STABILITY ANALYSIS OF THICK LAMINATED ANISOTROPIC PLATE USING THIRD ORDER ENERGY FUNCTIONAL

BY

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A THESIS SUBMITTED TO THE POSTGRADUATE SCHOOL FEDERAL UNIVERSITY OF TECHNOLOGY, OWERRI.

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August, 2024

CERTIFICATION

This is to certify that this work "Stability Analysis of Thick Laminated Anisotropic Plate using Third Order Energy Functional" was carried out by Sunny Chukwudum Uzoukwu (B. Eng., M. Eng.) with Reg. No. 20184144188 in partial fulfillment of the requirements for the award of Doctor of Philosophy (Ph. D) degree in Structural Engineering in the Department of Civil Engineering of Federal University of Technology Owerri, Imo State.

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DEDICATION

This work is dedicated to Jesus, for His mercies and grace.

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LIST OF SYMBOLS

Notations	Meaning
a	Length of the primary dimension of the plate
A	Amplitude
b	Width of the secondary dimension of the plate
C	Clamped support
t	Tertiary dimension (thickness) of the plate
x	The primary axis of the plate. That is the shorter of the two axes of the major plane of the plate
y	The secondary axis of the plate. That is the longer of the two axes of the major plane of the plate.
p	Aspect ratio. That is p = b/a
t	Thickness of the plate or the length of the tertiary dimension of the plate.
S	Simply support
π	*Total potential energy of the plate*
W_x	Deflection of plate on X-X axis
W_y	Deflection of plate on Y-Y axis
W	Deflection on both axis

h	Shape function
m	Number of lamina in the plate
n	Total number of laminas in the plate
N_X	Critical Buckling Load
N	Non-dimensional buckling load parameter
Q	Non-dimensional axis (quantity) parallel to x axis $Q = x/a$
G	Modulus of Rigidity of Thick Rectangular anisotropic Plate
E	Young modulus of elasticity of thick rectangular anisotropic plate
R	Non-dimensional axis (quantity) parallel to y axis $R = y/b$
V	External load
u	In- plane displacement along x-axis
v	In- plane displacement along y-axis
J_i	Ist Strain energy coefficient
g_i	2^{nd} Strain energy coefficient
w	In- plane displacement along z-axis
u_o, v_o	Middle Layer in-plane displacement

x, y, z	Orthogonal co-ordinates of the rectangular plate
$\emptyset_x$	Shear Rotation on x axis
$\emptyset_y$	Shear Rotation on y axis
ε_R	*Non dimensional* Strain component in x- durection
ε_Q	*Non dimensional* Strain component in y- durection
$\int$	Integration
P,L,Z	Amplitude Coefficients
∂	Differentiation
ε	Strain
σ	Stress
τ	Shear stress
γ	Shear strain
SSSS	Rectangular plate simply supported on the four edges
CCCC	Rectangular plate clamped on all edges.
CSCS	Rectangular plate clamped on two opposite edges and simply supported on the other two opposite edges.
CSSS	Rectangular plate clamped on the first edge and simply supported on the Remaining three edges
CCSS	Rectangular plate clamped on the first two edges and simply supported on the Remaining two edges
CCSC	Rectangular plate clamped on the first, second and third edges, but simply supported on the third edge

CCCS	Rectangular plate clamped on the first three edges and simply supported on the last edge.
SSSC	Rectangular plate simply supported on the first three edges, but clamped on the last edge.
SSCS	Rectangular plate simply supported on the first, second and last edges, clamped on the third edge.
CCSC	Rectangular plate clamped on the first, second and last edges, but simply supported on the third edge
SCSC	Rectangular plate simply supported on the first and third edges, clamped on the second and last edge
SCCS	Rectangular plate, simply supported on the first and last edges and clamped on the second and third edges.
CSSC	Rectangular plate simply supported on the second and third edge, clamped on the two edge supports
SSCS	Rectangular plate, simply supported on the first, second and last edge and clamped on the third edge

ABSTRACT

TheStability Analysis of Thick Laminated Anisotropic Plate using Third Order Energy Functional is presented. Fourteen different boundary conditions were considered in this research work thus SSSS, CCCC, SSCC, CCSS, CSSS, SSSC, SCSC, CSCS, SSSC, CCCS, CSSC, SCCS, CCSC and SSCS. The shape functions for all the fourteen plates were derived by considering the deflection and second derivative of deflection at the simple support edge while the deflection and first derivative of deflection were considered at the clamped support. The integral values of the differentiated shape functions, $(\bar{K} - values)$of the various boundary conditions were obtained. From these, the stiffness coefficients, k_i of the various boundary conditions were generated. Resolving from the first principle and by considering the relationship between the in-plane displacement and out of plane displacement, the strain and the stress functions for thick laminated laminated plate were obtained. The total potential energy was formulated by adding the work done by external load to the strain energy equation. Differentiating the total potential energy with respect to the deflection w, gave rise to the governing equation. Four compatible equations were formulated also by differentiating the total potential equation with respect to the two middle-layer in-plane displacements, ($-u_o$ and $-v_o$) and two shear rotations, ($\emptyset_x$and $\emptyset_y$).. Numerical analysis for the different plate laminas were conducted, considering different angles of arrangement The third order stiffnesses were formulated for thick laminated anisotropic plate and the results from this present work were compared with those from Ventsel & Krauthermmer (2001) The comparison showed acceptable percentage difference of 0.01% & 2% level of significance in statistics. The derived buckling loads for all the plates studied in this work, were compared with those from Reddy, Megson and Chajes (Previous Researchers) and the percentage differences obtained were within the range of 0.01% to 0.9%.. It is concluded that this method can be adopted in solving a laminated thick anisotropic plate using 3rd order energy functional

Key words: Boundary Condition, Stability Analysis, Critical Buckling Load, Polynomial Series, Third Order Strain Energy, Shape Functions, Third Order Energy Functional

CHAPTER ONE

INTRODUCTION

1.1 Background Information

Plate is an indispensable structural element which has been in use for a long period of time. Plates are used in many engineering applications such as in the building floors, satellites, construction of aircraft, ships, vehicles, bridges, shear walls, computer hard-disk drives, flat part of a table, manhole covering and panels, and other complex structures (Birman, 2011 and Volmir, 1974). There are properties that made plate element very unique and the variations in these properties plays important roles in classifying the plate. The impact of the use of this material cannot be over emphasized as it is seen in our everyday life. Due to the importance and wide application of this structural material, several researches have been carried out with the aim of maximizing its potentials for wider structural applications. Steel rectangular plates are widely used in buildings, bridges, automobiles, and ships. Rectangular plate has width, comparable to size of their lengths and so is considered as two dimensional plane element unlike beams and columns. A plate shows different material properties round about its shape and that include the poission's ratio, Young elastic modulus of elasticity and flexural rigidity.

1.1.1 Stability Analysis.

The buckling analysis, most times is referred to as stability study. Buckling analysis of anisotropic rectangular plate has been a center of focus for many researchers over the ages. Many exact solutions for isotropic plates have been developed, unlike in the case of anisotropic cases. Under compression, rectangular plate tends to buckle out of their plane. The buckled shape depends on the force or load effect and the pattern of the support adopted. In both cases, the plate element continues to carry loads in stable manner. Before now, the stability of rectangular plates has received the attention of many researchers for several centuries, its study has left much to be treated. The main purpose of stability analysis of a rectangular plate is to determine the critical buckling loads and the corresponding buckled configuration of equilibrium and the smallest value of the load producing the buckling is called the critical buckling load. The plate leading from the stable to unstable configurations of equilibrium always passes through the neutral state of equilibrium which can thus be considered as a border state (Iyengar, 1988). Buckling of a rectangular plate is of great importance in the initiation of a deflection pattern, which if loads are further increased above their critical values, rapidly leads to a very large lateral deflection. The plate leading from the stable to unstable configuration of equilibrium always passes through the neutral state of equilibrium. It also leads to large bending stresses which may cause complete failure of the plate.

1.1.2 Laminated Plate

The thicker the plate higher the critical buckling load needed to cause buckling effect. Figure 1.1 shows an example of When two or more plates are combined together to form a laminate, it is referred to as laminated plate. Each of the component is referred to as a lamina. The characteristics of a plate is simply the individual featuresof the various laminas.a laminated plate.

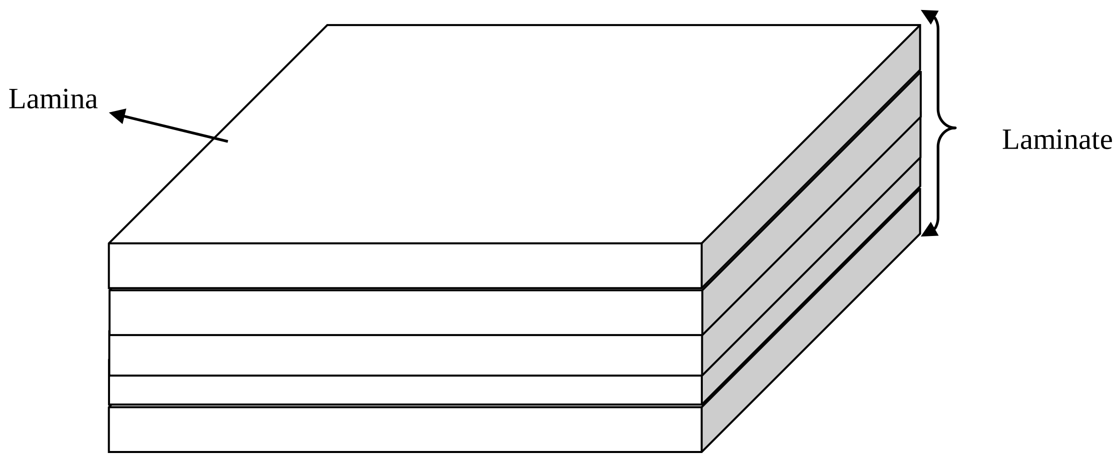


Figure 1.1 A Laminated Plate

Both the thin and the thick plate, whether isotropic or anisotropic, have the tendency to undergo lamination. Reissner and Stavsky (1961) formulated the laminated plate problem considering the stress function and the transverse deflection regulated by two coupled differential equations and got the solutions for an infinite two-layer plate subjected to different types of transverse and in-plane static loadings. The same formulation was used for a more general class of laminated plates. Dong, et al. (1962). Whitney and Leissa (1969) developed the general governing equations of laminated rectangular plates in terms of displacements including inertia terms and thermal stresses. In the analysis, closed form solutions to the linear differential equations, excluding external shear tractions, thermal effects as well as in-plane and rotary inertias, were obtained for static deflection, vibration and buckling of antisymmetric, cross-ply and angle-ply plates with all edges simply supported. Later investigations by Dong, et al. (1962); Azzi and Tsai (1965) and Whitney and Leissa (1969) confirmed that there exists coupling between transverse bending and in-plane stretching if laminates are layered up unsymmetrical about the middle plane.

1.1.3 Plate Orientation and Different Boundary Conditions.

The buckling of a typical rectangular plate a shown Figure 1.2 was considered in this work . These sides can either be Fixed, Clamped or Simply Supported. The naming of a plate is done considering the support conditions at the various edges. For example, a rectangular plate of SCCS means that the first edge is Simple, the second and third edges are Clamped supports and lastly the fourth edge is Fixed support. Usually a plate is named in an anti-clockwise fashion.

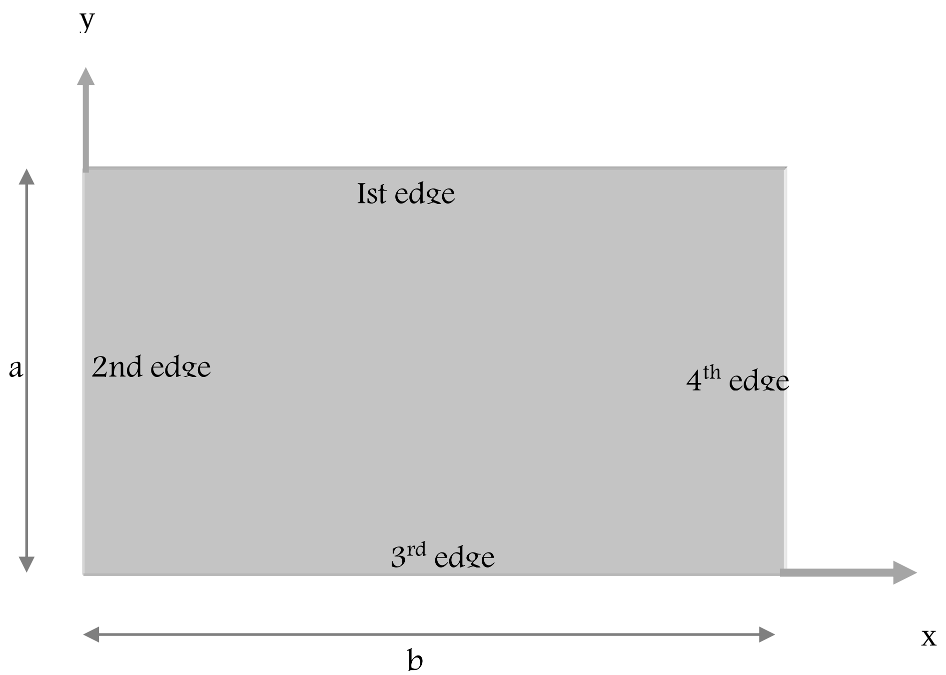
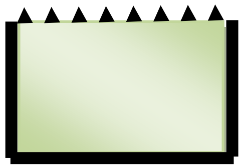
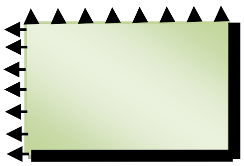
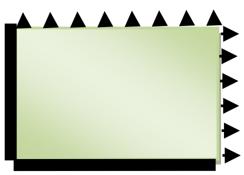


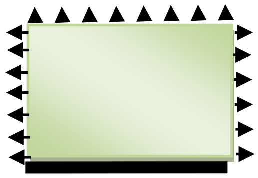
Figure 1.2 Boundary Orientation of a Rectangular Plate

Typical illustrations of supporting conditions of different plate members are: simple support edge, clamped edge, free edge, point support and elastic support. Table 1.1 shows the different supporting conditions a plate can possess at its edges (Szilard, 2004). The naming of the various plate sides, was done in the anti-clockwise direction.

Table 1.1 Symbols and Descriptions

	Different Boundary Orientation		
	Plate Name	**Plate Diagram**	**Plate Symbol**
	Simple-Simple-Simple-Simple		SSSS
	Clamped-Clamped-Clamped-Clamped		CCCC
	Clamped-Simple-Simple-Simple		CSSS
	Clamped-Simple-Clamped-Simple		CSCS
	Clamped-Clamped-Simple-Simple		CCSS

	Clamped-Clamped-Clamped-Simple		CCCS
	Simple-Simple-Simple-Clamped		SSSC
	Clamped-Simple-Clamped-Simple		SCSC
	Simple-Clamped-Clamped-Clamped		SCCC
	Simple-Simple-Simple-Clamped		SSSC
	Clamped-Simple-Simple-Clamped		CSSC

	Simple-Clamped- Clamped-Simple		SCCS
	Simple-Clamped-Clamped-Simple		SCCS
	Clamped-Clamped-Simple -Clamped		CCSC
	Simple -Simple-Clamped-Simple		SSCS

1.1.4 Third Order Energy Functional

Just like in the case of Ritz and Galerkin methods respectively, when the order of the polynomial deflection function is of two or four (fourth power polynomial) then fourth order (differentiable up to four times) energy functional can be admissible. But the situation changes when the order of polynomial is up to 3 for the analysis of laminated thick anisotropic plates.

Before now only second and fourth order energy functional have been used in the plate. buckling analyses, even though not in laminated situation of thick anisotropic plate.. Third order energy functional has not yet been employed for the this class of plate element. A laminated thick anisotropic plate problem may arise where the order of polynomial deflection is three (3). In this case the researcher shall be forced to use the second order energy functional (as fourth order energy functional cannot be used). Third order energy functional, remains the best option for this case where the order of polynomial deflection is 3.

1.2 Problem Statement

The buckling analysis of thick plate has not been given full attention, not minding the recent increase in the use of laminated composite members in structural works. Thorough research conducted in the course of this research work, shows that the reason behind this development is traceable to the complexity associated with the analysis of thick plate members. The analysis of isotropic plate members, where all material properties, like Poisson's ratio, Young's elastic modulus of elasticity and flexural rigidity are the same in every direction, is usually less cumbersome compared to the situation where there is no uniformity in those listed material properties. This is the case of anisotropic plate members. Where as isotropic plates have three engineering properties, of great interest anisotropic plates have five properties which produce more complex equations in the analysis.

Considering a thick anisotropic plate as a thin isotropic plate underestimates the stresses in the plate and most times this leads to serious error. Previous researchers have looked into various aspect of thick plate analysis: Stability analysis (Fazzolari and Carrera, 2011; Tran et al., 2017; Bouazza et al., 2016; Sahoo and Singh, 2015; Nali, Carrera and Lecca , 2011;Ibearugbulem, Ibeabuchi and Njoku, 2014; Avalos and Larondo, 1995; Kim, Thai and Lee, 2009; Wang, Xiang and Chakrabarty, 2001; Yang and He, 2018; etc).. The values of the buckling load of thick rectangular plate can be generated by integration of Euler- Bernoulli equation of forces. The complexities involved in derivations of these real displacement values, make the analysis seem extremely difficult. Few researchers who have done work on laminated plates considered thin Isotropic plate condition. Others who attempted laminated thick anisotropic plate condition tried the case where all the material properties are the same in all directions. A plate problem may arise where the buckling analysis of thick laminated anisotropic plate with say three polynomial shape function is the only solution. This is mostly in construction of bridges and helipads. And so provision of well detailed third order approach to this effect, will give much confidence in the analysis.

Also before now, the analysis of laminated thick anisotropic plate is usually done using, second order or fourth order. Third order has not yet been employed. When a case of third polynomial deflections of laminated thick plate arises, second other will be adopted. But if there is third order energy functional, the researcher will be left with better option and for less rigour will definitely choose third order functional. In view of this opinion, this research work tends to fill existing Literature with well detail approach.

1.3 Aim and Objectives

The aim of this work is Stability Analysis of Thick Laminated Anisotropic Plate, using Third Order Energy Functional. The objectives of the study are to;

(i) Derive strain equation of a thick laminated anisotropis plate.

(ii) Formulate the stress equation of the thick laminated anisotropic plate.

(iii) Determine Stress-Strain relation for a lamina of a thick laminated plate and translate from Local coordinate to Global coordinate system.

(iv) Obtain the Strain Energy, Total Potential Energy functional and Governing equation of equilibrium by minimizing the Total Potential Energy for a thick rectangular laminated anisotropic plate

(v) Carry out the numerical analysis of thick laminated Anisotropic plate considering different edge/boundary conditions.

1.4 Justification of The Study

The work coversna a very important aspect of structural engineering, which is useful structural Engineering. If the governing Equation is derived from the the Total Potential energy, on integrating that, the displacement, which satisfies the condition are obtained, one can confidently accept the results as more exact function for any arbitrary

challenge. The concise computer program developed in this work, can go a long way to save the stress of lengthy equations that will be needed in solving any Thick laminated palte situation.

(i) The availability of easier and more accurate approach will increase both the product and the demand for the thick laminated anisotropic plate

(ii) More researchers will be moved to work on the laminated thin anisotropic for the cases of third or any odd order polynomial function.

1.5 Scope of The Study.

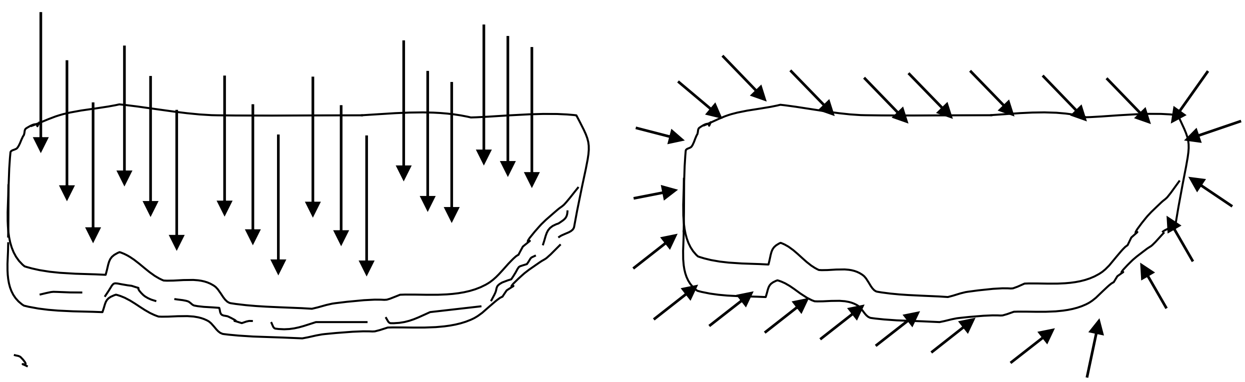
This study deal on thick rectangular anisotropic plate. The major area of concentration is on the stability analysis of a plate member with non-uniform material properties in different directions. The study is limited to stability analysis of thick anisotropic laminated plates using third order energy functional,with the plate having three laminas. Fourteen different boundary conditions shall be considered for the analysis. The following angle of orientation: 0^o and 90^o were considered . The laminated case is as shown in the Figure 1.1.

CHAPTER TWO

LITERATURE REVIEW

2.1 Structural Plate Element

According to Ibearugbulem et.al (2014) a plate is a solid which consists of two parallel plane surfaces separated by a small dimension called its thickness. Therefore, structural plates are plane elements with small thickness, compared to the planar dimensions, bounded either by straight or curved edges or boundaries. There are two types of loads applied to the namely transverse load and in –plane loads. They are loaded with forces that are normal to the center of the plate and distributed uniformly over either face of the plate (usually the top face). When a plate element is loaded with a force normal to its surface, it is considered as Transverse loading but when applied on the edges or boundary it is referred to as In –plane loading. Perpendicular load on the mid plane of a plate produces plate bending. When a plate is loaded beyond its bearing bearing capacity, it fails by buckling.



a) Tranverse Loading b) In-plane Loading

Figure 2.1: Loads on a plane

2.2 Classification of Plates

Considering t as the thickness of the plate with a and b as the in-plane dimensions, plates may be classified into three groups, with respect to their various ratios and taking the member to be a rectangular plate. (Ventsel and Krauthammer, (2001); Reddy, (2004)and Ugural,(1999)), Considering these ratios, a plate can be Thick Plate, Membrane plate or Thin Plate. Membrane plates, carry the lateral loads by axial tensile forces acting on the plate. These forces have the capacity to balance lateral loads applied to the plate membrane, due to their ability to produce projection on a vertical axis.

Thin plate is further divided into two different classes.

i. The Stiff Plates and

ii. Flexible plates.

2.2.1 Stiff Plates: Stiff plates are flexural rigid thin plates. Their loads are carried in two dimensions, mostly by twisting moments internal bending moments and transverse shear forces. For such a plate, the middle plane deformation and the membrane forces are negligible in Kirchhoff's linear buckling analysis.Theory of thin plates assumes, points of the plate lying initially on a normal to the center plane of the plate remains on the normal to the center surface of the plate after bending. The normal stresses in the direction transverse to the plate can be ignored. (Timoshenko &woinowsky-Krieger, 1987)

2.2.2 Flexible Plates: When a plate ratio of deflection to its thickness is greater than or equal to 0.3, it causes lateral deflections which is accompanied by the stretching of the middle surface. This type of plates are referred to as "flexible plates". These plates represent a combination of stiff plates and membranes. Flexible plates carry external loads by the combined action of shear forces, internal moments and membrane (axial) forces. Usually the membrane action predominates, when the magnitude of the maximum deflection is considerably greater than the plate's thickness.

2.2.3 Isotropic Plates

A structural plate element can be Isotropic or Anisotropic element. The isotropic case is characterized with uniformity in its material properties, like Young Elastic modulus of elasticity, Poisson's ratio and flexural rigidity, in all directions. The exact solution for the bending problem of fully clamped isotropic rectangular plates was obtained using the generalized integral transform technique, (An and Gu, 2011).

The results (values) for the transverse deflections and bending moments of the structural plates element (isotropic) under uniform loads were compared with the real solutions as detailed by Timoshenko & Woinowsky krieger (1959) and found to be in the same while the results of the transverse shear forces of the isotropic rectangular plates under influence of uniform loads corresponded to those, already in existence. Xu and Zhou (2010), proposed a method for determining the stress and displacement distribution of transversely loaded isotropic rectangular plates with continuously varying thickness and simply supported at four edges in which based on the three-dimensional elasticity theory, in their work on the three dimensional elasticity solution of transversely loaded isotropic rectangular plates with variable thickness. The general expressions for the displacements and stresses of the plates under static loads which exactly satisfies the governing differential equations and the edge conditions of the plate were derived. The double Fourier sinusoidal series expansions to the upper and lower surface of the plate, was used to determine the unknown coefficients in the solution. The use of software gave results from a 3-D finite element simulation with a high level of correlation, confirming the exactness and effectiveness of the method.

2.2.4 Thick Anisotropic Plate

While the plate whose ratio of the small dimension to its thickness (a/t) is greater that 20 is considered as thin, the case where the ratio is less than or equal to 20, is considered as thick plate. These facts were considered while deriving the governing equations and natural boundary conditions of an anisotropic laminated plate. Both the Ritz method of plate analysis and Galerkin approach are based on energy principles. The Ritz method of analysis came up with a more convenient approach for the derivation of the approximate solutions to edge value (boundary result) challenges. The method also was found to be very suitable in resolving the bending, buckling, and free vibration problems in plate element. Due to the fact that in the equations of equilibrium, or equations of motion for the case of dynamic challenges, are mainly satisfied approximately, the both the Ritz method of analysis and Galerkin methods are considered to a great extent as approximate solutions. The secret of deriving convergence to an exact solution for both the Ritz and Galerkin methods is simply by making suitable set of functions to represent the displacements. To great extent, both the Ritz and Galerkin methods often provide useful tools for obtaining solutions to complex boundary value problems.

When anisotropic materials are stressed in one of the principal directions, the lateral deformations in the other principal directions could be smaller or larger than the deformation in the direction of the applied stress depending on the material properties. For a general anisotropic material, the matrix of material constants contains twenty one (21) independent material constants, because of symmetry. This means that all the strains are coupled to all the stresses. Some materials such as wood, plywood, delta wood, and fiber-reinforced plastics, etc., fall into this category. These materials possess natural anisotropy. Besides the plates made of anisotropic materials, a number of manufactured plates made of isotropic materials also may fall into the category of anisotropic plates: examples include corrugated and stiffened plates, etc (Ventsel and Krauthammer, 2001). Such type of anisotropy is referred to as structural anisotropy. When an anisotropic material has three mutually perpendicular planes of symmetry with respect to its elastic properties, it is called orthotropic (i.e., orthogonally anisotropic). Practical applications of orthotropic plates in civil, marine, and aerospace engineering are numerous and include decks of contemporary steel bridges, composite-beam grid works, plates reinforced with closely spaced flexible ribs, and reinforced concrete plates.

2.3 Buckling of composite plate

The mode of buckling most of the time depends on, the loading pattern on the plate, the dimensions of the plate and the type of support, that is given to the structure . Buckling loads are usually lower than those that are likely to cause failure on the plate member and the simplest kind of buckling comes up whenever compressive loads are introduced on plates that are simply supported in opposing boundaries and the unloaded edges are free, as shown in Figure 2.2.

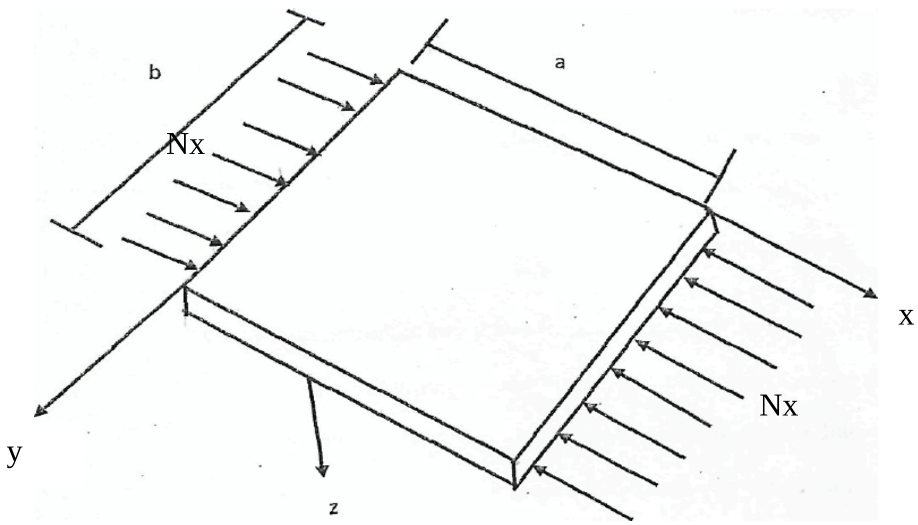


Figure 2.2: A plate with in-plane load

A plate member built in this way most times acts in the same way like the plate that has fixed edges. Here the critical load can be derived or obtained by applying the Euler theory. On attaining the critical load, the plate is unable to carry any further load. Plates of different shapes are often subjected to normal compressive and shearing loads acting in the middle plane of the plate (in-plane loads), when being applied in construction work like naval and aeronautical structures. This actually can give rise to buckling under certain conditions, as a result of the load. Buckling or elastic instability of plates is of great practical importance.

Szilard (2004) explained this state of stable equilibrium, adopting the simple analogy which considered the various states of equilibrium of a rigid sphere. He concluded that "if the sphere is supported on a large concave bowl, its equilibrium is said to be stable". Szilard (2004) added by recording that "If we try to stress the equilibrium condition by introducing a small displacement dx, the sphere, after some oscillations, comes back to its former shape.

In a situation where the plate is displaced from this equilibrium position by a small lateral load, the initial configuration of equilibrium is said to be unstable, the displacement goes on still further even when the acting force / load is withdrawn. Before now, many researchers have conducted several research works both on the buckling of plates and the conditions of composite plate elements using different approaches.

Hu & Lin (1995), carried out research on the buckling resistance of laminated plates element which loaded symmetrically. This they did using material system which was subjected to uniaxial compression. Their work was conducted using plates of different material properties. These includes thickness, aspect ratios, central circular cutouts and different edge orientations. Because of the differences in the properties mentioned, the optimal fiber arrangements, together with the related optimal buckling loads of symmetrically laminated plates were also examined.

