
& \left. \dots\} (U_m^{(30)} + U_{3m}^{(30)})_{,\hat{t}} + \delta(\hat{t}_0 + \delta \hat{t}_{01} + \dots) (U_m^{(30)} + U_{3m}^{(30)})_{,\tau} \right] \text{ at } \tau = 0 \tag{4.115}
\end{aligned}$$

$$\epsilon^3 \delta (U_m^{(31)} + U_{3m}^{(31)}) = \epsilon^3 \delta (U_m^{(31)} + U_{3m}^{(31)}) \text{ at } \tau = 0 \tag{4.116}$$

Note that all  $U_m^{(ij)}$  are evaluated at  $(\hat{t}_0, 0)$

Therefore,

$$\begin{aligned}
U_a = & 2\epsilon \left[ U_m^{(10)} + \delta \left\{ \hat{t}_0 U_{m,\hat{t}}^{(10)} + t_0 U_{m,\tau}^{(10)} + U_m^{(11)} \right\} + \dots \right] + 2\epsilon^2 \left[ \hat{t}_{10} U_{m,\hat{t}}^{(10)} + \right. \\
& \left. \delta \left\{ \hat{t}_{11} U_{m,\hat{t}}^{(10)} + t_{10} U_{m,\tau}^{(10)} + \hat{t}_{01} \hat{t}_{10} U_{m,\hat{t}\hat{t}}^{(10)} + \hat{t}_{10} t_0 U_{m,\hat{t}\tau}^{(10)} + \hat{t}_{10} U_{m,\hat{t}}^{(11)} + \dots \right\} + \dots \right] + 2\epsilon^3 \left[ \hat{t}_{20} U_{m,\hat{t}}^{(10)} + \right. \\
& \left. \frac{(\hat{t}_{10})^2}{2} U_{m,\hat{t}\hat{t}}^{(10)} + \hat{t}_{10} U_{m,\hat{t}}^{(20)} + (U_m^{(30)} + U_{3m}^{(30)}) + \delta \left\{ t_{21} U_{m,\hat{t}}^{(10)} + \hat{t}_{20} U_{m,\tau}^{(10)} + \hat{t}_{10} \hat{t}_{11} U_{m,\hat{t}\hat{t}}^{(10)} + \right. \right. \\
& \left. \left. \hat{t}_{20} t_0 U_{m,\hat{t}\tau}^{(10)} + \hat{t}_{10} t_{10} U_{m,\hat{t}\tau}^{(10)} + \hat{t}_{20} U_{m,\hat{t}}^{(11)} \dots \right\} + \frac{1}{2} (t_{10})^2 U_{m,\hat{t}\hat{t}}^{(11)} + \hat{t}_{11} U_{m,\hat{t}}^{(20)} + t_{10} U_{m,\tau}^{(20)} + \right.
\end{aligned}$$

$$\hat{t}_{10}t_0U_{m,\hat{t}\tau}^{(20)} + \hat{t}_{10}U_{m,\hat{t}}^{(21)} + \hat{t}_{01}\left(U_m^{(30)}+U_{3m}^{(30)}\right)_{\hat{t}} + t_0\left(U_m^{(30)}+U_{3m}^{(30)}\right)_{,\tau} + \dots \Big] at \tau = 0 \quad (4.117)$$

Therefore,

$$U_a = 2\epsilon\left[U_m^{(10)} + \delta t_0 U_{m,\tau}^{(10)} + \dots\right] + 2\epsilon^2\left[U_m^{(20)} + \delta t_{10} U_{m,\tau}^{(10)} + \dots\right] + 2\epsilon^3\left[\left(U_m^{(30)}+U_{3m}^{(30)}\right) + \delta t_{20} U_{m,\tau}^{(10)} + \delta \hat{t}_{20} U_{m,\hat{t}}^{(11)} + \delta t_{10} U_{m,\tau}^{(20)} + \delta t_0\left(U_m^{(30)}+U_{3m}^{(30)}\right)_{,\tau} + \dots\right] at \tau = 0 \quad (4.118)$$

In what follows, simplifications of the terms in equation (4.100) -(4.118) are carried out.

$$U_m^{(10)}(\hat{t}_0, 0) = 2B \quad (4.119.1)$$

$$U_{m,\tau}^{(10)}(\hat{t}_0, 0) = -B \quad (4.119.2)$$

$$\begin{aligned} U_m^{(20)}(\hat{t}_0, 0) &= -\alpha_4(0) + \frac{r_0(0)}{\theta^2} + \frac{r_1(0)}{3\theta^2} \\ &= \frac{-r_1(0)}{3\theta^2} + \frac{r_0(0)}{\theta^2} + \frac{r_0(0)}{\theta^2} - \frac{r_1(0)}{\theta^2} \\ &= 2\left[\frac{-r_1(0)}{3\theta^2} + \frac{r_0(0)}{\theta^2}\right] = 2\left[\frac{\alpha B^2}{3\theta^2} - \frac{3\alpha B^2}{\theta^2}\right] = \frac{-16\alpha B^2}{3\theta^2} \end{aligned} \quad (4.119.3)$$

$$\begin{aligned} U_m^{(30)}\left(\frac{\pi}{\theta}, 0\right) &= -\alpha_9(0) + \frac{r_{10}(0)}{\theta^2} - \frac{r_{11}(0)}{3\theta^2} + \frac{r_{12}(0)}{8\theta^2} - \frac{r_{13}(0)\cos\left\{(\varphi+\theta)\left(\frac{\pi}{\theta}\right)\right\}}{\varphi(2\theta+\varphi)} + \\ &\frac{r_{15}(0)\cos\left\{(\varphi-\theta)\left(\frac{\pi}{\theta}\right)\right\}}{\varphi(2\theta-\varphi)} \\ &= \alpha_9(0) + \frac{r_{10}(0)}{\theta^2} - \frac{r_{11}(0)}{3\theta^2} + \frac{r_{12}(0)}{8\theta^2} + \frac{r_{13}(0)\cos\left(\frac{\varphi\pi}{\theta}\right)}{\varphi(2\theta+\varphi)} - \frac{r_{15}(0)\left(\frac{\varphi\pi}{\theta}\right)}{\varphi(2\theta-\varphi)} \\ &= \frac{r_{10}(0)}{\theta^2} - \frac{r_{11}(0)}{3\theta^2} - \frac{r_{12}(0)}{8\theta^2} - \frac{r_{13}(0)}{\varphi(2\theta+\varphi)} + \frac{r_{10}(0)}{\theta^2} - \frac{r_{11}(0)}{3\theta^2} + \frac{r_{12}(0)}{8\theta^2} + \frac{r_{13}(0)\cos\left(\frac{\varphi\pi}{\theta}\right)}{\varphi(2\theta+\varphi)} \\ &\quad - \frac{r_{15}(0)\cos\left(\frac{\varphi\pi}{\theta}\right)}{\varphi(2\theta-\varphi)} \\ &= \frac{2r_{10}(0)}{\theta^2} - \frac{2r_{11}(0)}{3\theta^2} + \frac{r_{13}(0)\cos\left(\frac{\varphi\pi}{\theta}\right)}{\varphi(2\theta+\varphi)} - \frac{r_{15}(0)\cos\left(\frac{\varphi\pi}{\theta}\right)}{\varphi(2\theta-\varphi)} \\ &= \frac{2B^3}{\theta^2}\left(\frac{75\beta}{8} + \alpha^2 k_3\right) - \frac{2B^3}{3\theta^2}\left(\frac{-45\beta}{8} + \alpha^2 k_4\right) + \frac{B^3\alpha S_0\cos\left(\frac{\varphi\pi}{\theta}\right)}{\varphi(2\theta+\varphi)} - \frac{B^3\alpha S_0\cos\left(\frac{\varphi\pi}{\theta}\right)}{\varphi(2\theta-\varphi)} \end{aligned}$$

$$= \frac{135B^3\beta}{8\theta^2} + B^3\alpha S_0 \left[ \frac{1}{\varphi(2\theta+\varphi)} - \frac{1}{\varphi(2\theta-\varphi)} \right] \cos\left(\frac{\varphi\pi}{\theta}\right) + \frac{2B^3\alpha^2(3k_3-k_4)}{3\theta^2} \quad (4.120.1)$$

