CONTROLLABILITY OF IMPULSIVE DIFFERENTIAL SYSTEMS WITH PERTURBATION

BY

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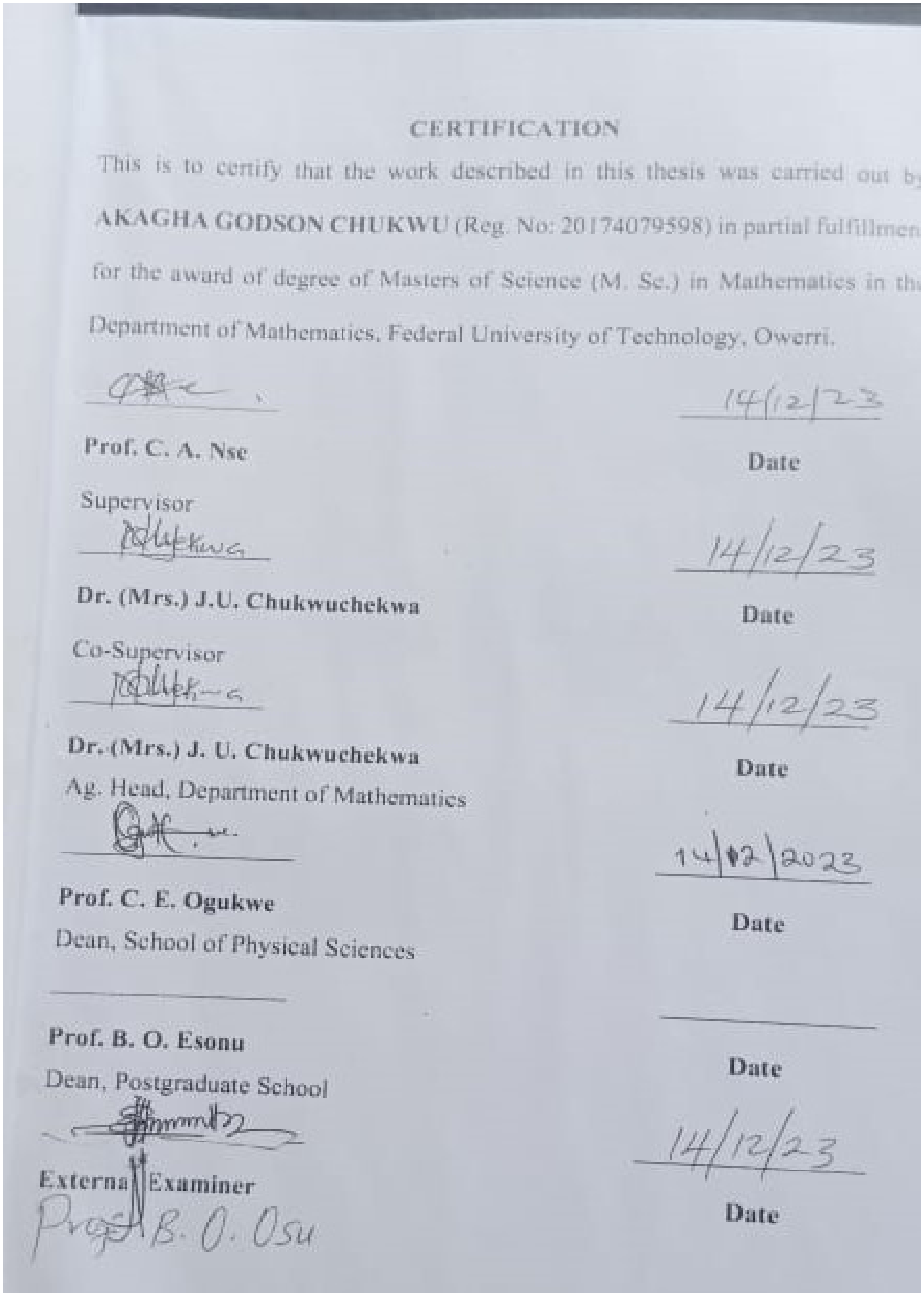

CERTIFICATION

This is to certify that the work described in this thesis was carried out b **AKAGHA GODSON CHUKWU** (Reg. No: 20174079598) in partial fulfillmen for the award of degree of Masters of Science (M. Sc.) in Mathematics in th Department of Mathematics, Federal University of Technology, Owerri.

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DEDICATION

I dedicate this work to my Almighty God who bestowed sufficient grace upon me to persevere to the end of this hurdle with shouts of Hallelujah Hosanna in the highest.

ACKNOWLEDGEMENT

I express my gratitude to my Almighty father for making it possible for me to complete this course. If not Him, I would have been of all men most miserable.

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ABSTRACT

This study is on controllability of impulsive differential systems with perturbation. This work tries to assess the various phenomena and specific properties of impulsive equations, especially problems related to (i) controllability of the solution after perturbation on the impulsive moments (ii) stability to solution of differential equation with impulsive effects (iii) modelling and investigating the properties of dynamic systems with impulsive effects which sharply changes its states from $x_0$to a desired state x_1. It was observed that the admissible control $u(t)$ adopted in this work is appropriate and rightly tailored to transform a given system from an initial state to a final state and at finite time. However, inadequacies are apparent especially control of impulsive differential system with perturbation clearly shows that their solutions have different behaviours from the solution of non-impulsive control differential system with perturbation. This inadequacies have led to the emergence of application of the admissible control $u(t)$ so as to achieve the desired result. Moreover, recommendations were made because of the dynamic nature of the system. However, we recommend that the work is open for further research especially to find out in particular whether controllability of impulsive system with perturbation will be naturally controllable (i.e. for non-autonomous impulsive optimal control system). Another area of further research may include impulsive differential system with variable moment of impulses.

Keywords: Control System, Perturbation, Differential Equation, Impulsive Effects, Dynamic System

CHAPTER ONE

INTRODUCTION

1.1 BACKGROUND OF THE STUDY

Suppose a hammer is used to strike a nail then the hammer will be in contact with the nail for a very short period of time, indeed almost instantaneously. A similar situation arises when a golfer strikes a golf ball. In both cases, the force applied during this short period of time builds up rapidly to a large value and then rapidly decreases to zero.

Such short sharp forces are known as impulsive forces and are of great interest in many engineering applications such as mechanics, population dynamics, optimal control, ecology, chemistry, physics, chemotherapy and so on.

Consequently the area of impulsive differential equation has been developing at a rapid rate, despite being a new study area and has gained wide recognition from various authors (Benchohra *et al.* 2012; Wright 2018).

Impulsive equation can be fixed moment of impulse, which is impulse occurring at a fixed time or variable moment of impulse, which is impulses occurring at a random time.

One major difference between impulsive differential equation at a fixed time and at variable times is the apparition of pulse phenomenon, that is, the solution of the differential equation may hit a given beating surface several times (Bajo, 1997).

Controllability deals with the question that if given a system of differential equation, we can get a control that will steer the system from an initial state x_0 to a final state $x_1$and in finite time (Nieto and Tisdel, 2010).

Mathematical control theory is an area of application – oriented mathematics that deals with basic principles underlining the analysis and design of control system. To control an object means to influence its behavior so as to achieve a desired result.

In order to implement this influence, engineers build devices that incorporated various mathematical techniques. These devices range from watt's steam engine, designed during England's Industrial Revolution to the sophisticated microprocessor controller found in consumer items such as CD player and automobiles, in industrial robots and airplane autopilots (Sofag, 1990).

Perturbation in this work is a term that models the disturbances on a given system as a result of the impulsive effect experienced by the system.