Taking into account the needfulness of buckling analysis of composite structures in various industrial applications, Darvizeh et al. (2004), carried out a comparative research on the buckling characteristics of composite plates. These they did using mathematical modeling developed in their work, for the case of generally laminated plate elements. The study also was done by adopting the Rayleigh-Ritz method and generalized differential quadrature rule (GDQR). Considering both the higher-order shear deformation theory and Ritz displacement functions, relating to an arbitrary edge support, Xie et al. (2005) carried out investigation on the buckling of laminated composite plates which are resting on the internal supports. They also investigated the buckling load under the influence of biaxial compressive load, taking into consideration the effect of angle of lamination, end conditions, aspect ratio of the plate, and the support, that is, the internal supports on critical buckling load.

In addition to the stability of plate element, Ni et al. (2005) studied on the buckling analysis for rectangular laminated composite plates with biaxial compressive load acting on them. They adopted higher-order shear deformation theory and a special displacement function, which could express an arbitrary boundary support. This was finally brought into the method adopted by Rayleigh-Ritz, and then the buckling modes were derived.

Zhong and Gu (2007) worked on stability of plates and also the exact solution for buckling of simply supported plate element. The plate element was considered to be supported symmetrically by cross-ply composite rectangular plates, under a linear boundary load. This was formulated for moderately thick laminated plates, considering the first-order shear deformation theory. Buckling loads of cross-ply rectangular plate elements were obtained for different aspect ratios. In this work also, investigation was carried out to determine the effects of load intensity variation and lay up configuration on the buckling load, followed by the verification of their, results with the aid of the computer code ABAQUS.

Also on unified higher-order models obtained within the frame work of the Carrera Unified formulation, Ibrahim et al. (2012), carried out a buckling analysis of composite thin wall beams. They adopted refined theory in their work and the buckling characteristics of flat panels and laminated composite beam were brought into study to completely demonstrate the efficacy of the work. It is however noted that shape functional, was assumed in their work.

Kshirsagar and Bhaskar (2008), illustrated more accurate free vibration analysis. Their work was on the clamped unsymmetric cross-ply/Antisymmetric angle-pIy/Functional graded rectangular plates, adopting superposition of truncated infinite series. The approach was used to analyze the case of all round clamped rectangular plate elements with different boundary conditions. Thin rectangular plate was considered to posses both the bottom Elastic modulus and the top Elastic modulus. Based on higher order zigzag theory, Singh and Chakrabarti (2012), developed stability analysis of laminated composite plates using an efficient C° FE model. Buckling of laminated composite plates were solved using finite element model, in their work. They also considered different features such as edge conditions, ply arrangement, the aspect ratio of the plate and the thickness of the plate element, with the thickness ratio, ranging from 5 to 100. With the use of finite element method, Osman et al. (2017), illustrated the buckling analysis of thin laminated composite plates, and considering various combinations of boundary conditions and aspect ratios the governing differential equation of rectangular plate was formulated. Their work appeared to be in line with other results from other sources, making the finite element method more reliable.

2.4 Existing works on stability of plates

The new version of differential quadratic method was used by Wang et al. (2006) to derive buckling loads of thin rectangular plates for the case of non-uniform distributed in-plane loading. Two different approaches were adopted by them in computing the results.

Firstly, by solving a problem in plane stress elasticity to derive the in-plane stress distribution

Secondly, by solving the buckling problem under the influence of the loads obtained in the first step. Yu and Schafer (2007) carried out an analysis on the effect of longitudinal stress gradients on the elastic buckling of thin isolated plate and two groups of thin rectangular plates. These includes a plate that is resting simply on all four boundaries or edges and also a plate element that is simply supported on the three edges with one longitudinal edge free. Here also the opposite longitudinal edge is restrained against any form of rotation.

Many researchers before now have done a lot of works on the buckling analysis of plates. Kobayashi and Sonala (1990) worked on the buckling of rectangular plate considering tapered thickness. This was done using the power series, employing coordinate transformation. The work gave an accurate buckling load parameter with rapid convergence, and they considered that the buckling load depends on the plate thickness variation to a very large extent. Usami (1993) generated a formular for effective-width of the plate element. This was used to predict the strength variations of various plate assembled members in combined compression and bending cases. He determined the ultimate strength of simply supported, initially imperfect steel plates in compression and bending. This he handled using an elastic-plastic large-displacement finite element approach. Not long after this, Ye (1994) adopted iterative boundary element and finite element approach to study the nonlinear behaviour of rectangular thin plates, with initial imperfections. The researcher detailed the imperfection in double Fourier series and iterative boundary element and finite element approach were adopted, in his work. Using the Spline function in both the iterative element and finite element method, the imperfection was expressed by double Fourier series. Navin et al. (1995) used Rayleigh-Ritz solution together with a variational formulation and a first-order transverse shear deformation theory to study the problem of buckling of arbitrary quadrilateral anisotropic plates with different types of boundary conditions under combined in-plane loading. Several values were gotten for isotropic, orthotropic and anisotropic plates with skewed geometries. Suemasu et al. (1996) conducted an experimental investigation on the compressive buckling behavior of orthotropic plate with a free edge delamination considering Rayleigh-Ritz approximation approach. Values of the buckling load of the geometrically admissible buckling mode obtained, shows that low local delamination buckling can occur at small load value when the lamination is closer to the surface and its size appears bigger when compared with that of the plate.

Xiang et al. (1996) conducted further work in this area by applying the first-order shear deformation plate theory in the analysis of elastic buckling of symmetric cross-multiple laminated rectangular plates with two parallel boundary. The researchers got the buckling load factors by using a generalized Levy solution in solving the differential equations that govern the buckling tendency of the laminates plates. Yao et al. (1997) studied the buckling strength of different categories of rectangular plates under combined pressure and thrust. They examined the influence of plate continuity and stiffeners on its buckling plastic collapse behavior and also examined the influence of loading sequence of pressure and thrust loads. They derived a semi-empirical formula to evaluate the elastic buckling strength of different range of rectangular plate. Their works considered the plates under the influence of both the lateral pressure and transverse thrust with the influence of plate continuity and stiffeners under consideration. The method of load control procedure with direct iterations was adopted for the continuation. The Rayleigh–Ritz kernel particle method for the post-buckling analysis of shear deformable laminated plates was presented and here, the parameter continuation and the arc-length continuation procedures were used for the solution of the resulting nonlinear algebraic equations. Before this, the nonlinear deformations of a long rectangular elastic plate clamped along its boundaries and subjected to in-plane biaxial compression has been resolved by other researchers. The perturbation technique combined with Rayleigh–Ritz method, was used for the local and global post-buckling analysis of stiffened rectangular plates. Capitalizing on the advantage of this development, a mixed load–displacement control approach was developed to allow the continuation through limit and turning points on a balanced scale. In the course of the work, different types of continuation algorithms were formulated based on the finite element approach.

(Sun and Harik, 2013) used strip method to analyse the buckling of stiffened anti symmetric laminated plates This they did by reducing the system of three equations of equilibrium governing the buckling response of anti-symmetric laminated composite rectangular plates to a single eighth-order partial differential equation in order to formulate the critical value of the in-plane loads. The application of differential quadrature and harmonic differential quadrature in buckling analysis of thin rectangular isotropic plates were compared by Omar (2004). From the work, it was discovered that not only that Harmonic Differential Quadrature approach gives more accurate values in terms of results but also needs less grid points than the Differential Quadrature approach. Next to this was Azhari et al. (2004), who developed a numerical approach, applicable both for the post-buckling analysis of skew and trapezoidal plates, considering the virtual work pattern together with the natural coordinates. The researchers in their work detailed how the post-buckling equilibrium path is controlled by the Hookean strains. Their work also gave account of how the total potential energy generated, is dependent on the compressive strain rather than stress. From the work carried out by Weaver (2005), the derivation of the general buckling loads equation for the case of long rectangular plates element with flexural anisotropic condition and simply supported boundary, adopting the finite element analysis, was made possible to minimize the difficulties associated with the derivation using bounded non-dimensional quantities. Azhari and Bradford (2005) did extensive work on the use of bubble functions for the post-local buckling of plate assemblies using the approach of finite strip. Their work shows that the use of bubble functions improves greatly the convergence of the method with previous researchers. The two different refined plate theories were adopted for stability analysis of both the isotropic and orthotropic plate elements. The use of shear correction factor was neglected since the theory takes account of transverse shear effects and parabolic distribution of the transverse shear strains through the entire thickness of the plate element. The governing equation was derived using the principle of virtual displacements.

Ahmed (2008) used numerical iteration approach to analyze the geometrical non-linear vibrations of clamped circular plates with different ranges of thickness by taking the effect of large amplitude motion. Taking the maximum thickness to be at the middle of the plate and twice the value of thickness at the boundary, he observed that as the ratio increases, the nonlinear frequency increases also and this modifies the corresponding mode shape. Namiranian et al. (2009) carried out a study on the compression of plate and buckling behavior of fused fabric composite. The study was done using a special designed clamp according to Dahlberg's test approach. It was discovered that the buckling behaviour of fused fabric composite against lay-up interlining direction is in line with interlining buckling characteristics. The work shows that fusible interlining lay-up angle have great effect on the buckling parameters. Using Navier Method in a closed form solution, Seung-Eock et al. (2009 derived the values of a simply supported rectangular plate subjected to in-plane loading. Xu and Zhou (2010) carried out a research work on the three dimensional elasticity solution of transversely isotropic rectangular plates with variable thickness. In their work, they proposed a method for deriving the stress and displacement distribution of transversely isotropic rectangular plates element having continuous varying thickness and simply supported at four boundaries. Considering the three-dimensional elastic theory, the general expressions for the displacements and stresses of the plates under static loads, that satisfies the governing equations and the boundary conditions of the plate are derived. Adopting the double Fourier sinusoidal series expansions both the upper and lower case of the plate element, the unknown coefficients in the solution are approximately determined. Using the two variable refined plate theories, Piscopo (2010) studied the refined for buckling analysis of rectangular plates under uniaxial and biaxial compression. His work considered for the case of one and two orthogonal directions. His results were compared with that of other researchers who adopted the classical thin plate theory, just to show the feasibility of his research.

Rakesh (2010) also adopting the principle of finite element procedure, modeled the elastic buckling of thin plate element together with cold-formed steel members in shear. From his work, a shear buckling energy solution and rotational springs were obtained. This rotational springs is of great importance in quantifying the influence of cross-section connectivity on the shear buckling stress. Ferreira et al. (2011) did a research work on the stability analysis of isotropic laminated plates. This was based on third-order shear deformation theory of Reddy using collocation with radial basis function to determine the buckling loads values of elastic plate members. This is the reason for parabolic distribution of the transverse strains through the thickness of the plate. Similar to this study, Tajdari. (2011) used finite element approach to work on the effects of plate-support conditions, aspect ratio, and hole-size on the mechanical buckling strength of the perforated plates. Here the plate is assumed to be subjected to linearly varying loading conditions. The study showed that the increase in the hole-size does not necessarily bring reduction on the mechanical buckling strength of the perforated plates. In addition that the mechanical buckling strength of the plate increases on clamped boundary condition more than on the place with simply supported boundary condition. The edge with the free boundary conditions improves the mechanical buckling strength of the perforated plate more effectively compared to boundary with the fixed support condition. Ali (2011) used equilibrium method to study the critical buckling load of thin plate. He also gave full detail of a finite element model for a simply supported and simply supported - simply supported – fixed – free rectangular plate. ABAQUS (V. 6.7) software was used for the study.

Sayyad and Ghugal (2012) in their study used the theory of exponential shear deformation to obtain the buckling analysis of thick isotropic plate member under the influence of both uniaxial and biaxial in-plane loads. This explains the parabolic distribution of the transverse shear strains across the thickness and satisfies the zero fraction boundary conditions on top and bottom areas of the plate without using shear correction factor. The principles of virtual work was finally adopted to derive the governing equation, together with the boundary conditions. Sandeep et al. (2012) using finite element approach, solved the buckling challenge of plate, using eight node quadrilateral element and plate kinematics. This they based on first order shear deformation approach and the study reveals that the buckling strength of the square plate is highly influenced by partial edge compression, as compared to plate subjected to uniform edge compression, but with increase in aspect ratio, influence of partial edge compression on plate buckling load decreases. Luiz et al. (2013) in their research work, used constructed design to optimize the geometry of simple supported, rectangular thin perforated plates subjected to the elastic buckling. They used three different centered whole shape like elliptical rectangle to get the critical buckling load. Bhaskara and Kameswara (2013) used the classical plate theory to derive the governing equation for annular plate with elastically restrained boundary support system to derive the critical buckling load parameters for ax symmetric and asymmetric buckling modes.

The investigation of the different loading patterns of the core plate high-mode buckling phenomenon was carried out by An-Chien et al. (2013) using cyclic loading test and finite element analysis. The values gotten from their research shows that the proposed buckling-restrained braces (BRBs) can sustain large cyclic strain reversals and cumulative plastic deformations in excess of 400 times the yield strain. In their conclusion, finite element analysis can be used to predict accurately the high-mode buckling wavelength. Srinivasa et al. (2012) carried out similar but on buckling of laminated composite skew plates. This was conducted using finite element method. From their results, it's deduced that the critical buckling load factor increases with skew angle and the variation of critical buckling load factor with the number of layer is not appreciable. Ibearugbulam and Ezeh (2013) adopting as a shape function Taylor-Mclaurin's series carried out the analysis of the instability of axially compressed Clamped-Clamped-Clamped-Clamped thin rectangular plate. Taylor-Maclaurin series was truncated at the fifth term, in deriving the shape function and this satisfied all the edge conditions of the plate. This later resulted into the critical buckling load of the plate shape functions they derived and the values were finally substituted into the total potential energy. Sayyad and Ghugal (2014) carried out research on buckling and free exponential shear deformation theory. Their work was also extended to vibration analysis of orthotropic plates. They produced the critical buckling loads and natural frequencies of orthotropic plate elements. Using present higher order shear deformation theory they obtained values, that were found to be in agreement with those derived by other several existing 15 higher order theories. This is applicable in the buckling and free vibration behavior of orthotropic plates. Using a two-dimensional improved Fourier series method,

Yufei et al. (2014) investigated a series solution for the case of in-plane vibration analysis of orthotropic rectangular plates. The study was narrowed down to elastically restrained boundaries. In their work, in-plane displacements were represented as a double Fourier cosine series and four supplementary functions in the form of the product of a polynomial function, with their values satisfying both governing differential equations and the boundary conditions on a point-wise basis. In the same year, Da-Guang (2014) adopted deformation theory to investigate the model of functionally graded material (FGM) rectangular plates having various supported edges. These boundaries are considered to be resting on two-parameter elastic foundations. In the study, the temperature was considered to be dependent and changes along the thickness of the plate while Poisson's ratio depends slightly on temperature. Based on Mindlin assumption, a simple mathematical approach for solving the differential equations, governing the buckling and bending analysis of thick rectangular plate was proposed by Fatemeh (2016). His work considered a case of simply supported plates resting on two-parametric foundation. The plate elastic modulus was assumed to differ according to a simple four parameter power law across the plate thickness. For the bending analysis both sinusoidal and uniform loads were introduced to the plates while for the case of buckling analysis, both uniaxial and biaxial in plane loads were applied to the plates. The differences of Functionally Graded, FG Material profile, thickness ratio and foundation parameters on buckling critical load and out of plane displacement were investigated. The values obtained both for the case of bending and buckling analysis proved to be similar with those presented by Civalek (2009) and Thai and Kun. (2015).

2.5 Study on uniaxial buckling

Bouazza et al. (2009), work on the buckling analysis of functionally graded plates, considering the case of edges that are simply supported. Their work covered, the use of the first order shear deformation theory in the study of the thermal buckling analysis of Functionally Graded Materials (FGM). In their study, the material properties were varied continuously in the thickness direction according to a sigmoid distribution, followed by the analysis of the thermal buckling behaviors under linear, uniform and sinusoidal temperature rise throughout the plate thickness. The effect of volume fraction distribution, system geometric parameters and temperature field were also studied. The values they obtained were compared with those got, applying the classical plate theory and it was found that in thermal buckling analysis, the rapid decrease of the critical temperature gradient is brought about by increased in the geometric parameter a/h. It was also observed that as the plate aspect ratio increases, the critical temperature also decreased and the plate element takes a thinner shape. Singh and Chakrabarti (2012) formulated an efficient C^o FE model. They based their work on higher zigzag theory for the buckling analysis of plates. In their work, the first model differential values of transverse displacements were treated as independent variables so as to overcome the problem of continuity associated with finite element implementation of the plate theory. Adopting the penalty parameter approach in their model, they compensated the C^o continuity of their Finite element model in a stiffness matrix calculations. The numerical values for the normalized critical uniaxial buckling loads with different modular ratios for a simply supported cross ply square plate for a thickness ratio of 100, 20 50 and 5 were compared with those presented by Fieldler et al. (2011). The results obtained were in good agreement. For the case of thickness ratio of 10, results derived for the normalized critical uniaxial buckling loads were also confirmed to be in very good agreement. Based on the outcome, it was therefore concluded that proposed model is very good and useful in predicting the buckling behaviour of laminated composites thin plates, for the situation of clamped-clamped-clamped-clamped edges and also uniaxially loaded plate element.

Ventsel & Krauthermmer (2001) explained that for the solution of a biaxially uniformly compressed simply supported square plate. According to them, if the square plate is compressed in two ways, by two equal system of forces, the critical value of these forces is two times less than that for the case of square plate compressed by the same force, but introduced in one side. Jayashankarbabu & Karisiddappa (2013) carried out research work on the buckling of thick plate. The plate under study having an eccentric cut out. The finite element method was adopted, in their work, they derived the elastic buckling load factor for square plates of different boundary conditions (Simple-Clamped-Simple-Clamped, Clamped-Clamped-Clamped-Clamped, Simple-Simple-Simple-Simple) containing square and circular cutouts subjected to uniaxial compression. Here the loads applied are considered to be applied both at the simply supported and at the clamped edges. Similar to this Srinivasa (2012) using the finite element method, worked extensively on the buckling of laminated composite skew plates. His work carried out thorough investigation on the effects of the aspect ratio, skew angle, fiber orientation angle, length to thickness ratio, and numbers of layers in the laminate and laminate sequence on the critical buckling load factor for anti-symmetric composite laminates. From their findings, it shows that the critical buckling factor increases with corresponding, increase in the skew angle. Another significant observation is that the variation of critical buckling load factor with the number of layers is not appreciable, when the number of layers in the laminate is much. Like et al. (2015) came up with style of plate analysis, but the case of simply supported on all edges. As detailed in their work "Semi-analytical finite strip transfer matrix method for buckling analysis of rectangular plates", this method was adopted in the discretization of rectangular thin plates, with loaded edges being simply supported by the semi-analytical finite strip technology. The global stiffness matrix of the system is not needed in this method, and this reduces the matrix order and thereby improving the computational efficiency. From comparism of values derived from both this method and other existing values, it shows that the method is not just reliable and but effective.

2.6 Energy methods in stability analysis

Most at times, the energy methods usually referred to as the direct method of calculus of variation. These are amongst the most important approximate method of mathematical physics. These methods are mostly adopted in resolution of difficult value problems. The main purpose of the variational approach is to obtain from a group of admissible functions those that represent the deflections of the elastic body, as it relates to its stable condition. Szilard (2004) proposed that the increasing difficulty in the geometrical arrangement, boundary conditions of plate and loadings on plate, the rigorous mathematical determination of the buckling load of the plate becomes progressively more complex and at the end unachievable. According to him, the differences in the solution of the buckling analysis needs the evaluation of certain simple and definite integration. To this extent, the required computational work can be resolved by choosing suitable orthogonal shape functions. According to (Iyengar, 1988), there is need to look for approximate techniques which have the ability to converge to the exact solution and one of such technique is the energy approach. Still on the subject, Reddy (2004) went on to explain that equations governing a physical problem are themselves approximate, stating that that the approximations were brought about through many sources, including the geometry, representation of detailed loads and various edge orientations.

2.7 Importance of energy method

According to Szilard (2004), energy method have very good advantage from the following:

i. The methods are usually easier, both in terms of conception and its mathematical applications.

ii. The Energy methods are extremely strong tool in deriving reusable analytical solution even for plate of arbitrary shape and edge arrangement.

iii. The Energy method gives a valuable preparation for good assimilation the principles of FEM (finite element methods).

iv In determining the critical buckling loads, energy methods helps to reduce that problem both to the determination of certain definite integrals and the solution of eigen values problems. Among the methods includes,

2.8 Rayleigh Ritz method

This energy method work with the principle of minimum potential energy. According to (Ventsel and Krauthammer, 2001), in Rayleigh-Ritz energy approach, all displacements that satisfy the edge conditions, those making the total potential energy of the structure to be minimum are the such deflections pertinent to the stable equilibrium state. After choosing the shape functions, substitute the edge conditions into it, to minimize it to a peculiar shape function and also putting this peculiar shape function into it to reduce it to potential energy functional. This will finally be integrated over the domain to reduce it to a function of generalized coordinates. The method has a good advantage over the equilibrium approach because it can resolve challenges (though approximate) that are very difficult to handle using equilibrium approach. It also has the ability to converge to exact solution as the number of terms in the assumed shape function tends to infinity.

Advantages and disadvantages of Ritz method

Ventsel and Krauthammer (2001) detailed the advantages of Ritz methods over
other methods as stated below;

i. The basic advantage lies on the fact that the coordinates functions Fi(x,y) must satisfy the kinematic (or geometrical) boundary conditions only. Therefore, the area of an application of the method to the plate bending problems is wider than that of classical analytical method to the plate bending. Thus, the Ritz method is very efficient for the analysis of plate having free edges and for plates with openings.
ii. The matrix of the linear algebraic equation is always symmetrical, resulting in stable and powerful logarithms for their numerical solution.
iii. Ritz method can be applied to rectangular plate of variable thickness successfully, because there is no difference between the expressions of for plates of constant and variable thicknesses.
iv. Ritz method serves as the basis for finite element method.

According to Ventsel and Krauthammer (2001), the disadvantages of Ritz methods are;

i. Ritz method can be applicable only to simple configurations of plates (rectangular, circular etc) because of the complexity of selecting the coordinate functions for domains of complex geometry.

ii. The Ritz method approximation results in the complex matrix of linear algebraic equations that produce some difficulties in its numerical implementation.

2.9 Galerkin's method

This method can be applied successfully to different types of problem as small and large deflections theories, linear and non-linear vibration and stability of plates and shells, provided that the differential equations of the problems under investigation have already been determined. Ibearugbulem et al (2014) stated that the following assumptions were made during the derivation of the Galerkin's equation:

(i) The governing differential equation is an equilibrium equation of all forces.

(ii) Multiplying the governing differential equation by the plate displacement shall result to work done on the plate.

(iii) Since continuity exist within the domain of the plate and throughout the loading period, then both the loads and the plate made the same displacement.

(iv) Since the load and the plate made the same displacement then the total work performed by them must be in equilibrium.

Comparing the Galerkin's method with the Rayleigh-Ritz method, Iyengar (1988), stated that "unlike the Rayleigh-Ritz technique the approximating function in Galerkin method must satisfy both the geometric and natural (Kinematics) boundary conditions. This restricts the choice of the approximate function. He further emphasized that "many a time, the results obtained by this method is better than that given by the Rayleigh-Ritz technique. Szilard (2004) in his own conclusion stated that the Galerkin's method is more general than the Ritz method. Other energy methods applied in the analysis of plate includes the Kantorovich method, in which Kantorovich introduced a solution procedure that falls between the exact solution of the plate differential equation and Galerkin's variational approach. He separated the variables and reduced the task of solving partial differential equations to solving ordinary differential equations of the fourth order. Also we have the Vlasov's method, the principle of conservation of energy, Finite Element Method, Morley-Trefftz method, the modified Castigliano's theorem, the complimentary energy method, weighted residual and the Hellinger-Reissner variational principles.

2.10 The classical plate theory

Classical plate theory is mainly governed by Kirchhoff's hypotheses. This theory is also known as small deflection theory. According Iyengar (1988), Ugural (1999), Ventseland Krauthammer (2001) and Ibearugbulem (2014). The idea of the hypotheses, are :

i. The deflection (w) which occurs at the mid-surface (mid-plane) is small when compared with the thickness of the plate, and in some cases, its assume the plate deflection is less than 20% of the plate thickness ($w \leq 0.2\ t$), considering the thickness of the plate as t. The plate deflection have a slope (Θ) that is so small, that its square is negligible when compared to one ($\Theta \leq 0.1$ radians).

ii. Before, during or after bending, it's observed that, the mid-surface (neutral surface/neutral plane) of the plate remains unstrained. It does neither compress nor stretch.

iii. A cross plane (cross surface) that is initially straight before bending shall remains the same after bending. Assume the plate is made of xy plane as the major horizontal plane and xz and yz planes as the two vertical planes of the plate. xz and yz planes that are straight (not curved) before bending shall remain straight after bending. Hence, the vertical planes remain unstrained (not stretched and not compressed). Consequently, the vertical normal strain ε_z and the vertical shear strains γ_{xz} and γ_{yz} are assumed to be zeros.

iv. According to Ibearugbulem et al. (2014), the stress (due to distributed load) applied perpendicular or normal to xy-plane σ_z is considered to be of no effect (negligible).

Plates with two opposite edges simply supported and the other two edges free subjected to a central line load were studied as a specific example. Three different thicknesses including thin, moderately thick and thick plates were considered in the research. It was shown that by employing a pattern, stress difference at the critical sections of the plate could be obtained. Numerical results were compared with those from a thin plate theory and a higher order thick plate theory and were found to be approximate. Guedes & Gordo (1995) investigated the compressive strength of rectangular plates under biaxial load and lateral pressure. Sebastian (1980) came up with detailed approach for investigating thick rectangular plates using three-dimensional photo-elasticity using the stress-freezing technique. They formulated equations to solve the strength of plates subject to biaxial compressive loads considering the effect of initial distortions and residual stresses. These equations to solve the strength of plates subject to biaxial compressive loads, were then extended to the case of simultaneous lateral pressure loads and were shown to be consistent with slenderness and aspect ratio of the plate.

Khamail (2011) further conducted research work on the out of changing ratio on geometrical non- linearity of thin plate element (steel) under transverse load, with the main aim of determining whether the structural efficiency of structural plates could be increased with different thickness. Also, the large displacement analysis of structural plate element (steel) considering variable thickness at the x-direction numerically, using the finite differences approach was conducted by him. The edge conditions of the plate element, alteration of the ratios, tapering of the equation and aspect ratio of the plate on large deflection behaviors of rectangular plates were also studied. From his research work it is observed that the large deflection is very sensitive for thickness variation (tapering ratio) where the maximum deflection will improve with about five percent for slenderness ratio and tapering ratio of simply supported plate.

Bejan’s contractual design to optimize the geometry of simply supported, rectangular thin perforated plates subject to elastic buckling, was adopted. Three different center hole shapes were considered. These holes includes elliptical, rectangular and diamond. They used an objective function to maximize the critical buckling load and kept the degree of freedom (ratio between the length of the plate) constant while the ratio between the characteristic dimension of the holes was optimized for several fractions. The Lanczos method, based on the finite element method was employed in conducting the analysis. The outcome of the numerical analysis showed that, for smaller values, the optimum geometry is the diamond hole and that for average or larger values of the elliptical and rectangular hole respectively.

2.11 Boundary conditions of a plate

In the analysis of a plate, that differential equation of equilibrium derived, during the distribution of stresses in a plate, is referred to as Boundary conditions. The boundary conditions of plate must be with respect to given stresses or displacement at the boundary of the plate element (Ugural 1999).These conditions provides enough information on the surfaces of a plate which must be well explained and resolved in order to generate suitable result. Depending on the condition of the plate whether simply, clamped or free edge support that causes force and moment or both, the slope and deflection or this normally take place at the edges of a plate. In the analysis of a rectangular plate, for plates with C, Clamped edge or F, Fixed edge, the deflection and slope are zero while in the case of S, Simple edge support, the deflection and bending moment Mx are both zero. In a situation where thee Free support at the boundary condition , no stress acts over this boundary, therefore stresses and the stress couples occurring at this boundary edge are equated to zero. For the case of an isotropic rectangular plate, the non- essential and essential boundary conditions are the two types of boundary conditions. Those conditions that are related to the geometry of a continuum at the boundaries (shape) are considered as essential conditions. Most times are called geometric or kinematic boundary conditions. Examples of kinematic boundary conditions are the deflections and slope (rotation) at the boundaries. The two examples are the conditions of bending moment and shear force at the boundary of the continuum. Those conditions that are related to the mechanical behavior of a continuum are referred to as Non-essential boundary conditions. They are also called the force boundary conditions or dynamic conditions.

2.12 Features of a thin rectangular plate

The fundamental assumptions of the elastic, linear, small – deflection theory of bending in thin plates element, can be detailed as follows, according to Ventsel and Krauthammer (2001)

(i) The plate element is elastic, homogeneous and isotropic. That means the flexural rigidity, poison ratio, young elastic modulus are uniform in all the directions of the plate.

(ii). The plate member is characterize with flat surface.

(iii) The thickness of the plate appears bigger than, the deflection at the mid plan of the plate. The slope of the deflected surface is therefore very small and the square of the slope is a negligible quantity in comparison with unity.

(iv) The "hypothesis of straight normals" is observed. This is a situation where the straight lines, initially normal to the middle plane before bending, maintains a straight shape and normal to the mid surface during the deformation, without any alteration in the length.

(v) The stress perpendicular to the mid plane surface of the plate element is of small value when compared with the other stress component and may be ignored in constitutive relations.

(vi) It’s assumed that the middle surface remains unrestrained, since the displacements of a plate is small after bending.

2.13 Features of a thick rectangular plate

From the look on the nature of a thick rectangular plate it can deduced that a plate has to be analyzed as a three dimensional case. This reason is simply because of the mathematical difficulties that may arise in the analysis of such plate. To overcome this, the plate is reduced either to two dimensional or one dimensional case, considering the following facts;

i. The thickness of the plate is very small as compared to the other two other dimensions (Iyengar, 1988). This assumption requires the satisfaction of only two boundary conditions, though there are three of such conditions at each boundary of a plate element.

ii. The transverse shear deformation is not put into consideration, in thick plate theory.

Table 2.1 contains the summary of the subject and facts together with the observed differences between the present work and that of the previous researchers.

Table 2.1: Summary of Previous Research Works on the Analysis of Thick Plates.