Therefore,

$$\begin{aligned} U_m^{(30)}\left(\frac{\pi}{\theta}, 0\right) &= \frac{135B^3\beta}{8\theta^2} \left[ 1 + \frac{8\theta^2}{135} \left\{ \left(\frac{\alpha}{\beta}\right) S_0 \left( \frac{1}{\varphi(2\theta+\varphi)} - \frac{1}{\varphi(2\theta-\varphi)} \right) \cos\left(\frac{\varphi\pi}{\theta}\right) \right\} \right. \\ &\quad \left. + \frac{2}{3\theta^2} \left(\frac{\alpha^2}{\beta}\right) \cdot \frac{8\theta^2}{135\beta} (3k_3 - k_4) \right] \\ &= \frac{135B^3\beta}{8\theta^2} (1 + A_{31}) \end{aligned} \quad (4.120.2)$$

Where,

$$\begin{aligned} A_{31} &= \left[ 1 + \frac{8\theta^2}{135} \left\{ \left(\frac{\alpha}{\beta}\right) S_0 \left( \frac{1}{\varphi(2\theta+\varphi)} - \frac{1}{\varphi(2\theta-\varphi)} \right) \cos\left(\frac{\varphi\pi}{\theta}\right) \right\} \right. \\ &\quad \left. + \frac{2}{3\theta^2} \left(\frac{\alpha^2}{\beta}\right) \cdot \frac{8\theta^2}{135\beta} (3k_3 - k_4) \right] \end{aligned}$$

$$S_0 = \left( \frac{\alpha}{\varphi^2-\theta^2} - \frac{\alpha}{4(\varphi^2-4\theta^2)} - \frac{3\alpha}{4\varphi^2} \right), k_3 = \left( \frac{10}{3\theta^2} - \frac{1}{(\varphi^2-\theta^2)} \right)$$

$$k_4 = \left( \frac{2}{3\theta^2} - \frac{1}{(\varphi^2-\theta^2)} + \frac{8}{3\theta\alpha} \right), k_5 = \left( \frac{1}{3\theta^2} + \frac{1}{4(\varphi^2-\theta^2)} \right)$$

Similarly,

$$\begin{aligned} U_{3m}^{(30)}\left(\frac{\pi}{\theta}, 0\right) &= \alpha_{11}(0) \cos\Omega\hat{t}_0 - \frac{r_{22}(0)}{\Omega^2-\theta^2} - \frac{r_{23}(0)}{\Omega^2-4\theta^2} + \frac{r_{24}(0)}{\Omega^2-9\theta^2} + \frac{r_{25}(0)\cos(\varphi+\theta)\left(\frac{\pi}{\theta}\right)}{\Omega^2-(\varphi+\theta)^2} + \\ &\quad \frac{r_{27}(0)\cos(\varphi-\theta)\left(\frac{\pi}{\theta}\right)}{\Omega^2-(\varphi-\theta)^2} \\ &= - \left\{ \frac{r_{22}(0)}{\Omega^2-\theta^2} + \frac{r_{23}(0)}{\Omega^2-4\theta^2} + \frac{r_{24}(0)}{\Omega^2-9\theta^2} + \frac{r_{25}(0)}{\Omega^2-(\varphi+\theta)^2} + \frac{r_{27}(0)}{\Omega^2-(\varphi-\theta)^2} \right\} \cos\Omega\hat{t}_0 \\ &\quad - \frac{r_{22}(0)}{\Omega^2-\theta^2} + \frac{r_{23}(0)}{\Omega^2-4\theta^2} - \frac{r_{24}(0)}{\Omega^2-9\theta^2} - \frac{r_{25}(0)\cos\left(\frac{\varphi\pi}{\theta}\right)}{\Omega^2-(\varphi+\theta)^2} \\ &\quad - \frac{r_{27}(0)\cos\left(\frac{\varphi\pi}{\theta}\right)}{\Omega^2-(\varphi-\theta)^2} \end{aligned}$$

$$\begin{aligned}
&= -\frac{r_{22}(0)}{\Omega^2 - \theta^2} (1 + \cos\Omega\hat{t}_0) + \frac{r_{23}(0)}{\Omega^2 - 4\theta^2} (1 - \cos\Omega\hat{t}_0) - \frac{r_{24}(0)}{\Omega^2 - 9\theta^2} (1 + \cos\Omega\hat{t}_0) \\
&\quad - \frac{r_{25}(0)}{\Omega^2 - (\varphi + \theta)^2} (1 + \cos\Omega\hat{t}_0) - \frac{r_{27}(0)}{\Omega^2 - (\varphi - \theta)^2} (1 + \cos\Omega\hat{t}_0) \\
&= \frac{-\frac{15B^3\beta}{16}\left(1 - \frac{16\alpha^2 k_{11}}{15\beta}\right)(1 + \cos\Omega\hat{t}_0)}{\Omega^2 - \theta^2} - \frac{\frac{3B^3\beta}{8}(1 - k_{12})(1 - \cos\Omega\hat{t}_0)}{\Omega^2 - 4\theta^2} - \frac{B^3\beta(1 - k_{13})(1 + \cos\Omega\hat{t}_0)}{16(\Omega^2 - 9\theta^2)} - \\
&\quad \frac{B^3\alpha S_0(1 + \cos\Omega\hat{t}_0)}{\Omega^2 - (\varphi + \theta)^2} - \frac{B^3\alpha S_0(1 + \cos\Omega\hat{t}_0)}{\Omega^2 - (\varphi - \theta)^2} \tag{4.121.1}
\end{aligned}$$

Therefore;

$$U_{3m}^{(30)} = -B^3\beta \left( A_{32} + \left(\frac{\alpha}{\beta}\right) S_0 A_{33} \right) \tag{4.121.2}$$

Where,

$$A_{32} = \left[ \frac{\frac{15}{16}\left(1 - \frac{16\alpha^2 k_{11}}{15\beta}\right)(1 + \cos\Omega\hat{t}_0)}{\Omega^2 - \theta^2} + \frac{\frac{3}{8}(1 - k_{12})(1 - \cos\Omega\hat{t}_0)}{\Omega^2 - 4\theta^2} + \frac{(1 - k_{13})(1 + \cos\Omega\hat{t}_0)}{16(\Omega^2 - 9\theta^2)} \right]$$

And

$$A_{33} = \left[ \frac{1 + \cos\Omega\hat{t}_0}{\Omega^2 - (\varphi + \theta)^2} - \frac{1 + \cos\Omega\hat{t}_0}{\Omega^2 - (\varphi - \theta)^2} \right], \quad k_{12} = \left[ -\frac{4}{3}\left(\frac{\alpha^2}{\beta}\right)\left(\frac{1}{\varphi^2 - \theta^2}\right) \right]$$

$$k_{13} = \left[ 2\left(\frac{\alpha^2}{\beta}\right)\left(\frac{1}{\varphi^2 - 4\theta^2}\right) \right],$$

$$U_{m,\tau}^{(20)}\left(\frac{\pi}{\theta}, 0\right) = -\alpha'_4(0) + \frac{r'_0(0)}{\theta^2} - \frac{r'_1(0)}{3\theta^2} \tag{4.122.2}$$

From (4.24.8),

$$\alpha'_4(0) = -\alpha'_4(0) + \frac{1}{2\theta} [\alpha B \beta_2(0) - 2\theta^2 \omega'_1(0) \beta_2(0) - \omega''_1(0) \theta \alpha_1(0) - 2\omega'_1(0) \alpha_1 \theta]$$

$$= \frac{-8\alpha B^2}{3\theta^2} + \frac{1}{2\theta} \left[ \alpha \beta \left(\frac{-B}{\theta}\right) - 2\theta^2 \left(\frac{B}{\theta^2}\right) \left(\frac{-B}{\theta}\right) - 2\left(\frac{B}{\theta^2}\right) (-B)\theta \right]$$

$$= \frac{-8\alpha B^2}{3\theta^2} + \frac{1}{2\theta} \left[ -\frac{B^2\alpha}{\theta} + \frac{2B^2}{\theta} + \frac{2B^2}{\theta} \right] = -\frac{26\alpha B^2}{6\theta^2} + \frac{4B^2}{\theta}$$

$$\alpha'_4(0) = \frac{-13\alpha B^2}{3\theta^2} + \frac{4B^2}{\theta} \tag{4.122.2}$$

$$\begin{aligned}
U_{m,\tau}^{(20)}\left(\frac{\pi}{\theta}, 0\right) &= \left(\frac{13\alpha B^2}{3\theta^2} + \frac{4B^2}{\theta}\right) + \frac{r'_0(0)}{\theta^2} - \frac{r'_1(0)}{3\theta^2} \\
&= \left(\frac{13\alpha B^2}{3\theta^2} + \frac{4B^2}{\theta}\right) + \frac{2\alpha B^2}{\theta^2} - \frac{2\alpha B^2}{3\theta^2} \\
&= \frac{17\alpha B^2}{3\theta^2} - \frac{4B^2}{\theta} = B^2 \left(\frac{17\alpha}{3\theta^2} - \frac{4}{\theta}\right)
\end{aligned} \tag{4.123}$$