This disturbance changes the output of the system from linear system to a non-linear system i.e. the disturbance that causes impulsive effect will reduce the functional efficiency and effectiveness of the system. For brevity sake, we shall denote perturbation as $f(t, u(t))$.

An impulsive system is a special kind of system consisting of differential system and difference system respectively that describe continuous evolutions and discrete events occurring in mathematical models of physical system (Boukhamla, 2018).

Problems involving differential equation with impulses that needs to be investigated upon can be categorized into several groups which includes;

(a) The transfer of classical results from the theory of equations without impulsive effects on the equation with impulses.

(b) Studying the various phenomena and specific properties of impulsive equations, especially problems related to;

    (i) Controllability of the solution after perturbation on the impulsive moments.

    (ii) Stability to solution of differential equations with impulsive effect.

    (iii) Modelling and investigating the properties of dynamic systems with impulsive effect which sharply changes its states.

    (iv) Solving optimization problems using impulsive equations to obtain the maximum yield of populations in limited environment (Dishliev *et al.* 2012).

1.2 AIM AND OBJECTIVES OF THE STUDY

The aim of the work is to formulate and prove sufficient conditions for the controllability of impulsive differential systems with perturbation.

The objectives of this work are to:

i. Obtain a solution of an impulsive control system with perturbation, and

ii. Use an admissible control u= $u(t)$ to transform a given impulsive control differential systems from an initial state of x_0 to a final state $x_1$at a finite time.

1.3 SIGNIFICANCE OF THE STUDY

Topics involving controllability of impulsive differential equation is a new innovation in mathematical research; in controllability of impulsive differential systems with perturbation we choose an arbitrary control u= $u(t)$ that transforms a given system from an initial state to a final state and at a finite time. This work also provides a method of solving impulsive differential systems with perturbation, which can further be extended in the areas of optimal control problems.

1.4 SCOPE OF THE STUDY

This work investigates perturbations that may occur at a given time $t = t_k$called the time points of moment of impact, the impulse is deemed to be instantaneous.

To achieve our objective in this work, we review some existing literatures.

1.5 DEFINITION OF TERMS

CONTROLLABILITY

A system is said to be controllable on $J = [0,T]$, if given any initial state x_0, there exist a continuous control $u(t)$that steers the initial state $x_0$to desirable final state x_1 in finite time.

IMPULSIVE DIFFERENTIAL EQUATION

Each system is given by an ordinary differential equation coupled with relations defining the jump condition. Assume that the law of evolution of the process is described by a differential equation.

$$\frac{dx}{dt} = f(t, x(t))$$

where$x \epsilon \Omega \subset \mathbb{R}^n$, $f: \mathbb{R} \text{x}\, \Omega \;\rightarrow\; \mathbb{R}^\text{n}$ are the moment of the impulse effect for the solution $x(t)$ which occurs at $t = T_K$ $(k \epsilon \mathbb{N})$

Denote $I(t, x)$: $\mathbb{R}\text{ x }\Omega \rightarrow \Omega$, where $(t, x) \rightarrow (t, x + I(t, x))$ is the mapping of the solution before the impulse, $x(T_k^-)$. Then

$x(T_k^+)$ is the impulse effect where $\Delta x T_k = x(T_k^+)$-$x(T_k^-)$ is the impulsive differential equation.

CONVOLUTION: Is a useful concept that has many applications in various fields of engineering. It is used to obtain the response of a linear system to any input in terms of the impulse response.

Given two piecewise-continuous functions $f(t)$ and $g(t)$, the convolution of $f(t)$ and $g(t)$, denoted by $(f \times g)(t)$, is defined as

$(f \times g)(t) = \int_{-\infty}^{\infty} f(\tau) g(t - \tau) d\tau$

(1.1)

In the particular case when $f(t)$ and $g(t)$ are casual functions:

$$f(\tau) = g(\tau) = 0 \ (\tau < 0). \, g(t - \tau) = 0 \ (\tau > t)$$

and we have

$$(f \times g)(t) = \int_{-\infty}^{\infty} f(\tau) g(t - \tau) d\tau \qquad (1.2)$$

The notation $f \times g$ indicates that the convolution $f \times g$ is a function of t; that is, it could also be written as $(f \times g)$(t). The integral $\int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau$ is called the convolution integral. We also observe that commutative law is satisfied in convolution so that $(f \times g)(t) = (g \times f)(t)$or for casual functions,

$$\int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau = \int_{-\infty}^{\infty} f(t-\tau)g(\tau)d\tau \qquad (1.3)$$

CLASSES OF IMPULSIVE DIFFERENTIAL EQUATIONS

There are three classes of impulsive differential equations:

Class 1: Equations with fixed moments of the impulsive effect.
For $k \epsilon \mathbb{N}$;

$$\left.\begin{array}{ll} \frac{dx}{dt} = f\big(t, x(t)\big) & t \neq T_k \\ \Delta x = I_k\big(x(t)\big) & t = T_k \end{array}\right\} \qquad (1.4)$$

The impulse is fixed beforehand by defining the sequence T_K : $T_K < T_{K+1} (k \epsilon\ \mathbb{Z})$. For $t \epsilon\ (T_K, T_{K+1})$the solution $x(t)$of equation (1.1) satisfies the equation $\frac{dx}{dt} = f(t,x)$, and for $t = T_K, x(t)$satisfies the relation $x(T_k^+) = x(T_k^-) + I_K(x(T_k^-))$.

Class 2: Equation with state-depended moments of the impulsive effect;

$$\left.\begin{array}{ll} \frac{dx}{dt} = f(f, x) & t \neq T_K(x) \\ \Delta x = I_K(x) & t = T_K(x) \end{array}\right\} \qquad (1.5)$$

where $\tau_K : \Omega \rightarrow \mathbb{R}$ and $\tau_K < \tau_{K+1}$ ($k \epsilon \mathbb{Z}, \epsilon \Omega$). The impulse occurs when the mapping point (t, x) meets some hypersurface σ_k of the equation $t = \tau_K(x)$.

Consider

$$\begin{cases} \frac{dx}{dt} = f(t,x) & t \neq \tau_K(x) \\ \Delta x = x^2 sgn(x) - x & t = \tau_K(x) \end{cases} \qquad (1.6)$$

where $t \geq 0$, $x \epsilon \mathbb{R}$and $\tau_K(x) = x + 6k$for $|x| < 3(k = 0,1,2,3 \dots)$

Starting at appoint $(0, x_0)$ with $|x| \geq 3$, there is no impulsive effect since the integral curve does not intersect the hypersurface σ.

Class 3: Autonomous impulsive equations

$$\left.\begin{cases} \frac{dx}{dt} = f(x) & x \notin \sigma \\ \Delta x = I_k(x) & x \epsilon \sigma, \end{cases}\right\} \qquad (1.7)$$

where σ is an $(n-1)$ – dimensional manifold contained in the phase space $\Omega \subset \mathbb{R}^n$. The impulse occurs when the solution $x(t)$ meets the manifold σ.

TIME INVARIANT LINEAR SYSTEM

Consider the plot of the input t and the corresponding response (output) of an arbitrary continuous system.

If the response pattern is retained but shifted wholesale through t_0 when the input is similarly shifted through t_0, then the system is said to be time invariant. In other words, it does not matter when we activate the system, we always obtain the same response for the same input, the response will be the same at any point in time.

That is, a system is said to be time-invariant if:$y(t) = L\{x(t)\}$ and $y(t \pm t_0) = L\{x(t \pm t_0)\}$ where t_0 is a constant.