S/No.	Author(s) and Period	Subject and Facts	Observed Diff.
1	Ali (2011)	Numerical Study of Buckling of Thin Plate The author used equilibrium method to study the critical buckling load of thin plate. He also gave full detail of a finite element model for a simply supported and simply supported - simply supported – fixed – free rectangular plate. ABAQUS (V. 6.7) software was used for the study	(i) The author adopted only software in the analysis of the work (ii) The case of thin rectangular plate was considered. (ii) Only Simply Supported and Fixed boundary was considered in his work.
	An-Chien et. Al.(2013)	High – Mode Buckling-restrained Brace Core Plates The authors adopted Cyclic Loading Test and Finite Element Method in their analysis. Their work yielded buckling wavelength but no account	(i) Polynomial shape functions were not considered in their theory. (ii)Their

2		of buckling coefficient.	analysis was computer based. (iii)Third order energyfunctional was not used.
3	An, C., & Gu, J.J, (2011).	Integral Transform Solution of Bending Problem of Clamped Orthotropic Rectangular Plates. The authors derived the exact solution for the Pure Bending problem of fully clamped Isotropic Rectangular Plates, using the generalized Integral transform technique.	(i) The authors studied the case of fully Clamped Plate. (ii) Their research didn't include the case of Anisotropic Thick plates. (iii)Their work did not cover buckling of laminated plates.
4	Abdul-Razzak, et.al.(2007)	Free Vibration Analysis of Rectangular Plates using Higher Order Ordinary finite Layer Method. The authors limited their work to free Vibration Analysis. External forces are assumed to be applied slowly in static analysis, in such a way that the loads and the resulting stresses and deformations are independent of time. Many components of machines and structures are subjected to dynamic effects, produced by time-dependent external forces or displacements, in engineering practice	(i)The authors limited their work to their Free Vibration Analysis . (ii)Third order energy functional was not adopted in their work
		Bending, Vibration and Buckling of Laminated Composite Plates Using a Simple Four Variable Plate Theory. They used simple trigonometric shear deformation theory for bending, buckling and free vibration of cross-ply laminated composite plates. The theory involved four unknown variables which were five in first order shear deformation theory or any other higher order theories. The	(i)The author's work was based on trigonometric function. (ii)Third Order Energy functional was not adopted in analysis.

5	Sayyaad et al. (2016)	in-plane displacement field in their work used sinusoidal function in terms of thickness coordinate to include the shear deformation effect. Their work satisfied the zero shear stress condition at the top and bottom surfaces of plate without using shear correction factor. Equations of motion associated with their theory were obtained using the dynamic version of virtual principle . A closed form solution was obtained using double trignometric series suggested by Navier.	
6	Gunjal et.al (2015)	Buckling Analysis of Thick Plates using Refined Trigonometric Shear Deformation Theory. Their work considered only two unknown variables. Their formulated theory does not require shear correction factor. The governing equations and boundary conditions were obtained using the principle of virtual work. Analytical solutions were obtained using Navier solution technique	(i)The polynomial shape function was not considered in their shear deformation theory (ii)Only one boundary condition(SSSS) was considered in their work for the investigation.
7	Ibrahim et al. (2012)	Buckling of composited thin walled beams by refined theory. The researchers worked on stability analysis of Laminated Composite thin walled structures by the ID finite element based on unified higher-order models obtained within the framework of the Carrera Unified Formulation.	(i)The author worked on thin walled beams. (ii)The work was based on Refined theory.
	Sayyadand	Buckling analysis of thick isotropic plates by using exponential shear deformation theory. Their analysis was done using displacement based and exponential shear deformation function also the	(i)The researchers didn't use polynomial displacement functions

8	Ghugal (2012b)	Governing equations and associated boundary conditions were derived from the principle of virtual work. The Navier's approach was adopted for solving the governing equations of square plates with all simply supported edges.	in their analysis (ii)They limited their work only one boundary condition (SSSS) for the investigation.
9	Singh and Chakarabarti (2012)	<u>Buckling Analysis of Laminated Composite Plates Using an Efficient C^o FE model</u> The author derived Buckling analysis of laminated composite plate using an efficient C^o FE model development based on higher order zigzag theory. In this model the first derivative of transverse displacement were treated an independent variables to overcome the problem of C^1 continually associated with the FE implementation of the theory.	(i)The authors used Finite element method and trigonometric functional. (ii) Their work was only on buckling.
10	Ivo, et.al. (2015)	<u>A new finite element formulation for vibration analysis of thick plates.</u> The authors adopted Bending deflection as a potential function for the definition of total deflection and angles of cross section rotations. Thisbrought about the introduction of interd=ependence among displacements, the shear locking problem, present and solved in known finite element formulations was avoided. Natural vibration analysis of rectangular plate, utilizing the four-node quadrilateral finite element, showed higher accuracy than the sophisticated finite elements incorporated in some commercial software.	(i) The polynomial shape function was not considered in their shear deformation theory (ii)Only three different boundary conditions; SSSS, CSCS and CFSS for the investigation, in their work
11	Ibearugbulem et al (2011)	<u>Buckling Analysis of axially compressed SSSS thin rectangular plate using Talor-Mclaurin shape function</u>. They presented a comprehensive	(i)Their work was limited to thin plate but not laminated plate.

		buckling analysis axially compressed rectangular thin plate with simply supported edges, applying Taylor-Mclaurin's series and Ritz Method.	(ii)Their work did not handle up to four compatible equations.
12	Fatemeh (2016)	A Theoretical Analysis of Static Response in Functionally Graded (FG) Rectangular Thick Plates with a Four-Parameter Power-Law Distribution. The author adopted a simple mathematical approach in resolving the differential equations governing the buckling and bending analysis of FG thick rectangular plates resting on two parametric foundation based on Mindlin assumption. In their work, the plate boundaries were considered as simply supported. He assumed the Young modulus of the FG plate to vary according to a simple four-parameter power law across the thickness direction. For bending analysis, he subjected the plate to two kinds of loading: sinusoidal and uniform loading. For bucking analysis, uniaxial and biaxial in-plane loading were applied to the plate.	(i)The author used a trigonometric function and not orthogonal polynomial as shape function. (ii)The authors used only one boundary condition (SSSS) for the investigation.
13	Lin (1973)	Free Transverse Vibration of Rectangular Laminated Plates. The author adopted was Kirchhoff assumptions in conjunction with the small deflection theory, governing equations of laminated plates were derived directly from energy principles	(i)The author used a trigonometric function base on assumed displacement function. (ii)The Stability analysis was not considered .
		A two variable refined plate theory for orthotropic plate analysis. The authors formulated a theory which gave rise to two governing	(i)Their polynomial Function is

14	Shimpi and Patel (2006)	equations which are completely uncoupled for static analysis, and are only initially coupled (i.e., no elastic coupling at all) for dynamic analysis. Also the number of unknown values involved is two, unlike in the case of simple shear deformation theories which is three.	expressed mathematically as $F(z) = 1/4(z/h) - 5/3(z/h)^3$ and not to third order.
15	Sayyad& Ghugal(2014)	Buckling and free-vibration analysis of orthotropic platesby using exponential shear deformation theory. The Navier approach was adopted in resolving the governing equations for the case of simply supported square orthotropic plates, also formulating their Governing equations and boundary conditions from the principle of virtual work. The values gotten using their theory were found to agree well with those gotten from other several existing higher order theories for analyzing the buckling and free vibration behaviour of orthotropic plates.	(i)In their work, the author did not use polynomial displacement functions as shape functions. (ii)Their Analysis was limited to Simple Simple Simple Simple boundary condition only.
16	Bhaskar and Kanshik (2004b)	Analysis of clamped unsymmetric cross-ply rectangular plates by superposition of simple exact double Fourier series solutions. The researcher adopted virtual work in deriving double Fourier series that was applied in solving the deflection of simple supported arbitrary laminated cross-ply plate. The deflection equation was formulated using the derived series.	(i)The work was for simply supported edges. (ii) The author did not work on polynomial shape functional.

17	Osman et al. (2017)	Buckling Analysis of Thin Laminated Composite Plates using Finite Element Method. The Fixed End Moment was used to obtain numerical solution of the governing differential equations. Stability analysis of Laminated plates with rectangular cross-section for variation combinations of boundary conditions and aspect ratios was treated.	(i)The author treated thin plate and void of Third order functional. (ii)The author did not work with polynomial shape functional.
18	Matsunaga, (2000)	Vibration and Stability of Thick Plates on Elastic Foundations. He derived a set of fundamental dynamic equations of a two-dimensional, higher-order theory for thick rectangular plates subjected to in-plane stresses adopting the method of power series expansion of the displacement components, through Hamilton's principle. The author also used several sets of truncated approximate principles in resolution of the eigenvalue problems for the case of simply supported thick elastic plate. The integration the three dimensional equations of motion in the thickness direction, gave rise to the distribution of modal transverse stresses	(i)In their work, the researchers did not adopt polynomial displacement functions, instead power series expansion functions was considered as shape functions (ii)Their Analysis also was limited to only to Simple Simple Simple Simple boundary condition.
19	Osman et al (2019)	Effect of boundary Conditions on buckling Load for Laminated Composite Plate. The Finite element method was used by the author to get the numerical solution of the governing differential equations. Effect of differential boundary conditions were observed for buckling modes. The assumed material parameters and modes are E/E2=5,10 ,20,25,40; G12 = G13 = G23 = 0.5E2; v12 = 0.25	(i) Only the buckling loads for the cases of SSSS, CCCC, CSCS were considered. (ii)The author did not work with polynomial shape functional.

Materials concluding from the extensive research done on the subject,none od the researchers have conducted Stability analysis of thick anisotropic plate using third energy functional. From the outcome of this present work, buckling analysis of thick laminated anisotropic having polynomial deflection of 3rd order can be resolved not just with third energy functional but with computer base program that reduce the long period of time needed for the needed for the analysis. The present is thereby justified in filling this vacuum in the structural world under the tittle “Stability Analysis of Thick Laminated Anisotropic Plate Using Third Order Energy Functional”.

CHAPTER THREE

MATERIALS AND METHODS

3.1 Materials

Several equations were adopted as fundamentals in carrying out this project. In the list include Deflection Equation, Total Potential Energy Equations, Governing Equation, Excel Program and Stiffness Coefficients. The polynomial rules were adopted in deriving the shape functions. The integration of the differential values of the shape function gave the needed stiffness coefficients. Gauss Elimination Formular was adopted in the resolution of the compatibility equations. The Lenghty equations derived, were further programmed using excel spread sheet.

3.2 Methods

The shape functions were derived by analyzing the shape of the various plate arrangements.
The stiffness coefficients were derived as well by integrating the derivatives of the shape functions. The constituent relations were generated and this formed the basis for the formulation of the Governing Equations. The 3rd Order Energy Functional formulated was added to that of external load to give the total potential energy equation. Further resolutions led to the formulation of the four compatibility equations.

In summary the entire research work were in two stages. The first stage of the work which is purely calculations, involves manually resolution while the later stage was done using computer program.

3.2.1 Derivation of Deflection Equations of a Thick Plate

The shape functions were obtained by considering two major support conditions, namely Simple Support edge, *S* and Clamped Support edge, *C*. In the case of a Simple Support, the deflection equation W and the 2nd order derivative of the deflection equation W^{11}, were equated to zero. This will give rise to simultaneous equations by taking R = 0 at the left hand support for X axis and Q = 0 at the top of the support for Y axis while $R = 1$ at the right hand support X axis and $Q = 1$ at the bottom support for Y axis. Also in the case of the Clamped edge, the deflection equation, W and 1st order derivative of the deflection equation, W^1, were equated to zero and simultaneous equations were formed by considering $R = 0$ at the left hand support for the X axis and Q = 0 at the top support for the Y axis, while at the Right hand support, $R = 1$ for X axis while $Q = 1$ at the bottom support for Y axis.

The generated equations were solved simultaneously, the results shall be the various values of the primary and secondary dimensions (a_1, b_1, a_2, b_2 a_3, b_3, a_4 and b_4), where R and Q are non-dimensional axis parallel to X and Y axis respectively. The deflection is finally expressed as the product of the shape function and the amplitude, as demonstrated in the case of Clamped Simple Clamped Simple plate element.

For the Horizontal axis, Fig 3.1 shows the horizontal component of the CSCS plate, with the two edges resting on simple supports. In line with Ibearugbulem et al (2014), deflection equation for plates can be expressed in a polynomial function as shown in Equation (3.1). The value R was considered as zero at the left hand support and one at the right hand support. This is due to the fact of the simple on both edges.

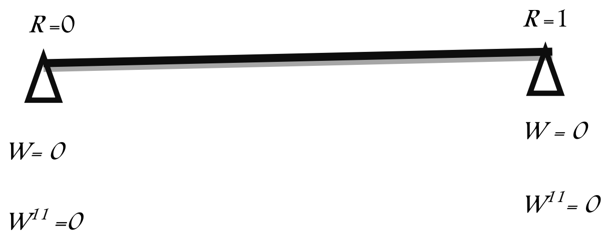


Fig 3.1 Horizontal component of the plate

Considering the deflection on the horizontal component, which can be expressed mathematically as shown in Equation (3.1).

$$W_x = a_o + a_1R + a_2 R^2 + a_3R^3 + a_4R^4 \qquad (3.1)$$

In non dimensional coordinate

Differentiating W_x gives:

$$W_x^1 = a_1 + 2a_2R + 3a_3R^2 + 4a_4R^3 \qquad (3.2)$$

Differentiation Equation (3.2) gives:

$$W_x^{11} = 2a_2 + 6a_2R + 12a_4R^2 \qquad (3.3)$$

Taking the left hand support as the origin with the deflection on the

X-component as zero

$$W_x = 0 = a_o + 0 + 0 + 0 + 0 \quad (3.4)$$

From Equation (3.4) it was deduced that

$a_o = 0$

Also when the value of R on the left hand support is substituted into the

Equation (3.3), that gives

$$W_x^{11} = 0 = 2a_2 + 6a_3(0) + 12a^4(R)^2 = 0 \quad (3.5)$$

therefore leaving Equation (3.5) as

$a_2 = 0$

Similarly, taking the right hand support as the origin, with the deflection

as zero and based on the various edge conditions as shown in Fig 3.1,

the deflection on x-component is as shown in Equation (3.6)

$$W_x = 0 = a_o + a_1(1) + a_2 (1)^2 + a_3(1)^3 + a4(1)^4 \quad (3.6)$$

further minimizing gives

$$W_x = a_o + a_1 + a_2 + a_3 + a_4 \quad (3.7)$$

From the first and second derivative of the deflection, it was deduced that

$a_o = a_2 = 0$ and substituting these values in to Equation (3.1) gives Equation (3.8),

considering R and W_x as 1 and 0 respcetively.

$$0 = a_1 + a_3 + a_4 \quad (3.8)$$

and further minimizing Equation (3.8) gives

$$a_1 + a_3 = - a_4 \quad (3.9)$$

Also when the second derivative of the deflection is zero, the equation is expressed as

$$W_x^{11} = 0 = 0 + 6a_3 + 12a_4 \quad (3.10)$$

At the edge R =1

therefore leaving Equation (3.10) as

$$a_3 = -2a_4 \quad (3.11)$$

Substituting $-2a_4$ for a_3 in Equation (3.9) gives

$$a_1 + (-2a_4) = -a_4 \quad (3.12)$$

further minimizing Equation (3.12) gives

$$a_1 = a_4$$

Putting the derived values of a_1, a_2 and a_3 into Equation (3.1) gives

$$W_x = 0 + a_4R + 0 + (-2a_4) R^3 + a_4R^4.$$

and further factorization gives

$$W_x = a_4(R-2R^3+R^4) \quad (3.13)$$

Similarly, for the case of the vertical component, Fig 3.2 shows the vertical component of the *CSCS* plate, with the two edges resting on clamped supports. According to Ibearugbulem et al (2014), deflection equation for plates can be expressed in a polynomial function as shown in Equation (3.14). The value Q was considered as zero at the top support and one at the bottom support.

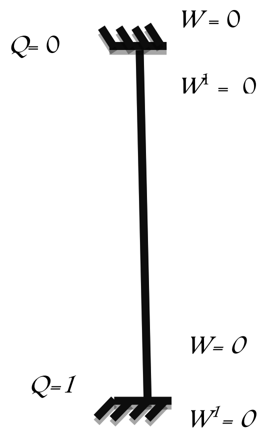


Fig 3.2 Vertical component of the plate

Considering the deflection on the vertical component, which can be expressed mathematically as shown in Equation (3.14).

$$W_y = b_o + b_1Q + b_2 Q^2 + b_3Q^3 + b_4Q^4 \quad (3.14)$$

Differentiating Equatio+n (3.14) gives:

$$W_y^1 = b_1 + 2b_2Q + 3b_3Q^2 + 4b_4Q^3 \quad (3.15)$$

Taking the top support as the origin and considering the edge conditions

as shown in Fig 3.2, when $W_y = 0$ gives

$$W_y = 0 = b_o + 0 + 0 + 0 + 0$$

which is further reduced to

$$b_o = 0 \quad (3.16)$$

also considering the top support with the Q and W_y^1 as 1 and 0 respectively gives

$$W_y^1 = 0 = b_1 + 0 + 0 + 0$$

and then leaving

$$b_1 = 0 \quad (3.17)$$

Taking the bottom support as the origin and considering the edge conditions as shown in Fig 3.2, when the deflection, $W_y = 0$ Equation (3.14) becomes

$$0 = b_o + b_1 + b_2 + b_3 + b_4 \quad (3.18)$$

Substituting Equations (3.16) and (3.17) into Equation (3.14) gives

$$0 = b_2 + b_3 + b_4 \quad (3.19)$$

That means

$$b_2 + b_3 = - b_4 \quad (3.20)$$

Also when the first derivative of deflection is considered, at the bottom support with under the same bottom edge conditions as shown in Fig 3.2, that gives

$$W_y^1 = 0 = b_1 + 2b_2Q + 3b_3Q^2 + 4b_3Q^3 \quad (3.21)$$

Factorizing Equation (3.21) further gives

$$2b_2 + 3b_3 = - 4b_4 \quad (3.22)$$

Solving Equations (3.20) and (3.22) simultaneously, yields

$$b_2 = b_4 \quad (3.23)$$

and $b_3 = -2b_4$ (3.24)

Substituting Equation (3.23) and (3.24) into Equation (3.14) gives Equation (3.25) after several minimization.

$$W_y = b_4(Q^2-2Q^3+Q^4) \quad (3.25)$$

Bringing the deflection on both the x-component and y-component together gives

$$W = a_4\, b_4\, (R-2R^3+R^4)\, (Q^2-2Q^3+Q^4) \quad (3.26)$$

Expressing Equation (3.26) in terms of amplitude gives

$W = Ah$ (3.27)

where

$A = a_4 b_4$ (3.28)

and

$h = (R\text{-}2R^3\text{+}R^4)\ (Q^2\text{-}2Q^3\text{+}Q^4).$ (3.29)

Equation (3.29) and (3.26) are the shape function and the deflection equations for the *CSCS* plate respectively. The same approach was adopted in deriving the shape functions of the remaining plates boundary conditions.

3.2.2 Formulation of The Stiffness Coefficients of a Thick Plate.

The derived shape functions of the plates were further differentiated at different degrees to give the $K -$ values. From these $K -$ values obtained, the stiffness coefficients were formulated by integrating the product of the differential values of the shape functions. These differential values include K1, K2, K3 and K6. The shape function for *CSCS* was given as

$$h = (R - 2R^3 + R^4)(Q^2 - 2Q^3 + Q^4) \quad (3.30)$$

Differentiating the shape function with respect to R gives

$$\frac{\partial h}{\partial R} = (1 - 6R^2 + 4R^3)(Q^2 - 2Q^3 + Q^4) \quad (3.31)$$

Differentiating again gives

$$\frac{\partial^2 h}{\partial R^2} = (-12R + 12R^2)(Q^2 - 2Q^3 + Q^4) \quad (3.32)$$

Differentiating the further gives

$$\frac{\partial^3 h}{\partial R^3} = (-12 + 24R)(Q^2 - 2Q^3 + Q^4) \quad (3.33)$$

Similarly differentiating the shape function with respect to Q gives

$$\frac{\partial h}{\partial Q} = (R - 2R^3 + R^4)(2Q - 6QR^2 + 4Q^3) \quad (3.34)$$

Differentiating Equation (3.34) gives

$$\frac{\partial^2 h}{\partial Q^2} = (R - 2R^3 + R^4)(2 - 12Q + 12Q^2) \quad (3.35)$$

Differentiating the last equation with respect to Q gives

$$\frac{\partial^3 h}{\partial Q^3} = (R - 2R^3 + R^4)(-12 + 24Q) \tag{3.36}$$

Also differentiating Equation (3.31) with respect to Q gives

$$\frac{\partial^2 h}{\partial R \partial Q} = (1 - 6R^2 + 4R^3)(2Q - 6QR^2 + 4Q^3) \tag{3.37}$$

Differentiating Equation (3.37) with respect to Q gives

$$\frac{\partial^3 h}{\partial R \partial Q^2} = (1 - 6R^2 + 4R^3)(2 - 12Q + 12Q^2) \tag{3.38}$$

The differentiated shape functions were multiplied at different conditions and that gave for the case of the first differential values,

$$K1 = \frac{\partial^3 h}{\partial R^3} * \frac{\partial h}{\partial R} \tag{3.39}$$

Putting the values

$$K1 = (-12 + 24R)(Q^2 - 2Q^3 + Q^4)x\,(1 - 6R^2 + 4R^3)(Q^2 - 2Q^3 + Q^4) \tag{3.40}$$

Collecting the like terms together yields

$$K1 = (-12 + 24R)(1 - 6R^2 + 4R^3)\;x\;(Q^2 - 2Q^3 + Q^4)(Q^2 - 2Q^3 + Q^4) \tag{3.41}$$

and multiplying out each bracket gives

$$K1 = -12\,(1 - 6R^2 + 4R^3) + 24R(1 - 6R^2 + 4R^3)\;x\;Q^2(Q^2 - 2Q^3 + Q^4)$$

$$-2Q^3(Q^2 - 2Q^3 + Q^4) + Q^4(Q^2 - 2Q^3 + Q^4) \tag{3.42}$$

Further minimizing Equation (3.42) gives

$$K1 = (-12 + 24R + 72R^2 - 192R^3 + 96R^4)x\,(Q^4 - 4Q^5 + 6Q^6 - 4Q^7 + Q^8) \tag{3.43}$$

Therefore

$$\frac{\partial^3 h}{\partial R^3} \times \frac{\partial h}{\partial R} =$$

$$(-12 + 24R + 72R^2 - 192R^3 + 96R^4) \times (Q^4 - 4Q^5 + 6Q^6 - 4Q^7 + Q^8)$$

But the first stiffness coefficient is gotten by executing the expression in

Equation (3.44). That is

$$k_1 = \int_0^1 \int_0^1 K dR dQ \tag{3.44}$$

Substituting Equation (3.43) into Equation (3.44) gives

$$k_1 = [(-12 + 24R + 72R^2 - 192R^3 + 96R^4)_0^1 (Q^4 - 4Q^5 + 6Q^6 - 4Q^7 + Q^8)_0^1] \quad dRdQ \tag{3.45}$$

$$k_1 = \left[\left(\frac{-12R}{1} + \frac{24R^2}{2} + \frac{72R^2}{3} - \frac{192R^4}{4} + \frac{96R^5}{5}\right)_0^1 \left(\frac{Q^5}{5} - \frac{4Q^6}{6} + \frac{6Q^7}{7} - \frac{4Q^8}{8} + \frac{Q^9}{9}\right)_0^1\right] \tag{3.46}$$

Substituting the upper and lower limit into the Equation (3.46) gives

$$k_1 = \left[\left(\frac{-12}{1} + \frac{24}{2} + \frac{72}{3} - \frac{192}{4} + \frac{96}{5}\right)\left(\frac{1}{5} - \frac{4}{6} + \frac{6}{7} - \frac{4}{8} + \frac{1}{9}\right)\right] \tag{3.47}$$

Reducing Equation (3.47) further gives

$$k_1 = \left[\left(-4\frac{4}{5}\right)\left(\frac{1}{630}\right)\right] = \frac{-4}{525} \tag{3.48}$$

Similarly for the case of the second differential value,

$$K2 = \frac{\partial^3 h}{\partial R \partial Q^2} \times \frac{\partial h}{\partial R} \tag{3.46}$$

Substituting Equation (3.38) and (3.31) in Equation (3,46) gives

$$K2 = (1 - 6R^2 + 4R^3)(2 - 12Q + 12Q^2)x(1 - 6R^2 + 4R^3)(Q^2 - 2Q^3 + Q^4)$$

Collecting the like terms together yields

$$K2 = (1 - 6\mathrm{R}^2 + 4\mathrm{R}^3)(1 - 6\mathrm{R}^2 + 4\mathrm{R}^3) \text{ x } (2 - 12\mathrm{Q} + 12\mathrm{Q}^2)(\mathrm{Q}^2 - 2\mathrm{Q}^3 + \mathrm{Q}^4)$$

and multiplying out each bracket gives

$$K2 = (1 - 6\mathrm{R}^2 + 4\mathrm{R}^3 - 6\mathrm{R}^2 + 36\mathrm{R}^4 - 24\mathrm{R}^5 + 4\mathrm{R}^3 - 24\mathrm{R}^5 + 16\mathrm{R}^6)\text{x } (2\mathrm{Q}^2 - 4\mathrm{Q}^3 + 2\mathrm{Q}^4 - 12\mathrm{Q}^3 + 24\mathrm{Q}^4 - 12\mathrm{Q}^5 + 12\mathrm{Q}^4 - 24\mathrm{Q}^5 + 12\mathrm{Q}^6) \tag{3.47}$$

Further minimizing Equation (3.47) gives

$$K2 = (1 - 12R^2 + 8R^3 + 36R^4 - 48R^5 + 16R^6) \times$$

$$(2Q^2 - 16Q^3 + 38Q^4 - 36Q^5 + 12Q^6) \quad (3.48)$$

But the second stiffness coefficient is gotten using Equation (3.49) . That is

$$k_2 = \int_0^1 \int_0^1 \overline{K2} dRdQ \quad (3.49)$$

putting Equation (3.48) into Equation (3.49) gives

$$k_2 = \int_0^1 \int_0^1 [(1 - 12R^2 + 8R^3 + 36R^4 - 48R^5 + 16R^6)_0^1 (2Q^2 - 16Q^3 + 38Q^4 - 36Q^5 + 12Q^6)_0^1] \; dRdQ \quad (3.50)$$

Further integrating Equation (3.50) gives

$$k_2 = \left[\left(\frac{R}{1} + \frac{12R^3}{3} + \frac{8R^4}{4} - \frac{36R^5}{5} + \frac{48R^6}{6} + \frac{16R^7}{7}\right)_0^1 \left(\frac{2Q^5}{3} - \frac{16Q^6}{4} + \frac{38Q^7}{5} - \frac{36Q^8}{6} + \frac{12Q^9}{7}\right)_0^1\right] \quad (3.51)$$

Substituting the upper and lower limit into the Equation (3.46) gives

$$k_2 = \left[\left(\frac{1}{1} + \frac{12}{3} + \frac{8}{4} - \frac{36}{5} + \frac{48}{6} + \frac{16}{7}\right)\left(\frac{2}{3} - \frac{16}{4} + \frac{38}{5} - \frac{36}{6} + \frac{12}{7}\right)\right] \quad (3.52)$$

Therefore

$$k_2 = \left(\frac{17}{35}\right)\left(\frac{-2}{105}\right) = \frac{-34}{3675} \quad (3.53)$$

Also for the third differential coefficient,

$$K3 = \frac{\partial^3 h}{\partial Q^3} * \frac{\partial h}{\partial Q} \quad (3.54)$$

Further substitution gives

$$K3 = (R - 2R^3 + R^4)(-12 + 24Q)x(R - 2R^3 + R^4)(2Q - 6Q^2 + 4Q^3) \quad (3.55)$$

Collecting the like terms together yields

$$K3 = (R - 2R^3 + R^4)(R - 2R^3 + R^4) \; x \; (-12 + 24Q)(2Q - 6QR^2 + 4Q^3) \quad (3.56)$$

and multiplying out each bracket Equation (3.56) gives

$$K3 = (R^2 - 4R^4 + 2R^5 + 4R^6 - 4R^7 + R^8)x\,(-24Q + 120Q^2 - 192Q^3 + 96Q^4)$$

Therefore

$$K3 = (R^2 - 4R^4 + 2R^5 + 4R^6 - 4R^7 + R^8)x\,(-24Q + 120Q^2 - 192Q^3 + 96Q^4) \quad (3.57)$$

Just like the first two cases of stiffnesses,

$$k_3 = \int_0^1 \int_0^1 \overline{K3}\,dRdQ$$

That implies

$$k_3 = \int_0^1 \int_0^1 (R^2 - 4R^4 + 2R^5 + 4R^6 - 4R^7 + R^8)\,x\,(-24Q + 120Q^2 - 192Q^3 + 96Q^4)\,dRdQ \quad (3.58)$$

integrating gives

$$k_3 = \left[\left(\frac{R^3}{3} - \frac{4R^5}{5} + \frac{2R^6}{6} + \frac{4R^7}{7} - \frac{4R^8}{8} + \frac{R^9}{9}\right)_0^1 \left(\frac{-24Q^2}{2} - \frac{120Q^3}{3} + \frac{192Q^4}{4} - \frac{96Q^5}{5}\right)_0^1\right] \quad (3.59)$$

Putting the upper and the lower limts values gives

$$k_3 = \left[\left(\frac{1}{3} - \frac{4}{5} + \frac{2}{6} + \frac{4}{7} - \frac{4}{8} + \frac{1}{9}\right)\left(\frac{-24}{2} - \frac{120}{3} + \frac{192}{4} - \frac{96}{5}\right)\right] \quad (3.60)$$

That means

$$k_3 = \left(\frac{31}{630}\right)\left(\frac{-4}{5}\right) = \left(\frac{-62}{1575}\right) \quad (3.61)$$

Also for the sixth differential value,

$$K6 = \left(\frac{\partial h}{\partial R}\right)\left(\frac{\partial h}{\partial R}\right) \quad (3.62)$$

That means

$$(\frac{\partial h}{\partial R})(\frac{\partial h}{\partial R}) = (1-6R^2+4R^3)(Q^2-2Q^3+Q^4)x(1-6R^2+4R^3)(Q^2-2Q^3+Q^4)$$