Also,

$$\begin{aligned}
\omega_2'' &= -\frac{1}{2\theta^2} \left[ \frac{(\omega_1')^2 \theta^2}{\alpha_1} + \frac{2\omega_1' \theta^2}{\alpha_1} - 2\alpha \left( \frac{r_0}{\theta^2} - \frac{r_1}{6\theta^2} + 3 \left( \frac{\alpha_4}{\alpha_1} \right) \right) - \left\{ \frac{\left( \frac{\alpha_1^2}{2} + B^2 \right)}{\varphi^2} + \frac{\alpha_1^2 \alpha^2}{4(\varphi^2 - 4\theta^2)} \right\} - \frac{45\beta}{4} \left( \frac{\alpha_1^2}{4} + B^2 \right) \right] \\
\omega_2''(0) &= -\frac{1}{2\theta^2} \left[ \frac{\theta^2 \{ \alpha_1'(0)(\omega_1'(0))^2 - 2\alpha_1 \omega_1''(0) \omega_1'(0) \}}{\alpha_1^2(0)} + 2\theta^2 \{ \alpha_1'(0)(\omega_1'(0)\alpha_4(0)) - \alpha_1(0)(\omega_1''(0)\alpha_4(0) + \omega_1'(0)\alpha_4'(0)) \} - 2\alpha \left( \frac{r_0(0)}{\theta^2} - \frac{r_1'(0)}{6\theta^2} + B \left( \frac{\alpha_1(0)\alpha_4'(0) - \alpha_4(0)\alpha_1'(0)}{\alpha_1^2(0)} \right) \right) - \left\{ \frac{\alpha_1'(0)\alpha_1(0)\alpha^2}{\varphi^2} + \frac{\alpha^2 \alpha_1(0)\alpha_1'(0)}{4(\varphi^2 - 4\theta^2)} - \frac{45\beta}{4} \left( \frac{\alpha_1(0)\alpha_1'(0)}{2} \right) \right\} \right]
\end{aligned} \tag{4.124}$$

$$\begin{aligned}
&= -\frac{1}{2\theta^2} \left[ \theta^2 \left\{ \frac{B^3}{\theta^4 B^2} \right\} + 2\theta^2 \left\{ \frac{B^2}{\theta^2} \cdot \frac{8\alpha B^2}{3\theta^2} + B \left( \frac{B}{\theta^2} \cdot B^2 S_{51} \right) \right\} - 2\alpha \left\{ \frac{2\alpha B^2}{\theta^2} - \frac{2\alpha B^2}{6\theta^2} + B \left( \frac{-B \cdot B^2 S_{51} - \frac{8\alpha B^3}{3\theta^2}}{B^2} \right) \right\} - \left\{ \alpha^2 \left( \frac{-B^2}{\varphi^2} \right) + \frac{\alpha^2 B(-B)}{2(\varphi^2 - 4\theta^2)} - \frac{45\beta(-B^2)}{8} \right\} \right] \\
&= -\frac{1}{2\theta^2} \left[ \frac{B}{\theta^2} + 2 \left\{ \frac{B^4 \alpha}{3\theta^2} + B^4 S_{51} \right\} - 2\alpha^2 \left\{ \frac{5B^2}{3\theta^2} - B^2 \left( \frac{S_{51}}{\alpha} + \frac{8}{3\theta^2} \right) \right\} + \alpha^2 B^2 \left( \frac{1}{\varphi^2} - \frac{1}{2(\varphi^2 - 4\theta^2)} \right) + \frac{45\beta B^2}{8} \right]
\end{aligned} \tag{4.125}$$

Therefore,

$$\begin{aligned}
\omega_2''(0) &= -\frac{1}{2\theta^2} \left[ \frac{B}{\theta^2} - 2\alpha^2 B^2 \left\{ \frac{5}{3\theta^2} - \left( \frac{S_{51}}{\alpha} + \frac{8}{3\theta^2} \right) \right\} + \alpha^2 B^2 \left( \frac{1}{\varphi^2} - \frac{1}{2(\varphi^2 - 4\theta^2)} \right) + \frac{45\beta B^2}{8} + 2B^4 \alpha \left( \frac{1}{3\theta^2} + \frac{S_{51}}{\alpha} \right) \right]
\end{aligned} \tag{4.126}$$

From equation (4.68.1), the following is obtained,

$$\alpha_9'(0) = h_1(0) - \alpha_9(0)$$

But, it is noted that,

$$\begin{aligned}
h_1(0) &= -\frac{1}{2\theta^2} \left[ 2\omega'_1(0)\theta\alpha'_6(0) + \omega''_2(0)\alpha_1(0)\theta + \omega'_1(0)\theta\alpha_4(0) + 2\omega'_1(0)\theta\alpha_4(0) \right. \\
&\quad \left. + 2\omega'_2(0)\theta\alpha_1(0) + \left\{ \frac{2\alpha(B\beta_6(0) - \alpha_1(0)r_4(0))}{6\theta^2} \right\} \right] \\
&= -\frac{1}{2\theta^2} \left[ \omega'_1(0)\{2\theta\alpha'_6(0) + 3\theta\alpha_4(0)\} + 2\omega'_2(0)\theta\alpha_1(0) \right. \\
&\quad \left. - \frac{\alpha\alpha_1(0)r_4(0)}{3\theta^2} + \omega''_2(0)\alpha_1(0)\theta \right] \\
&= -\frac{1}{2\theta^2} \left[ \frac{B}{\theta^2} \left\{ 2\theta \left( \frac{-\alpha B^2}{\theta^2} \right) + 3\theta \cdot \frac{8\alpha B^2}{3\theta^2} \right\} + 2\theta(-B) \cdot B^2 S_{49} \right. \\
&\quad \left. - \frac{\alpha(-B) \left( \frac{\alpha B^2}{\theta} \right)}{3\theta^2} - B\theta\omega''_2(0) \right]
\end{aligned}$$

Therefore,

$$\begin{aligned}
h_1(0) &= -\frac{1}{2\theta^2} \left[ \frac{-\alpha B^3}{\theta^2} \left( \frac{-2\alpha}{\theta} + \frac{8\alpha}{\theta} \right) - 2\theta B^3 S_{49} + \frac{\alpha^2 B^3}{3\theta^3} - B\theta\omega''_2(0) \right] \\
&= -\frac{1}{2\theta^2} \left[ \frac{6\alpha^2 B^3}{\theta^3} - 2\theta B^3 S_{49} + \frac{\alpha^2 B^3}{3\theta^3} - B\theta\omega''_2(0) \right] = B^3 S_{62} \quad (4.127)
\end{aligned}$$

Where,

$$S_{62} = -\frac{1}{2\theta^2} \left[ \frac{6\alpha^2}{\theta^3} - 2\theta S_{49} + \frac{\alpha^2}{3\theta^3} - \frac{\theta\omega''_2(0)}{B^2} \right], h_1(0) = B^3 S_{62}$$

$$\alpha_9(0) = B^3 S_{63}, S_{63} = \left( \frac{-S_3}{\theta^2} + \frac{S_4}{3\theta^2} + \frac{S_5}{8\theta^2} + \frac{\alpha S_0}{\varphi(2\theta - \varphi)} \right)$$

Therefore,

$$\alpha'_9(0) = h_1(0) - \alpha_9(0) = B^3 S_{62} - B^3 S_{63} = B^3 (S_{62} - S_{63}) = B^3 S_{64}$$

Where,

$$S_{64} = S_{62} - S_{63}$$

Similarly,

$$\begin{aligned}
&U_{m,\tau}^{(30)} \left( \frac{\pi}{\theta}, 0 \right) \\
&= -B^3 S_{65} + \frac{B^3 S_{20}}{\theta^2} + \frac{B^3 S_{20}}{\theta^2} + \frac{B^3 S_{21}}{3\theta^2} + \frac{2\alpha B^3 S_0}{\theta^2} \cos \left( \frac{\varphi\pi}{\theta} \right) \left[ \frac{1}{(2\theta - \varphi)} - \frac{1}{(2\theta + \varphi)} \right]
\end{aligned}$$

Therefore,

$$U_{m,\tau}^{(30)}\left(\frac{\pi}{\theta}, 0\right) = B^3 S_{65} + \frac{B^3 S_{20}}{\theta^2} + \frac{B^3 S_{20}}{\theta^2} + \frac{B^3 S_{21}}{3\theta^2} + \frac{2\alpha B^3 S_0}{\theta^2} \cos\left(\frac{\varphi\pi}{\theta}\right) \left[\frac{1}{(2\theta-\varphi)} - \frac{1}{(2\theta+\varphi)}\right] \quad (4.129)$$