To see this analysis more clearly, consider the output in two different ways. Firstly consider it as the response of the system acting on the input $y_1(t) = L\{x(t)\}$and secondly consider it as a dependent variable equated to an expression involving the input as an independent variable, that is $y_2(t) = L\{x(t)\}$ where $y_1(t) = y_2(t)$

The time delay through to then results in $y_1(t - t_0) = L\{x(t - t_0)\}$ and $y_2(t - t_0) = L\{x(t - t_0)\}$. If the delay response is the same as a delayed version of the original response, that is $y_1(t - t_0) = y_2(t - t_0)$, the system L is time-invariant but if $y_1(t - t_0) \neq y_2(t - t_0)$,, the system L is not time-invariant.

CHAPTER TWO

LITERATURE REVIEW

We review some woks done by some authors who have made immense contribution to the study of impulsive differential systems.

Lakshmikantam *et al.* (1989) in a book titled "Theory of impulsive differential equation" considers the following impulsive control system.

$$\left.\begin{aligned} &x'(t) = Ax + Bu \\ &x(t_k^+) = I + D^k\, u(t_k),\ k = 1,2,3,\dots,m \\ &x(t_0) = x_0 \end{aligned}\right\} \qquad (2.1)$$

A and B are $n \times n$ and $n \times r$ matrices respectively, also $0 = t_0 < t_1 < t_2 < t_3, \dots\dots. < t_{p-1} < t_p < t_{p+1} = t_p$ are the interval in which the impulse may occur, for each $k = 1,2,\dots\dots,m$.

$D^k u(t_k)$ is an $n \times n$ diagonal matrix, I is an identity matrix on $\mathbb{R}^n$. But at a fixed t, and with a control$u = u(t)$, the system was transformed from an initial state of $x(0) = x_0$to a desired final state of x_1 at a finite time, and gave the solution to the system (2.1) as

$$x(t) = e^{AT} \prod_{0<t_k<t} [I + D^k u(t_k)] x_0 \quad + \int_0^T \prod_{s<t_k<1} [I + D^k u(t_k)] e^{A(t-s)} ds \qquad (2.2)$$

for $0 \leq t \leq T$and where $\prod_{s<t_k<1}[I + D^k u(t_k)]$models the impulsive regions of the solution and e^{AT} is a fundamental matrix of the homogeneous part of the system (2.1).

Leela et al (1993) also considered the system (2.1) and studied of impulsive control relative to impulsive differential equations without perturbation and obtained a result to demonstrate the importance of employing impulsive controls to the system (2.1). They examined the condition for controllability of the system (2.1) and without loss of generality, it was assumed that the trial state of the system (2.1) is the origin in $\mathbb{R}^n$. The control $u = u(t)$ is said to be an impulsive control if at $t = t_0$, the impulses are regulated and the rest of the given domain of definition u= $u(t)$ is arbitrarily chosen. The solution to the system (2.1) is given by (2.2). Suppose that the final time of T > 0, then the controllability of the system (2.1) implies that

$$x(t) = e^{AT} \prod_{0<t_k<t} [I + D^k u(t_k)] x_0 + \int_0^T \prod_{s<t_k<1} [I + D^k u(t_k)] e^{A(t-s)} B u(s)\, ds \tag{2.3}$$

George et al (2000) investigated the controllability of a system having a transfer function $G(s)$ with the response $x(t)$.

Initially the quiescent state of the system to an input $u(t)$ is determined by the transformed relationship: $X(t) = G(s)u(t)$.

They observed that if the input $u(t)$ is taken to be the unit impulse function $\delta(t)$, then the system response will be determined by

$$X(t) = G(s)L[\delta(t)] = G(s) \tag{2.4}$$

Taking inverse laplace transform leads to the corresponding time response $h(t)$, which is called the impulse response of the system.

They concluded that since the impulse response is the inverse Laplace transform of the transfer function, it follows that both the impulse response and transfer function carry the same information about the dynamics of the controllability of linear system of equation.

Theoretically therefore, it is possible to determine the complete information about the system by exciting it with an impulse and measuring the response. For this reason, it is common practice in engineering to regard the transfer function as being the Laplace transform of the impulse, since this places greater emphasis on the parameter of the system when considering system control design.

Bajo. I. (1997) Studied Pulse Accumulation in impulse differential equation with variable times. He observed that in transient and steady state response analysis, one part of the response known as the transient (corresponding to the complementary function), which reduces to zero as time increase, the other part, known as steady state (corresponding to the particular integral), is the response of the system when time tends to infinity, accumulates at long-run.

Benchora et al. (2006) studied impulsive Differential equation and included contemporary mathematics and its application. They applied blood-flow model to illustrate Navier-stokes equation for viscous flow.

Dishev et al. (2012) they studied specific Asymptotic properties of solution of impulsive differential equation methods and application using Lyapunov stability analysis.

Marvel (2017) studied in paper titled non – resonance impulsive equation. He used first order method to solve periodic problems which yielded resonance impulsive differential system.

Liing and Shen (2019) studied periodic boundary value problem for the first order differential equation with impulse and established by fixed point theorem that the solution is unique.

Jovita and Lauretta (2020) studied impulsive differential equation. They examined the condition for controllability without perturbation. Some conditions were derived which showed the existence and solution from impulsive control system which led to advances to difference equation.

Snesham (2019) studied qualitative investigation and approximation method for impulsive equation. He observed that obtaining a numerical solution of first order ordinary differentialequations using Euler's method, he was able to define direction field that is contained in impulsive equation.

Nieto (1997) studied the existence of the solution for a non-linear ordinary differential equation of first order with periodic boundary values conditions that is subject to impulsive actions at a fixed moment. He developed basic result for the linear situation and then extended the result when the impulses are absent. He developed equations and presented a periodic boundary value problem for first order differential equation with impulses effects. He then considered the system of the form

$$\left.\begin{aligned}
&\dot{U}(t) + \lambda u(t) + t\big(t, u(t)\big), t = t, \quad t \in J = [0, T], \quad j = 1,2, \ldots p \\
&U(0) = U(T) \\
&U(t^{+},) = U(t^{-},) + I_j\Big(U\big(t_j\big)\Big), j = 1, 2, \ldots p
\end{aligned}\right\} \qquad (2.6)$$

When $\lambda \in \mathbb{R}, J = [0, T], T > 0, 0 = t_0 < t_1 < t_2 < t_3 < \cdots . < t_{p-1} < t_p = T,$

$$I_j \in \mathrm{C}(\mathbb{R}, \mathbb{R}), \mathrm{j} = 1, \ldots \mathrm{p}$$

$f: J \times \mathbb{R} \rightarrow \mathbb{R}$is such that f is continuous at every point. He obtained a result for a case when the impulse are present and also when the impulses are absent and made a remark that these two results have a different behavior to dynamic system and noted that when impulses are absent, the system in (2.6) is a non-resonance problem since the linear part of the system (2.6) is invertible.

Nieto and Tisdell (2010) studied the exact controllability of the impulsive system and arbitrarily chosen admissible control u= $u(t)$ and showed that the impulsive system was controllable. It was also noted that for a finite dimensional system such as (2.6), the system is controllable.

Zanda K. M (2009) gave the condition for uniqueness of the solution to the system (2.1). It was noted that in the case of impulses on a given sets for a variable moment, the moments of impulses are known initially. The impulse depends not only on the right part of the system (2.1), but also on the initial condition $x(t_0)$ that defined the set of impulses.

Hinpang et al (2016) studied the existence of a periodic solution for impulsive control system with perturbation of infinite dimensional system but the system itself is not controllable.