Collecting the like terms together gives

$$K6 = (1-6R^2+4R^3)(1-6R^2+4R^3)x(Q^2-2Q^3+Q^4)(Q^2-2Q^3+Q^4) \quad (3.63)$$

Further factorization after multiplying out each bracket in Equation (3.63) yields

$$K6 = (1-12R^2+8R^3+36R^4-48R^5+16R^6)x(Q^4-4Q^5+6Q^6-4Q^7+Q^8) \quad (3.64)$$

Therefore

$$\left(\frac{\partial h}{\partial R}\right)\left(\frac{\partial h}{\partial R}\right) = (1-12R^2+8R^3+36R^4-48R^5+16R^6)x(Q^4-4Q^5+6Q^6-4Q^7+Q^8) \quad (3.65)$$

Similarly the sixth stiffness coefficient is gotten by using Equation (3.66). That is

$$k_6 = \int_0^1\int_0^1 \overline{K6}\,dRdQ \quad (3.66)$$

That implies that

$$k_6 = \int_0^1\int_0^1 (1-12R^2+8R^3+36R^4-48R^5+16R^6)\,x(Q^4-4Q^5+6Q^6-4Q^7+Q^8)dRdQ \quad (3.67)$$

Integrating Equation (3.67) gives

$$k_6 = \left[\left(\frac{R}{1}-\frac{12R^3}{3}+\frac{8R^4}{4}+\frac{36R^5}{5}-\frac{48R^6}{6}+\frac{16R^7}{7}\right)_0^1\left(\frac{Q^5}{5}-\frac{4Q^6}{6}+\frac{6Q^7}{7}-\frac{4Q^8}{8}+\frac{Q^9 1}{9}\right)_0^1\right] \quad (3.68)$$

Introducing the upper and lower limits values gives

$$k_6 = \left[\left(\frac{1}{1}-\frac{12}{3}+\frac{8}{4}+\frac{36}{5}-\frac{48}{6}+\frac{16}{7}\right)_0^1\left(\frac{1}{5}-\frac{4}{6}+\frac{6}{7}-\frac{4}{8}+\frac{11}{9}\right)_0^1\right] \quad (3.69)$$

Therefore

$$k_6 = (\frac{17}{35})\text{ x}(\frac{19}{630}) = \frac{17}{22050} \quad (3.70)$$

Further more for the seventh coefficient,

$$K7 = \left(\frac{\partial h}{\partial Q}\right) * \left(\frac{\partial h}{\partial Q}\right) \quad (3.71)$$

but

$$\left(\frac{\partial h}{\partial Q}\right) = (R - 2R^3 + R^4)(2Q - 6Q^2 + 4Q^3) \quad (3.72)$$

That implies

$$K7 = (R - 2R^3 + R^4)(2Q - 6Q^2 + 4Q^3) * (R - 2R^3 + R^4)(2Q - 6Q^2 + 4Q^3) \quad (3.73)$$

Collecting the like terms and multiplying Equation (3.73) gives

$$K7 = (R^2 - 2R^4 + R^5 - 2R^4 + 4R^6 - 2R^7 + R^5 - 2R^7 + R^8)*(4Q^2 - 12Q^3 + 8Q^4 - 12Q^3 + 36Q^4 - 24Q^5 + 8Q^4 - 24Q^5 + 16Q^6) \quad (3.74)$$

minimizing further gives

$$K7 = (R^2 - 4R^4 + 2R^5 + 4R^6 - 4R^7 + R^8) \times (4Q^2 - 24Q^3 + 52Q^4 - 48Q^5 + 16Q^6) \quad (3.75)$$

But

$$k_7 = \int_0^1 \int_0^1 \overline{K7} dRdQ$$

That means

$$k_7 = \left[\left(\frac{R^3}{3} - \frac{4R^5}{5} + \frac{2R^6}{6} + \frac{4R^7}{7} - \frac{4R^8}{8} + \frac{R^9}{9}\right)_0^1 \left(\frac{4Q^3}{3} - \frac{24Q^4}{4} + \frac{52Q^5}{5} - \frac{48Q^6}{6} + \frac{16Q^7}{7}\right)_0^1\right] \quad (3.76)$$

Introducing the upper and lower limits gives

$$k_7 = \left[\left(\frac{1}{3} - \frac{4}{5} + \frac{2}{6} + \frac{4}{7} - \frac{4}{8} + \frac{1}{9}\right)_0^1 \left(\frac{4}{3} - \frac{24}{4} + \frac{52}{5} - \frac{48}{6} + \frac{16}{7}\right)_0^1\right] \quad (3.77)$$

And so

$$k_7 = (-0.11746)*(0.01948) = (-0.00224) \qquad (3.78)$$

The next differentiatial value is gotten using Equation (3.79). That is

$$\overline{K8} = h \qquad (3.79)$$

But

$$h = (R - 2R^3 + R^4)(Q^2 - 2Q^3 + Q^4) \qquad (3.80)$$

That means, the eighth stiffness coefficients can be expressed as

$$k_8 = \int_0^1 \int_0^1 \overline{K8} dRdQ \qquad (3.81)$$

Substituting Equation (3.80) into (Equation (3.81) and integrating gives

$$k_8 = \left[\left(\frac{R^2}{2} - \frac{2R^4}{4} + \frac{R^5}{5} \right)_0^1 \left(\frac{Q^2}{2} - \frac{2Q^4}{4} + \frac{Q^5}{5} \right)_0^1 \right] \qquad (3.82)$$

Putting the various integral limits into Equation (3.82) gives

$$k_8 = \left(\frac{1}{2} - \frac{2}{4} + \frac{1}{5} \right) \left(\frac{1}{3} - \frac{2}{4} + \frac{1}{5} \right) \qquad (3.83)$$

further reducing gives

$$k_8 = \left(\frac{1}{5} \right) * \left(\frac{1}{3} - \frac{2}{4} + \frac{1}{5} \right) = (-0.0667) \qquad (3.84)$$

The next differential vale is given as

$$K11 = \left(\frac{\partial^3 h}{\partial R^3} \right) dRdQ \qquad (3.85)$$

where

$$\left(\frac{\partial^3 h}{\partial R^3} \right) = (-12 + 24R)(Q^2 - 2Q^3 + Q^4) \qquad (3.86)$$

And so its corresponding stiffness coefficient is given as

$$k_{11} = \int_0^1 \int_0^1 \overline{K11} dRdQ \tag{3.87}$$

That means

$$k_{11} = \left(-\frac{12R}{1} + \frac{24R^2}{2}\right)\left(\frac{Q^3}{3} - \frac{2Q^4}{4} + \frac{Q^5}{5}\right) \tag{3.88}$$

substituting the values of R and Q in to Equation (3.88) gives

$$k_{11} = (-12 + 12)\left(\frac{1}{3} - \frac{2}{4} + \frac{1}{5}\right) = 0 \tag{3.89}$$

The last differential values was gotten adopting Equation (3.90). That is

$$K12 = \left(\frac{\partial^3 h}{\partial Q^3}\right) dRdQ \tag{3.90}$$

where

$$\left(\frac{\partial^3 h}{\partial Q^3}\right) = (R - 2R^3 + R^4)(-12 + 24Q) \tag{3.91}$$

Similar to other stiffnesses,

$$k_{12} = \int_0^1 \int_0^1 \overline{K12} dRdQ \tag{3.92}$$

Integrating Equation (3.92)

$$k_{12} = \left[\left(\frac{R^2}{2} - \frac{2R^4}{4} + \frac{R^5}{5}\right)_0^1 \left(-\frac{12Q}{1} + \frac{24Q^2}{2}\right)_0^1\right] \tag{3.93}$$

substituting the values of *R* and *Q* in to Equation (3.93) gives

$$k_{12} = \left[\left(\frac{1}{2} - \frac{2}{4} + \frac{1}{5}\right)\left(-\frac{12}{1} + \frac{24}{2}\right)\right] = 0 \tag{3.94}$$

The rest of the diffential values for all the chape functions and the stiffnesses coefficients for all the plates treated, were derived following the same approach.

3.2.3 **Determination of the Stress-Strain Relation for a Lamina of The thick Laminated Plate and Translate From Local Coordinate to Global Coordinate System**.

The strain-displacement relations suitable for small deflection of anisotropic rectangular plates will be considered. In this case of laminated thick plate, the middle layer in-plane displacements u_0 and v_0 are not treated as constants as in the case of ordinary anisotropic plate. Herein they are treated as functions. So differentiating them will not result to zeros. Taking The in-plane displacements are defined as:

$$u = -z\frac{dw}{dx} + u_0 \tag{3.95}$$

$$v = -z\frac{dw}{dy} + v_0 \tag{3.96}$$

Where w is the out of plane displacement (or deflection). It should be noted that the in-plane displacements, u and v are functions of the three coordinates, x, y and z whereas the deflection is only a function of the in-plane coordinates, x and y. Another important note is that the displacements discussed so far are for one lamina in the laminated plate. Unlike isotropic plate, the in-plane displacements of the middle surface are not constants. The refined plate theory (RPT) in-plane displacements, u and v are defined mathematically from Figure (3.3) as presented in Equations (3.97) and (3.98).

$$u = u_c + u_s + u_0 \tag{3.97}$$

$$v = v_c + v_s + v_0 \tag{3.98}$$

Where u and v are the in-plane displacement in x direction and y direction respectively and the out of plane displacement (deflection) is taken as *w*. Figure 3.3 shows a typical section of a deformed thick plate .

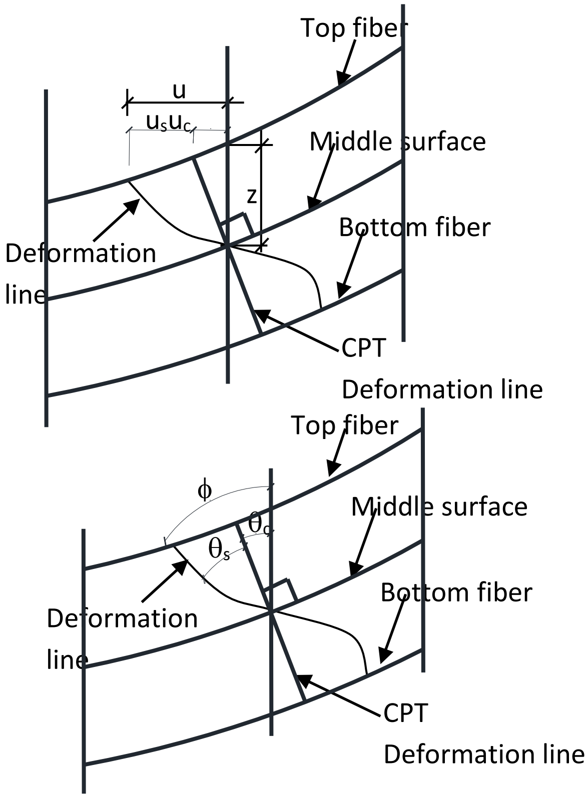


Figure 3.2: Horizontal section of a deformed thick plate

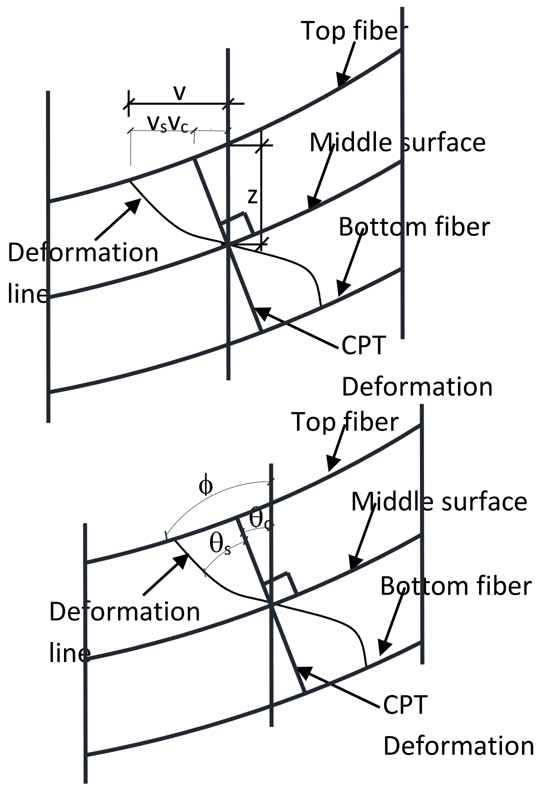


Figure 3.3: Vertical section of a deformed thick plate

Where:

CPT: Classical Plate Theory

ϕ:	Total rotation of the middle surface
θ_{cx} and θ_{cy}:	Classical plate theorem rotation of the middle surface.
θ_{sx} and θ_{sy}:	Angle between the CPT deformation line and the shear deformation line.
u_c and v_c:	In-plane displacement due to classical plate theory.
u_s and v_s:	In-plane displacement due to shear deformation theory.

The classical part of the in-plane displacements u_c and v_c are defined as follows:

$$u_c = -z\theta_{cx} = -z\frac{dw}{dx} \tag{3.99}$$

$$v_c = -z\theta_{cy} = -z\frac{dw}{dy} \tag{3.100}$$

Analogously, the shear deformation part of the in-plane displacements u_s and v_s are defined as:

$$u_s = F(z)\theta_{sx} \tag{3.101}$$

$$v_s = F(z)\theta_{sy} \qquad (3.102)$$

Where F(z) is the shear deformation profile defined as:

$$F(z) = z - \frac{4}{3}.\frac{z^3}{t^2} = z\left(1 - \frac{4}{3}\left[\frac{z}{t}\right]^2\right) \qquad (3.103)$$

In non-dimensional coordinate (S = z / t) term, the shear deformation profile is defined as:

$$F = F(s) = t\left(S - \frac{4}{3}S^3\right) \qquad (3.104)$$

That is:

$$F = tH \qquad (3.105)$$

where:

$$H = S - \frac{4}{3}S^3 \qquad (3.106)$$

Substituting Equations (3.99) and (3.101) into Equation (3.97) gives:

$$u = -Z\frac{\partial w}{\partial x} + F(z).\emptyset_x + u_0 \qquad (3.107)$$

Substituting Equations (3.100) and (3.102) into Equation (3.98) gives:

$$v = -Z\frac{\partial w}{\partial y} + F(z).\emptyset_y + v_0 \qquad (3.108)$$

In terms of non-dimensional coordinates (*R = x/a*, *Q = y/b and S = z/t*) and aspect ratio *(β = b/a*), Equations (3.107) and (3.108) are rewritten respectively

$$u = \frac{t}{a}\left[-S\frac{\partial w}{\partial R} + Ha.\emptyset_x\right] + u_0 \qquad (3.109)$$

$$v = \frac{t}{a\beta}\left[-S\frac{\partial w}{\partial Q} + Ha\beta.\emptyset_y\right] + v_0 \qquad (3.110)$$

3.2.4 Strain Energy, Total Potential Energy Functional and Governing Equation of Equilibrium

The in-plane strains are defined as:

$$\varepsilon_x = \frac{du}{dx} \tag{3.111}$$

$$\varepsilon_y = \frac{dv}{dy} \tag{3.112}$$

$$\gamma_{xy} = \varepsilon_{xy} + \varepsilon_{yx} = \frac{du}{dy} + \frac{dv}{dx} \tag{3.113}$$

Substituting Equation (3.109) into Equation (3.111) (having in mind that the middle surface in-plane displacements are not constants) gives the equation for the relationship between normal strain in x coordinate and deflection as:

$$\varepsilon_R = \frac{\partial u}{\partial x} = \frac{\partial u}{a\,\partial R} = \varepsilon_{xi} + \varepsilon_{x0} = \frac{t}{a^2}\left[-S\frac{\partial^2 w}{\partial R^2} + Ha.\frac{\partial \phi_x}{\partial R}\right] + \frac{du_0}{adR} \tag{3.114}$$

Similarly, substituting Equation (3.16) into Equation (3.18) gives the relationship between normal strain in y coordinate and deflection as:

$$\varepsilon_Q = \frac{\partial v}{\partial y} = \frac{\partial v}{a\beta\,\partial Q} = \varepsilon_{yi} + \varepsilon_{y0} = \frac{t}{\beta^2 a^2}\left[-S\frac{\partial^2 w}{\partial Q^2} + Ha\beta.\frac{\partial \phi_y}{\partial Q}\right] + \frac{dv_0}{a\beta\,\partial Q} \tag{3.115}$$

Substituting Equations (3.109) and (3.110) into Equation (3.113) gives the relationship between engineering shear strain in x-y plane and deflection as:

$$\gamma_{RQ} = \left(\varepsilon_{RQ} + \varepsilon_{QR}\right) = \gamma_{xyi} + \gamma_{xy0}$$
$$= \frac{t}{\beta a^2}\left[-2S\frac{\partial^2 w}{\partial R\,\partial Q} + Ha.\left(\frac{\partial \phi_x}{\partial Q} + \beta.\frac{\partial \phi_y}{\partial R}\right)\right] + \frac{du_0}{a\beta\,\partial Q} + \frac{dv_0}{adR} \tag{3.116}$$

It is assumed that the vertical strain ε_z, is equal to zero. Thus, the remaining two engineering strain components are derived as follows:

$$\varepsilon_{RS} = \frac{\partial u}{\partial z} = \frac{\partial u}{t\,\partial S} = \varepsilon_{xzi} + \varepsilon_{xz0} = \frac{1}{a}\left[-\frac{\partial w}{\partial R} + a\frac{\partial H}{\partial S}.\varnothing_x\right] + \frac{du_0}{tdS} \tag{3.117}$$

$$\varepsilon_{SR} = \frac{\partial w}{\partial x} = \varepsilon_{zxi} + \varepsilon_{zx0} = \frac{1}{a}\frac{\partial w}{\partial R} + \frac{1}{a}\frac{dw_0}{\partial R} = \frac{1}{a}\frac{\partial w}{\partial R} + 0 \tag{3.118}$$

$$\varepsilon_{SR} = \frac{1}{a}\frac{\partial w}{\partial R} \tag{3.119}$$

Adding Equations (3.117) and (3.119) gives the x-z engineering plane shear strain:

$$\gamma_{RS} = \varepsilon_{RS} + \varepsilon_{SR} = \frac{1}{a}\left[-\frac{\partial w}{\partial R} + a\frac{\partial H}{\partial S}.\emptyset_x\right] + \frac{du_0}{tdS} + \frac{1}{a}\frac{\partial w}{\partial R}$$

That is:

$$\gamma_{RS} = -\frac{1}{a}\frac{\partial w}{\partial R} + \frac{\partial H}{\partial S}.\emptyset_x + \frac{du_0}{tdS} + \frac{1}{a}\frac{\partial w}{\partial R}$$

$$\gamma_{RS} = \frac{\partial H}{\partial S}.\emptyset_x + \frac{du_0}{tdS} \tag{3.120}$$

In a similar way, the y-z complementary plane shear strains, obtained as:

$$\gamma_{QS} = \frac{\partial H}{\partial S}.\emptyset_y + \frac{dv_0}{tdS} \tag{3.121}$$

Summarizing Equations (3.114), (3.115), (3.116), (3.117) and (3.119) gives the strain vector as:

$$\varepsilon = \begin{bmatrix} \left(\frac{du_0}{adR} - \frac{tS}{a^2}\frac{\partial^2 w}{\partial R^2} + \frac{tH}{a}.\frac{\partial \phi_x}{\partial R}\right) \\ \left(\frac{dv_0}{a\beta\,\partial Q} - \frac{tS}{\beta^2 a^2}\frac{\partial^2 w}{\partial Q^2} + \frac{tH}{a\beta}.\frac{\partial \phi_y}{\partial Q}\right) \\ \left(\frac{du_0}{a\beta\,\partial Q} + \frac{dv_0}{adR} - \frac{2tS}{\beta a^2}\frac{\partial^2 w}{\partial R\,\partial Q} + \frac{tH}{\beta a}.\left(\frac{\partial \phi_x}{\partial Q} + \beta.\frac{\partial \phi_y}{\partial R}\right)\right) \\ \left(\frac{du_0}{tdS} + \frac{\partial H}{\partial S}.\emptyset_x\right) \\ \left(\frac{dv_0}{tdS} + \frac{\partial H}{\partial S}.\emptyset_y\right) \end{bmatrix} \tag{3.122}$$

3.2.4.1 Establishment of Constitutive Relations

The constitutive relation was formulated using Hook's and Poisson's theorems in deriving the stress - strain relations. The five stress components together with their corresponding five strain components forms the bases for stress-strain relation. The five engineering components are as follows

$$\varepsilon_1 = \frac{\sigma_1}{E_1} - \mu_{21}\frac{\sigma_2}{E_2} \tag{3.123}$$

$$\varepsilon_2 = \frac{\sigma_2}{E_2} - \mu_{12}\frac{\sigma_1}{E_1} \tag{3.124}$$

$$\varepsilon_3 = \frac{1}{G_{12}} \sigma_3 \tag{3.125}$$

$$\varepsilon_4 = \frac{1}{G_{13}} \sigma_4 \tag{3.126}$$

$$\varepsilon_5 = \frac{1}{G_{23}} \sigma_5 \tag{3.127}$$

.

3.2.4.2 Global to local transformation of plate

On both Global and Local cordinate, the stress values are considered as being equal. That is $\varepsilon_x = \varepsilon_x$ and $\varepsilon_y = \varepsilon_y$. The shear stresses γ_{xy} and γ_{yz} are $2\varepsilon_{xz}$ and $2\varepsilon_{yz}$ respectively. Considereing T as the transformation matrix, the transformation of strain from Global to Local cordinate for an anisotropic plates is given that

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \end{bmatrix} = [T] \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{bmatrix} \tag{3.128}$$

But the relation between strain vector and strain tensor vector can be given as

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_{xy} \\ \varepsilon_{xz} \\ \varepsilon_{yz} \end{bmatrix} = [\Delta] \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_{xy} \\ \varepsilon_{xz} \\ \varepsilon_{yz} \end{bmatrix} \tag{3.129}$$

But the relation between strain vector and the strain tensor vector can be given as

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix} * \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_{xy} \\ \varepsilon_{xz} \\ \varepsilon_{yz} \end{bmatrix} = [\Delta] \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_{xy} \\ \varepsilon_{xz} \\ \varepsilon_{yz} \end{bmatrix} \tag{3.130}$$

Also Equation (3.130) can be expressed as

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{bmatrix} = [\Delta] \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_{xy} \\ \varepsilon_{xz} \\ \varepsilon_{yz} \end{bmatrix} \tag{3.131}$$

Rearranging Equation (3.131) gives

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{bmatrix} [\Delta]^{-1} = \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_{xy} \\ \varepsilon_{xz} \\ \varepsilon_{yz} \end{bmatrix} \quad (3.132)$$

Putting back Equation (3.131) into Equation (3.128)

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \end{bmatrix} = [T] x [\Delta] \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_{xy} \\ \varepsilon_{xz} \\ \varepsilon_{yz} \end{bmatrix} \quad (3.133)$$

Substituting Equation (3.132) into (3.133) gives

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \end{bmatrix} = [T] x [\Delta] x [\Delta]^{-1} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_{xy} \\ \varepsilon_{xz} \\ \varepsilon_{yz} \end{bmatrix} \quad (3.134)$$

In the same way, stress components are transformed from Global coordinates to Local coordinates shown in Equatin (3.136), considering

$$m = \text{Cos}\theta \quad (3.135a)$$

$$n = \text{Sin}\theta \quad (3.135b)$$

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \end{bmatrix} = \begin{bmatrix} m^2 & n^2 & 2mn & 0 & 0 \\ n^2 & m^2 & -2mn & 0 & 0 \\ -mn & mn & ((m^2 - n^2) & 0 & 0 \\ 0 & 0 & 0 & m & n \\ 0 & 0 & 0 & -n & m \end{bmatrix} . \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{bmatrix} \quad (3.136)$$

and reducing Equation (3.136) further gives:

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \end{bmatrix} = [T] \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{bmatrix} \quad (3.137)$$

where

$$[T] = \begin{bmatrix} m^2 & n^2 & 2mn & 0 & 0 \\ n^2 & m^2 & -2mn & 0 & 0 \\ -mn & mn & ((m^2 - n^2) & 0 & 0 \\ 0 & 0 & 0 & m & n \\ 0 & 0 & 0 & -n & m \end{bmatrix} \quad (3.138)$$

Rearranging Equation (3.139) gives:

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{bmatrix} = [T]^{-1} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \end{bmatrix} \tag{3.139}$$

Equation (3.123), (3.124), (3.125), (3.126) and (3.127) are summarized in matrix form as:

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \end{bmatrix} = \begin{bmatrix} 1/E_1 & -\mu_{21}/E_2 & 0 & 0 & 0 \\ -\mu_{12}/E_1 & 1/E_2 & 0 & 0 & 0 \\ 0 & 0 & 1/G_{12} & 0 & 0 \\ 0 & 0 & 0 & 1/G_{13} & 0 \\ 0 & 0 & 0 & 0 & 1/G_{23} \end{bmatrix} . \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \end{bmatrix} \tag{3.140}$$

That is:

$$[E] \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \end{bmatrix} = \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \end{bmatrix} \tag{3.141}$$

where

$$G_{12}{}^{*} = G_{12}(1 - \mu_{12}\mu_{21}) \tag{3.142}$$

$$G_{13}{}^{*} = G_{13}(1 - \mu_{12}\mu_{21}) \tag{3.143}$$

$$G_{23}{}^{*} = G_{23}(1 - \mu_{12}\mu_{21}) \tag{3.144}$$

Substituting Equation (3.141) into Equation (3.139) gives:

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{bmatrix} = [T]^{-1}[E] \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \end{bmatrix} \tag{3.145}$$

Substituting Equation (3.132) into Equation (3.145) gives:

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{bmatrix} = [T]^{-1}[E].[\Delta][T][\Delta]^{-1} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{bmatrix} \tag{3.146}$$

But:

$$[\Delta].[T].[\Delta]^{-1} = [T]^{-T} \tag{3.147}$$

Substituting Equation (3.147) into Equation (3.146) gives:

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{bmatrix} = [T]^{-1}[E].[T]^{-T} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{bmatrix} \tag{3.148}$$

That is:

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{bmatrix} = [EE] \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{bmatrix} \tag{3.149}$$

where:

$$[EE] = [T]^{-1}[E].[T]^{-T} \tag{3.150}$$

$$[E] = \frac{1}{1-\mu_{12}\mu_{21}} \begin{bmatrix} E_1 & E_2 \,.\, \mu_{12} & 0 & 0 & 0 \\ E_1 \,.\, \mu_{21} & E_2 & 0 & 0 & 0 \\ 0 & 0 & G_{12}{}^{*} & 0 & 0 \\ 0 & 0 & 0 & G_{13}{}^{*} & 0 \\ 0 & 0 & 0 & 0 & G_{23}{}^{*} \end{bmatrix} \tag{3.151}$$

Expressing Equation (3.151) in terms of E values gives:

$$= \frac{1}{1-\mu_{12}\mu_{21}} \begin{bmatrix} E_{11} & E_{12} & 0 & 0 & 0 \\ E_{21} & E_{22} & 0 & 0 & 0 \\ 0 & 0 & E_{33} & 0 & 0 \\ 0 & 0 & 0 & E_{44} & 0 \\ 0 & 0 & 0 & 0 & E_{55} \end{bmatrix} \tag{3.152}$$

Equation (3.152) can be rewritten as:

$$[E] = \frac{E_0}{1-\mu_{12}\mu_{21}} \begin{bmatrix} d_{11} & d_{12} & 0 & 0 & 0 \\ d_{21} & d_{22} & 0 & 0 & 0 \\ 0 & 0 & d_{33} & 0 & 0 \\ 0 & 0 & 0 & d_{44} & 0 \\ 0 & 0 & 0 & 0 & d_{55} \end{bmatrix} \tag{3.153}$$

$$E_{11} = E_1 = d_{11}E_0 \tag{3.154}$$

$$E_{12} = E_2 \,.\, \mu_{12} = d_{11}E_0 \tag{3.155}$$

$$E_{21} = E_1 . \mu_{21} = d_{11}E_0 \tag{3.156}$$

$$E_{22} = E_2 = d_{11}E_0 \tag{3.157}$$

$$E_{33} = G_{12}(1 - \mu_{12}\mu_{21}) = d_{11}E_0 \tag{3.158}$$

$$E_{44} = G_{13}(1 - \mu_{12}\mu_{21}) = d_{11}E_0 \tag{3.159}$$

$$E_{55} = G_{23}(1 - \mu_{12}\mu_{21}) = d_{11}E_0 \tag{3.160}$$

$$[E] = \frac{1}{1 - \mu_{12}\mu_{21}} \begin{bmatrix} B_{11} & B_{12} & 0 & 0 & 0 \\ B_{21} & B_{22} & 0 & 0 & 0 \\ 0 & 0 & B_{33} & 0 & 0 \\ 0 & 0 & 0 & B_{44} & 0 \\ 0 & 0 & 0 & 0 & B_{55} \end{bmatrix} \tag{3.161}$$

From Equations (3.136) and (3.153) , Equation (3.161) was formulated, producing the values as stated in Equation (3.162) to (3.169)

$$B_{11} = m^4 d_{11} + 2m^2n^2(d_{12} + 2d_{33}) + n^4 d_{22} \tag{3.162}$$

$$B_{12} = d_{12}(n^4 + m^4) + m^2n^2(d_{11} + d_{22} - 4d_{33}) \tag{3.163}$$

$$B_{13} = m^3n(d_{11} - d_{12} - 2d_{33}) + mn^3(d_{12} - d_{22} + 2d_{33}) \tag{3.164}$$

$$B_{22} = n^4 d_{11} + 2m^2n^2(d_{12} + 2d_{33}) + m^4 d_{22} \tag{3.165}$$

$$B_{23} = mn^3 d_{11} - m^3 n d_{22} + (m^3 n - mn^3)(d_{12} + 2d_{33}) \tag{3.166}$$

$$B_{33} = m^2n^2(d_{11} - 2d_{12} + d_{22} - 2d_{33}) + d_{33}(m^4 + n^4) \tag{3.167}$$

$$B_{21} = B_{12}, \qquad B_{31} = B_{13} \text{ and } B_{32} = B_{23}$$

$$B_{44} = d_{44} \tag{3.168}$$

$$B_{55} = d_{55} \tag{3.169}$$

Substituting Equation (3.161) into Equation (3.149) yields the expressions from Equation (3.170a) to (3.173):

$$\sigma_x = \frac{E_0}{1 - \mu_{12}\mu_{21}}\left(B_{11} . \varepsilon_x + B_{12} . \varepsilon_y + B_{13} . \gamma_{xy}\right) \tag{3.170a}$$

$$\sigma_y = \frac{E_0}{1 - \mu_{12}\mu_{21}}\left(B_{21} . \varepsilon_x + B_{22} . \varepsilon_y + B_{23} . \gamma_{xy}\right) \tag{3.170b}$$

$$\tau_{xy} = \frac{E_0}{1-\mu_{12}\mu_{21}}\left(B_{31}\,.\,\varepsilon_x + B_{32}\,.\,\varepsilon_y + B_{33}\,.\,\gamma_{xy}\right) \quad (3.171)$$

$$\tau_{xz} = \frac{E_0}{1-\mu_{12}\mu_{21}} B_{44}.\gamma_{xz} \quad (3.172)$$

$$\tau_{yz} = \frac{E_0}{1-\mu_{12}\mu_{21}} B_{55}.\gamma_{yz} \quad (3.173)$$

Substituting Equations (3.114), (3.115)and (3.116) into Equation (3.170) gives:

$$\sigma_R = \frac{E_0 t}{[1-\mu_{12}\mu_{21}]a^2}\Bigg(B_{11}\,.\left[-S\frac{\partial^2 w}{\partial R^2} + Ha.\frac{\partial \phi_x}{\partial R} + \frac{adu_0}{tdR}\right] + \frac{B_{12}}{\beta^2}\,.\left[-S\frac{\partial^2 w}{\partial Q^2} + Ha\beta.\frac{\partial \phi_y}{\partial Q} + \frac{a\beta dv_0}{t\,\partial Q}\right] + \frac{B_{13}}{\beta}\,.\left[-2S\frac{\partial^2 w}{\partial R\,\partial Q} + Ha\left(\frac{\partial \phi_x}{\partial Q} + \beta.\frac{\partial \phi_y}{\partial R}\right) + \frac{adu_0}{t\,\partial Q} + \frac{a\beta dv_0}{dR}\right]\Bigg) \quad (3.174)$$