Where,

$$S_{65} = -S_{64} + \frac{S_{20}}{\theta^2} + \frac{S_{20}}{\theta^2} + \frac{S_{21}}{3\theta^2} + \frac{2\alpha S_0}{\theta^2} \cos\left(\frac{\varphi\pi}{\theta}\right) \left[\frac{1}{(2\theta-\varphi)} - \frac{1}{(2\theta+\varphi)}\right]$$

Also,

$$U_{3m,\tau}^{(30)} = \alpha'_{11}(0) \cos \Omega \hat{t}_0 + \beta'_{11}(0) \sin \Omega \hat{t}_0 + \frac{r'_{22}(0) \cos \theta \hat{t}_0}{\Omega^2 - \theta^2} + \frac{r'_{23}(0) \cos 2\theta \hat{t}_0}{\Omega^2 - 4\theta^2} + \frac{r'_{24}(0) \cos 3\theta \hat{t}_0}{\Omega^2 - 9\theta^2} + \frac{r'_{25}(0) \cos(\varphi + \theta) \hat{t}_0 + r'_{26}(0) \sin(\varphi + \theta) \hat{t}_0}{\Omega^2 - (\varphi + \theta)^2} + \frac{r'_{27}(0) \cos(\varphi - \theta) \hat{t}_0 + r'_{28}(0) \sin(\varphi - \theta) \hat{t}_0}{\Omega^2 - (\varphi - \theta)^2} \quad (4.130.1)$$

But,

$$h_6(0) = \frac{\alpha \beta B^2 S_{43}}{4\Omega} = \frac{\alpha B^3 S_{43}}{4\Omega} \quad (4.131.2)$$

Therefore,

$$\alpha'_{11}(0) = h_6(0) - \alpha_{11}(0) = \frac{\alpha B^3 S_{43}}{4\Omega} - B^3 S_{48} = B^3 S_{66} \quad (4.131.3)$$

Where,

$$S_{66} = \frac{\alpha S_{43}}{4\Omega} - S_{48}$$

It follows that,

$$\beta'_{11}(0) = h_5(0) - \beta_{11}(0)$$

That is,

$$\beta'_{11}(0) = -\beta_{11}(0) = 0, \quad \text{since } h_5(0) = 0 \quad (4.131.4)$$

Therefore,

$$U_{3m,\tau}^{(30)}\left(\frac{\pi}{\theta}, 0\right) = B^3 S_{66} \cos \Omega \left(\frac{\pi}{\theta}\right) - \frac{B^3 S_{17}}{\Omega^2 - \theta^2} + \frac{B^3 S_{18}}{\Omega^2 - 4\theta^2} - \frac{B^3 S_{19}}{\Omega^2 - 9\theta^2} - \frac{2\alpha B^3 S_0 \cos\left(\frac{\varphi\pi}{\theta}\right)}{\Omega^2 - (\varphi + \theta)^2} - \frac{2\alpha B^3 S_0 \cos\left(\frac{\varphi\pi}{\theta}\right)}{\Omega^2 - (\varphi - \theta)^2}$$

That is,

$$U_{3m,\tau}^{(30)}\left(\frac{\pi}{\theta}, 0\right) = B^3 S_{67} \quad (4.132)$$

Where,

$$S_{67} = S_{66} \cos \Omega \left(\frac{\pi}{\theta}\right) - \frac{S_{17}}{\Omega^2 - \theta^2} + \frac{S_{18}}{\Omega^2 - 4\theta^2} - \frac{S_{19}}{\Omega^2 - 9\theta^2} - 2\alpha S_0 \cos\left(\frac{\varphi\pi}{\theta}\right) \left[ \frac{1}{\Omega^2 - (\varphi + \theta)^2} - \frac{1}{\Omega^2 - (\varphi - \theta)^2} \right] \quad (4.133)$$

Therefore, the maximum displacement is

$$U_a\left(\frac{\pi}{\theta}, 0\right) = 2\epsilon [2B - t_0 B \delta + \dots] + 2\epsilon^2 \left[ \frac{-16\alpha B^2}{3\theta^2} - \frac{t_0 B(-B)\delta}{\theta^2} + \dots \right] + 2\epsilon^3 \left[ \frac{135B^3\beta(1+A_{31})}{8\theta^2} - B^3\beta \left( A_{32} + \frac{\alpha}{\beta} S_0 A_{33} \right) \right] + \delta \left[ -t_{20}B - \hat{t}_{20}B + t_{10}B^2 \left( \frac{17\alpha}{3\theta^2} - \frac{4}{\theta^2} \right) + t_0 B^2 (S_{65} + S_{67}) + \dots \right] \quad (4.134)$$

That is,

$$U_a\left(\frac{\pi}{\theta}, 0\right) = \left[ 4B\epsilon \left( 1 - \frac{t_0\delta}{2} \right) - \frac{32\alpha B^2 \epsilon^2}{3\theta^2} \left( 1 - \frac{3\delta t_0}{16\alpha} + \dots \right) + \frac{135\beta(1+A_{31})B^3 \epsilon^3}{4\theta^2} \left\{ 1 - \frac{8\theta^2(A_{32} + \frac{\alpha}{\beta} S_0 A_{33})}{135(1+A_{31})} \right\} + \frac{8\delta\theta^2}{135\beta(1+A_{31})} \left\{ \frac{-t_{20}}{B^2} - \frac{\hat{t}_{20}}{B^2} + \frac{t_{10}}{B} \left( \frac{17\alpha}{3\theta^2} - \frac{4}{\theta} \right) + t_0(S_{65} + S_{67}) \right\} \right] \quad (4.135)$$

Where,

$$\begin{aligned} D_1 &= 1 - \frac{t_0\delta}{2} \\ D_2 &= 1 - \frac{3t_0\delta}{16\alpha} \\ D_3 &= 1 - \frac{8\theta^2(A_{32} + \frac{\alpha}{\beta} S_0 A_{33})}{135(1+A_{31})} \\ D_4 &= \frac{8\delta\theta^2}{135\beta(1+A_{31})} \left\{ \frac{-t_{20}}{B^2} - \frac{\hat{t}_{20}}{B^2} + \frac{t_{10}}{B} \left( \frac{17\alpha}{3\theta^2} - \frac{4}{\theta} \right) + t_0(S_{65} + S_{67}) \right\} \end{aligned}$$

A further simplification of (4.135) yields

$$U_a\left(\frac{\pi}{\theta}, 0\right) \equiv U_a = 4B\epsilon D_1 - \frac{32\alpha B^2 D_2 \epsilon^2}{3\theta^2} + \frac{135\beta(1+A_{31})B^3 \epsilon^3}{4\theta^2} [D_3 + D_4] + \dots \quad (4.136)$$

Equation (4.136) can be rewrite as

$$U_a = 4B\epsilon D_1 - \frac{32\alpha B^2 D_2 \epsilon^2}{3\theta^2} + \frac{135\beta(1+A_{31})B^3 D_3 \epsilon^3}{4\theta^2} \left[ 1 + \frac{D_4}{D_3} \right] + \dots \quad (4.137)$$

To reverse the series (4.137), needs to recast  $U_a$  as,

$$U_a = \epsilon c_1 + \epsilon^2 c_2 + \epsilon^2 c_3 + \dots \quad (4.138.1)$$

Where,

$$c_1 = 4BD_1, c_2 = -\frac{32\alpha B^2 D_2}{3\theta^2},$$

$$c_3 = \frac{135\beta(1+A_{31})B^3 D_3}{4\theta^2} \left(1 + \frac{D_4}{D_3}\right) = \frac{135\beta(1+A_{31})B^3 D_3(1+D_5)}{4\theta^2} \quad (4.138.2)$$

Let,

$$D_5 = \left(\frac{D_4}{D_3}\right)$$

To reverse the series (4.138.1) as in Ette (2007), it follows thus

$$\epsilon = d_1 U_a + d_2 U_a^2 + d_3 U_a^3 + \dots \quad (4.138.3)$$

By substituting for  $U_a$  in (4.138.3) and equating the coefficients of powers of  $\epsilon$  to obtained

$$\epsilon = d_1(\epsilon c_1 + \epsilon^2 c_2 + \epsilon^3 c_3 + \dots) + d_2(\epsilon c_1 + \epsilon^2 c_2 + \epsilon^3 c_3 + \dots)^2 + d_3(\epsilon c_1 + \epsilon^2 c_2 + \epsilon^3 c_3 + \dots)^3 \quad (4.139.1)$$

$$O(\epsilon): 1 = d_1 c_1$$

Therefore,

$$d_1 = \frac{1}{c_1} \quad (4.139.2)$$

$$O(\epsilon^2): 0 = d_1 c_1 + d_2 c_1^2$$

Therefore,

$$d_2 = \frac{d_1 c_2}{c_1^2} = -\frac{c_2}{c_1^3} \quad (4.139.3)$$

$$O(\epsilon^3): 0 = d_1 c_3 + 2d_2 c_1 c_2 + d_3 c_1^3$$

Therefore,

$$d_3 = \frac{-(d_1 c_3 + 2d_2 c_1 c_2)}{c_1^3} = \frac{2c_2^2 - c_1 c_3}{c_1^5} \quad (4.139.4)$$

#### 4.4 THE DYNAMIC BUCKLING LOAD, $\lambda_D$ OF THE COLUMN

As in (3.1) the dynamic buckling load  $\lambda_D$  is now obtained from the maximization,  $\frac{d\lambda}{dU_a} = 0$ . This is easily done from (4.138.1).