Fang and Sun (2012) studied the existence and uniqueness of solution to complex valued non-linear impulsive differential system and established by fixed point theorem that the solution is unique. They also used the successive approximation method in obtaining the solution of the complex value non-linear impulsive differential system.

Bervabali (2014), in a paper titled “contribution to impulsive equations” used method of upper and lower solution to obtain an existence result for first order impulsive differential equation with variable moments.

Boukhamla (2018) studied controllability of differential equations under impulsive conditions and applied Hilbert uniqueness method to obtain the impulsive control in the case when the initial state space is a Hilbert space. He noted that the problem of controllability is not all that simple and in general case there is no universal method to get it explicitly.

On the other hand, he obtained necessary and sufficient conditions for the null controllability of the impulsive system. The Hilbert uniqueness method is a technique which exists in the continuous case for finding a control that steers the solution to a given final state.

Wei and Mi (2015) studied periodic boundary value problem for the first order differential equation with impulse. Consider the system of the form.

$$\left.\begin{array}{l}
u(t) = g\left(t, u(t), u\left(\theta(t)\right)\right), t\neq \mathrm{t_j}\ , t\epsilon J = [0, T],\ \ j = 1,2, \dots . p \\
\Delta \mathrm{u}(t) = \mathrm{I_k}\ \left(u\left(\mathrm{t}_k\ \right)\right),\ \ k = 1,2, \dots p, \\
\text{Where } \mathrm{g} \in (J\mathrm{x}\ \mathbb{R}^2, \mathbb{R}), 0 \leq t, t \in \mathrm{J}, \\
\mathrm{t_0}\ < \mathrm{t_1}\ < \mathrm{t_2}\ < \mathrm{t_3}\ < \cdots . < \mathrm{t_{p+1}} < t_p\ = T, \\
\mathrm{I_k}\ \subset (\mathbb{R}, \mathbb{R})
\end{array}\right\} \qquad (2.7)$$

$$\Delta \mathrm{u}(\mathrm{t_k}\ ) = u(\mathrm{t_k^-}), k = 1,2, \dots . . p$$

By using monotone interactive method they obtained the lower and upper solution to the system (2.7).

Liang and Shan (2019) studied the existence and approximation of solutions of the system (2.7). A new comparison result was presented and the result obtained by Wei and Mi (2015) was extended. The Banach fixed point theorem was used to prove for the existence and uniqueness of the system (2.7).

The monotone interactive method and the method of upper and lower solutions were used to obtain the solution to the system (2.7).

Lie et al (2015) studied impulsive differential equations; periodic solution and application, by using Lyapunov stability method and contraction mapping principle.

Some condition ensuring the existence and solution were derived, which are given from impulsive control and impulsive perturbation point view.

Randelovic et al (2020) discussed numerical solution of differential equation. They considered an impulsive differential equation of the form;

$$\left.\begin{array}{l}\frac{dy}{dx} = f(t,x),\ (t \neq t_x) \\ \Delta U_k(x) = 0 = I_k\big(x\Delta U_k(x)\big) = 0 \\ x_0 = x(t_0)\end{array}\right\} \quad (2.8)$$

called stoke impulsive differential equation (SIDE), and developed an algorithm based on well-known numerical methods, known as algorithm for solving impulsive differential equations (ASIDE). They called their method Algorithm for solving stoke-impulsive differential equation (ASIDE).

With the above literature, we now extend the work done by Nieto Xavi (1997) and include a non-linear function $f(t, x(t))$ to the system (2.6).

This function models the perturbation which is a disturbance caused by the impulse at a fixed moment. We consider a necessary condition for the perturbed impulsive differential system to be controllable.

CHAPTER THREE

METHODOLOGY

In this chapter, we consider impulsive control system with perturbation.

For this case, we choose a control $u(t)$ that steers the perturbed system from an initial state x_0 to a final state $x_1$in finite time.

3.1 CONTROLLABILITY WITH IMPULSIVE DIFFERENTIAL EQUATION

Consider, the following control impulsive system with perturbation given as follows;

$$\left.\begin{array}{l} x(t) = Ax(t) + Bu(t) + f\big(t, x(t)\big), t \neq t_k, t \in \mathrm{J} = [t_k, T] \\ x(t_k^+) = x(t_k) + I_k\big(x(t_k)\big), k = 1, 2, 3, \\ x(t_0) = x_0 \end{array}\right\} \quad (3.1)$$

where A is an $n \times n$ real matrix, $f: \mathrm{J} \mathrm{x} \mathbb{R}^n \rightarrow \mathbb{R}^n$is a non-linear continuous function, B is an operator defined on a set of admissible control $u$and $I_k: \mathbb{R}^n \rightarrow \mathbb{R}^n, k = 1, 2, \ldots. m$, is called impulse.

Let $J = [0, T] \subset \mathbb{R}, T > 0$ and $0 = t_0 < t_1 < t_2 < t_3 < \cdots . < t_{m-1} < t_m < t_{m-1} = T$ be the time points at which the impulse occurred.

$x\left(t_k^+\right)$ and $x\left(t_k^-\right)$represents the right limit and the left limit respectively of the state at$t = t_k$, as usual, $x\left(t_k^+\right) = \lim_{t \to t^+} x(t)$. In this work, we assume the moment to be fixed.

3.2 SOLUTION TO IMPULSIVE DIFFERENTIAL SYSTEMS WITH PERTURBATION

To find the solution of the impulsive differential system given by:

$$\left.\begin{aligned} &x(t) = Ax(t) + Bu(t) + f\big(t, x(t)\big), t_k \neq t_{k,} t \in \mathrm{J} = [0, T] \\ &x(t^+) = x\big(t_k^-\big) + I_k\big(x(t_k)\big), k = 1, 2, \ldots \ldots m \\ &x(t_0) = x_0 \end{aligned}\right\} \qquad (3.2)$$

We adopt the method of successive approximation to arrive at our solution

LEMMA 3.1

If a linear time invariant system has impulse response sequence $\{y\delta x\}$, and we wish to find the system response $\{y_n\}$ to an input sequence $\{U_n\}$, with the system initially quiescent state.

$$\{U_n\} = \{U_0, U_1, U_2, \ldots U_n\}$$

as

$$\{U_k\} \quad = U_0\{\delta_k\} + U_1\{\delta_{k-1}\} + U_2\{\delta_{k-2}\} + \cdots U_n\{\delta_{k-n}\} \qquad (3.3)$$

where

$\delta_{k-j} = \begin{cases} 0 & (k \neq \mathrm{j}) \\ 1 & (k = j) \end{cases}$ is the required solution

Proof:

Let $\{y_k\} = \sum_{j=0}^{h} U_j\{y\delta_{k-j}\}$ (3.4)

as the response of the system to the input sequence $\{U_k\}$ satisfying (3.3)

By method of successive approximation, we have

$$\left.\begin{aligned}\{y_k\} &= U_0\{y\delta_k\} + U_1\{y\delta_{k-1}\} + \cdots + U_j\{y\delta_{k-j}\}\\ &= U_0\{y\delta_0, y\delta_{1,} y\delta_2, \ldots y\delta_h \ldots.\}\\ &+U_1\{0, y\delta_0, \ldots y\delta_{h-1,} \ldots.\}\\ &+U_2\{0, 0, y\delta_0, \ldots y\delta_{h-2,} \ldots.\}\\ &.\\ &.\\ &+U_h\{0, 0, 0, \ldots 0, y\delta_0, \ldots y\delta_{1,} \ldots.\}\\ &+\cdots \ldots\ldots\ldots\ldots\ldots\ldots\ldots. \text{hth } position\end{aligned}\right\}$$

From the expansion, we find that the hthterm of the output sequence is determined by

$$y_h = \{\textstyle\sum_{j=0}^{h} U_j y\delta_{k-j}\} \tag{3.5}$$

That is

$$\{y_k\} = \left\{\textstyle\sum_{j=0}^{h} U_j y\delta_{k-j}\right\} \tag{3.6}$$

The expansion is called convolution sum.