Similarly, substituting Equations (3.20), (3.21) and (3.22) into Equation (3.76) gives:

$$\sigma_Q = \frac{E_0 t}{[1-\mu_{xy}\mu_{yx}]a^2}.\Bigg(B_{21}\,.\left[-S\frac{\partial^2 w}{\partial R^2} + Ha.\frac{\partial \phi_x}{\partial R} + \frac{adu_0}{tdR}\right] + \frac{B_{22}}{\beta^2}\,.\left[-S\frac{\partial^2 w}{\partial Q^2} + Ha\beta.\frac{\partial \phi_y}{\partial Q} + \frac{a\beta dv_0}{t\,\partial Q}\right] + \frac{B_{23}}{\beta}\,.\left[-2S\frac{\partial^2 w}{\partial R\,\partial Q} + Ha.\left(\frac{\partial \phi_x}{\partial Q} + \beta.\frac{\partial \phi_y}{\partial R}\right) + \frac{adu_0}{t\,\partial Q} + \frac{a\beta dv_0}{dR}\right]\Bigg) \quad (3.175)$$

Substituting Equations (3.114), (3.1151) and (3.116) into Equation (3.171) gives:

$$\tau_{RQ} = \frac{E_0 t}{[1-\mu_{xy}\mu_{yx}]a^2}.\Bigg(B_{31}\,.\left[-S\frac{\partial^2 w}{\partial R^2} + Ha.\frac{\partial \phi_x}{\partial R} + \frac{adu_0}{tdR}\right] + \frac{B_{32}}{\beta^2}\,.\left[-S\frac{\partial^2 w}{\partial Q^2} + Ha\beta.\frac{\partial \phi_y}{\partial Q} + \frac{a\beta dv_0}{t\,\partial Q}\right] + \frac{B_{33}}{\beta}\,.\left[-2S\frac{\partial^2 w}{\partial R\,\partial Q} + Ha.\left(\frac{\partial \phi_x}{\partial Q} + \beta.\frac{\partial \phi_y}{\partial R}\right) + \frac{adu_0}{t\,\partial Q} + \frac{a\beta dv_0}{dR}\right]\Bigg) \quad (3.176)$$

Substituting Equation (3.119) into Equation (3.172) gives:

$$\tau_{RS} = \frac{E_0}{1-\mu_{xy}\mu_{yx}}.B_{44}.\left[\frac{\partial H}{\partial S}.\emptyset_x + \frac{du_0}{tdS}\right] = \frac{E_0 t}{[1-\mu_{xy}\mu_{yx}]a^2}.B_{44}.\left[\frac{a^2}{t}.\frac{\partial H}{\partial S}.\emptyset_x + \frac{a^2}{t}.\frac{du_0}{tdS}\right]$$

$$= \frac{E_0 t}{[1-\mu_{xy}\mu_{yx}]a^2}.B_{44}.\left[\frac{a^2}{t}.\frac{\partial H}{\partial S}.\emptyset_x + \frac{a^2}{t}.\frac{du_0}{tdS}\right] \tag{3.177}$$

Substituting Equation (3.120) into Equation (3.173) gives:

$$\tau_{QS} = \frac{E_0}{1-\mu_{xy}\mu_{yx}}.B_{55}.\left[\frac{\partial H}{\partial S}.\emptyset_y + \frac{dv_0}{tdS}\right]$$

$$= \frac{E_0 t}{[1-\mu_{xy}\mu_{yx}]a^2}.B_{55}.\left[\frac{a^2}{t}.\frac{\partial H}{\partial S}.\emptyset_y + \frac{a^2}{t}.\frac{dv_0}{tdS}\right] \tag{3.178}$$

Considering that the stress on a body is directly and substituting Equations (3.114) and (3.153) to (3.121) in to Equation (3.149) gives

$$\begin{bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\\ \tau_{xz}\\ \tau_{yz}\end{bmatrix} = \frac{E_0}{1-\mu_{12}\mu_{21}}\begin{bmatrix}B_{11} & B_{12} & B_{13} & 0 & 0\\ B_{21} & B_{22} & B_{23} & 0 & 0\\ B_{31} & B_{32} & B_{33} & 0 & 0\\ 0 & 0 & 0 & B_{44} & 0\\ 0 & 0 & 0 & 0 & B_{55}\end{bmatrix}$$

$$\begin{bmatrix}\frac{du_0}{adR} - \frac{tS}{a^2}\frac{\partial^2 w}{\partial R^2} + \frac{tH}{a}.\frac{\partial \phi_x}{\partial R}\\ \frac{dv_0}{a\beta\,\partial Q} - \frac{tS}{\beta^2 a^2}\frac{\partial^2 w}{\partial Q^2} + \frac{tH}{a\beta}.\frac{\partial \phi_y}{\partial Q}\\ \frac{du_0}{a\beta\,\partial Q} + \frac{dv_0}{adR} - \frac{2tS}{\beta a^2}\frac{\partial^2 w}{\partial R\,\partial Q} + \frac{tH}{\beta a}.\left(\frac{\partial \phi_x}{\partial Q} + \beta.\frac{\partial \phi_y}{\partial R}\right)\\ \frac{du_0}{tdS} + \frac{\partial H}{\partial S}.\emptyset_x\\ \frac{dv_0}{tdS} + \frac{\partial H}{\partial S}.\emptyset_y\end{bmatrix} \tag{3.179}$$

That means Equation (3.122) and (3.179) represents the STRAIN and STRESS in a thick laminated plate respectively.

3.2.4.3 Formulation of Strain Energy and Total Potential Energy Functional for a Laminated Thick Rectangular Plate

The total potential energy functional for a Laminated thick plate is expressed mathematically as shown in Equation (3.181), considering an external load, V given as

$$V = \iint (\frac{N_x}{2}\left(\frac{dw}{dx}\right)^2 dxdy \tag{3.180}$$

$$\pi = \frac{1}{2}\iiint [\sigma][\varepsilon]\, dxdydz - \iint (0 + \frac{N_x}{2}\left(\frac{dw}{dx}\right)^2 + 0) + 0)dxdy \tag{3.181}$$

Then bringing Equation (3.122) and (3.179) into Equation (3.181) gives

$$\pi =$$

$$\frac{E_0}{2}\iiint \begin{bmatrix} \frac{du_0}{adR} - \frac{tS}{a^2}\frac{\partial^2 w}{\partial R^2} + \frac{tH}{a}.\frac{\partial \phi_x}{\partial R} \\ \frac{dv_0}{a\beta\,\partial Q} - \frac{tS}{\beta^2 a^2}\frac{\partial^2 w}{\partial Q^2} + \frac{tH}{a\beta}.\frac{\partial \phi_y}{\partial Q} \\ \frac{du_0}{a\beta\,\partial Q} + \frac{dv_0}{adR} - \frac{2tS}{\beta a^2}\frac{\partial^2 w}{\partial R\,\partial Q} + \frac{tH}{\beta a}.\left(\frac{\partial \phi_x}{\partial Q} + \beta.\frac{\partial \phi_y}{\partial R}\right) \\ \frac{du_0}{tdS} + \frac{\partial H}{\partial S}.\emptyset_x \\ \frac{dv_0}{tdS} + \frac{\partial H}{\partial S}.\emptyset_y \end{bmatrix}^T . \begin{bmatrix} B_{11} & B_{12} & B_{13} & 0 & 0 \\ B_{21} & B_{22} & B_{23} & 0 & 0 \\ B_{31} & B_{32} & B_{33} & 0 & 0 \\ 0 & 0 & 0 & B_{44} & 0 \\ 0 & 0 & 0 & 0 & B_{55} \end{bmatrix}$$

$$\begin{bmatrix} \frac{du_0}{adR} - \frac{tS}{a^2}\frac{\partial^2 w}{\partial R^2} + \frac{tH}{a}.\frac{\partial \phi_x}{\partial R} \\ \frac{dv_0}{a\beta\,\partial Q} - \frac{tS}{\beta^2 a^2}\frac{\partial^2 w}{\partial Q^2} + \frac{tH}{a\beta}.\frac{\partial \phi_y}{\partial Q} \\ \frac{du_0}{a\beta\,\partial Q} + \frac{dv_0}{adR} - \frac{2tS}{\beta a^2}\frac{\partial^2 w}{\partial R\,\partial Q} + \frac{tH}{\beta a}.\left(\frac{\partial \phi_x}{\partial Q} + \beta.\frac{\partial \phi_y}{\partial R}\right) \\ \frac{du_0}{tdS} + \frac{\partial H}{\partial S}.\emptyset_x \\ \frac{dv_0}{tdS} + \frac{\partial H}{\partial S}.\emptyset_y \end{bmatrix} dxdydz - \iint (0 + \frac{N_x}{2}\left(\frac{dw}{dx}\right)^2 + 0)dxdy \tag{3.182}$$

Rearranging and factorizing Equation (3.182) gives

$$\pi = \frac{t^2}{a^4}\iiint \begin{bmatrix} -S\frac{\partial^2 w}{\partial R^2} + aH.\frac{\partial \phi_x}{\partial R} + \frac{a}{t}\frac{du_0}{dR} \\ -\frac{S}{\beta^2}\frac{\partial^2 w}{\partial Q^2} + \frac{aH}{\beta}.\frac{\partial \phi_y}{\partial Q} + \frac{a}{t}\frac{dv_0}{\beta\,\partial Q} \\ -\frac{2S}{\beta}\frac{\partial^2 w}{\partial R\,\partial Q} + \frac{aH}{\beta}.\left(\frac{\partial \phi_x}{\partial Q} + \beta.\frac{\partial \phi_y}{\partial R}\right) + \frac{a}{t}\frac{du_0}{\beta\,\partial Q} + \frac{a}{t}\frac{dv_0}{dR} \\ \frac{a^2}{t}\frac{\partial H}{\partial S}.\emptyset_x + \frac{a^2}{t^2}\frac{du_0}{dS} \\ \frac{a^2}{t}\frac{\partial H}{\partial S}.\emptyset_y + \frac{a^2}{t^2}\frac{dv_0}{dS} \end{bmatrix}^T . \begin{bmatrix} B_{11} & B_{12} & B_{13} & 0 & 0 \\ B_{21} & B_{22} & B_{23} & 0 & 0 \\ B_{31} & B_{32} & B_{33} & 0 & 0 \\ 0 & 0 & 0 & B_{44} & 0 \\ 0 & 0 & 0 & 0 & B_{55} \end{bmatrix}$$

$$\begin{bmatrix} -S\frac{\partial^2 w}{\partial R^2} + aH.\frac{\partial \phi_x}{\partial R} + \frac{a}{t}\frac{du_0}{dR} \\ -\frac{S}{\beta^2}\frac{\partial^2 w}{\partial Q^2} + \frac{aH}{\beta}.\frac{\partial \phi_y}{\partial Q} + \frac{a}{t}\frac{dv_0}{\beta\,\partial Q} \\ -\frac{2S}{\beta}\frac{\partial^2 w}{\partial R\,\partial Q} + \frac{aH}{\beta}.\left(\frac{\partial \phi_x}{\partial Q} + \beta.\frac{\partial \phi_y}{\partial R}\right) + \frac{a}{t}\frac{du_0}{\beta\,\partial Q} + \frac{a}{t}\frac{dv_0}{dR} \\ \frac{a^2}{t}\frac{\partial H}{\partial S}.\emptyset_x + \frac{a^2}{t^2}\frac{du_0}{dS} \\ \frac{a^2}{t}\frac{\partial H}{\partial S}.\emptyset_y + \frac{a^2}{t^2}\frac{dv_0}{dS} \end{bmatrix} \tag{3.183}$$

Further evaluating Equation (3.89) gives the total product of σε

$$\pi = B_{11}\left(-S\frac{\partial^2 w}{\partial R^2} + aH.\frac{\partial \phi_x}{\partial R} + \frac{a}{t}\frac{du_0}{dR}\right)^2 + B_{12}\left(-S\frac{\partial^2 w}{\partial R^2} + aH.\frac{\partial \phi_x}{\partial R} + \frac{a}{t}\frac{du_0}{dR}\right)$$

$$\left(-\frac{S}{\beta^2}\frac{\partial^2 w}{\partial Q^2} + \frac{aH}{\beta}.\frac{\partial \phi_y}{\partial Q} + \frac{a}{t}\frac{dv_0}{\beta\,\partial Q}\right) + B_{13}\left(-S\frac{\partial^2 w}{\partial R^2} + aH.\frac{\partial \phi_x}{\partial R} + \frac{a}{t}\frac{du_0}{dR}\right)$$

$$\left(-\frac{2S}{\beta}\frac{\partial^2 w}{\partial R\,\partial Q} + \frac{aH}{\beta}.\left(\frac{\partial \phi_x}{\partial Q} + \beta.\frac{\partial \phi_y}{\partial R}\right) + \frac{a}{t}\frac{du_0}{\beta\,\partial Q} + \frac{a}{t}\frac{dv_0}{dR}\right)$$

$$+B_{21}\left(-\frac{S}{\beta^2}\frac{\partial^2 w}{\partial Q^2} + \frac{aH}{\beta}.\frac{\partial \phi_y}{\partial Q} + \frac{a}{t}\frac{dv_0}{\beta\,\partial Q}\right)\left(-S\frac{\partial^2 w}{\partial R^2} + aH.\frac{\partial \phi_x}{\partial R} + \frac{a}{t}\frac{du_0}{dR}\right).$$

$$+B_{22}\left(-\frac{S}{\beta^2}\frac{\partial^2 w}{\partial Q^2} + \frac{aH}{\beta}.\frac{\partial \phi_y}{\partial Q} + \frac{a}{t}\frac{dv_0}{\beta\,\partial Q}\right)^2 + B_{23}\left(-\frac{S}{\beta^2}\frac{\partial^2 w}{\partial Q^2} + \frac{aH}{\beta}.\frac{\partial \phi_y}{\partial Q} + \frac{a}{t}\frac{dv_0}{\beta\,\partial Q}\right)$$

$$\left(-\frac{2S}{\beta}\frac{\partial^2 w}{\partial R\,\partial Q} + \frac{aH}{\beta}.\left(\frac{\partial \phi_x}{\partial Q} + \beta.\frac{\partial \phi_y}{\partial R}\right) + \frac{a}{t}\frac{du_0}{\beta\,\partial Q} + \frac{a}{t}\frac{dv_0}{dR}\right)$$

$$+B_{31}\left(-\frac{2S}{\beta}\frac{\partial^2 w}{\partial R\,\partial Q} + \frac{aH}{\beta}.\left(\frac{\partial \phi_x}{\partial Q} + \beta.\frac{\partial \phi_y}{\partial R}\right) + \frac{a}{t}\frac{du_0}{\beta\,\partial Q} + \frac{a}{t}\frac{dv_0}{dR}\right)$$

$$\left(-S\frac{\partial^2 w}{\partial R^2} + aH.\frac{\partial \phi_x}{\partial R} + \frac{a}{t}\frac{du_0}{dR}\right)$$

$$+B_{32}\left(-\frac{2S}{\beta}\frac{\partial^2 w}{\partial R\,\partial Q} + \frac{aH}{\beta}.\left(\frac{\partial \phi_x}{\partial Q} + \beta.\frac{\partial \phi_y}{\partial R}\right) + \frac{a}{t}\frac{du_0}{\beta\,\partial Q} + \frac{a}{t}\frac{dv_0}{dR}\right)$$

$$\left(-\frac{S}{\beta^2}\frac{\partial^2 w}{\partial Q^2} + \frac{aH}{\beta}.\frac{\partial \phi_y}{\partial Q} + \frac{a}{t}\frac{dv_0}{\beta\,\partial Q}\right)$$

$$+B_{33}\left(-\frac{2S}{\beta}\frac{\partial^2 w}{\partial R\,\partial Q}+\frac{aH}{\beta}.\left(\frac{\partial\phi_x}{\partial Q}+\beta.\frac{\partial\phi_y}{\partial R}\right)+\frac{a}{t}\frac{du_0}{\beta\,\partial Q}+\frac{a}{t}\frac{dv_0}{dR}\right)^2$$

$$+B_{44}\left(\frac{a^2}{t}\frac{\partial H}{\partial S}.\emptyset_x+\frac{a^2}{t^2}\frac{du_0}{dS}\right)^2+B_{55}\left(\frac{a^2}{t}\frac{\partial H}{\partial S}.\emptyset_y+\frac{a^2}{t^2}\frac{dv_0}{dS}\right)^2 \tag{3.184}$$

where

$$\sigma_R\varepsilon_R=B_{11}\left(-S\frac{\partial^2 w}{\partial R^2}+aH.\frac{\partial\phi_x}{\partial R}+\frac{a}{t}\frac{du_0}{dR}\right)\left(-S\frac{\partial^2 w}{\partial R^2}+aH.\frac{\partial\phi_x}{\partial R}+\frac{a}{t}\frac{du_0}{dR}\right)$$

$$+B_{12}\left(-S\frac{\partial^2 w}{\partial R^2}+aH.\frac{\partial\phi_x}{\partial R}+\frac{a}{t}\frac{du_0}{dR}\right)\left(-\frac{S}{\beta^2}\frac{\partial^2 w}{\partial Q^2}+\frac{aH}{\beta}.\frac{\partial\phi_y}{\partial Q}+\frac{a}{t}\frac{dv_0}{\beta\partial Q}\right)$$

$$\left(-\frac{2S}{\beta}\frac{\partial^2 w}{\partial R\partial Q}+\frac{aH}{\beta}.\left(\frac{\partial\phi_x}{\partial Q}+\beta.\frac{\partial\phi_y}{\partial R}\right)+\frac{a}{t}\frac{du_0}{\beta\partial Q}+\frac{a}{t}\frac{dv_0}{dR}\right)$$

$$+B_{13}\left(-S\frac{\partial^2 w}{\partial R^2}+aH.\frac{\partial\phi_x}{\partial R}+\frac{a}{t}\frac{du_0}{dR}\right) \tag{3.185}$$

$$\sigma_Q\varepsilon_Q=B_{21}\left(-\frac{S}{\beta^2}\frac{\partial^2 w}{\partial Q^2}+\frac{aH}{\beta}.\frac{\partial\phi_y}{\partial Q}+\frac{a}{t}\frac{dv_0}{\beta\partial Q}\right)\left(-S\frac{\partial^2 w}{\partial R^2}+aH.\frac{\partial\phi_x}{\partial R}+\frac{a}{t}\frac{du_0}{dR}\right).$$

$$+B_{22}\left(-\frac{S}{\beta^2}\frac{\partial^2 w}{\partial Q^2}+\frac{aH}{\beta}.\frac{\partial\phi_y}{\partial Q}+\frac{a}{t}\frac{dv_0}{\beta\partial Q}\right)^2+B_{23}\left(-\frac{S}{\beta^2}\frac{\partial^2 w}{\partial Q^2}+\frac{aH}{\beta}.\frac{\partial\phi_y}{\partial Q}+\frac{a}{t}\frac{dv_0}{\beta\partial Q}\right)$$

$$\left(-\frac{2S}{\beta}\frac{\partial^2 w}{\partial R\partial Q}+\frac{aH}{\beta}.\left(\frac{\partial\phi_x}{\partial Q}+\beta.\frac{\partial\phi_y}{\partial R}\right)+\frac{a}{t}\frac{du_0}{\beta\partial Q}+\frac{a}{t}\frac{dv_0}{dR}\right) \tag{3.186}$$

$$\tau_{RQ}\gamma_{RQ}=B_{31}\left(-\frac{2S}{\beta}\frac{\partial^2 w}{\partial R\partial Q}+\frac{aH}{\beta}.\left(\frac{\partial\phi_x}{\partial Q}+\beta.\frac{\partial\phi_y}{\partial R}\right)+\frac{a}{t}\frac{du_0}{\beta\partial Q}+\frac{a}{t}\frac{dv_0}{dR}\right)$$

$$\left(-S\frac{\partial^2 w}{\partial R^2}+aH.\frac{\partial\phi_x}{\partial R}+\frac{a}{t}\frac{du_0}{dR}\right)$$

$$+B_{32}\left(-\frac{2S}{\beta}\frac{\partial^2 w}{\partial R\partial Q}+\frac{aH}{\beta}.\left(\frac{\partial\phi_x}{\partial Q}+\beta.\frac{\partial\phi_y}{\partial R}\right)+\frac{a}{t}\frac{du_0}{\beta\partial Q}+\frac{a}{t}\frac{dv_0}{dR}\right)$$

$$\left(-\frac{S}{\beta^2}\frac{\partial^2 w}{\partial Q^2}+\frac{aH}{\beta}.\frac{\partial\phi_y}{\partial Q}+\frac{a}{t}\frac{dv_0}{\beta\partial Q}\right)$$

$$+B_{33}\left(-\frac{2S}{\beta}\frac{\partial^2 w}{\partial R\partial Q}+\frac{aH}{\beta}.\left(\frac{\partial\phi_x}{\partial Q}+\beta.\frac{\partial\phi_y}{\partial R}\right)+\frac{a}{t}\frac{du_0}{\beta\partial Q}+\frac{a}{t}\frac{dv_0}{dR}\right)^2 \tag{3.187}$$

$$\tau_{RS}\gamma_{RS}=B_{44}\left(\frac{a^2}{t}\frac{\partial H}{\partial S}.\emptyset_x+\frac{a^2}{t^2}\frac{du_0}{dS}\right)^2 \tag{3.188}$$

$$\tau_{QS}\gamma_{QS}=B_{55}\left(\frac{a^2}{t}\frac{\partial H}{\partial S}.\emptyset_y+\frac{a^2}{t^2}\frac{dv_0}{dS}\right)^2 \tag{3.189}$$

Expanding the coefficient of B_{11} in Equation (3.184) gives:

$$\sigma_R\varepsilon_R=B_{11}\left(-S\frac{\partial^2 w}{\partial R^2}+aH.\frac{\partial\phi_x}{\partial R}+\frac{a}{t}\frac{du_0}{dR}\right)\left(-S\frac{\partial^2 w}{\partial R^2}+aH.\frac{\partial\phi_x}{\partial R}+\frac{a}{t}\frac{du_0}{dR}\right)$$

$$\sigma_R\varepsilon_R=B_{11}\left(S\frac{\partial^2 w}{\partial R^2}.S\frac{\partial^2 w}{\partial R^2}-aH.\frac{\partial\phi_x}{\partial R}.S\frac{\partial^2 w}{\partial R^2}-\frac{a}{t}\frac{du_0}{dR}.S\frac{\partial^2 w}{\partial R^2}\right)$$

$$+\left(-S\frac{\partial^2 w}{\partial R^2}.aH.\frac{\partial\phi_x}{\partial R}+aH.\frac{\partial\phi_x}{\partial R}.aH.\frac{\partial\phi_x}{\partial R}+\frac{a}{t}\frac{du_0}{dR}.aH.\frac{\partial\phi_x}{\partial R}\right)$$

$$+\left(-S\frac{\partial^2 w}{\partial R^2}.\frac{a}{t}\frac{du_0}{dR}+aH.\frac{\partial\phi_x}{\partial R}.\frac{a}{t}\frac{du_0}{dR}+\frac{a}{t}\frac{du_0}{dR}.\frac{a}{t}\frac{du_0}{dR}\right)$$

$$=B_{11}$$

$$\left(S^2\left(\frac{\partial^2 w}{\partial R^2}\right)^2-aHS\frac{\partial^2 w}{\partial R^2}.\frac{\partial\phi_x}{\partial R}-\frac{a}{t}S.\frac{\partial^2 w}{\partial R^2}.\frac{du_0}{dR}-aHS.\frac{\partial^2 w}{\partial R^2}.\frac{\partial\phi_x}{\partial R}+\frac{a^2}{t^2}.\left(\frac{du_0}{dR}\right)^2\right)$$

$$+a^2H^2\left(\frac{\partial\phi_x}{\partial R}\right)^2+\frac{a^2}{t}H.\frac{du_0}{dR}.\frac{\partial\phi_x}{\partial R} \tag{3.190}$$

and finally expanding the coefficient of B_{12} in Equation (3.184) gives

$$\sigma_Q\varepsilon_Q=B_{12}\left(-S\frac{\partial^2 w}{\partial R^2}+aH.\frac{\partial\phi_x}{\partial R}+\frac{a}{t}\frac{du_0}{dR}\right)\left(-\frac{S}{\beta^2}\frac{\partial^2 w}{\partial Q^2}+\frac{aH}{\beta}.\frac{\partial\phi_y}{\partial Q}+\frac{a}{t}\frac{dv_0}{\beta\partial Q}\right)$$

$$\sigma_Q\varepsilon_Q=B_{12}\left(-\frac{S}{\beta^2}\frac{\partial^2 w}{\partial Q^2}.-S\frac{\partial^2 w}{\partial R^2}+\frac{aH}{\beta}.\frac{\partial\phi_y}{\partial Q}.-S\frac{\partial^2 w}{\partial R^2}+\frac{a}{t}\frac{dv_0}{\beta\partial Q}.-S\frac{\partial^2 w}{\partial R^2}-\frac{S}{\beta^2}\frac{\partial^2 w}{\partial Q^2}.aH.\frac{\partial\phi_x}{\partial R}\right.$$
$$+\frac{aH}{\beta}.\frac{\partial\phi_y}{\partial Q}.aH.\frac{\partial\phi_x}{\partial R}+\frac{a}{t}\frac{dv_0}{\beta\partial Q}.aH.\frac{\partial\phi_x}{\partial R}-\frac{S}{\beta^2}\frac{\partial^2 w}{\partial Q^2}.\frac{a}{t}\frac{du_0}{dR}+\frac{aH}{\beta}.\frac{\partial\phi_y}{\partial Q}.\frac{a}{t}\frac{du_0}{dR}$$
$$\left.+\frac{a}{t}\frac{dv_0}{\beta\partial Q}.\frac{a}{t}\frac{du_0}{dR}\right)$$

$$= B_{12}\left(\frac{S}{\beta^2}\frac{\partial^2 w}{\partial Q^2}.S\frac{\partial^2 w}{\partial R^2}-\frac{aH}{\beta}.\frac{\partial\phi_y}{\partial Q}.S\frac{\partial^2 w}{\partial R^2}-\frac{a}{t}\frac{dv_0}{\beta\,\partial Q}.S\frac{\partial^2 w}{\partial R^2}-\frac{S}{\beta^2}\frac{\partial^2 w}{\partial Q^2}.aH.\frac{\partial\phi_x}{\partial R}+\frac{aH}{\beta}.\frac{\partial\phi_y}{\partial Q}.aH.\frac{\partial\phi_x}{\partial R}\right.$$
$$\left.+\frac{a}{t}\frac{dv_0}{\beta\,\partial Q}.aH.\frac{\partial\phi_x}{\partial R}-\frac{S}{\beta^2}\frac{\partial^2 w}{\partial Q^2}.\frac{a}{t}\frac{du_0}{dR}+\frac{aH}{\beta}.\frac{\partial\phi_y}{\partial Q}.\frac{a}{t}\frac{du_0}{dR}+\frac{a}{t}\frac{dv_0}{\beta\,\partial Q}.\frac{a}{t}\frac{du_0}{dR}\right)$$

$$= B_{12}\left(\frac{S^2}{\beta^2}.\frac{\partial^2 w}{\partial R^2}.\frac{\partial^2 w}{\partial Q^2}-\frac{aHS}{\beta}.\frac{\partial^2 w}{\partial R^2}.\frac{\partial\phi_y}{\partial Q}-\frac{a}{t\beta}S.\frac{\partial^2 w}{\partial R^2}.\frac{dv_0}{\partial Q}-\frac{aHS}{\beta^2}\frac{\partial^2 w}{\partial Q^2}.\frac{\partial\phi_x}{\partial R}+\frac{a^2H^2}{\beta}.\frac{\partial\phi_y}{\partial Q}.\frac{\partial\phi_x}{\partial R}\right.$$
$$\left.+\frac{a^2H}{t\beta}\frac{dv_0}{\partial Q}.\frac{\partial\phi_x}{\partial R}-\frac{aS}{t\beta^2}\frac{\partial^2 w}{\partial Q^2}.\frac{du_0}{dR}+\frac{a^2H}{t\beta}.\frac{\partial\phi_y}{\partial Q}.\frac{du_0}{dR}+\frac{a^2}{t^2\beta}\frac{du_0}{dR}.\frac{dv_0}{\partial Q}\right) \tag{3.190}$$

Expanding the coefficient of B_{13} in Equation (3.184) gives

$$\tau_{RQ}\gamma_{RQ} = B_{13}\left(-S\frac{\partial^2 w}{\partial R^2}+aH.\frac{\partial\phi_x}{\partial R}+\frac{a}{t}\frac{du_0}{dR}\right)$$
$$\left(-\frac{2S}{\beta}\frac{\partial^2 w}{\partial R\,\partial Q}+\frac{aH}{\beta}.\left(\frac{\partial\phi_x}{\partial Q}+\beta.\frac{\partial\phi_y}{\partial R}\right)+\frac{a}{t}\frac{du_0}{\beta\,\partial Q}+\frac{a}{t}\frac{dv_0}{dR}\right)$$
$$= B_{13}\left(-S\frac{\partial^2 w}{\partial R^2}+aH.\frac{\partial\phi_x}{\partial R}+\frac{a}{t}\frac{du_0}{dR}\right)$$
$$\left(-\frac{2S}{\beta}\frac{\partial^2 w}{\partial R\,\partial Q}+.\frac{\partial\phi_x}{\partial Q}.\frac{aH}{\beta}+\beta.\frac{\partial\phi_y}{\partial R}.\frac{aH}{\beta}+\frac{a}{t}\frac{du_0}{\beta\,\partial Q}+\frac{a}{t}\frac{dv_0}{dR}\right)$$