Therefore,

$$\begin{aligned} \frac{d\epsilon}{dU_a} &= \left( \frac{d\epsilon}{d\lambda} \cdot \frac{d\lambda}{dU_{aD}} \right) = 0 \\ &= \frac{d}{dU_a} (d_1 U_a + d_2 U_a^2 + d_3 U_a^3 + \dots) = 0 \end{aligned} \quad (4.140)$$

This implies that,

$$\frac{d(d_1) \cdot U_a}{dU_a} + d_1 \frac{d(U_a)}{dU_a} + \frac{d(d_2) \cdot U_a^2}{dU_a} + d_2 \frac{d(U_a^2)}{dU_a} + \frac{d(d_3) \cdot U_a^3}{dU_a} + d_3 \frac{d(U_a^3)}{dU_a} = 0 \quad (4.141)$$

That is,

$$U_a \left( \frac{d(d_1)}{d\lambda} \frac{d\lambda}{dU_a} \right) + d_1 + \frac{d(d_2)}{d\lambda} \cdot \frac{d\lambda}{dU_a} \cdot U_a^2 + 2U_a d_2 + \frac{d}{d\lambda} \frac{d\lambda}{dU_a} U_a^3 + 3d_3 U_a^2 = 0 \quad (4.142)$$

Then, from (4.142),

$$d_1 + 2U_{aD} d_2 + 3d_3 U_{aD}^2 = 0 \quad (4.143)$$

Where  $U_{aD}$  is the value of  $U_a$  at buckling and solving (4.143) yields,

$$U_{aD} = \frac{1}{3d_3} \left\{ -d_2 \pm (d_2^2 - 3d_1 d_3)^{\frac{1}{2}} \right\} \quad (4.144)$$

The negative root sign in (4.144) is considered because the positive root sign is of no physical significance.

Therefore,

$$U_{aD} = \frac{1}{3d_3} \left\{ -d_2 - (d_2^2 - 3d_1 d_3)^{\frac{1}{2}} \right\} \quad (4.145)$$

In what follows, first simplify the terms in (4.145),

$$(d_2^2 - 3d_1 d_3)^{\frac{1}{2}} = \left[ \frac{c_2^2}{c_1^6} + 3 \left( \frac{1}{c_1} \right) \left( \frac{2c_2^2 - c_1 c_3}{c_1^5} \right) \right]^{\frac{1}{2}} = \sqrt{\frac{3c_3}{c_1^5} \left( 1 - \frac{5c_2^2}{3c_1 c_3} \right)} \quad (4.146)$$

But,

$$\begin{aligned}
-d_2 - (d_2^2 - 3d_1d_3)^{\frac{1}{2}} &= -(d_2^2 - 3d_1d_3)^{\frac{1}{2}} \left[ 1 + \frac{d_2}{(d_2^2 - 3d_1d_3)^{\frac{1}{2}}} \right] \\
&= -\sqrt{\frac{3c_3}{c_1^5} \left( 1 - \frac{5c_2^2}{3c_1c_3} \right)} \left[ 1 + \frac{\left( \frac{-c_2}{c_1^3} \right)}{\sqrt{\frac{3c_3}{c_1^5} \left( 1 - \frac{5c_2^2}{3c_1c_3} \right)}} \right] = -\sqrt{\frac{3c_3}{c_1^5} \left( 1 - \frac{5c_2^2}{3c_1c_3} \right)} \left[ 1 - \frac{c_2}{\sqrt{3c_1c_3} \left( 1 - \frac{5c_2^2}{3c_1c_3} \right)^{\frac{1}{2}}} \right]
\end{aligned}$$

It can be recalled that,

$$d_3 = \frac{2c_2^2 - c_1c_3}{c_1^3} = \frac{-c_3}{c_1^4} \left( 1 - \frac{2c_2^2}{c_1c_3} \right) \quad (4.147)$$

It follows from (4.144) that,

$$U_{aD} = \frac{1}{\frac{-c_3}{c_1^4} \left( 1 - \frac{2c_2^2}{c_1c_3} \right)} \left[ -\sqrt{\frac{3c_3}{c_1^5} \left( 1 - \frac{5c_2^2}{3c_1c_3} \right)} \left\{ \left\{ 1 - \frac{c_2}{\sqrt{3c_1c_3} \left( 1 - \frac{5c_2^2}{3c_1c_3} \right)^{\frac{1}{2}}} \right\} \right\} \right] \quad (4.148)$$

That is,

$$U_{aD} = \sqrt{\frac{c_1^3}{3c_3}} \left[ \sqrt{\left( 1 - \frac{5c_2^2}{3c_1c_3} \right)} \left\{ \frac{1 - \frac{c_2}{\sqrt{3c_1c_3} \left( 1 - \frac{5c_2^2}{3c_1c_3} \right)^{\frac{1}{2}}}}{\frac{2c_2^2}{c_1c_3}} \right\} \right] \quad (4.149)$$

But,

$$\begin{aligned}
\sqrt{\frac{c_1^3}{3c_3}} &= \frac{1}{\sqrt{3}} \left\{ \frac{\{4BD_1\}^3}{3[135\beta(1+A_{31})B^3D_3(1+D_5)]} \right\}^{\frac{1}{2}} = \frac{1}{\sqrt{3}} \left\{ \frac{64B^3D_1^3 \cdot 4\theta^2}{405\beta B^3D_3(1+D_5)(1+A_{31})} \right\}^{\frac{1}{2}} = \\
\frac{16\theta D_1^{\frac{3}{2}}}{9\sqrt{15}\beta D_3(1+D_5)(1+A_{31})} &= \frac{16\theta}{9\sqrt{15}\beta(1+D_5)(1+A_{31})} \left( \frac{D_1^{\frac{3}{2}}}{D_3} \right) = \frac{16\theta D_6}{9\sqrt{15}\beta} = \frac{16\theta D_6}{9\sqrt{15}\beta^{\frac{1}{2}}}
\end{aligned} \quad (4.150)$$

Where,

$$D_6 = \frac{\left( \frac{D_1^3}{D_3} \right)^{\frac{1}{2}}}{\sqrt{(1+D_5)(1+A_{31})}}$$

Simplifying  $1 - \frac{5c_2^2}{3c_1c_3}$  yields,

$$\begin{aligned}
1 - \frac{5c_2^2}{3c_1c_3} &= 1 - \frac{5 \left( \frac{-32\alpha B^2 D_2}{3\theta^2} \right)^2}{3(4BD_1) \left\{ \frac{135\beta(1+A_{31})B^3 D_3(1+D_5)}{4\theta^2} \right\}} \\
&= \left[ 1 + \frac{5120\alpha^2 D_2^2 B^4}{9\theta^4} \cdot \frac{4\theta^2}{1620B^4\beta(1+D_5)(1+A_{31})D_1 D_3} \right] \\
&= \left[ 1 + \frac{1024 \left( \frac{\alpha^2}{\beta} \right) D_2^2}{729\theta^2 D_1 D_3 (1+D_5)(1+A_{31})} \right] = D_7
\end{aligned}$$