THEOREM 3.1

[Glyn, (2019)] If$f(t)$ and g(t) are of exponential order σ, piecewise continuous on $t \geq 0$ and have Laplace transforms $F(s)$ and $G(s)$ respectively, then, for $s > 0$

$$L\left\{\int_0^t 6(t)g(t-\tau)\mathrm{d}\tau\right\} = L\{fg(t)\} = F(s)G(s) \tag{3.7}$$

Proof:

Let $F(s)G(s) = L\{f(t)\}L\{g(t)\} = \left[\int_0^\infty e^{-sx}f(x)dx\right]\left[\int_0^\infty e^{-sy}g(y)dy\right]$ where we have used the dummy variables x and y, rather that t, in the integration to avoid confusion. This may now be expressed in the form of double integral

$$F(s)G(s) = \int_0^\infty \int_0^\infty e^{-s(x+y)} f(x)g(y)dxdy = \iint e^{-s(x+y)} f(x)g(y)dxdy$$

Where R is the first quadrant in the (x, y) plane as shown below. On making the substitution

$$x + y = t, y = \tau$$

The double integral is transformed into

$$F(s)G(s) = \iint e^{-s}\ f(t-\tau)g(\tau)dtd\tau$$

Where $R_1$is the semi-infinite region in the (τ, t) plane bounded by the lines $\tau = 0$ and $\tau = t$, a shown below. This may be written as $F(s)G(s) = \int_0^\infty e^{-s} \left[\int_0^t f(t-\tau)g(\tau)\right] dt = \int_0^\infty e^{-s}\ [(g \times f)(t)dt =$ $L\{(g \times f)(t)\}\ \ ]$

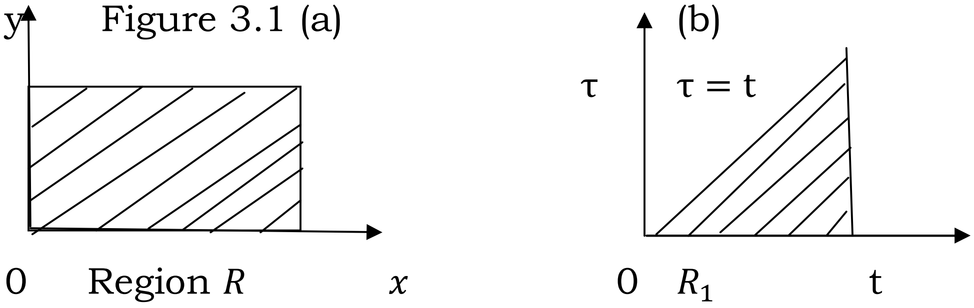
Figure 3.1 (a)
(b)

CHAPTER FOUR

ANALYSIS OF RESULTS AND APPLICATION

4.1 CONTROLLABILITY OF IMPULSIVE DIFFERENTIAL SYSTEMS WITH PERTURBATION

At this point, we choose our control to be

$$u(t) = \frac{1}{T}e^{A(T-t)}\left[x_1 - e^{At}x_0 - \int_0^t e^{(T-s)}f(s,x(s))\right]ds \tag{4.1}$$

Now we define the following operations

Let $\psi\colon \mathrm{p} \subset [O,T],\mathbb{R}^n \mathrm{x} p \subset [O,T],\mathbb{R}^n$be an operator defined by:

$$[\psi(x,\mathrm{u})](t) = e^{At}x_0 + \int_0^t e^{A(t-s)}[u(s) + f(s,x(t))]ds$$
$$\sum_{k=1}^{m} e^{A(T-t_k)}I_k(x(t_k)) \tag{4.2}$$

and $\mu\colon p \subset [O,T],\mathbb{R}^n \to \mathrm{p} \subset [O,T],\mathbb{R}^n$ be an operator that defines the right hand side of (4.1)

$$[\mu(x)](t)\colon \frac{1}{T}e^{A(T-t)}\left[x_1 - e^{At}x_0 - \int_0^t e^{(T-s)}f(s,x(s))\right]ds \tag{4.3}$$
$$\sum_{k=1}^{m} e^{A(T-t_k)}I_k(x(t_k))$$

Let ω be an operator such that

$$\omega = \psi(\mathrm{x},\mathrm{u})\colon \mathrm{p} \subset C[O,T] \to \mathrm{p} \subset C[O,T]$$

has a fixed point for any initial condition x_0 and a final condition x_1

Substituting (4.1) into (4.2) and integrating over the interval [0.T].

$$[\omega(\mathrm{x})](T) = [\psi(\mathrm{x}),u(x)](T) - e^{At}x_0$$
$$+ \int_0^t e^{A(t-s)}[u(s) + f(s,x(t))]ds + \sum_{k=1}^{m} e^{A(T-t_k)}I_k(x(t_k))$$
$$= e^{At}x_0 + \int_0^t e^{A(T-S)}[u(s)(T) + f(s,x(t))]ds + \sum_{k=1}^{m} e^{A(T-t_k)}I_k(x(t_k))$$

$$= e^{At}x_0 + \int_0^t e^{A(t-s)}\left[\frac{1}{T}e^{-A(t-s)}\right]x_1 - e^{At}x_0 - \int_0^T e^{A(T-S)}f\big(s,x(s)\big)ds$$

$$-\sum_{k=1}^{m} e^{A(T-t_k)}I_k(x(\,t_k)) + f\big(s,x(s)\big)ds + \sum_{k=1}^{m} e^{A(T-t_k)}I_k(x(\,t_k))$$

$$= e^{At}x_0 + \frac{x_1}{T}\int_0^T ds - \frac{e^{At}x_0}{T}\int_0^T ds$$

$$-\frac{1}{T}\int_0^T \big[e^{A(T-\tau)}f\big(\tau,u(t)\big)d\tau\big]ds$$

$$-\frac{1}{T}\sum_{k=1}^{m} e^{A(T-t_k)}I_k(x(\,t_k))\int_0^T ds + \left[\int_0^T e^{A(T-\tau)}f\big(\tau,u(t)\big)d\tau\right]ds$$

$$-\frac{1}{T}\sum_{k=1}^{m} e^{A(T-t_k)}I_k(x(\,t_k))\int_0^T ds + \left[\int_0^T e^{A(T-S)}f(s,x(s0)ds\right]$$

$$+\sum_{k=1}^{m} e^{A(T-t_k)}I_k(x(\,t_k))$$

Let $[\int_0^T e^{A(t-s)}f(s,x(s))\,ds] =$ m, then we have

$e^{At}\,x_0 + \frac{x_1}{T}\,.T - e^{AT}x_0\,.\,T - \frac{1}{T}$ M.T

$-\frac{1}{T}\sum_{k=1}^{m} e^{A(T-t_k)}I_k\,(x,(t_k))\,.T + M + \sum_{k=1}^{m} e^{A(T-t_k)}I_k$

$(x(t_k)) = x_1$

This shows that with the control of (4.1) the system (4.2) is controllable.