Also expanding the coefficient of B_{13} in Equation (3.184)gives

$$\tau_{RS}\gamma_{RS} = \left(-\frac{2S}{\beta}\frac{\partial^2 w}{\partial R\,\partial Q}.-S\frac{\partial^2 w}{\partial R^2}+\frac{\partial\phi_x}{\partial Q}.\frac{aH}{\beta}.-S\frac{\partial^2 w}{\partial R^2}+\beta.\frac{\partial\phi_y}{\partial R}.\frac{aH}{\beta}.-S\frac{\partial^2 w}{\partial R^2}+\frac{a}{t}\frac{du_0}{\beta\,\partial Q}.-S\frac{\partial^2 w}{\partial R^2}\right.$$
$$+\frac{a}{t}\frac{dv_0}{dR}.-S\frac{\partial^2 w}{\partial R^2}-\frac{2S}{\beta}\frac{\partial^2 w}{\partial R\,\partial Q}.aH.\frac{\partial\phi_x}{\partial R}+\frac{\partial\phi_x}{\partial Q}.\frac{aH}{\beta}.aH.\frac{\partial\phi_x}{\partial R}$$
$$+\beta.\frac{\partial\phi_y}{\partial R}.\frac{aH}{\beta}.aH.\frac{\partial\phi_x}{\partial R}+\frac{a}{t}\frac{du_0}{\beta\,\partial Q}.aH.\frac{\partial\phi_x}{\partial R}+\frac{a}{t}\frac{dv_0}{dR}.aH.\frac{\partial\phi_x}{\partial R}-\frac{2S}{\beta}\frac{\partial^2 w}{\partial R\,\partial Q}.\frac{a}{t}\frac{du_0}{dR}$$
$$\left.+\frac{\partial\phi_x}{\partial Q}.\frac{aH}{\beta}.\frac{a}{t}\frac{du_0}{dR}+\beta.\frac{\partial\phi_y}{\partial R}.\frac{aH}{\beta}.\frac{a}{t}\frac{du_0}{dR}+\frac{a}{t}\frac{du_0}{\beta\,\partial Q}.\frac{a}{t}\frac{du_0}{dR}+\frac{a}{t}\frac{dv_0}{dR}.\frac{a}{t}\frac{du_0}{dR}\right)$$

That means

$\tau_{RS}\gamma_{RS}$

$$= B_{13}\left(\left(\frac{2S^2}{\beta}\frac{\partial^2 w}{\partial R\,\partial Q}\cdot\frac{\partial^2 w}{\partial R^2} - \cdot\frac{aH}{\beta}S\frac{\partial^2 w}{\partial R^2}\cdot\frac{\partial\phi_x}{\partial Q} - aHS\frac{\partial^2 w}{\partial R^2}\cdot\frac{\partial\phi_y}{\partial R} - \frac{a}{t\beta}S\frac{\partial^2 w}{\partial R^2}\cdot\frac{du_0}{\partial Q}\right.\right.$$
$$\left. - \frac{a}{t}S\frac{\partial^2 w}{\partial R^2}\cdot\frac{dv_0}{dR}\right)$$
$$+\left(-\frac{2aHS}{\beta}\frac{\partial^2 w}{\partial R\,\partial Q}\cdot\frac{\partial\phi_x}{\partial R} + \frac{a^2H^2}{\beta}\cdot\frac{\partial\phi_x}{\partial R}\cdot\frac{\partial\phi_x}{\partial Q} + a^2H^2\cdot\frac{\partial\phi_y}{\partial R}\cdot\frac{\partial\phi_x}{\partial R} + \frac{a^2H}{t}\frac{du_0}{\beta\,\partial Q}\cdot\frac{\partial\phi_x}{\partial R} + \frac{a^2H}{t}\frac{dv_0}{dR}\cdot\frac{\partial\phi_x}{\partial R}\right)$$
$$+\left(-\frac{2aS}{t\beta}\frac{\partial^2 w}{\partial R\,\partial Q}\cdot\frac{du_0}{dR} + \frac{a^2H}{t\beta}\frac{\partial\phi_x}{\partial Q}\cdot\frac{du_0}{dR} + \frac{a^2H}{t\beta}\cdot\frac{\partial\phi_y}{\partial R}\cdot\frac{du_0}{dR} + \frac{a^2}{t^2}\frac{du_0}{\beta\,\partial Q}\cdot\frac{du_0}{dR}\right.$$
$$\left.\left. + \frac{a^2}{t^2}\cdot\frac{dv_0}{dR}\cdot\frac{du_0}{dR}\right)\right) \tag{3.191}$$

Expanding the coefficient of B_{21}in Equation (3.184) is the same as that of the cofficient B_{12}

That means

$$B_{21} = B_{12}. \tag{3.192a}$$

Expanding the coefficient of B_{22} in Equation (3.184)gives

$$B_{22}\left(-\frac{S}{\beta^2}\frac{\partial^2 w}{\partial Q^2} + \frac{aH}{\beta}\cdot\frac{\partial\phi_y}{\partial Q} + \frac{a}{t}\frac{dv_0}{\beta\,\partial Q}\right)^2$$

$$= B_{22}\left(-\frac{S}{\beta^2}\frac{\partial^2 w}{\partial Q^2} + \frac{aH}{\beta}\cdot\frac{\partial\phi_y}{\partial Q} + \frac{a}{t}\frac{dv_0}{\beta\,\partial Q}\right)\left(-\frac{S}{\beta^2}\frac{\partial^2 w}{\partial Q^2} + \frac{aH}{\beta}\cdot\frac{\partial\phi_y}{\partial Q} + \frac{a}{t}\frac{dv_0}{\beta\,\partial Q}\right)$$

$$= B_{22}\left(\frac{S}{\beta^2}\frac{\partial^2 w}{\partial Q^2}\cdot\frac{S}{\beta^2}\frac{\partial^2 w}{\partial Q^2} - \frac{aH}{\beta}\cdot\frac{\partial\phi_y}{\partial Q}\cdot\frac{S}{\beta^2}\frac{\partial^2 w}{\partial Q^2} - \frac{a}{t}\frac{dv_0}{\beta\,\partial Q}\cdot\frac{S}{\beta^2}\frac{\partial^2 w}{\partial Q^2} - \frac{S}{\beta^2}\frac{\partial^2 w}{\partial Q^2}\cdot\frac{aH}{\beta}\cdot\frac{\partial\phi_y}{\partial Q}\right.$$
$$+ \frac{aH}{\beta}\cdot\frac{\partial\phi_y}{\partial Q}\cdot\frac{aH}{\beta}\cdot\frac{\partial\phi_y}{\partial Q} + \frac{a}{t}\frac{dv_0}{\beta\,\partial Q}\cdot\frac{aH}{\beta}\cdot\frac{\partial\phi_y}{\partial Q} - \frac{S}{\beta^2}\frac{\partial^2 w}{\partial Q^2}\cdot\frac{a}{t}\frac{dv_0}{\beta\,\partial Q} + \frac{aH}{\beta}\cdot\frac{\partial\phi_y}{\partial Q}\cdot\frac{a}{t}\frac{dv_0}{\beta\,\partial Q}$$
$$\left. + \frac{a}{t}\frac{dv_0}{\beta\,\partial Q}\cdot\frac{a}{t}\frac{dv_0}{\beta\,\partial Q}\right)$$

$$= B_{22}\left(\frac{S^2}{\beta^4}\left(\frac{\partial^2 w}{\partial Q^2}\right)^2 - \frac{aHS}{\beta^3}\cdot\frac{\partial^2 w}{\partial Q^2}\cdot\frac{\partial\phi_y}{\partial Q} - \frac{aS}{t\beta^3}\cdot\frac{\partial^2 w}{\partial Q^2}\cdot\frac{dv_0}{\partial Q} - \frac{aHS}{\beta^3}\frac{\partial^2 w}{\partial Q^2}\cdot\frac{\partial\phi_y}{\partial Q} + \frac{a^2H^2}{\beta^2}\cdot\left(\frac{\partial\phi_y}{\partial Q}\right)^2\right.$$
$$\left. + \frac{a^2H}{t\beta^2}\frac{dv_0}{\partial Q}\cdot\frac{\partial\phi_y}{\partial Q} - \frac{aS}{t\beta^3}\frac{\partial^2 w}{\partial Q^2}\cdot\frac{dv_0}{\partial Q} + \frac{a^2H}{t\beta^2}\cdot\frac{\partial\phi_y}{\partial Q}\cdot\frac{dv_0}{\partial Q} + \frac{a^2}{t^2\beta^2}\cdot\left(\frac{dv_0}{\partial Q}\right)^2\right) \tag{3.192b}$$

Expanding the the coefficient of B_{23} in Equation (3.184)gives

$$B_{23}\left(-\frac{S}{\beta^2}\frac{\partial^2 w}{\partial Q^2}+\frac{aH}{\beta}.\frac{\partial\phi_y}{\partial Q}+\frac{a}{t}\frac{dv_0}{\beta\partial Q}\right)\left(-\frac{2S}{\beta}\frac{\partial^2 w}{\partial R\partial Q}+\frac{aH}{\beta}.\left(\frac{\partial\phi_x}{\partial Q}+\beta.\frac{\partial\phi_y}{\partial R}\right)+\frac{a}{t}\frac{du_0}{\beta\partial Q}+\frac{a}{t}\frac{dv_0}{dR}\right)$$

$$=B_{23}\left(-\frac{S}{\beta^2}\frac{\partial^2 w}{\partial Q^2}+\frac{aH}{\beta}.\frac{\partial\phi_y}{\partial Q}+\frac{a}{t}\frac{dv_0}{\beta\partial Q}\right)\left(-\frac{2S}{\beta}\frac{\partial^2 w}{\partial R\partial Q}+\frac{aH}{\beta}.\frac{\partial\phi_x}{\partial Q}+\beta\frac{aH}{\beta}.\frac{\partial\phi_y}{\partial R}+\frac{a}{t}\frac{du_0}{\beta\partial Q}\right.$$
$$\left.+\frac{a}{t}\frac{dv_0}{dR}\right)$$

$$=B_{23}\left(-\frac{2S}{\beta}\frac{\partial^2 w}{\partial R\partial Q}.-\frac{S}{\beta^2}\frac{\partial^2 w}{\partial Q^2}+\frac{aH}{\beta}.\frac{\partial\phi_x}{\partial Q}.-\frac{S}{\beta^2}\frac{\partial^2 w}{\partial Q^2}+\beta\frac{aH}{\beta}.\frac{\partial\phi_y}{\partial R}.-\frac{S}{\beta^2}\frac{\partial^2 w}{\partial Q^2}\right.$$
$$+\frac{a}{t}\frac{du_0}{\beta\partial Q}.-\frac{S}{\beta^2}\frac{\partial^2 w}{\partial Q^2}+\frac{a}{t}\frac{dv_0}{dR}.-\frac{S}{\beta^2}\frac{\partial^2 w}{\partial Q^2}-\frac{2S}{\beta}\frac{\partial^2 w}{\partial R\partial Q}.\frac{aH}{\beta}.\frac{\partial\phi_y}{\partial Q}$$
$$+\frac{aH}{\beta}.\frac{\partial\phi_x}{\partial Q}.\frac{aH}{\beta}.\frac{\partial\phi_y}{\partial Q}+\beta\frac{aH}{\beta}.\frac{\partial\phi_y}{\partial R}.\frac{aH}{\beta}.\frac{\partial\phi_y}{\partial Q}+\frac{a}{t}\frac{du_0}{\beta\partial Q}.\frac{aH}{\beta}.\frac{\partial\phi_y}{\partial Q}+\frac{a}{t}\frac{dv_0}{dR}.\frac{aH}{\beta}.\frac{\partial\phi_y}{\partial Q}$$
$$-\frac{2S}{\beta}\frac{\partial^2 w}{\partial R\partial Q}.\frac{a}{t}\frac{dv_0}{\beta\partial Q}+\frac{aH}{\beta}.\frac{\partial\phi_x}{\partial Q}.\frac{a}{t}\frac{dv_0}{\beta\partial Q}+\beta\frac{aH}{\beta}.\frac{\partial\phi_y}{\partial R}.\frac{a}{t}\frac{dv_0}{\beta\partial Q}+\frac{a}{t}\frac{du_0}{\beta\partial Q}.\frac{a}{t}\frac{dv_0}{\beta\partial Q}$$
$$\left.+\frac{a}{t}\frac{dv_0}{dR}.\frac{a}{t}\frac{dv_0}{\beta\partial Q}\right)$$

$$=B_{23}\left(\frac{2S^2}{\beta^3}\frac{\partial^2 w}{\partial R\partial Q}.\frac{\partial^2 w}{\partial Q^2}-\frac{aSH}{\beta^3}.\frac{\partial\phi_x}{\partial Q}.\frac{\partial^2 w}{\partial Q^2}-\frac{aSH}{\beta^2}.\frac{\partial\phi_y}{\partial R}.\frac{\partial^2 w}{\partial Q^2}-\frac{aS}{t\beta^3}\frac{du_0}{\partial Q}.\frac{\partial^2 w}{\partial Q^2}-\frac{aS}{t\beta^2}\frac{dv_0}{dR}.\frac{\partial^2 w}{\partial Q^2}\right.$$
$$-\frac{2aHS}{\beta^2}\frac{\partial^2 w}{\partial R\partial Q}.\frac{\partial\phi_y}{\partial Q}+\frac{a^2H^2}{\beta^2}.\frac{\partial\phi_x}{\partial Q}.\frac{\partial\phi_y}{\partial Q}+\frac{a^2H^2}{\beta}.\frac{\partial\phi_y}{\partial R}.\frac{\partial\phi_y}{\partial Q}+\frac{a^2H}{t\beta^2}\frac{du_0}{\partial Q}.\frac{\partial\phi_y}{\partial Q}$$
$$+\frac{a^2H}{t\beta}\frac{dv_0}{dR}.\frac{\partial\phi_y}{\partial Q}-\frac{2aS}{t\beta^2}\frac{\partial^2 w}{\partial R\partial Q}.\frac{dv_0}{\partial Q}+\frac{a^2H}{t\beta^2}.\frac{\partial\phi_x}{\partial Q}.\frac{dv_0}{\partial Q}+\frac{a^2H}{t\beta}.\frac{\partial\phi_y}{\partial R}.\frac{dv_0}{\partial Q}$$
$$\left.+\frac{a^2}{t^2\beta^2}\frac{du_0}{\partial Q}\frac{dv_0}{\partial Q}+\frac{a^2}{t^2\beta}\frac{dv_0}{dR}.\frac{dv_0}{\partial Q}\right) \qquad (3.193)$$

Expanding the coefficient of B_{31} in Equation (3.184) gives the same as that of B_{13}. That means

$$B_{31}=B_{13} \qquad (3.194)$$

Similarly expanding the coefficient of B_{32} in Equation (3.184) gives the same as that of

B_{32}. Meaning that

$$B_{32}=B_{23} \qquad (3.195)$$

Expanding the he coefficient of B_{33} in Equation (3.184) gives

$$B_{33}\left(-\frac{2S}{\beta}\frac{\partial^2 w}{\partial R\,\partial Q}+\frac{aH}{\beta}.\left(\frac{\partial\phi_x}{\partial Q}+\beta.\frac{\partial\phi_y}{\partial R}\right)+\frac{a}{t}\frac{du_0}{\beta\,\partial Q}+\frac{a}{t}\frac{dv_0}{dR}\right)^2$$

$$=B_{33}\left(-\frac{2S}{\beta}\frac{\partial^2 w}{\partial R\,\partial Q}+\frac{aH}{\beta}.\frac{\partial\phi_x}{\partial Q}+aH\frac{\partial\phi_y}{\partial R}+\frac{a}{t}\frac{du_0}{\beta\,\partial Q}+\frac{a}{t}\frac{dv_0}{dR}\right)$$

This gives

$$=B_{33}\left(\frac{2S}{\beta}\frac{\partial^2 w}{\partial R\,\partial Q}.\frac{2S}{\beta}\frac{\partial^2 w}{\partial R\,\partial Q}-\frac{aH}{\beta}.\frac{\partial\phi_x}{\partial Q}.\frac{2S}{\beta}\frac{\partial^2 w}{\partial R\,\partial Q}-aH\frac{\partial\phi_y}{\partial R}.\frac{2S}{\beta}\frac{\partial^2 w}{\partial R\,\partial Q}-\frac{a}{t}\frac{du_0}{\beta\,\partial Q}.\frac{2S}{\beta}\frac{\partial^2 w}{\partial R\,\partial Q}\right.$$
$$\left.-\frac{a}{t}\frac{dv_0}{dR}.\frac{2S}{\beta}\frac{\partial^2 w}{\partial R\,\partial Q}\right)+$$

$$B_{33}\left(-\frac{2S}{\beta}\frac{\partial^2 w}{\partial R\,\partial Q}.\frac{aH}{\beta}.\frac{\partial\phi_x}{\partial Q}+\frac{aH}{\beta}.\frac{\partial\phi_x}{\partial Q}.\frac{aH}{\beta}.\frac{\partial\phi_x}{\partial Q}+aH\frac{\partial\phi_y}{\partial R}.\frac{aH}{\beta}.\frac{\partial\phi_x}{\partial Q}+\frac{a}{t}\frac{du_0}{\beta\,\partial Q}.\frac{aH}{\beta}.\frac{\partial\phi_x}{\partial Q}+\frac{a}{t}\frac{dv_0}{dR}.\frac{aH}{\beta}.\frac{\partial\phi_x}{\partial Q}\right)+$$

$$B_{33}\left(-\frac{2S}{\beta}\frac{\partial^2 w}{\partial R\,\partial Q}.aH\frac{\partial\phi_y}{\partial R}+\frac{aH}{\beta}.\frac{\partial\phi_x}{\partial Q}.aH\frac{\partial\phi_y}{\partial R}+aH\frac{\partial\phi_y}{\partial R}.aH\frac{\partial\phi_y}{\partial R}+\frac{a}{t}\frac{du_0}{\beta\,\partial Q}.aH\frac{\partial\phi_y}{\partial R}+\frac{a}{t}\frac{dv_0}{dR}.aH\frac{\partial\phi_y}{\partial R}\right)+$$

$$B_{33}\left(-\frac{2S}{\beta}\frac{\partial^2 w}{\partial R\,\partial Q}.\frac{a}{t}\frac{du_0}{\beta\,\partial Q}+\frac{aH}{\beta}.\frac{\partial\phi_x}{\partial Q}.\frac{a}{t}\frac{du_0}{\beta\,\partial Q}+aH\frac{\partial\phi_y}{\partial R}.\frac{a}{t}\frac{du_0}{\beta\,\partial Q}+\frac{a}{t}\frac{du_0}{\beta\,\partial Q}.\frac{a}{t}\frac{du_0}{\beta\,\partial Q}\right.$$
$$\left.+\frac{a}{t}\frac{dv_0}{dR}.\frac{a}{t}\frac{du_0}{\beta\,\partial Q}\right)+$$

$$B_{33}\left(-\frac{2S}{\beta}\frac{\partial^2 w}{\partial R\,\partial Q}.\frac{a}{t}\frac{dv_0}{dR}+\frac{aH}{\beta}.\frac{\partial\phi_x}{\partial Q}.\frac{a}{t}\frac{dv_0}{dR}+aH\frac{\partial\phi_y}{\partial R}.\frac{a}{t}\frac{dv_0}{dR}+\frac{a}{t}\frac{du_0}{\beta\,\partial Q}.\frac{a}{t}\frac{dv_0}{dR}+\frac{a}{t}\frac{dv_0}{dR}.\frac{a}{t}\frac{dv_0}{dR}\right)$$

Simplifying further gives

$$=B_{33}\left(\frac{4S^2}{\beta^2}\left(\frac{\partial^2 w}{\partial R\,\partial Q}\right)^2-\frac{2aHS}{\beta^2}.\frac{\partial^2 w}{\partial R\,\partial Q}.\frac{\partial\phi_x}{\partial Q}-\frac{2aHS}{\beta}.\frac{\partial^2 w}{\partial R\,\partial Q}.\frac{\partial\phi_y}{\partial R}-\frac{2aS}{t\beta^2}.\frac{\partial^2 w}{\partial R\,\partial Q}.\frac{du_0}{\partial Q}\right.$$
$$\left.-\frac{2aS}{t\beta}.\frac{\partial^2 w}{\partial R\,\partial Q}.\frac{dv_0}{dR}\right)+B_{33}$$

$$\left(-\frac{2aHS}{\beta^2}\frac{\partial^2 w}{\partial R\,\partial Q}.\frac{\partial\phi_x}{\partial Q}+\frac{a^2H^2}{\beta^2}.\left(\frac{\partial\phi_x}{\partial Q}\right)^2+\frac{a^2H^2}{\beta}\frac{\partial\phi_y}{\partial R}.\frac{\partial\phi_x}{\partial Q}+\frac{a^2H}{t\beta^2}\frac{du_0}{\partial Q}.\frac{\partial\phi_x}{\partial Q}+\frac{a^2H}{t\beta}\frac{dv_0}{dR}.\frac{\partial\phi_x}{\partial Q}\right)+$$

$$\left(-\frac{2aHS}{\beta}\frac{\partial^2 w}{\partial R\,\partial Q}.\frac{\partial\phi_y}{\partial R}+\frac{a^2H^2}{\beta}.\frac{\partial\phi_x}{\partial Q}.\frac{\partial\phi_y}{\partial R}+a^2H^2\left(\frac{\partial\phi_y}{\partial R}\right)^2+\frac{a^2H}{t\beta}\frac{du_0}{\partial Q}.\frac{\partial\phi_y}{\partial R}+\frac{a^2H}{t}\frac{dv_0}{dR}.\frac{\partial\phi_y}{\partial R}\right)+$$

$$B_{33}\left(-\frac{2aS}{t\beta^2}\frac{\partial^2 w}{\partial R\,\partial Q}.\frac{du_0}{\partial Q}+\frac{a^2H}{t\beta^2}.\frac{\partial\phi_x}{\partial Q}.\frac{du_0}{\partial Q}+\frac{a^2H}{t\beta}\frac{\partial\phi_y}{\partial R}.\frac{du_0}{\partial Q}+\frac{a^2}{t^2\beta^2}.\left(\frac{du_0}{\partial Q}\right)^2+\frac{a^2}{t^2\beta}\frac{dv_0}{dR}.\frac{du_0}{\partial Q}\right)+$$

$$B_{33}\left(-\frac{2aS}{t\beta}\frac{\partial^2 w}{\partial R\,\partial Q}.\frac{dv_0}{dR}+\frac{a^2H}{t\beta}.\frac{\partial\phi_x}{\partial Q}.\frac{dv_0}{dR}+\frac{a^2H}{t}\frac{\partial\phi_y}{\partial R}.\frac{dv_0}{dR}+\frac{a^2}{t^2\beta}\frac{du_0}{\partial Q}.\frac{dv_0}{dR}\right.$$
$$\left.+\frac{a^2}{t^2}\left(\frac{dv_0}{dR}\right)^2\right) \qquad (3.196)$$

Expanding the coefficient of B_{44} in Equation (3.184) gives

$$B_{44}\left(\frac{a^2}{t}\frac{\partial H}{\partial S}.\emptyset_x+\frac{a^2}{t^2}\frac{du_0}{dS}\right)\left(\frac{a^2}{t}\frac{\partial H}{\partial S}.\emptyset_x+\frac{a^2}{t^2}\frac{du_0}{dS}\right)$$

$$=B_{44}\left(\frac{a^2}{t}\frac{\partial H}{\partial S}.\emptyset_x.\frac{a^2}{t}\frac{\partial H}{\partial S}.\emptyset_x+\frac{a^2}{t^2}\frac{du_0}{dS}.\frac{a^2}{t}\frac{\partial H}{\partial S}.\emptyset_x+\frac{a^2}{t}\frac{\partial H}{\partial S}.\emptyset_x.\frac{a^2}{t^2}\frac{du_0}{dS}+\frac{a^2}{t^2}\frac{du_0}{dS}.\frac{a^2}{t^2}\frac{du_0}{dS}\right)$$

$$=B_{44}\frac{a^4}{t^2}\left(\left(\emptyset_x.\frac{\partial H}{\partial S}\right)^2+\emptyset_x.\frac{du_0}{dS}.\frac{\partial H}{\partial S}+\emptyset_x.\frac{\partial H}{\partial S}.\frac{du_0}{dS}+\left(\frac{du_0}{dS}\right)^2\right) \qquad (3.197)$$

and finally expanding the coefficient of B_{55} in Equation (3.184) gives

$$B_{55}\left(\frac{a^2}{t}\frac{\partial H}{\partial S}.\emptyset_y+\frac{a^2}{t^2}\frac{dv_0}{dS}\right)\left(\frac{a^2}{t}\frac{\partial H}{\partial S}.\emptyset_y+\frac{a^2}{t^2}\frac{dv_0}{dS}\right)$$

$$=B_{55}\left(\frac{a^2}{t}\frac{\partial H}{\partial S}.\emptyset_y+\frac{a^2}{t^2}\frac{dv_0}{dS}\right)\left(\frac{a^2}{t}\frac{\partial H}{\partial S}.\emptyset_y+\frac{a^2}{t^2}\frac{dv_0}{dS}\right)$$

$$=B_{55}\left(\frac{a^2}{t}\frac{\partial H}{\partial S}.\emptyset_y.\frac{a^2}{t}\frac{\partial H}{\partial S}.\emptyset_y+\frac{a^2}{t^2}\frac{dv_0}{dS}.\frac{a^2}{t}\frac{\partial H}{\partial S}.\emptyset_y+\frac{a^2}{t}\frac{\partial H}{\partial S}.\emptyset_y.\frac{a^2}{t^2}\frac{dv_0}{dS}+\frac{a^2}{t^2}\frac{dv_0}{dS}.\frac{a^2}{t^2}\frac{dv_0}{dS}\right)$$

$$=B_{55}\frac{a^4}{t^2}\left(\left(\emptyset_y.\frac{\partial H}{\partial S}\right)^2+.\emptyset_y.\frac{dv_0}{dS}.\frac{\partial H}{\partial S}+\emptyset_y.\frac{\partial H}{\partial S}.\frac{dv_0}{dS}+\left(\frac{dv_0}{dS}\right)^2\right) \qquad (3.198)$$

Equation (3.199) was gotten by combining Equations (3.190a) to (3.191)

$$\sigma_R\varepsilon_R=B_{11}\left(S^2\left(\frac{\partial^2 w}{\partial R^2}\right)^2-aHS\frac{\partial^2 w}{\partial R^2}.\frac{\partial\phi_x}{\partial R}-\frac{a}{t}S.\frac{\partial^2 w}{\partial R^2}.\frac{du_0}{dR}-aHS.\frac{\partial^2 w}{\partial R^2}.\frac{\partial\phi_x}{\partial R}+a^2H^2.\left(\frac{\partial\phi_x}{\partial R}\right)^2\right.$$
$$\left.+\frac{a^2}{t}H.\frac{du_0}{dR}.\frac{\partial\phi_x}{\partial R}-\frac{aS}{t}.\frac{\partial^2 w}{\partial R^2}.\frac{du_0}{\partial R}+\frac{a^2}{t}H.\frac{\partial\phi_x}{\partial R}.\frac{du_0}{dR}+\frac{a^2}{t^2}.\left(\frac{du_0}{dR}\right)^2\right)+$$

$$+B_{12}\left(\frac{S^2}{\beta^2}.\frac{\partial^2 w}{\partial R^2}.\frac{\partial^2 w}{\partial Q^2}-\frac{aHS}{\beta}.\frac{\partial^2 w}{\partial R^2}.\frac{\partial\phi_y}{\partial Q}-\frac{a}{t\beta}S.\frac{\partial^2 w}{\partial R^2}.\frac{dv_0}{\partial Q}-\frac{aHS}{\beta^2}\frac{\partial^2 w}{\partial Q^2}.\frac{\partial\phi_x}{\partial R}+\frac{a^2H^2}{\beta}.\frac{\partial\phi_y}{\partial Q}.\frac{\partial\phi_x}{\partial R}\right.$$
$$\left.+\frac{a^2H}{t\beta}\frac{dv_0}{\partial Q}.\frac{\partial\phi_x}{\partial R}-\frac{aS}{t\beta^2}\frac{\partial^2 w}{\partial Q^2}.\frac{du_0}{dR}+\frac{a^2H}{t\beta}.\frac{\partial\phi_y}{\partial Q}.\frac{du_0}{dR}+\frac{a^2}{t^2\beta}\frac{du_0}{dR}.\frac{dv_0}{\partial Q}\right)$$

$$+B_{13}\left(\frac{2S^2}{\beta}\frac{\partial^2 w}{\partial R\,\partial Q}.\frac{\partial^2 w}{\partial R^2}-.\frac{aH}{\beta}S\frac{\partial^2 w}{\partial R^2}.\frac{\partial \phi_x}{\partial Q}-aHS\frac{\partial^2 w}{\partial R^2}.\frac{\partial \phi_y}{\partial R}-\frac{a}{t\beta}S\frac{\partial^2 w}{\partial R^2}.\frac{du_0}{\partial Q}+\frac{a}{t}S\frac{\partial^2 w}{\partial R^2}.\frac{dv_0}{dR}\right.$$
$$-\frac{2aHS}{\beta}\frac{\partial^2 w}{\partial R\,\partial Q}.\frac{\partial \phi_x}{\partial R}+\frac{a^2H^2}{\beta}.\frac{\partial \phi_x}{\partial R}.\frac{\partial \phi_x}{\partial Q}+a^2H^2.\frac{\partial \phi_y}{\partial R}.\frac{\partial \phi_x}{\partial R}+\frac{a^2H}{t}\frac{du_0}{\beta\,\partial Q}.\frac{\partial \phi_x}{\partial R}$$
$$+\frac{a^2H}{t}\frac{dv_0}{dR}.\frac{\partial \phi_x}{\partial R}-\frac{2aS}{t\beta}\frac{\partial^2 w}{\partial R\,\partial Q}.\frac{du_0}{dR}+\frac{a^2H}{t\beta}\frac{\partial \phi_x}{\partial Q}.\frac{du_0}{dR}+\frac{a^2H}{t\beta}.\frac{\partial \phi_y}{\partial R}.\frac{du_0}{dR}$$
$$\left.+\frac{a^2}{t^2}\frac{du_0}{\beta\,\partial Q}.\frac{du_0}{dR}+\frac{a^2}{t^2}.\frac{dv_0}{dR}.\frac{du_0}{dR}\right) \qquad (3.199)$$