That is,

$$D_7 = 1 - \frac{5c_2^2}{3c_1c_3}$$

It follows that;

$$\begin{aligned}
1 - \frac{c_2}{\sqrt{3c_1c_3} \left( 1 - \frac{5c_2^2}{3c_1c_3} \right)^{\frac{1}{2}}} &= 1 - \frac{\left( \frac{-32\alpha B^2 D_2}{3\theta^2} \right)}{\sqrt{3c_1c_3} D_7} = 1 + \frac{32\alpha B^2 D_2}{3\theta^2 \sqrt{3(4BD_1) \left\{ \frac{135\beta(1+A_{31})B^3 D_3(1+D_5)}{4\theta^2} \right\} D_7}} = \\
1 + \frac{32 \left( \frac{\alpha}{\beta^{\frac{1}{2}}} \right) D_2}{27\sqrt{5}\theta \sqrt{D_1 D_3 D_7 (1+D_5)(1+A_{31})}} &= D_8 \\
\left( 1 - \frac{2c_2^2}{c_1c_3} \right) &= 1 - \frac{2 \left( \frac{-32\alpha B^2 D_2}{3\theta^2} \right)^2}{(4BD_1) \left\{ \frac{135\beta(1+A_{31})B^3 D_3(1+D_5)}{4\theta^2} \right\}} \\
&= 1 - \frac{2048\alpha^2 B^4 D_2^2}{9\theta^4} \cdot \frac{4\theta^2}{(4BD_1)(135\beta(1+A_{31})B^3 D_3(1+D_5))} \\
&= 1 - \frac{2048 D_2^2 \left( \frac{\alpha^2}{\beta} \right)}{1215\theta^2 D_1 D_3 (1+A_{31})(1+D_5)} = D_9
\end{aligned}$$

Thus, simplifying (4.150)

$$U_{aD} = \frac{16\theta D_6}{9\sqrt{15}\beta^{\frac{1}{2}}} \left[ D_7^{\frac{1}{2}} \left\{ \frac{1-D_8}{D_9} \right\} \right] = \frac{16\theta D_6 D_{10}}{9\sqrt{15}\beta^{\frac{1}{2}}} \quad (4.151)$$

Where,

$$D_{10} = \left[ D_7^{\frac{1}{2}} \left\{ \frac{1-D_8}{D_9} \right\} \right]$$

Therefore,

$$U_{aD} = \frac{16\theta D_6 D_{10}}{9\sqrt{15}\beta^{\frac{1}{2}}}$$

To determine the dynamic buckling load,  $\lambda_D$  needs evaluating (4.138) at buckling and get,

$$\epsilon = d_1 U_{aD} + d_2 U_{aD}^2 + d_3 U_{aD}^3 + \dots \quad (4.152)$$

Multiplying equation (4.152) by 3 and get

$$\begin{aligned} 3\epsilon &= 3d_1 U_{aD} + 3d_2 U_{aD}^2 + 3d_3 U_{aD}^3 + \dots \\ &= 3(d_1 U_{aD} + d_2 U_{aD}^2) + U_{aD}(3d_3 U_{aD}^2) + \dots \end{aligned} \quad (4.153)$$

But from (4.143),

$$3d_3 U_{aD}^2 = -d_1 - 2d_2 U_{aD} \quad (4.154)$$

Substituting (4.154) for  $3d_3 U_{aD}^2$  in (4.153) yields,

$$\begin{aligned} 3\epsilon &= 3(d_1 U_{aD} + d_2 U_{aD}^2) + U_{aD}(-d_1 - 2d_2 U_{aD}) + \dots \\ &= 2d_1 U_{aD} + d_2 U_{aD}^2 = 2d_1 U_{aD} \left(1 + \frac{d_2 U_{aD}}{2d_1}\right) \end{aligned} \quad (4.155)$$

On substituting for  $d_1, d_2$  in equation (4.155) and obtain

$$3\epsilon = \frac{2}{c_1} U_{aD} \left(1 - \frac{c_2 U_{aD}}{2c_1^2}\right) \quad (4.156)$$

On substituting for  $c_1, c_2$  and  $U_{aD}$  in equation (4.156) yields

$$\begin{aligned} 3\epsilon &= \frac{2 \left( \frac{16\theta D_6 D_{10}}{9\sqrt{15}\beta^2} \right)}{4BD_1} \left[ 1 - \frac{\left( \frac{-32\alpha B^2 D_2}{3\theta^2} \right) \left\{ \frac{16\theta D_6 D_{10}}{9\sqrt{15}\beta^2} \right\}}{2(4BD_1)^2} \right] \\ &= \frac{8\theta D_6 D_{10}}{9\sqrt{15}D_1 \beta^2 B} \left[ 1 + \left( \frac{\alpha D_2}{(D_1 \theta)^2} \right) \left( \frac{16\theta D_6 D_{10}}{9\sqrt{15}\beta^2} \right) \right] = \frac{8\theta D_6 D_{10}}{9\sqrt{15}D_1 \beta^2 B} \left[ 1 + \frac{16 \left( \frac{\alpha}{\beta^2} \right) D_2 D_6 D_{10}}{27\sqrt{15}D_1^2 \theta} \right] \end{aligned} \quad (4.157)$$

This implies that,

$$3\epsilon = \frac{8(16m^4 - 8\lambda_D m^2 + 1)^{\frac{1}{2}} D_6 D_{10} (16m^4 - 8\lambda_D m^2 + 1)}{9\sqrt{15}D_1 \beta^2 \cdot 8\lambda_D m^2 \bar{a}_m} \left[ 1 + \frac{16 \left( \frac{\alpha}{\beta^2} \right) D_2 D_6 D_{10}}{27\sqrt{15}D_1^2 \theta} \right]$$

That is,

$$3\epsilon = \frac{(16m^4 - 8\lambda_D m^2 + 1)^{\frac{3}{2}} D_6 D_{10}}{9\sqrt{15} D_1 \beta^{\frac{1}{2}} (\lambda_D m^2 \bar{a}_m)} \left[ 1 + \frac{16 \left( \frac{\alpha}{\beta^{\frac{1}{2}}} \right) D_2 D_6 D_{10}}{27\sqrt{15} D_1^2 \theta(\lambda_D)} \right]$$

Therefore,

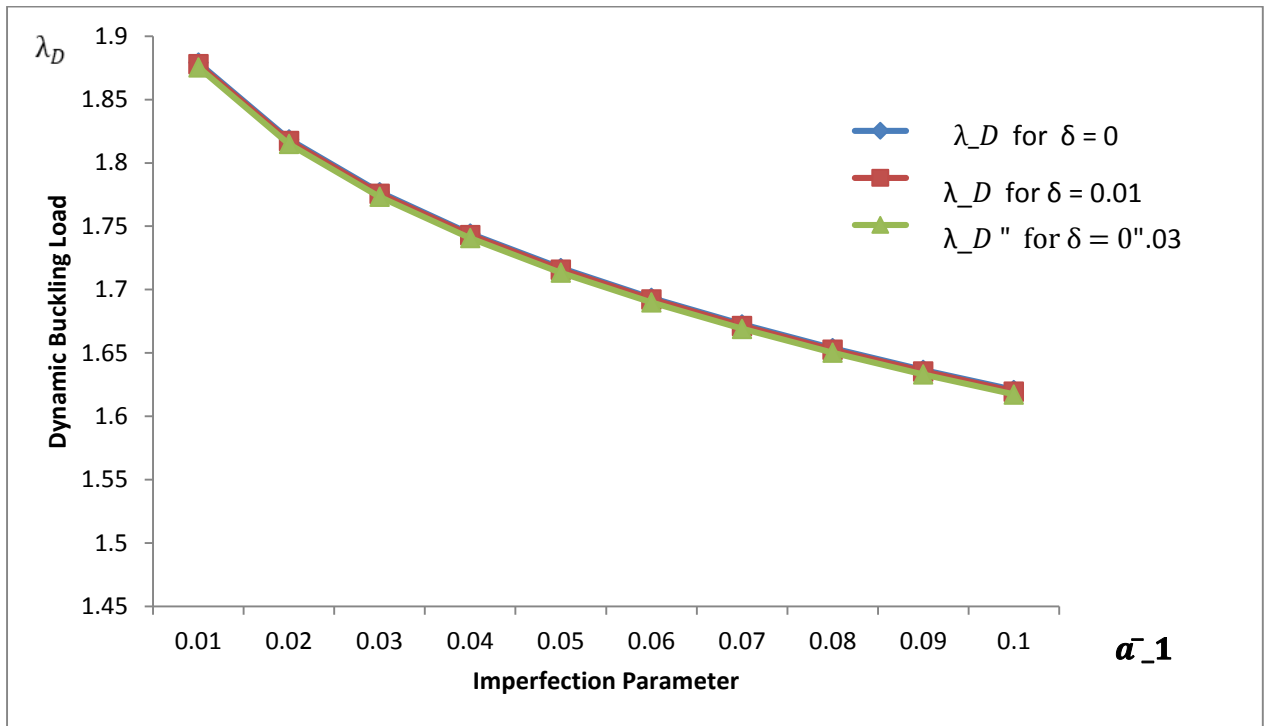
$$(16m^4 - 8\lambda_D m^2 + 1)^{\frac{3}{2}} = 27\sqrt{15} D_1 (\lambda_D \epsilon) m^2 \bar{a}_m \left[ 1 + \frac{16 \left( \frac{\alpha}{\beta^{\frac{1}{2}}} \right) D_2 D_6 D_{10}}{27\sqrt{15} D_1^2 \theta(\lambda_D)} \right]^{-1} \quad (4.158)$$

#### 4.5 RESULTS FROM ANALYTIC PROCEDURES

A simple computer simulation using Matlab software, gives the values of the dynamic buckling load,  $\lambda_D$ , at different values of  $\epsilon$  and  $\delta$  using equation (4.158).