We now consider a two dimensional system with

k = 1, $J = [0,T]$,

$\dot{x}(t)\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$, A = $\begin{pmatrix} -1 & 2 \\ -3 & 4 \end{pmatrix}$, B = $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$x_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $x_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, f = $f(t,x(t) = \begin{pmatrix} f_1,(t,x(t)) \\ f_2(t,x(t)) \end{pmatrix}$

We now present a non-impulsive control differential equation with perturbation of the form

$$\left.\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ -3 & 4 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} u\right] (\mathrm{t}) + \begin{pmatrix} f_1, (t, x(t)) \\ f_2(t, x(t)) \end{pmatrix} \quad \right\} \qquad (4.4)$$

$$\left.\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ -3 & 4 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} + \begin{pmatrix} f_1, (t, x(t)) \\ f_2(t, x(t)) \end{pmatrix} \quad \right\} \qquad (4.5)$$

Since |Q| = Q then the rank of the 2 x 2 Matrix is 1

Therefore, Rank Q = 1 < 2

We conclude that the given system (4.4) is not controllable.

For the system (4.4) to be controllable, we first find the solution to (4.4).

The characteristics equation of the system (4.4) is

$|\mathrm{A} - \mathrm{I}\lambda| = 0$

$\Longrightarrow \left|\begin{pmatrix} -1 & 2 \\ -3 & 4 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right| = \begin{vmatrix} -1-\lambda & 2 \\ -3 & 4-\lambda \end{vmatrix}$

$\Longrightarrow (1+\lambda)\ (4-\lambda) + 6 = 0$

$\Longrightarrow + 4\lambda - \lambda^2 + 6 = 0$

$\Longrightarrow \lambda^2 - 3\lambda + 10 = 0$

$\Longrightarrow (-1-\lambda)(4 - \lambda) + 6 = 0$

$\Longrightarrow -4 + \lambda - 4\lambda + \lambda^2 + 6 = 0$

$\Longrightarrow \lambda^2 - 3\lambda + 2 = 0$

For $\lambda_1 = 1$, we have

$\begin{pmatrix} -2 & 2 \\ -3 & 3 \end{pmatrix} \begin{pmatrix} \xi_1^0 \\ \xi_2^0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ i.e $\begin{pmatrix} -1-1 & 2 \\ -3 & 4-1 \end{pmatrix}$

$\Longrightarrow \xi_1^0 = \xi_2^0$ $\quad\quad$ $\begin{pmatrix} -2 & 2 \\ -3 & 3 \end{pmatrix}\begin{pmatrix} \xi_1^0 \\ \xi_2^0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Therefore, $x^{(0)}(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t$

For $\lambda_2 = 2$, we have

$\begin{pmatrix} -3 & 2 \\ -3 & 2 \end{pmatrix}\begin{pmatrix} \xi_1^1 \\ \xi_2^1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\quad\quad$ i.e $\begin{pmatrix} -1-2 & 2 \\ -3 & 4-2 \end{pmatrix}$

$-3\xi_1^1 + 2\xi_2^0 = 0$ $\quad\quad$ $\begin{pmatrix} -2 & 2 \\ -3 & 3 \end{pmatrix}\begin{pmatrix} \xi_1^0 \\ \xi_2^0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$-3\xi_1^1 + 2\xi_2^0$

$x^{(1)}(t) = \begin{pmatrix} 2 \\ 3 \end{pmatrix} e^{2t}$

The fundamental matrix becomes

$$e^{At} = \begin{pmatrix} e^t \, 2e^{2t} \\ e^t \, 3e^{2t} \end{pmatrix} \tag{4.6}$$

Hence the solution to the system (4.4) becomes

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} e^t \, 2e^{2t} \\ e^t \, 3e^{2t} \end{pmatrix}\begin{pmatrix} 0 \\ 0 \end{pmatrix} +$$

$$\int_0^1 \begin{pmatrix} 3e^{(t-s)} - 2e^{2(t-s)} - \; 2e^{(t-s)} + \; 2e^{2(t-s)} \\ 3e^{(t-s)} - 3e^{2(t-s)} - \; 2e^{(t-s)} + \; 3e^{2(t-s)} \end{pmatrix}\left[\begin{pmatrix} u_1(s) \\ u_2(s) \end{pmatrix} + \begin{pmatrix} f_1(s, x(s)) \\ f_2(s, x(s)) \end{pmatrix}\right] ds \tag{4.7}$$

Clearly, by choosing a control u such that

$$\left.\begin{aligned} &u = \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} = \begin{pmatrix} 3e^{-(-T-t)} - 2e^{-2(T-t)} - \; 2e^{-(T-t)} + \; 2e^{-2(T-t)} \\ 3e^{-(-T-t)} - 3e^{-2(T-t)} - \; 2e^{-(T-t)} + \; 3e^{-2(T-t)} \end{pmatrix} \\ &\left[\begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{pmatrix} e^t \, 2e^{2t} \\ e^t \, 3e^{2t} \end{pmatrix}\begin{pmatrix} 0 \\ 0 \end{pmatrix} - \int_0^1 \begin{pmatrix} 3e^{(-T-s)} - 2e^{-2(T-s)} - \; 2e^{(T-s)} + \; 2e^{-2(T-s)} \\ 3e^{(-T-s)} - 3e^{-2(T-s)} - \; 2e^{(T-s)} + \; 3e^{-2(T-s)} \end{pmatrix}\begin{pmatrix} f_1(s, x(s)) \\ f_2(s, x(s)) \end{pmatrix}\right] \end{aligned}\right\} \tag{4.8}$$

On substituting (4.8) into (4.7), we have

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} e^t & 2e^{2t} \\ e^t & 3e^{2t} \end{pmatrix}\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left[+\int_0^1 \begin{pmatrix} 3e^{(t-s)} - 2e^{2(t-s)} - 2e^{(t-s)} + 2e^{2(t-s)} \\ 3e^{(t-s)} - 3e^{2(t-s)} - 2e^{(t-s)} + 3e^{2(t-s)} \end{pmatrix} \begin{pmatrix} 3e^{(-T-s)} - 2e^{-2(T-s)} - 2e^{-(T-s)} + 2e^{-2(T-s)} \\ 3e^{(T+s)} - 3e^{-2(T-s)} - 2e^{(T+s)} + 3e^{-2(T-s)} \end{pmatrix}\right]$$

$$\begin{pmatrix} 3e^{-(T-s)} - 2e^{-2(T-s)} - 2e^{-(T-s)} + 2e^{-2(T-s)} \\ 3e^{-(-T-s)} - 3e^{-2(T-s)} - 2e^{-(-T-s)} + 3e^{-2(T-s)} \end{pmatrix}$$

$$\left[\int_0^1 \begin{pmatrix} 3e^{(T-t)} - 2e^{-2(T-t)} - 2e^{(T-t)} + 2e^{-2(T-t)} \\ 3e^{(-T-t)} - 3e^{-2(T-t)} - 2e^{(-T-t)} + 3e^{-2(T-t)} \end{pmatrix} \begin{pmatrix} f_1(t, x(t)) \\ f_2(t, x(t)) \end{pmatrix}\right] \mathrm{ds} \tag{4.9}$$

$$+\int_0^1 \begin{pmatrix} 3e^{(t-s)} - 2e^{2(t-s)} - 2e^{(t-s)} + 2e^{2(t-s)} \\ 3e^{(t-s)} - 3e^{2(t-s)} - 2e^{(t-s)} + 3e^{2(t-s)} \end{pmatrix} \begin{pmatrix} f_1(\mathrm{s}, x(s)) \\ f_2(\mathrm{s}, x(s)) \end{pmatrix} \mathrm{ds}$$