Similarly Equation (3.200) was gotten by combining Equations (3.192) to (3.193). That is

$$\sigma_Q\varepsilon_Q = B_{21}\frac{S^2}{\beta^2}.\frac{\partial^2 w}{\partial R^2}.\frac{\partial^2 w}{\partial Q^2}-\frac{aHS}{\beta}.\frac{\partial^2 w}{\partial R^2}.\frac{\partial \phi_y}{\partial Q}-\frac{a}{t\beta}S.\frac{\partial^2 w}{\partial R^2}.\frac{dv_0}{\partial Q}-\frac{aHS}{\beta^2}\frac{\partial^2 w}{\partial Q^2}.\frac{\partial \phi_x}{\partial R}$$
$$+\frac{a^2H^2}{\beta}.\frac{\partial \phi_y}{\partial Q}.\frac{\partial \phi_x}{\partial R}+\frac{a^2H}{t\beta}\frac{dv_0}{\partial Q}.\frac{\partial \phi_x}{\partial R}-\frac{aS}{t\beta^2}\frac{\partial^2 w}{\partial Q^2}.\frac{du_0}{dR}+\frac{a^2H}{t\beta}.\frac{\partial \phi_y}{\partial Q}.\frac{du_0}{dR}+\frac{a^2}{t^2\beta}\frac{du_0}{dR}.\frac{dv_0}{\partial Q}$$
$$+B_{22}\left(\frac{S^2}{\beta^4}\left(\frac{\partial^2 w}{\partial Q^2}\right)^2-\frac{aHS}{\beta^3}.\frac{\partial^2 w}{\partial Q^2}.\frac{\partial \phi_y}{\partial Q}-\frac{aS}{t\beta^3}.\frac{\partial^2 w}{\partial Q^2}.\frac{dv_0}{\partial Q}-\frac{aHS}{\beta^3}\frac{\partial^2 w}{\partial Q^2}.\frac{\partial \phi_y}{\partial Q}+\frac{a^2H^2}{\beta^2}.\left(\frac{\partial \phi_y}{\partial Q}\right)^2\right.$$
$$\left.+\frac{a^2H}{t\beta^2}\frac{dv_0}{\partial Q}.\frac{\partial \phi_y}{\partial Q}-\frac{aS}{t\beta^3}\frac{\partial^2 w}{\partial Q^2}.\frac{dv_0}{\partial Q}+\frac{a^2H}{t\beta^2}.\frac{\partial \phi_y}{\partial Q}.\frac{dv_0}{\partial Q}+\frac{a^2}{t^2\beta^2}.\left(\frac{dv_0}{\partial Q}\right)^2\right)$$
$$+B_{23}\left(\frac{2S^2}{\beta^3}\frac{\partial^2 w}{\partial R\,\partial Q}.\frac{\partial^2 w}{\partial Q^2}-\frac{aSH}{\beta^3}.\frac{\partial \phi_x}{\partial Q}.\frac{\partial^2 w}{\partial Q^2}-\frac{aSH}{\beta^2}.\frac{\partial \phi_y}{\partial R}.\frac{\partial^2 w}{\partial Q^2}-\frac{aS}{t\beta^3}\frac{du_0}{\partial Q}.\frac{\partial^2 w}{\partial Q^2}-\frac{aS}{t\beta^2}\frac{dv_0}{dR}.\frac{\partial^2 w}{\partial Q^2}\right.$$
$$-\frac{2aHS}{\beta^2}\frac{\partial^2 w}{\partial R\,\partial Q}.\frac{\partial \phi_y}{\partial Q}+\frac{a^2H^2}{\beta^2}.\frac{\partial \phi_x}{\partial Q}.\frac{\partial \phi_y}{\partial Q}+\frac{a^2H^2}{\beta}.\frac{\partial \phi_y}{\partial R}.\frac{\partial \phi_y}{\partial Q}+\frac{a^2H}{t\beta^2}\frac{du_0}{\partial Q}.\frac{\partial \phi_y}{\partial Q}$$
$$+\frac{a^2H}{t\beta}\frac{dv_0}{dR}.\frac{\partial \phi_y}{\partial Q}-\frac{2aS}{t\beta^2}\frac{\partial^2 w}{\partial R\,\partial Q}.\frac{dv_0}{\partial Q}+\frac{a^2H}{t\beta^2}.\frac{\partial \phi_x}{\partial Q}.\frac{dv_0}{\partial Q}+\frac{a^2H}{t\beta}.\frac{\partial \phi_y}{\partial R}.\frac{dv_0}{\partial Q}$$
$$\left.+\frac{a^2}{t^2\beta^2}\frac{du_0}{\partial Q}\frac{dv_0}{\partial Q}+\frac{a^2}{t^2\beta}\frac{dv_0}{dR}.\frac{dv_0}{\partial Q}\right) \qquad (3.200)$$

Also Equation (3.201) was gotten by combining Equations (3.194) to (3.196). That is

$$\tau_{RQ}\gamma_{RQ} = B_{31}\left(\left(\frac{2S^2}{\beta}\frac{\partial^2 w}{\partial R\,\partial Q}.\frac{\partial^2 w}{\partial R^2} - \frac{aH}{\beta}S\frac{\partial^2 w}{\partial R^2}.\frac{\partial\phi_x}{\partial Q} - aHS\frac{\partial^2 w}{\partial R^2}.\frac{\partial\phi_y}{\partial R} - \frac{a}{t\beta}S\frac{\partial^2 w}{\partial R^2}.\frac{du_0}{\partial Q} - \frac{a}{t}S\frac{\partial^2 w}{\partial R^2}.\frac{dv_0}{dR}\right) + \left(-\frac{2aHS}{\beta}\frac{\partial^2 w}{\partial R\,\partial Q}.\frac{\partial\phi_x}{\partial R} + \frac{a^2H^2}{\beta}.\frac{\partial\phi_x}{\partial R}.\frac{\partial\phi_x}{\partial Q} + a^2H^2.\frac{\partial\phi_y}{\partial R}.\frac{\partial\phi_x}{\partial R} + \frac{a^2H}{t}\frac{du_0}{\beta\,\partial Q}.\frac{\partial\phi_x}{\partial R} + \frac{a^2H}{t}\frac{dv_0}{dR}.\frac{\partial\phi_x}{\partial R}\right) + \left(-\frac{2aS}{t\beta}\frac{\partial^2 w}{\partial R\,\partial Q}.\frac{du_0}{dR} + \frac{a^2H}{t\beta}\frac{\partial\phi_x}{\partial Q}.\frac{du_0}{dR} + \frac{a^2H}{t\beta}.\frac{\partial\phi_y}{\partial R}.\frac{du_0}{dR} + \frac{a^2}{t^2}\frac{du_0}{\beta\,\partial Q}.\frac{du_0}{dR} + \frac{a^2}{t^2}.\frac{dv_0}{dR}.\frac{du_0}{dR}\right)\right) + B_{32}$$

$$\left(\frac{2S^2}{\beta^3}\frac{\partial^2 w}{\partial R\,\partial Q}.\frac{\partial^2 w}{\partial Q^2} - \frac{aSH}{\beta^3}.\frac{\partial\phi_x}{\partial Q}.\frac{\partial^2 w}{\partial Q^2} - \frac{aSH}{\beta^2}.\frac{\partial\phi_y}{\partial R}.\frac{\partial^2 w}{\partial Q^2} - \frac{aS}{t\beta^3}\frac{du_0}{\partial Q}.\frac{\partial^2 w}{\partial Q^2} - \frac{aS}{t\beta^2}\frac{dv_0}{dR}.\frac{\partial^2 w}{\partial Q^2} - \frac{2aHS}{\beta^2}\frac{\partial^2 w}{\partial R\,\partial Q}.\frac{\partial\phi_y}{\partial Q} + \frac{a^2H^2}{\beta^2}.\frac{\partial\phi_x}{\partial Q}.\frac{\partial\phi_y}{\partial Q} + \frac{a^2H^2}{\beta}.\frac{\partial\phi_y}{\partial R}.\frac{\partial\phi_y}{\partial Q} + \frac{a^2H}{t\beta^2}\frac{du_0}{\partial Q}.\frac{\partial\phi_y}{\partial Q} + \frac{a^2H}{t\beta}\frac{dv_0}{dR}.\frac{\partial\phi_y}{\partial Q} - \frac{2aS}{t\beta^2}\frac{\partial^2 w}{\partial R\,\partial Q}.\frac{dv_0}{\partial Q} + \frac{a^2H}{t\beta^2}.\frac{\partial\phi_x}{\partial Q}.\frac{dv_0}{\partial Q} + \frac{a^2H}{t\beta}.\frac{\partial\phi_y}{\partial R}.\frac{dv_0}{\partial Q} + \frac{a^2}{t^2\beta^2}\frac{du_0}{\partial Q}\frac{dv_0}{\partial Q} + \frac{a^2}{t^2\beta}\frac{dv_0}{dR}.\frac{dv_0}{\partial Q}\right)$$

$$+B_{33}\left(\frac{4S^2}{\beta^2}\left(\frac{\partial^2 w}{\partial R\,\partial Q}\right)^2 - \frac{2aHS}{\beta^2}.\frac{\partial^2 w}{\partial R\,\partial Q}.\frac{\partial\phi_x}{\partial Q} - \frac{2aHS}{\beta}.\frac{\partial^2 w}{\partial R\,\partial Q}.\frac{\partial\phi_y}{\partial R} - \frac{2aS}{t\beta^2}.\frac{\partial^2 w}{\partial R\,\partial Q}.\frac{du_0}{\partial Q} - \frac{2aS}{t\beta}.\frac{\partial^2 w}{\partial R\,\partial Q}.\frac{dv_0}{dR}\right) +$$

$$B_{33}\left(-\frac{2aHS}{\beta^2}\frac{\partial^2 w}{\partial R\,\partial Q}.\frac{\partial\phi_x}{\partial Q} + \frac{a^2H^2}{\beta^2}.\left(\frac{\partial\phi_x}{\partial Q}\right)^2 + \frac{a^2H^2}{\beta}\frac{\partial\phi_y}{\partial R}.\frac{\partial\phi_x}{\partial Q} + \frac{a^2H}{t\beta^2}\frac{du_0}{\partial Q}.\frac{\partial\phi_x}{\partial Q} + \frac{a^2H}{t\beta}\frac{dv_0}{dR}.\frac{\partial\phi_x}{\partial Q}\right) +$$

$$B_{33}\left(-\frac{2aHS}{\beta}\frac{\partial^2 w}{\partial R\,\partial Q}.\frac{\partial\phi_y}{\partial R} + \frac{a^2H^2}{\beta}.\frac{\partial\phi_x}{\partial Q}.\frac{\partial\phi_y}{\partial R} + a^2H^2\left(\frac{\partial\phi_y}{\partial R}\right)^2 + \frac{a^2H}{t\beta}\frac{du_0}{\partial Q}.\frac{\partial\phi_y}{\partial R} + \frac{a^2H}{t}\frac{dv_0}{dR}.\frac{\partial\phi_y}{\partial R}\right) +$$

$$B_{33}\left(-\frac{2aS}{t\beta^2}\frac{\partial^2 w}{\partial R\,\partial Q}.\frac{du_0}{\partial Q} + \frac{a^2H}{t\beta^2}.\frac{\partial\phi_x}{\partial Q}.\frac{du_0}{\partial Q} + \frac{a^2H}{t\beta}\frac{\partial\phi_y}{\partial R}.\frac{du_0}{\partial Q} + \frac{a^2}{t^2\beta^2}.\left(\frac{du_0}{\partial Q}\right)^2 + \frac{a^2}{t^2\beta}\frac{dv_0}{dR}.\frac{du_0}{\partial Q}\right) +$$

$$B_{33}\left(-\frac{2aS}{t\beta}\frac{\partial^2 w}{\partial R\,\partial Q}\cdot\frac{dv_0}{dR}+\frac{a^2H}{t\beta}\cdot\frac{\partial\phi_x}{\partial Q}\cdot\frac{dv_0}{dR}+\frac{a^2H}{t}\frac{\partial\phi_y}{\partial R}\cdot\frac{dv_0}{dR}+\frac{a^2}{t^2\beta}\frac{du_0}{\partial Q}\cdot\frac{dv_0}{dR}+\frac{a^2}{t^2}\left(\frac{dv_0}{dR}\right)^2\right) \quad (3.201)$$

where as

$$\tau_{RS}\gamma_{RS}=B_{44}\frac{a^4}{t^2}\left(\left(\emptyset_x.\frac{\partial H}{\partial S}\right)^2+\emptyset_x.\frac{du_0}{dS}\cdot\frac{\partial H}{\partial S}+\emptyset_x.\frac{\partial H}{\partial S}\cdot\frac{du_0}{dS}+\left(\frac{du_0}{dS}\right)^2\right) \quad (3.202)$$

and

$$\tau_{QS}\gamma_{QS}=B_{55}\frac{a^4}{t^2}\left(\left(\emptyset_y.\frac{\partial H}{\partial S}\right)^2+.\,\emptyset_y.\frac{dv_0}{dS}\cdot\frac{\partial H}{\partial S}+\emptyset_y.\frac{\partial H}{\partial S}\cdot\frac{dv_0}{dS}+\left(\frac{dv_0}{dS}\right)^2\right) \quad (3.\,203)$$

The Strain energy functional is gotten by by bring together Equations (3.199) ,(3.200), (3.201), (3.202) and (3.203) and that gave Equation (3.204)

$$\int_{-0.5}^{0.5}\sigma.\,\varepsilon dS=\frac{E_0t^3}{12[1-\mu_{XY}\mu_{YX}]a^4}B_{11}\left(S^2\left(\frac{\partial^2 w}{\partial R^2}\right)^2-aHS\frac{\partial^2 w}{\partial R^2}\cdot\frac{\partial\phi_x}{\partial R}-\frac{a}{t}S.\frac{\partial^2 w}{\partial R^2}\cdot\frac{du_0}{dR}\right.$$
$$-aHS.\frac{\partial^2 w}{\partial R^2}\cdot\frac{\partial\phi_x}{\partial R}+a^2H^2.\left(\frac{\partial\phi_x}{\partial R}\right)^2+\frac{a^2}{t}H.\frac{du_0}{dR}\cdot\frac{\partial\phi_x}{\partial R}-\frac{aS}{t}\cdot\frac{\partial^2 w}{\partial R^2}\cdot\frac{du_0}{\partial R}$$
$$\left.+\frac{a^2}{t}H.\frac{\partial\phi_x}{\partial R}\cdot\frac{du_0}{dR}+\frac{a^2}{t^2}.\left(\frac{du_0}{dR}\right)^2\right)+$$

$$2B_{12}\left(\frac{S^2}{\beta^2}\cdot\frac{\partial^2 w}{\partial R^2}\cdot\frac{\partial^2 w}{\partial Q^2}-\frac{aHS}{\beta}\cdot\frac{\partial^2 w}{\partial R^2}\cdot\frac{\partial\phi_y}{\partial Q}-\frac{a}{t\beta}S.\frac{\partial^2 w}{\partial R^2}\cdot\frac{dv_0}{\partial Q}-\frac{aHS}{\beta^2}\frac{\partial^2 w}{\partial Q^2}\cdot\frac{\partial\phi_x}{\partial R}+\frac{a^2H^2}{\beta}\cdot\frac{\partial\phi_y}{\partial Q}\cdot\frac{\partial\phi_x}{\partial R}\right.$$
$$\left.+\frac{a^2H}{t\beta}\frac{dv_0}{\partial Q}\cdot\frac{\partial\phi_x}{\partial R}-\frac{aS}{t\beta^2}\frac{\partial^2 w}{\partial Q^2}\cdot\frac{du_0}{dR}+\frac{a^2H}{t\beta}\cdot\frac{\partial\phi_y}{\partial Q}\cdot\frac{du_0}{dR}+\frac{a^2}{t^2\beta}\frac{du_0}{dR}\cdot\frac{dv_0}{\partial Q}\right)$$

$$+2B_{13}\Bigg(\bigg(\frac{2S^2}{\beta}\frac{\partial^2 w}{\partial R\,\partial Q}.\frac{\partial^2 w}{\partial R^2}-\frac{aH}{\beta}S\frac{\partial^2 w}{\partial R^2}.\frac{\partial\phi_x}{\partial Q}-aHS\frac{\partial^2 w}{\partial R^2}.\frac{\partial\phi_y}{\partial R}-\frac{a}{t\beta}S\frac{\partial^2 w}{\partial R^2}.\frac{du_0}{\partial Q}$$

$$-\frac{a}{t}S\frac{\partial^2 w}{\partial R^2}.\frac{dv_0}{dR}\bigg)$$

$$+\bigg(-\frac{2aHS}{\beta}\frac{\partial^2 w}{\partial R\,\partial Q}.\frac{\partial\phi_x}{\partial R}+\frac{a^2H^2}{\beta}.\frac{\partial\phi_x}{\partial R}.\frac{\partial\phi_x}{\partial Q}+a^2H^2.\frac{\partial\phi_y}{\partial R}.\frac{\partial\phi_x}{\partial R}+\frac{a^2H}{t}\frac{du_0}{\beta\,\partial Q}.\frac{\partial\phi_x}{\partial R}$$

$$+\frac{a^2H}{t}\frac{dv_0}{dR}.\frac{\partial\phi_x}{\partial R}\bigg)$$

$$+\bigg(-\frac{2aS}{t\beta}\frac{\partial^2 w}{\partial R\,\partial Q}.\frac{du_0}{dR}+\frac{a^2H}{t\beta}\frac{\partial\phi_x}{\partial Q}.\frac{du_0}{dR}+\frac{a^2H}{t\beta}.\frac{\partial\phi_y}{\partial R}.\frac{du_0}{dR}+\frac{a^2}{t^2}\frac{du_0}{\beta\,\partial Q}.\frac{du_0}{dR}$$

$$+\frac{a^2}{t^2}.\frac{dv_0}{dR}.\frac{du_0}{dR}\bigg)\Bigg)$$

$$+B_{22}\Bigg(\frac{S^2}{\beta^4}\left(\frac{\partial^2 w}{\partial Q^2}\right)^2-\frac{aHS}{\beta^3}.\frac{\partial^2 w}{\partial Q^2}.\frac{\partial\phi_y}{\partial Q}-\frac{aS}{t\beta^3}.\frac{\partial^2 w}{\partial Q^2}.\frac{dv_0}{\partial Q}-\frac{aHS}{\beta^3}\frac{\partial^2 w}{\partial Q^2}.\frac{\partial\phi_y}{\partial Q}+\frac{a^2H^2}{\beta^2}.\left(\frac{\partial\phi_y}{\partial Q}\right)^2$$

$$+\frac{a^2H}{t\beta^2}\frac{dv_0}{\partial Q}.\frac{\partial\phi_y}{\partial Q}-\frac{aS}{t\beta^3}\frac{\partial^2 w}{\partial Q^2}.\frac{dv_0}{\partial Q}+\frac{a^2H}{t\beta^2}.\frac{\partial\phi_y}{\partial Q}.\frac{dv_0}{\partial Q}+\frac{a^2}{t^2\beta^2}.\left(\frac{dv_0}{\partial Q}\right)^2\Bigg)$$

$$+2B_{23}\Bigg(\frac{2S^2}{\beta^3}\frac{\partial^2 w}{\partial R\,\partial Q}.\frac{\partial^2 w}{\partial Q^2}-\frac{aSH}{\beta^3}.\frac{\partial\phi_x}{\partial Q}.\frac{\partial^2 w}{\partial Q^2}-\frac{aSH}{\beta^2}.\frac{\partial\phi_y}{\partial R}.\frac{\partial^2 w}{\partial Q^2}-\frac{aS}{t\beta^3}\frac{du_0}{\partial Q}.\frac{\partial^2 w}{\partial Q^2}-\frac{aS}{t\beta^2}\frac{dv_0}{dR}.\frac{\partial^2 w}{\partial Q^2}$$

$$-\frac{2aHS}{\beta^2}\frac{\partial^2 w}{\partial R\,\partial Q}.\frac{\partial\phi_y}{\partial Q}+\frac{a^2H^2}{\beta^2}.\frac{\partial\phi_x}{\partial Q}.\frac{\partial\phi_y}{\partial Q}+\frac{a^2H^2}{\beta}.\frac{\partial\phi_y}{\partial R}.\frac{\partial\phi_y}{\partial Q}+\frac{a^2H}{t\beta^2}\frac{du_0}{\partial Q}.\frac{\partial\phi_y}{\partial Q}$$

$$+\frac{a^2H}{t\beta}\frac{dv_0}{dR}.\frac{\partial\phi_y}{\partial Q}-\frac{2aS}{t\beta^2}\frac{\partial^2 w}{\partial R\,\partial Q}.\frac{dv_0}{\partial Q}+\frac{a^2H}{t\beta^2}.\frac{\partial\phi_x}{\partial Q}.\frac{dv_0}{\partial Q}+\frac{a^2H}{t\beta}.\frac{\partial\phi_y}{\partial R}.\frac{dv_0}{\partial Q}$$

$$+\frac{a^2}{t^2\beta^2}\frac{du_0}{\partial Q}\frac{dv_0}{\partial Q}+\frac{a^2}{t^2\beta}\frac{dv_0}{dR}.\frac{dv_0}{\partial Q}\Bigg)$$

$$+B_{33}\Bigg(\frac{4S^2}{\beta^2}\left(\frac{\partial^2 w}{\partial R\,\partial Q}\right)^2-\frac{2aHS}{\beta^2}.\frac{\partial^2 w}{\partial R\,\partial Q}.\frac{\partial\phi_x}{\partial Q}-\frac{2aHS}{\beta}.\frac{\partial^2 w}{\partial R\,\partial Q}.\frac{\partial\phi_y}{\partial R}-\frac{2aS}{t\beta^2}.\frac{\partial^2 w}{\partial R\,\partial Q}.\frac{du_0}{\partial Q}$$

$$-\frac{2aS}{t\beta}.\frac{\partial^2 w}{\partial R\,\partial Q}.\frac{dv_0}{dR}\Bigg)+$$

$$B_{33}\left(-\frac{2aHS}{\beta^2}\frac{\partial^2 w}{\partial R\,\partial Q}.\frac{\partial\phi_x}{\partial Q}+\frac{a^2H^2}{\beta^2}.\left(\frac{\partial\phi_x}{\partial Q}\right)^2+\frac{a^2H^2}{\beta}\frac{\partial\phi_y}{\partial R}.\frac{\partial\phi_x}{\partial Q}+\frac{a^2H}{t\beta^2}\frac{du_0}{\partial Q}.\frac{\partial\phi_x}{\partial Q}+\frac{a^2H}{t\beta}\frac{dv_0}{dR}.\frac{\partial\phi_x}{\partial Q}\right)+$$

$$B_{33}\left(-\frac{2aHS}{\beta}\frac{\partial^2 w}{\partial R\,\partial Q}.\frac{\partial\phi_y}{\partial R}+\frac{a^2H^2}{\beta}.\frac{\partial\phi_x}{\partial Q}.\frac{\partial\phi_y}{\partial R}+a^2H^2\left(\frac{\partial\phi_y}{\partial R}\right)^2+\frac{a^2H}{t\beta}\frac{du_0}{\partial Q}.\frac{\partial\phi_y}{\partial R}+\frac{a^2H}{t}\frac{dv_0}{dR}.\frac{\partial\phi_y}{\partial R}\right)+$$

$$B_{33}\left(-\frac{2aS}{t\beta^2}\frac{\partial^2 w}{\partial R\,\partial Q}.\frac{du_0}{\partial Q}+\frac{a^2H}{t\beta^2}.\frac{\partial\phi_x}{\partial Q}.\frac{du_0}{\partial Q}+\frac{a^2H}{t\beta}\frac{\partial\phi_y}{\partial R}.\frac{du_0}{\partial Q}+\frac{a^2}{t^2\beta^2}.\left(\frac{du_0}{\partial Q}\right)^2+\frac{a^2}{t^2\beta}\frac{dv_0}{dR}.\frac{du_0}{\partial Q}\right)+$$

$$B_{33}\left(-\frac{2aS}{t\beta}\frac{\partial^2 w}{\partial R\,\partial Q}.\frac{dv_0}{dR}+\frac{a^2H}{t\beta}.\frac{\partial\phi_x}{\partial Q}.\frac{dv_0}{dR}+\frac{a^2H}{t}\frac{\partial\phi_y}{\partial R}.\frac{dv_0}{dR}+\frac{a^2}{t^2\beta}\frac{du_0}{\partial Q}.\frac{dv_0}{dR}+\frac{a^2}{t^2}\left(\frac{dv_0}{dR}\right)^2\right)$$

$$+B_{44}\frac{a^4}{t^2}\left(\left(\emptyset_x.\frac{\partial H}{\partial S}\right)^2+\emptyset_x.\frac{du_0}{dS}.\frac{\partial H}{\partial S}+\emptyset_x.\frac{\partial H}{\partial S}.\frac{du_0}{dS}+\left(\frac{du_0}{dS}\right)^2\right)$$

$$+B_{55}\frac{a^4}{t^2}\left(\left(\emptyset_y.\frac{\partial H}{\partial S}\right)^2+.\emptyset_y.\frac{dv_0}{dS}.\frac{\partial H}{\partial S}+\emptyset_y.\frac{\partial H}{\partial S}.\frac{dv_0}{dS}\left(\frac{dv_0}{dS}\right)^2\right) \qquad (3.204)$$

From Equation (3.204), extractions were made, considered as first strain coefficients which are represented in Equations (3.205) to Equation (3.208) and also second strain coefficients which are expressed Equations (3.209) to (3.211) as

$$J_1=\int_{-0.5}^{0.5}S^2 dS=\frac{1}{12} \qquad (3.205)$$

$$J_2=\int_{-0.5}^{0.5}SH\,dS=\int_{-0.5}^{0.5}\left[S^2-\frac{4}{3}S^4\right]dS$$

$$=\left[\frac{S^3}{3}-\frac{4}{15}S^5\right]_{-05}^{0.5}$$

And finally

$$J_2=\frac{1}{15} \qquad (3.206)$$

Also

$$J_3=\int_{-0.5}^{0.5}H^2\,dS=\int_{-0.5}^{0.5}\left[S^2-\frac{8}{3}S^4+\frac{16}{9}S^6\right]dS$$

$$=\left[\frac{S^2}{3}-\frac{8}{15}S^5+\frac{16}{63}S^7\right]_{-0.5}^{0.5}$$

and that gives

$$J_3=\frac{17}{315} \qquad (3.207)$$

Furthermore

$$J_4=\int_{-0.5}^{0.5}\left[\frac{\partial H}{\partial S}\right]^2 dS=\int_{-0.5}^{0.5}[1-8S^2+16S^4]dS=\left[S-8\frac{S^3}{3}+16\frac{S^5}{5}\right]_{-0.5}^{0.5}$$

That is

$$J_4 = \frac{8}{15} \tag{3.208}$$

also

$$g_2 = \frac{J_2}{J_1} = \frac{1}{15}\Big/\frac{1}{12} = \frac{4}{5} \tag{3.209}$$

$$g_3 = \frac{J_3}{J_1} = \frac{17}{315}\Big/\frac{1}{12} = \frac{68}{105} \tag{3.210}$$

$$g_4 = \frac{J_4}{J_1} = \frac{8}{15}\Big/\frac{1}{12} = \frac{32}{5} \tag{3.211}$$