$\bar{a}_1 \epsilon$	$\lambda_D$ for $\delta = 0$	$\lambda_D$ for $\delta = 0.01$	$\lambda_D$ for $\delta = 0.03$
<b>0.01</b>	1.87913	1.87789	1.87548
<b>0.02</b>	1.81858	1.81736	1.81496
<b>0.03</b>	1.77694	1.77571	1.77332
<b>0.04</b>	1.74427	1.74306	1.74068
<b>0.05</b>	1.71702	1.71582	1.71345
<b>0.06</b>	1.69344	1.69225	1.68989
<b>0.07</b>	1.67257	1.67138	1.66903
<b>0.08</b>	1.65376	1.65257	1.65023
<b>0.09</b>	1.63659	1.63541	1.63307
<b>0.1</b>	1.62076	1.61959	1.61726

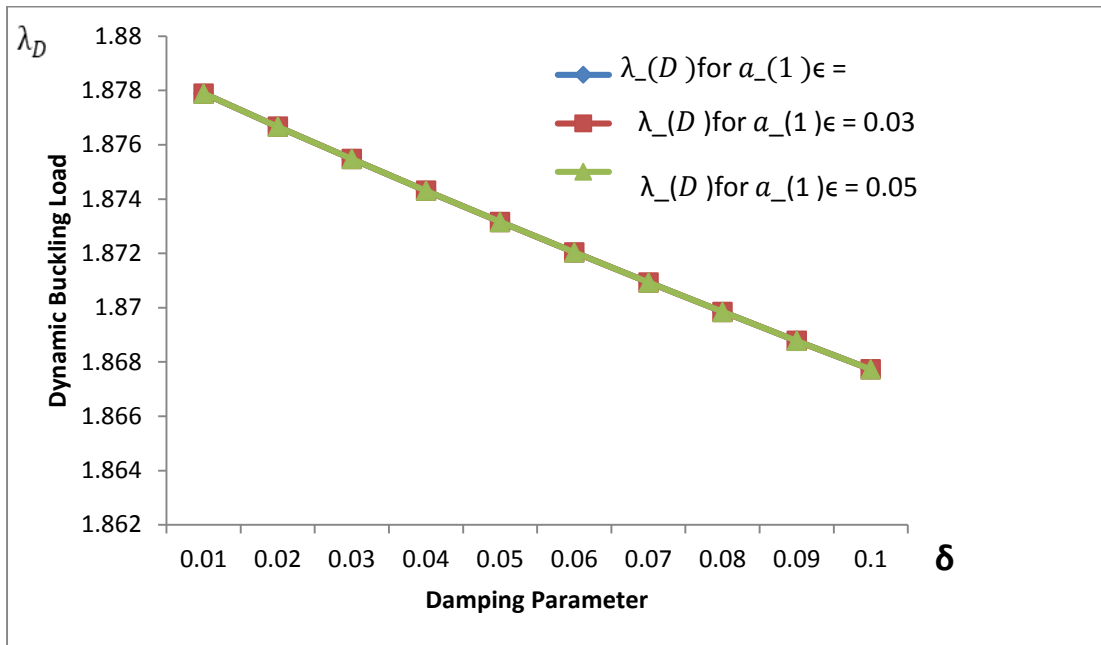
**Figure 4.1 Table of imperfection parameters  $\bar{a}_1 \epsilon$  and dynamic buckling  $\lambda_D$  loads obtained at different damping coefficients  $\delta$  using equation (4.158).**



**Figure 4.2:** The graph of dynamic buckling load  $\lambda_D$  against imperfection parameters  $\bar{\alpha}_1$ , at various values of damping  $\delta$  using equation (4.158).

$\Delta$	$\lambda_D$ for $\bar{\alpha}_1 \epsilon = 0.01$	$\lambda_D$ for $\bar{\alpha}_1 \epsilon = 0.03$	$\lambda_D$ for $\bar{\alpha}_1 \epsilon = 0.05$
<b>0.01</b>	1.87789	1.87789	1.87789
<b>0.02</b>	1.87667	1.87667	1.87667
<b>0.03</b>	1.87548	1.87548	1.87548
<b>0.04</b>	1.87431	1.87431	1.87431
<b>0.05</b>	1.87316	1.87316	1.87316
<b>0.06</b>	1.87204	1.87204	1.87204
<b>0.07</b>	1.87093	1.87093	1.87093
<b>0.08</b>	1.86985	1.86985	1.86985
<b>0.09</b>	1.86878	1.86878	1.86878
<b>0.1</b>	1.86773	1.86773	1.86773

**Figure 4.3.** Table of damping  $\delta$  with dynamic buckling loads  $\lambda_D$  at various imperfection parameters  $\bar{\alpha}_1 \epsilon$  using equation (4.158).



**Figure 4.4: Graphical plot of relationship between damping parameter  $\delta$  and dynamic buckling load  $\lambda_D$  for  $m = 1$ ,  $\alpha = 1$ ,  $\beta = 1$  and some values of  $\bar{a}_1\epsilon$  from equation (4.158).**

#### 4.6 COMPUTATIONAL PROCEDURE BY BLOCK UNIFICATION METHOD (BUM)

Assume the following boundary conditions for the governing equation.

$$u_0 = \alpha_{00}, u_N = \alpha_{0N}, u'_0 = \alpha_{10}, U'_N = \beta_{1N} \quad (4.159)$$

And the vector of unknown  $u$  is given by:

$$u = (u_1, \dots, u_{N-1}, u'_1, \dots, u'_{N-1}, u''_0, \dots, u''_N, u, \dots, u)^T \quad (4.160)$$

This makes a total of  $(N - 1) + (N - 1) + (N + 1) + (N + 1) = 4N$  unknowns variables.

On the other hand, we have two formulae in equation (3.51) which for  $n = 0(5)N - 5$  make a total of  $\frac{2N}{5}$  formulas.

Also, there are six formulae in equation (3.52) and taking  $n = 0(5)N - 5$  gives  $\frac{6N}{5}$  formulae. In equation (3.53) and (3.54) similarly, there are  $\frac{6N}{5}$  each. Therefore, the total number of equations is  $4N$ .

Hence, we have systems with  $4N$  equations and  $4N$  unknowns whose solutions provide a set of approximate values of BVP in equation (3.44).

Now, the equation (3.51), (3.52), (3.53) and (3.54) in the form of equation (3.44) to form a block and solved simultaneously using codes written in Matlab software.

Let consider equation (3.6),

$$\omega_{,tt} + 2\delta\omega_{,t} + \omega_{,xxxx} + 2\lambda f(t)\omega_{,xx} + \omega - \alpha\omega^2 - \beta\omega^3 = -2\epsilon\lambda f(t) \frac{d^2\bar{\omega}}{dx^2}, \quad t > 0, \quad 0 < X < \pi$$

Subject to the initial conditions,

$$\omega(x, 0) = 0 = \omega_{,t}(x, 0), \quad 0 < x < \pi$$

And appropriate boundary conditions,

$$\omega = \omega_{,x} = 0 \text{ at } x = 0, \pi$$

Where,

$$f(t) = 1.$$

Here, the problem on semi-discretizing the time variable, becomes

$$\frac{\omega_{m+1} - 2\omega_m + \omega_{m-1}}{(\Delta t)^2} + 2\delta \left( \frac{\omega_{m+1} - \omega_{m-1}}{\Delta t} \right) + \frac{d\omega_m^4}{dx^4} - 2\lambda f(t) \frac{d\omega_m^2}{dx^2} = f_m, \quad 0 \leq x \leq 1, \quad m = 1, \dots, M-1 \quad (4.161)$$

Where,

$$\Delta t = \frac{L_4 - L_3}{M}, \quad t_m = L_3 + m\Delta t, \quad m = 0, 1, \dots, M, \quad \omega = [\omega_1(x), \dots, \omega_M(x)]^T, \quad \omega_m(x) \approx \omega(x, t_m),$$

$$g = [g_1(x), \dots, g_m(x)]^T$$

And,

$$g_m(x) \approx g(x, t_m) = 2\lambda\epsilon f(t) \frac{d^2\omega}{dx^2} - \omega + \alpha\omega^2 + \beta\omega^3 = 0$$

This is expressed in the form:

$$\omega^{(4)} = f(x, \omega, \omega', \omega'', \omega''') = A\omega + g \quad (4.162)$$

Where, A is a  $(M-1) \times (M-1)$  matrix arising from the discretizing system, and g is a vector of constants.