$$\left.\begin{aligned}
&= \left[\int_0^1 \begin{pmatrix} 3e^{(t-s)} - 2e^{2(t-s)} - 2e^{(t-s)} + 2e^{2(t-s)} \\ 3e^{(t-s)} - 3e^{2(t-s)} - 2e^{(t-s)} + 3e^{2(t-s)} \end{pmatrix} \begin{pmatrix} 3e^{(T+s)} - 2e^{-2(T-s)} - 2e^{-(T-s)} + 2e^{-2(T-s)} \\ 3e^{(T+s)} - 3e^{-2(T-s)} - 2e^{(T-s)} + 3e^{-2(T-s)} \end{pmatrix}\right] \\
&- \begin{pmatrix} 3e^{(T-s)} - 2e^{-2(T-s)} - 2e^{(T-s)} + 2e^{-2(T-s)} \\ 3e^{(T-s)} - 3e^{-2(T-s)} - 2e^{(T-s)} + 3e^{-2(T-s)} \end{pmatrix} \\
&\left[\int_0^1 \begin{pmatrix} 3e^{(T+t)} - 2e^{-2(T-t)} - 2e^{(T-t)} + 2e^{-2(T-t)} \\ 3e^{(T+t)} - 3e^{2(T-t)} - 2e^{(T+t)} + 3e^{-2(T-t)} \end{pmatrix} \begin{pmatrix} f_1(t, x(t)) \\ f_2(t, x(t)) \end{pmatrix}\right] \mathrm{ds} \\
&\left[+\int_0^1 \begin{pmatrix} 3e^{(t-s)} - 2e^{2(t-s)} - 2e^{(t-s)} + 2e^{2(t-s)} \\ 3e^{(t-s)} - 3e^{2(t-s)} - 2e^{(t-s)} + 3e^{2(t-s)} \end{pmatrix} \begin{pmatrix} f_1(\mathrm{s}, x(s)) \\ f_2(\mathrm{s}, x(s)) \end{pmatrix}\right] \mathrm{ds}
\end{aligned}\right\} \qquad (4.10)$$

Using the above computation reduces to

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad (4.11)$$

This shows that the system (4.4) can be steered from an initial state of (4.10) to a desired state of (4.8) given the control u of (4.11).

Consider an impulsive differential system with perturbation and using the previous examples.

Let $x\,(t^+) - x\,(t^-) = |_{\mathrm{k}}(x\,(t_k))$ be the impulse and denoted by

$|_{\mathrm{k}}\,(x\,(t_k)) = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$, for k = 1. Then the system 4.4 becomes

$$\left.\begin{aligned}
&\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ -3 & 4 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} \mathrm{u}\right] (t) + \begin{pmatrix} t_1, (t, x(t)) \\ t_2(t, x(t)) \end{pmatrix} \\
&x\,(t^+) - x\,(t^-) = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \\
&x(\mathrm{o}) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\end{aligned}\right\} \qquad (4.12)$$

The solution of the non-impulsive differential system of (4.11) is given by (4.7), to obtain solution of the system (4.9), we substitute (4.7) into (3.4) to have

$$\left.\begin{aligned}
&\begin{pmatrix}x_1(t)\\x_2(t)\end{pmatrix} = \begin{pmatrix}e^t\,2e^{2t}\\e^t\,3e^{2t}\end{pmatrix}\begin{pmatrix}0\\0\end{pmatrix} + \begin{pmatrix}3e^{(t-s)} - 2e^{2(t-s)} - 2e^{(t-s)} + 2e^{2(t-s)}\\3e^{(t-s)} - 3e^{2(t-s)} - 2e^{(t-s)} + 3e^{2(t-s)}\end{pmatrix}\\
&\left[\begin{pmatrix}u_1(s)\\u_2(s)\end{pmatrix} + \begin{pmatrix}f_1(s,x(s))\\f_2(s,x(s))\end{pmatrix}\right] ds\\
&+ \begin{pmatrix}3e^{(T+t_1)} - 2e^{-2(T-t_1)} - 2e^{-(T-t_1)} + 2e^{-2(T-t_1)}\\3e^{(T+t_1)} - 3e^{-2(T-t_1)} - 2e^{(T+t_1)} + 3e^{-2(T-t_1)}\end{pmatrix}\begin{pmatrix}1 & 0\\0 & 2\end{pmatrix}
\end{aligned}\right\}\ (4.13)$$

This can be written as

$$\left.\begin{aligned}
&\begin{pmatrix}x_1(t)\\x_2(t)\end{pmatrix} = \int_0^1 \begin{pmatrix}3e^{(t-s)} - 2e^{2(t-s)} - 2e^{(t-s)} + 2e^{2(t-s)}\\3e^{(t-s)} - 3e^{2(t-s)} - 2e^{(t-s)} + 3e^{2(t-s)}\end{pmatrix}\left[\begin{pmatrix}u_1(s)\\u_2(s)\end{pmatrix} + \begin{pmatrix}f_1(s,x(s))\\f_2(s,x(s))\end{pmatrix}\right]\\
&ds \qquad (4.14)\\
&+ \begin{pmatrix}3e^{(T+t_1)} - 2e^{-2(T-t_1)} - 4e^{-(T-t_1)} + 4e^{-2(T-t_1)}\\3e^{(T+t_1)} - 3e^{-2(T-t_1)} - 4e^{(T+t_1)} + 6e^{-2(T-t_1)}\end{pmatrix}
\end{aligned}\right\}$$

By choosing a control u such that

$$\left.\begin{aligned}
&u = \begin{pmatrix}u_1(t)\\u_2(t)\end{pmatrix} = \begin{pmatrix}3e^{(T+t)} - 2e^{-2(T-t)} - 2e^{-(T-t)} + 2e^{-2(T-t)}\\3e^{(T+t)} - 3e^{-2(T-t)} - 2e^{(T+t)} + 3e^{-2(T-t)}\end{pmatrix}\\
&\begin{pmatrix}1\\0\end{pmatrix} - \begin{pmatrix}e^t\,2e^{2t}\\e^t\,3e^{2t}\end{pmatrix}\begin{pmatrix}0\\0\end{pmatrix}\\
&- \int_0^1 \begin{pmatrix}3e^{(T+s)} - 2e^{-2(T-s)} - 2e^{(T-s)} + 2e^{-2(T-s)}\\3e^{(T+s)} - 3e^{-2(T-s)} - 2e^{(T+s)} + 3e^{-2(T-s)}\end{pmatrix}\begin{pmatrix}f_1(s,x(s))\\f_2(s,x(s))\end{pmatrix} ds\\
&- \begin{pmatrix}3e^{(T+t_1)} - 2e^{-2(T-t_1)} - 2e^{-(T-t_1)} + 2e^{-2(T-t_1)}\\3e^{(T+t_1)} - 3e^{-2(T-t_1)} - 2e^{(T+t_1)} + 3e^{-2(T-t_1)}\end{pmatrix}\begin{pmatrix}1 & 0\\0 & 2\end{pmatrix}
\end{aligned}\right\}\ (4.15)$$