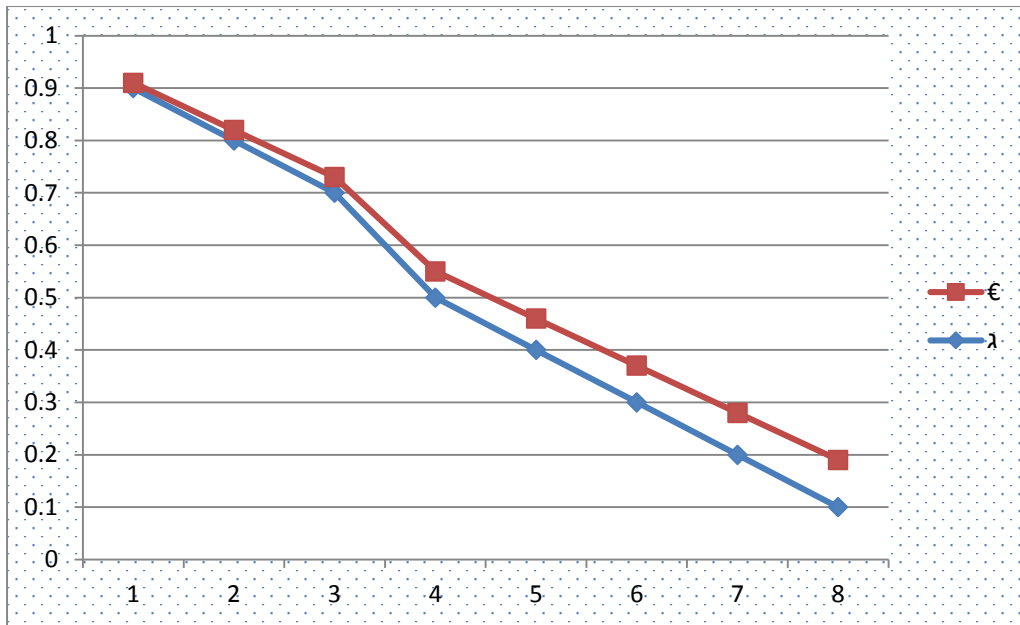
#### 4.7 NUMERICAL RESULTS

A simple computer programme, written in Matlab software, gives the values of the dynamic buckling load,  $\lambda_D$ , at different values of  $\epsilon$  and  $\delta$  using equation (3.51), (3.52), (3.53) and (3.54) in the form of equation (3.44) to form a block and solved simultaneously.

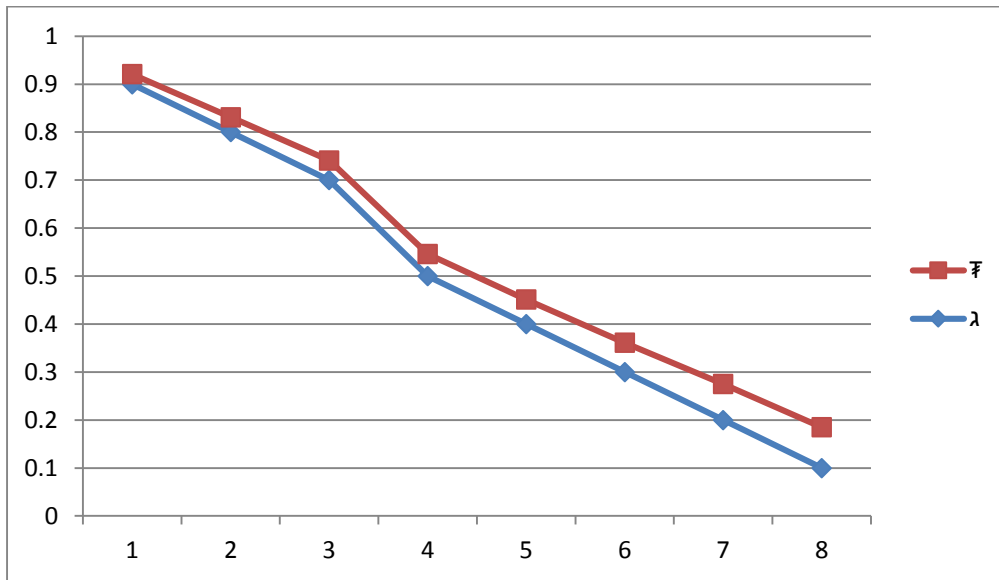
Assuming  $h = 0.05$  and  $k = 10$ .

$x$	$\lambda_D$	$\epsilon$	$\omega$	$U_a$
0.1	0.9	0.01	0.021	$4.85 \times 10^{-10}$
0.2	0.8	0.02	0.031	$8.56 \times 10^{-10}$
0.3	0.7	0.03	0.041	$1.28 \times 10^{-9}$
0.4	0.5	0.05	0.046	$1.39 \times 10^{-9}$
0.5	0.4	0.06	0.051	$10.58 \times 10^{-9}$
0.6	0.3	0.07	0.061	$1.9 \times 10^{-9}$
0.7	0.2	0.08	0.075	$2.1 \times 10^{-8}$
0.8	0.1	0.09	0.085	$2.46 \times 10^{-8}$

**Figure 4.5: The values of dynamic buckling load  $\lambda_D$  against, imperfection factor  $\omega$  and maximum displacement  $U_a$  at various values of damping  $\delta$ .**



**Figure 4.7: The graph of dynamic buckling load  $\lambda_D$  against amplitude of imperfection  $\epsilon$ , at various values of damping coefficients  $\delta$ .**



**Figure 4.8: The graph of dynamic buckling load  $\lambda_D$  against amplitude of loading  $\lambda$ , at various values of damping  $\delta$ .**

#### 4.8 DISCUSSION OF RESULTS

Equation (4.158) gives an implicit formula for determining the dynamic buckling load  $\lambda_D$ . The results of analysis show that the dynamic buckling load decreases with increased imperfection amplitude and damping. The result is also asymptotic and valid as the small parameters  $\delta$  and  $\epsilon$  become increasing small relative to unity.

The analysis of Equation (3.44) and (3.51) - (3.54) with appropriate boundary conditions enable numerical analysis on equation (3.6). This also show that the dynamic buckling load decreases with increased imperfection amplitude and damping.

#### 4.9 ANALYSIS OF THE RESULT

The analysis of the result of the nonlinear elastic model column structure trapped by a step load and lying on a quadratic-cubic foundation is hereby presented. The results show that the dynamic buckling load decreases with increased imperfection amplitude and vice-versa. This is equivalent to saying that, the nearer the structure is to a perfect nature, the more stable it is for a step load. Besides, we clearly observe that, within the limit of accuracy retained in this work, there is no marked difference in the values of  $\lambda_D$  for the different cases of  $\delta$ .

# CHAPTER FIVE

## SUMMARY, CONCLUSION AND RECOMMENDATIONS

### 5.1 SUMMARY

The dynamic buckling of a viscously damped but finite column lying on the quadratic cubic foundation that is trapped by a step load using perturbation and Block Unification Methods (BUM) have been judiciously carried out. The effect of light damping and imperfections have been highlighted and discussed.

Graphical plots help to elucidate the results and all results obtained are of course asymptotic and valid within the limit.

### 5.2 CONCLUSION

The results obtained indicate that the dynamic buckling load decreases with increased imperfection, and decreases with increased in damping. More so, it was shown that the results obtained are strictly asymptotic and valid in the limit as the small parameters become increasingly small relative to unity.

### 5.3 RECOMMENDATION

Analysis in this work can be applied in problems related to cylindrical shells, toroidal shells, plates and other engineering structures with a quadratic-cubic nonlinear elastic foundation that are trapped by step loading, impulsive loading or periodic loading.

### 5.4 CONTRIBUTION TO KNOWLEDGE

We analytically and numerically solved the governing equation (nonlinear fourth order partial differential equations), which in most cases are usually approached by numerical analysis using Finite Element Method. Thus, similar nonlinear problems can now be solved through perturbation and Block Unification Methods.

We have improved upon perturbation problems generally by being able to devise a perturbation scheme that solves a two-small parameter problem in a dynamic system.

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