Substituting (4.15) into (4.14), we have

$$
\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \int_0^1 \begin{pmatrix} 3e^{(t-s)} - 2e^{2(t-s)} - 2e^{(t-s)} + 2e^{2(t-s)} \\ 3e^{(t-s)} - 3e^{2(t-s)} - 2e^{(t-s)} + 3e^{2(t-s)} \end{pmatrix}
$$

$$
\left.\begin{matrix}
\begin{pmatrix} 3e^{(T+t)} - 2e^{-2(T-t)} - 2e^{-(T-t)} + 2e^{-2(T-t)} \\ 3e^{(T+t)} - 3e^{-2(T-t)} - 2e^{(T+t)} + 3e^{-2(T-t)} \end{pmatrix} \\
\left.\begin{matrix}
\begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} e^{t} & 2e^{2t} \\ e^{t} & 3e^{2t} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\
- \int_0^1 \begin{pmatrix} 3e^{(T+s)} - 2e^{-2(T-s)} - 2e^{(T-s)} + 2e^{-2(T-s)} \\ 3e^{(T+s)} - 3e^{-2(T-s)} - 2e^{(T+s)} + 3e^{-2(T-s)} \end{pmatrix} \\
\begin{pmatrix} f_1(\mathrm{s}, x(s)) \\ f_2(\mathrm{s}, x(s)) \end{pmatrix} \mathrm{ds} \\
- \begin{pmatrix} 3e^{(T+t_1)} - 2e^{-2(T-t_1)} - 2e^{-(T-t_1)} + 2e^{-2(T-t_1)} \\ 3e^{(T+t_1)} - 3e^{-2(T-t_1)} - 2e^{(T+t_1)} + 3e^{-2(T-t_1)} \end{pmatrix} \\
\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}
\end{matrix}\right\} \\
+ \begin{pmatrix} 3e^{(T+t_1)} - 2e^{-2(T-t_1)} - 2e^{-(T-t_1)} + 2e^{-2(T-t_1)} \\ 3e^{(T+t_1)} - 3e^{-2(T-t_1)} - 2e^{(T+t_1)} + 3e^{-2(T-t_1)} \end{pmatrix}
\end{matrix}\right\} \begin{pmatrix} f_1(\mathrm{s}, x(s)) \\ f_2(\mathrm{s}, x(s)) \end{pmatrix} \mathrm{ds} \quad (4.16)
$$

Continuing this computation, the above gives

$$
\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}
$$

This shows that with the control as in (4.15), the system (4.12) can be steered into system (4.16) which is the desirable state.

Hence, the system is controllable.

CHAPTER FIVE

SUMMARY, CONCLUSION AND RECOMMENDATION

5.1 SUMMARY

The investigation of an impulsive control differential system with perturbation clearly shows that their solutions have different behaviours from the solutions of non-impulsive control differential system with perturbation. However, the controllability, of impulsive control differential system was based on an arbitrary control choice so as to generate the controllability of the system.

The control also takes care of the non-linear perturbation function which is subject to further research.

5.2 CONCLUSION

System of the impulsive control differential equation for an autonomous system and with perturbation is only controllable depending on the control choice. In some cases, the non-linear perturbation $f(t, x(t))$ will have to be generated on some certain smooth conditions.

5.2 RECOMMENDATION

Since evolutions are subjected to dynamism and can take different forms and shapes and avert the nature and state of the system in which they affect the impulsive differential system, we however recommend that this work is opened for further research especially to find out in particular, whether controllability of impulsive control system with perturbation will be naturally controllable (i.e. for non-autonomous impulsive optimal control system).

Another area of further research may include: impulsive differential system with variable moment of impulses.

5.3 CONTRIBUTION TO KNOWLEDGE

We succeeded in presenting a necessary criterion for controllability of impulsive differential systems with perturbation; we formulated a control $u(t)$ that transformed an impulsive differential system with perturbation into a controllability system.

REFERENCES

Bajo, I., (1997). Pulse Accumulation in impulsive differential equation with variable times, *Journal of Mathematical Analysis and Application* (Online) Vol 216, Pg 211-217.

Benchohra, M., Henderson, J., and Ntouyas, S. (2006). Impulsive Differential Equation and Included Contemporary Mathematics and its application. Vol 2, Pg 5-33. Available from http://www.hindawill.com

Bervabali, F. E., (2014). Contribution to Impulsive Equations. *European Scientific Journal.* Vol. 3, Pg 393-397. ISBN: 1857-7887.

Boukhamla, R., (2018). Controllability of Impulsive Equation under Impulsive Conditions. *Springer Open Joiurnal.* Vol. 2, Pg 266-280.

Dishev, A., Dishlieva, K. and Nenov, S. (2012). Specific Asymptotic Properties of Solution of Impulsive Differential Equation Methods and Application. Academic Publication Ltd, Sofia.

Fang and Sun (2012). Existence and uniqueness of solutions to complex-valued non-linear impulsive differential systems. *Springer Open Journal* (Online) Vol. 20 12(115), Pg 1-9. Available from http://www.advanceindifferenceequation.com

George, R. K., Nandakumeran, A. K. and Arapostathics, A., (2010). A note in controllability of impulsive system. *Journal of Mathematical Analysis and Application* (Online). Vol. 241, Pg 276-283. Available from http://www.idealibrary.com.

George et al. (2000). Controllability of a system having a transfer function. International journal of transfer function. Vol 9 pg 81-91.

Glyn James (2019) Convolution of Impulsive effects, 5th edition, p 115.

Jovita M., and Lauretta, F., (2020). Impulsive differential equation. Advances in Difference equation. Vol. 2020 Pg 1-9.

Huinpang et al. (2006) Existence of a positive solution of impulsive control system.

Hinpang, S., Xiang, X. and Sattlayaiioram, P., (2016). Periodic control system with perturbation. *International Journal of Differential system and Application.* Vol. 12(2) Pg 121-129.

Lakshmikantam, V., Bainor, D. and Sumeonor, P. S., (1989). Theory of impulsive differential equation. World Scientific Singapore.

Leela, S., Farzena, A. M. and Sivenindara, S. (1999). Controllability of impulsive differential equations. *Journal of Mathematical Analysis and Application.* Vol. 177, Pg 24-30.

Liang and shan(2019), Existence and approximation of a solution of a system.

Lie R., Miron K and Steven D. (2015), impulsive differential equation, periodic solutions and applications.

Liang and Shan. (2009). Existence and approximation of a solution of a system.

Lie et al. (2015). Impulsive differential equation, periodic solutions and solution

Liing, R. X. and Shen, J. H., (2019). Periodic Boundary value problem for the first order differential equation with impulse. *Journal of Applied Mathematical Computation.* Vol. 24, Pg 27-35.

Marvel, J. J., (2017). Basic theory for non resonance impulsive periodic problem of first order. *Journal of Mathematical Analysis and Application.* Vol. 205 Pg 223-233.

Nieto Xavi (1997) Existence of the solution on nonlinear ordinary differential equation.

Nieto and Tisdel (2010) exact Controllability of the impulsive system.

Randelovic, B. M., Stegenuric, L. V. and Dancovic, B. M. (2020). Numerical solution of impulse differential equation. *Ser. Math. InJournal.* Vol. 5, Pg 109-111.

Sontes E. D., (1990). Mathematical Control theory. Springer Verlag, Berlin. Available online from: http://www.math.rutsers.edu/sontes.

Sofong Arantes (1990) Microprocessor controller and automobiles.

Snesham, G. H., (2019). Qualitative investigation and Approximate method for impulsive equation. Nova Science Publication Inc. Switzerland.

Wei, D. and Mi, J. R., (2015). Periodic boundary value problem for the first order Differential equation with impulse. *Journal of Applied Mathematical Computations.* Vol. 165 Pg 433-446.

Xiaidi, L., Bohner, M. and Way, L. K., (2013). Impulsive Differential Equation. Periodic Solution and Application. *Automatics Journal* Vol 52 Pg 173-178.

Zanda K. M.(2009) Uniqueness of the impulse of variable moments.