Further introduction of this J_i and g_i into the Total Potential Energy will reduce the functional into lowest possible expression as shall be discissed later part fo the work. Reduciung Equation (3.204) gives

$$\int_{-0.5}^{0.5} \sigma.\varepsilon dS = \frac{E_0 t^3}{12[1-\mu_{XY}\mu_{YX}]a^4} B_{11}\left(S^2\left(\frac{\partial^2 w}{\partial R^2}\right)^2 - aHS.\frac{\partial^2 w}{\partial R^2}.\frac{\partial \phi_x}{\partial R} - \frac{a}{t}S.\frac{\partial^2 w}{\partial R^2}.\frac{du_0}{dR} - aHS.\frac{\partial^2 w}{\partial R^2}.\frac{\partial \phi_x}{\partial R} + a^2S^2.\left(\frac{\partial \phi_x}{\partial R}\right)^2 + \frac{a^2}{t}H.\frac{du_0}{dR}.\frac{\partial \phi_x}{\partial R} - \frac{aS}{t}.\frac{\partial^2 w}{\partial R^2}.\frac{du_0}{\partial R} + \frac{a^2}{t}H\frac{\partial \phi_x}{\partial R}.\frac{du_0}{dR} + \frac{a^2}{t^2}.\left(\frac{du_0}{dR}\right)^2\right) +$$

$$2B_{12}\left(\frac{S^2}{\beta^2}.\frac{\partial^2 w}{\partial R^2}.\frac{\partial^2 w}{\partial Q^2} - \frac{aHS}{\beta}.\frac{\partial^2 w}{\partial R^2}.\frac{\partial \phi_y}{\partial Q} - \frac{a}{t\beta}S.\frac{\partial^2 w}{\partial R^2}.\frac{dv_0}{\partial Q} - \frac{aHS}{\beta^2}\frac{\partial^2 w}{\partial Q^2}.\frac{\partial \phi_x}{\partial R} + \frac{a^2H^2}{\beta}.\frac{\partial \phi_y}{\partial Q}.\frac{\partial \phi_x}{\partial R} + \frac{a^2H}{t\beta}\frac{dv_0}{\partial Q}.\frac{\partial \phi_x}{\partial R} - \frac{aS}{t\beta^2}\frac{\partial^2 w}{\partial Q^2}.\frac{du_0}{dR} + \frac{a^2H}{t\beta}.\frac{\partial \phi_y}{\partial Q}.\frac{du_0}{dR} + \frac{a^2}{t^2\beta}\frac{du_0}{dR}.\frac{dv_0}{\partial Q}\right)$$

$$+2B_{13}\left(\frac{2S^2}{\beta}\frac{\partial^2 w}{\partial R\,\partial Q}.\frac{\partial^2 w}{\partial R^2} - \frac{aHS}{\beta}\frac{\partial^2 w}{\partial R^2}.\frac{\partial \phi_x}{\partial Q} - aHS\frac{\partial^2 w}{\partial R^2}.\frac{\partial \phi_y}{\partial R} - \frac{a}{t\beta}S\frac{\partial^2 w}{\partial R^2}.\frac{du_0}{\partial Q} - \frac{a}{t}S\frac{\partial^2 w}{\partial R^2}.\frac{dv_0}{dR}\right)$$

$$+\left(-\frac{2aHS}{\beta}\frac{\partial^2 w}{\partial R\,\partial Q}.\frac{\partial \phi_x}{\partial R} + \frac{a^2H^2}{\beta}.\frac{\partial \phi_x}{\partial R}.\frac{\partial \phi_x}{\partial Q} + a^2H^2.\frac{\partial \phi_y}{\partial R}.\frac{\partial \phi_x}{\partial R} + \frac{a^2H}{t}\frac{du_0}{\beta\,\partial Q}.\frac{\partial \phi_x}{\partial R} + \frac{a^2H}{t}\frac{dv_0}{dR}.\frac{\partial \phi_x}{\partial R}\right)$$

$$+\left(-\frac{2aS}{t\beta}\frac{\partial^2 w}{\partial R\,\partial Q}.\frac{du_0}{dR} + \frac{a^2H}{t\beta}\frac{\partial \phi_x}{\partial Q}.\frac{du_0}{dR} + \frac{a^2H}{t\beta}.\frac{\partial \phi_y}{\partial R}.\frac{du_0}{dR} + \frac{a^2}{t^2}\frac{du_0}{\beta\,\partial Q}.\frac{du_0}{dR} + \frac{a^2}{t^2}.\frac{dv_0}{dR}.\frac{du_0}{dR}\right)$$

$$+B_{22}\left(\frac{S^2}{\beta^4}\left(\frac{\partial^2 w}{\partial Q^2}\right)^2-\frac{aHS}{\beta^3}.\frac{\partial^2 w}{\partial Q^2}.\frac{\partial\phi_y}{\partial Q}-\frac{aS}{t\beta^3}.\frac{\partial^2 w}{\partial Q^2}.\frac{dv_0}{\partial Q}-\frac{aHS}{\beta^3}\frac{\partial^2 w}{\partial Q^2}.\frac{\partial\phi_y}{\partial Q}+\frac{a^2H^2}{\beta^2}.\left(\frac{\partial\phi_y}{\partial Q}\right)^2\right.$$
$$\left.+\ \frac{a^2H}{t\beta^2}\frac{dv_0}{\partial Q}.\frac{\partial\phi_y}{\partial Q}-\frac{aS}{t\beta^3}\frac{\partial^2 w}{\partial Q^2}.\frac{dv_0}{\partial Q}+\frac{a^2H}{t\beta^2}.\frac{\partial\phi_y}{\partial Q}.\frac{dv_0}{\partial Q}+\frac{a^2}{t^2\beta^2}.\left(\frac{dv_0}{\partial Q}\right)^2\right)$$

$$+2B_{23}\left(\frac{2S^2}{\beta^3}\frac{\partial^2 w}{\partial R\,\partial Q}.\frac{\partial^2 w}{\partial Q^2}-\frac{aSH}{\beta^3}.\frac{\partial\phi_x}{\partial Q}.\frac{\partial^2 w}{\partial Q^2}-\frac{aSH}{\beta^2}.\frac{\partial\phi_y}{\partial R}.\frac{\partial^2 w}{\partial Q^2}-\frac{aS}{t\beta^3}\frac{du_0}{\partial Q}.\frac{\partial^2 w}{\partial Q^2}-\frac{aS}{t\beta^2}\frac{dv_0}{dR}.\frac{\partial^2 w}{\partial Q^2}\right.$$
$$-\ \frac{2aHS}{\beta^2}\frac{\partial^2 w}{\partial R\,\partial Q}.\frac{\partial\phi_y}{\partial Q}+\frac{a^2H^2}{\beta^2}.\frac{\partial\phi_x}{\partial Q}.\frac{\partial\phi_y}{\partial Q}+\frac{a^2H^2}{\beta}.\frac{\partial\phi_y}{\partial R}.\frac{\partial\phi_y}{\partial Q}+\frac{a^2H}{t\beta^2}\frac{du_0}{\partial Q}.\frac{\partial\phi_y}{\partial Q}$$
$$+\frac{a^2H}{t\beta}\frac{dv_0}{dR}.\frac{\partial\phi_y}{\partial Q}-\ \frac{2aS}{t\beta^2}\frac{\partial^2 w}{\partial R\,\partial Q}.\frac{dv_0}{\partial Q}+\frac{a^2H}{t\beta^2}.\frac{\partial\phi_x}{\partial Q}.\frac{dv_0}{\partial Q}+\frac{a^2H}{t\beta}.\frac{\partial\phi_y}{\partial R}.\frac{dv_0}{\partial Q}$$
$$\left.+\frac{a^2}{t^2\beta^2}\frac{du_0}{\partial Q}\frac{dv_0}{\partial Q}+\frac{a^2}{t^2\beta}\frac{dv_0}{dR}.\frac{dv_0}{\partial Q}\right)$$

$$+B_{33}\left(\frac{4S^2}{\beta^2}\left(\frac{\partial^2 w}{\partial R\,\partial Q}\right)^2-\frac{2aHS}{\beta^2}.\frac{\partial^2 w}{\partial R\,\partial Q}.\frac{\partial\phi_x}{\partial Q}-\frac{2aHS}{\beta}.\frac{\partial^2 w}{\partial R\,\partial Q}.\frac{\partial\phi_y}{\partial R}-\frac{2aS}{t\beta^2}.\frac{\partial^2 w}{\partial R\,\partial Q}.\frac{du_0}{\partial Q}\right.$$
$$\left.-\frac{2aS}{t\beta}.\frac{\partial^2 w}{\partial R\,\partial Q}.\frac{dv_0}{dR}\right)+$$

$$B_{33}\left(-\frac{2aHS}{\beta^2}\frac{\partial^2 w}{\partial R\,\partial Q}.\frac{\partial\phi_x}{\partial Q}+\frac{a^2H^2}{\beta^2}.\left(\frac{\partial\phi_x}{\partial Q}\right)^2+\frac{a^2H^2}{\beta}\frac{\partial\phi_y}{\partial R}.\frac{\partial\phi_x}{\partial Q}+\frac{a^2H}{t\beta^2}\frac{du_0}{\partial Q}.\frac{\partial\phi_x}{\partial Q}+\frac{a^2H}{t\beta}\frac{dv_0}{dR}.\frac{\partial\phi_x}{\partial Q}\right)$$

$$B_{33}\left(-\frac{2aHS}{\beta}\frac{\partial^2 w}{\partial R\,\partial Q}.\frac{\partial\phi_y}{\partial R}+\frac{a^2H^2}{\beta}.\frac{\partial\phi_x}{\partial Q}.\frac{\partial\phi_y}{\partial R}+a^2H^2\left(\frac{\partial\phi_y}{\partial R}\right)^2+\frac{a^2H}{t\beta}\frac{du_0}{\partial Q}.\frac{\partial\phi_y}{\partial R}+\frac{a^2H}{t}\frac{dv_0}{dR}.\frac{\partial\phi_y}{\partial R}\right)+$$

$$B_{33}\left(-\frac{2aS}{t\beta^2}\frac{\partial^2 w}{\partial R\,\partial Q}.\frac{du_0}{\partial Q}+\frac{a^2H}{t\beta^2}.\frac{\partial\phi_x}{\partial Q}.\frac{du_0}{\partial Q}+\frac{a^2H}{t\beta}\frac{\partial\phi_y}{\partial R}.\frac{du_0}{\partial Q}+\frac{a^2}{t^2\beta^2}.\left(\frac{du_0}{\partial Q}\right)^2+\frac{a^2}{t^2\beta}\frac{dv_0}{dR}.\frac{du_0}{\partial Q}\right)+$$

$$B_{33}\left(-\frac{2aS}{t\beta}\frac{\partial^2 w}{\partial R\,\partial Q}.\frac{dv_0}{dR}+\frac{a^2H}{t\beta}.\frac{\partial\phi_x}{\partial Q}.\frac{dv_0}{dR}+\frac{a^2H}{t}\frac{\partial\phi_y}{\partial R}.\frac{dv_0}{dR}+\frac{a^2}{t^2\beta}\frac{du_0}{\partial Q}.\frac{dv_0}{dR}+\frac{a^2}{t^2}\left(\frac{dv_0}{dR}\right)^2\right)$$

$$+B_{44}\frac{a^4}{t^2}\left(\left(\emptyset_x.\frac{\partial H}{\partial S}\right)^2+\emptyset_x.\frac{du_0}{dS}.\frac{\partial H}{\partial S}+\emptyset_x.\frac{\partial H}{\partial S}.\frac{du_0}{dS}+\left(\frac{du_0}{dS}\right)^2\right))$$

$$+B_{55}\frac{a^4}{t^2}\left(\left(\emptyset_y.\frac{\partial H}{\partial S}\right)^2+.\,\emptyset_y.\frac{dv_0}{dS}.\frac{\partial H}{\partial S}+\emptyset_y.\frac{\partial H}{\partial S}.\frac{dv_0}{dS}+\left(\frac{dv_0}{dS}\right)^2\right) \tag{3.212}$$

The Strain Energy was further collapsed into Bending Stiffness U_b, Coupling Stiffness U_c and Axial/Membrane Stiffness U_m as shown in Equation (3.213) , (3.214) and (3.215) respectively. For the case of Bending Stiffness gives

$$U_B = \frac{E_0 t^3}{[1-\mu_{XY}\mu_{YX}]} B_{11}\left(S^2\left(\frac{\partial^2 w}{\partial R^2}\right)^2 - aHS.\frac{\partial^2 w}{\partial R^2}.\frac{\partial \phi_x}{\partial R} - aHS.\frac{\partial^2 w}{\partial R^2}.\frac{\partial \phi_x}{\partial R} + a^2S^2.\left(\frac{\partial \phi_x}{\partial R}\right)^2\right)$$

$$+2B_{12}\left(\frac{S^2}{\beta^2}.\frac{\partial^2 w}{\partial R^2}.\frac{\partial^2 w}{\partial Q^2} - \frac{aHS}{\beta}.\frac{\partial^2 w}{\partial R^2}.\frac{\partial \phi_y}{\partial Q} - \frac{aHS}{\beta^2}\frac{\partial^2 w}{\partial Q^2}.\frac{\partial \phi_x}{\partial R} + \frac{a^2H^2}{\beta}.\frac{\partial \phi_y}{\partial Q}.\frac{\partial \phi_x}{\partial R}\right)$$

$$+2B_{13}\left(-\frac{2S^2}{\beta}\frac{\partial^2 w}{\partial R\,\partial Q}.\frac{\partial^2 w}{\partial R^2} - \frac{aHS}{\beta}\frac{\partial^2 w}{\partial R^2}.\frac{\partial \phi_x}{\partial Q} - aHS\frac{\partial^2 w}{\partial R^2}.\frac{\partial \phi_y}{\partial R} - \frac{2aHS}{\beta}\frac{\partial^2 w}{\partial R\,\partial Q}.\frac{\partial \phi_x}{\partial R} + \frac{a^2H^2}{\beta}.\frac{\partial \phi_x}{\partial R}.\frac{\partial \phi_x}{\partial Q} + a^2H^2.\frac{\partial \phi_y}{\partial R}.\frac{\partial \phi_x}{\partial R}\right)$$

$$+B_{22}\left(\frac{S^2}{\beta^4}\left(\frac{\partial^2 w}{\partial Q^2}\right)^2 - \frac{aHS}{\beta^3}.\frac{\partial^2 w}{\partial Q^2}.\frac{\partial \phi_y}{\partial Q} - \frac{aHS}{\beta^3}\frac{\partial^2 w}{\partial Q^2}.\frac{\partial \phi_y}{\partial Q} + \frac{a^2H^2}{\beta^2}.\left(\frac{\partial \phi_y}{\partial Q}\right)^2\right)$$

$$+2B_{23}\left(\frac{2S^2}{\beta^3}\frac{\partial^2 w}{\partial R\,\partial Q}.\frac{\partial^2 w}{\partial Q^2} - \frac{aSH}{\beta^3}.\frac{\partial \phi_x}{\partial Q}.\frac{\partial^2 w}{\partial Q^2} - \frac{aSH}{\beta^2}.\frac{\partial \phi_y}{\partial R}.\frac{\partial^2 w}{\partial Q^2} - \frac{2aHS}{\beta^2}\frac{\partial^2 w}{\partial R\,\partial Q}.\frac{\partial \phi_y}{\partial Q} + \frac{a^2H^2}{\beta^2}.\frac{\partial \phi_x}{\partial Q}.\frac{\partial \phi_y}{\partial Q} + \frac{a^2H^2}{\beta}.\frac{\partial \phi_y}{\partial R}.\frac{\partial \phi_y}{\partial Q}\right)$$

$$+B_{33}\left(\frac{4S^2}{\beta^2}\left(\frac{\partial^2 w}{\partial R\,\partial Q}\right)^2 - \frac{2aHS}{\beta^2}.\frac{\partial^2 w}{\partial R\,\partial Q}.\frac{\partial \phi_x}{\partial Q} - \frac{2aHS}{\beta}.\frac{\partial^2 w}{\partial R\,\partial Q}.\frac{\partial \phi_y}{\partial R}\right)$$

$$+B_{33}\left(-\frac{2aHS}{\beta^2}\frac{\partial^2 w}{\partial R\,\partial Q}.\frac{\partial \phi_x}{\partial Q} + \frac{a^2H^2}{\beta^2}.\left(\frac{\partial \phi_x}{\partial Q}\right)^2 + \frac{a^2H^2}{\beta}\frac{\partial \phi_y}{\partial R}.\frac{\partial \phi_x}{\partial Q}\right)$$

$$+B_{33}\left(-\frac{2aHS}{\beta}\frac{\partial^2 w}{\partial R\,\partial Q}.\frac{\partial \phi_y}{\partial R} + \frac{a^2H^2}{\beta}.\frac{\partial \phi_x}{\partial Q}.\frac{\partial \phi_y}{\partial R} + a^2H^2\left(\frac{\partial \phi_y}{\partial R}\right)^2\right) \tag{3.213}$$

For the case of Coupling Stiffness gives

$$U_C = \frac{E_0 t^3}{[1-\mu_{XY}\mu_{YX}]a^4} B_{11}\left(-\frac{a}{t}S.\frac{\partial^2 w}{\partial R^2}.\frac{du_0}{dR} + \frac{a^2}{t}H.\frac{du_0}{dR}.\frac{\partial \phi_x}{\partial R} - \frac{aS}{t}.\frac{\partial^2 w}{\partial R^2}.\frac{du_0}{\partial R} + \frac{a^2}{t}H\frac{\partial \phi_x}{\partial R}.\frac{du_0}{dR}\right)$$

$$+2B_{12}\left(-\frac{a}{t\beta}S.\frac{\partial^2 w}{\partial R^2}.\frac{dv_0}{\partial Q} + \frac{a^2H}{t\beta}\frac{dv_0}{\partial Q}.\frac{\partial \phi_x}{\partial R} - \frac{aS}{t\beta^2}\frac{\partial^2 w}{\partial Q^2}.\frac{du_0}{dR} + \frac{a^2H}{t\beta}.\frac{\partial \phi_y}{\partial Q}.\frac{du_0}{dR}\right)$$

$$+2B_{13}\left(-\frac{a}{t\beta}S\frac{\partial^2 w}{\partial R^2}.\frac{du_0}{\partial Q} - \frac{a}{t}S\frac{\partial^2 w}{\partial R^2}.\frac{dv_0}{dR} + \frac{a^2H}{t}\frac{du_0}{\beta\,\partial Q}.\frac{\partial \phi_x}{\partial R} + \frac{a^2H}{t}\frac{dv_0}{dR}.\frac{\partial \phi_x}{\partial R} - \frac{2aS}{t\beta}\frac{\partial^2 w}{\partial R\,\partial Q}.\frac{du_0}{dR} + \frac{a^2H}{t\beta}\frac{\partial \phi_x}{\partial Q}.\frac{du_0}{dR} + \frac{a^2H}{t\beta}.\frac{\partial \phi_y}{\partial R}.\frac{du_0}{dR}\right)$$

$$+B_{22}\left(-\frac{aS}{t\beta^3}.\frac{\partial^2 w}{\partial Q^2}.\frac{dv_0}{\partial Q}+\frac{a^2H}{t\beta^2}\frac{dv_0}{\partial Q}.\frac{\partial\phi_y}{\partial Q}-\frac{aS}{t\beta^3}\frac{\partial^2 w}{\partial Q^2}.\frac{dv_0}{\partial Q}+\frac{a^2H}{t\beta^2}.\frac{\partial\phi_y}{\partial Q}.\frac{dv_0}{\partial Q}\right)$$

$$+2B_{23}\left(-\frac{aS}{t\beta^3}\frac{du_0}{\partial Q}.\frac{\partial^2 w}{\partial Q^2}-\frac{aS}{t\beta^2}\frac{dv_0}{dR}.\frac{\partial^2 w}{\partial Q^2}+\frac{a^2H}{t\beta^2}\frac{du_0}{\partial Q}.\frac{\partial\phi_y}{\partial Q}+\frac{a^2H}{t\beta}\frac{dv_0}{dR}.\frac{\partial\phi_y}{\partial Q}\right.$$
$$\left.-\frac{2aS}{t\beta^2}\frac{\partial^2 w}{\partial R\,\partial Q}.\frac{dv_0}{\partial Q}+\frac{a^2H}{t\beta^2}.\frac{\partial\phi_x}{\partial Q}.\frac{dv_0}{\partial Q}+\frac{a^2H}{t\beta}.\frac{\partial\phi_y}{\partial R}.\frac{dv_0}{\partial Q}\right)$$

$$+B_{33}\left(-\frac{2aS}{t\beta^2}.\frac{\partial^2 w}{\partial R\,\partial Q}.\frac{du_0}{\partial Q}-\frac{2aS}{t\beta}.\frac{\partial^2 w}{\partial R\,\partial Q}.\frac{dv_0}{dR}\right)$$
$$+B_{33}\left(\frac{a^2H}{t\beta^2}\frac{du_0}{\partial Q}.\frac{\partial\phi_x}{\partial Q}+\frac{a^2H}{t\beta}\frac{dv_0}{dR}.\frac{\partial\phi_x}{\partial Q}\right)$$
$$+B_{33}\left(+\frac{a^2H}{t\beta}\frac{du_0}{\partial Q}.\frac{\partial\phi_y}{\partial R}+\frac{a^2H}{t}\frac{dv_0}{dR}.\frac{\partial\phi_y}{\partial R}\right)$$
$$+B_{33}\left(-\frac{2aS}{t\beta^2}\frac{\partial^2 w}{\partial R\,\partial Q}.\frac{du_0}{\partial Q}+\frac{a^2H}{t\beta^2}.\frac{\partial\phi_x}{\partial Q}.\frac{du_0}{\partial Q}+\frac{a^2H}{t\beta}\frac{\partial\phi_y}{\partial R}.\frac{du_0}{\partial Q}\right)$$
$$+B_{33}\left(-\frac{2aS}{t\beta}\frac{\partial^2 w}{\partial R\,\partial Q}.\frac{dv_0}{dR}+\frac{a^2H}{t\beta}.\frac{\partial\phi_x}{\partial Q}.\frac{dv_0}{dR}+\frac{a^2H}{t}\frac{\partial\phi_y}{\partial R}.\frac{dv_0}{dR}\right) \qquad (3.214)$$

Also for the case of the Axial/Membrane Stiffness, gives

$$U_M=\frac{E_0t^3}{[1-\mu_{XY}\mu_{YX}]a^4}B_{11}\left(+\frac{a^2}{t^2}.\left(\frac{du_0}{dR}\right)^2\right)+2B_{12}\left(+\frac{a^2}{t^2\beta}\frac{du_0}{dR}.\frac{dv_0}{\partial Q}\right)$$

$$+2B_{13}\left(\frac{a^2}{t^2}\frac{du_0}{\beta\,\partial Q}.\frac{du_0}{dR}+\frac{a^2}{t^2}.\frac{dv_0}{dR}.\frac{du_0}{dR}\right)+B_{22}\left(\frac{a^2}{t^2\beta^2}.\left(\frac{dv_0}{\partial Q}\right)^2\right)$$

$$+2B_{23}\left(\frac{a^2}{t^2\beta^2}\frac{du_0}{\partial Q}\frac{dv_0}{\partial Q}+\frac{a^2}{t^2\beta}\frac{dv_0}{dR}.\frac{dv_0}{\partial Q}\right)+B_{33}\left(\frac{a^2}{t^2\beta^2}\left(\frac{du_0}{\partial Q}\right)^2+\frac{a^2}{t^2\beta}\frac{dv_0}{dR}.\frac{du_0}{\partial Q}\right)$$

$$+B_{33}\left(\frac{a^2}{t^2\beta}\frac{du_0}{\partial Q}.\frac{dv_0}{dR}+\frac{a^2}{t^2}\left(\frac{dv_0}{dR}\right)^2\right)$$

$$+B_{44}\frac{a^4}{t^2}\left(\left(\emptyset_x.\frac{\partial H}{\partial S}\right)^2+\emptyset_x.\frac{du_0}{dS}.\frac{\partial H}{\partial S}+\emptyset_x.\frac{\partial H}{\partial S}.\frac{du_0}{dS}+\left(\frac{du_0}{dS}\right)^2\right)$$

$$+B_{55}\frac{a^4}{t^2}\left(\left(\emptyset_y.\frac{\partial H}{\partial S}\right)^2+.\,\emptyset_y.\frac{dv_0}{dS}.\frac{\partial H}{\partial S}+\emptyset_y.\frac{\partial H}{\partial S}.\frac{dv_0}{dS}+\left(\frac{dv_0}{dS}\right)^2\right) \qquad (3.215)$$

Adding Equation (3.213) (3.214)and (3.215) together gives

$$U = \frac{abt}{2}\int_0^1 \int_0^1 \int_{-0.5}^{0.5} (\sigma.\varepsilon)\, dR\, dQ\, dS$$

This can be written in terms of Bending, Coupling and Axial stiffness and

$$\sigma.\varepsilon = (\sigma.\varepsilon)_B + (\sigma.\varepsilon)_C + (\sigma.\varepsilon)_M \tag{3.216}$$

or generally as

$$(\sigma.\varepsilon) = \sigma_R\varepsilon_R + \sigma_Q\varepsilon_Q + \tau_{RQ}\gamma_{RQ} + \tau_{RS}\gamma_{RS} + \tau_{QS}\gamma_{QS} \tag{3.217}$$

and substituting the derived values of $(\sigma.\varepsilon)_B$, $(\sigma.\varepsilon)_C$ and $(\sigma.\varepsilon)_M$ as ealier established gives

$$\begin{aligned}
\int_{-0.5}^{0.5} \sigma.\varepsilon dS = \frac{E_0 t^3}{[1-\mu_{XY}\mu_{YX}]} B_{11}\Bigg(& S^2\left(\frac{\partial^2 w}{\partial R^2}\right)^2 - aHS.\frac{\partial^2 w}{\partial R^2}.\frac{\partial \phi_x}{\partial R} - aHS.\frac{\partial^2 w}{\partial R^2}.\frac{\partial \phi_x}{\partial R} \\
& + a^2S^2.\left(\frac{\partial \phi_x}{\partial R}\right)^2\Bigg) \\
& +2B_{12}\left(\frac{S^2}{\beta^2}.\frac{\partial^2 w}{\partial R^2}.\frac{\partial^2 w}{\partial Q^2} - \frac{aHS}{\beta}.\frac{\partial^2 w}{\partial R^2}.\frac{\partial \phi_y}{\partial Q} - \frac{aHS}{\beta^2}\frac{\partial^2 w}{\partial Q^2}.\frac{\partial \phi_x}{\partial R} + \frac{a^2H^2}{\beta}.\frac{\partial \phi_y}{\partial Q}.\frac{\partial \phi_x}{\partial R}\right) \\
& +2B_{13}\Bigg(-\frac{2S^2}{\beta}\frac{\partial^2 w}{\partial R\,\partial Q}.\frac{\partial^2 w}{\partial R^2} - \frac{aHS}{\beta}\frac{\partial^2 w}{\partial R^2}.\frac{\partial \phi_x}{\partial Q} - aHS\frac{\partial^2 w}{\partial R^2}.\frac{\partial \phi_y}{\partial R} - \frac{2aHS}{\beta}\frac{\partial^2 w}{\partial R\,\partial Q}.\frac{\partial \phi_x}{\partial R} \\
& + \frac{a^2H^2}{\beta}.\frac{\partial \phi_x}{\partial R}.\frac{\partial \phi_x}{\partial Q} + a^2H^2.\frac{\partial \phi_y}{\partial R}.\frac{\partial \phi_x}{\partial R}\Bigg) \\
& + B_{22}\left(\frac{S^2}{\beta^4}\left(\frac{\partial^2 w}{\partial Q^2}\right)^2 - \frac{aHS}{\beta^3}.\frac{\partial^2 w}{\partial Q^2}.\frac{\partial \phi_y}{\partial Q} - \frac{aHS}{\beta^3}\frac{\partial^2 w}{\partial Q^2}.\frac{\partial \phi_y}{\partial Q} + \frac{a^2H^2}{\beta^2}.\left(\frac{\partial \phi_y}{\partial Q}\right)^2\right) \\
& +2B_{23}\Bigg(\frac{2S^2}{\beta^3}\frac{\partial^2 w}{\partial R\,\partial Q}.\frac{\partial^2 w}{\partial Q^2} - \frac{aSH}{\beta^3}.\frac{\partial \phi_x}{\partial Q}.\frac{\partial^2 w}{\partial Q^2} - \frac{aSH}{\beta^2}.\frac{\partial \phi_y}{\partial R}.\frac{\partial^2 w}{\partial Q^2} - \frac{2aHS}{\beta^2}\frac{\partial^2 w}{\partial R\,\partial Q}.\frac{\partial \phi_y}{\partial Q} \\
& + \frac{a^2H^2}{\beta^2}.\frac{\partial \phi_x}{\partial Q}.\frac{\partial \phi_y}{\partial Q} + \frac{a^2H^2}{\beta}.\frac{\partial \phi_y}{\partial R}.\frac{\partial \phi_y}{\partial Q}\Bigg) \\
& +B_{33}\left(\frac{4S^2}{\beta^2}\left(\frac{\partial^2 w}{\partial R\,\partial Q}\right)^2 - \frac{2aHS}{\beta^2}.\frac{\partial^2 w}{\partial R\,\partial Q}.\frac{\partial \phi_x}{\partial Q} - \frac{2aHS}{\beta}.\frac{\partial^2 w}{\partial R\,\partial Q}.\frac{\partial \phi_y}{\partial R}\right) \\
& +B_{33}\left(-\frac{2aHS}{\beta^2}\frac{\partial^2 w}{\partial R\,\partial Q}.\frac{\partial \phi_x}{\partial Q} + \frac{a^2H^2}{\beta^2}.\left(\frac{\partial \phi_x}{\partial Q}\right)^2 + \frac{a^2H^2}{\beta}\frac{\partial \phi_y}{\partial R}.\frac{\partial \phi_x}{\partial Q}\right) \\
& +B_{33}\left(-\frac{2aHS}{\beta}\frac{\partial^2 w}{\partial R\,\partial Q}.\frac{\partial \phi_y}{\partial R} + \frac{a^2H^2}{\beta}.\frac{\partial \phi_x}{\partial Q}.\frac{\partial \phi_y}{\partial R} + a^2H^2\left(\frac{\partial \phi_y}{\partial R}\right)^2\right)
\end{aligned} \tag{3.218a}$$

Resolving Equation (3.218a) and gives

$$U = \frac{E_0 t^3}{[1-\mu_{XY}\mu_{YX}]a^4} B_{11}\left(-\frac{a}{t}S.\frac{\partial^2 w}{\partial R^2}.\frac{du_0}{dR} + \frac{a^2}{t}H.\frac{du_0}{dR}.\frac{\partial \phi_x}{\partial R} - \frac{aS}{t}.\frac{\partial^2 w}{\partial R^2}.\frac{du_0}{\partial R} + \frac{a^2}{t}H\frac{\partial \phi_x}{\partial R}.\frac{du_0}{dR}\right)$$
$$+ 2B_{12}\left(-\frac{a}{t\beta}S.\frac{\partial^2 w}{\partial R^2}.\frac{dv_0}{\partial Q} + \frac{a^2 H}{t\beta}\frac{dv_0}{\partial Q}.\frac{\partial \phi_x}{\partial R} - \frac{aS}{t\beta^2}\frac{\partial^2 w}{\partial Q^2}.\frac{du_0}{dR} + \frac{a^2 H}{t\beta}.\frac{\partial \phi_y}{\partial Q}.\frac{du_0}{dR}\right)$$
$$+ 2B_{13}\left(-\frac{a}{t\beta}S\frac{\partial^2 w}{\partial R^2}.\frac{du_0}{\partial Q} - \frac{a}{t}S\frac{\partial^2 w}{\partial R^2}.\frac{dv_0}{dR} + \frac{a^2 H}{t}\frac{du_0}{\beta\,\partial Q}.\frac{\partial \phi_x}{\partial R} + \frac{a^2 H}{t}\frac{dv_0}{dR}.\frac{\partial \phi_x}{\partial R}\right.$$
$$\left. + \frac{2aS}{t\beta}\frac{\partial^2 w}{\partial R\,\partial Q}.\frac{du_0}{dR} + \frac{a^2 H}{t\beta}\frac{\partial \phi_x}{\partial Q}.\frac{du_0}{dR} + \frac{a^2 H}{t\beta}.\frac{\partial \phi_y}{\partial R}.\frac{du_0}{dR}\right)$$

$$+ B_{22}\left(-\frac{aS}{t\beta^3}.\frac{\partial^2 w}{\partial Q^2}.\frac{dv_0}{\partial Q} + \frac{a^2 H}{t\beta^2}\frac{dv_0}{\partial Q}.\frac{\partial \phi_y}{\partial Q} - \frac{aS}{t\beta^3}\frac{\partial^2 w}{\partial Q^2}.\frac{dv_0}{\partial Q} + \frac{a^2 H}{t\beta^2}.\frac{\partial \phi_y}{\partial Q}.\frac{dv_0}{\partial Q}\right)$$

$$+2B_{23}\left(-\frac{aS}{t\beta^3}\frac{du_0}{\partial Q}.\frac{\partial^2 w}{\partial Q^2} - \frac{aS}{t\beta^2}\frac{dv_0}{dR}.\frac{\partial^2 w}{\partial Q^2} + \frac{a^2 H}{t\beta^2}\frac{du_0}{\partial Q}.\frac{\partial \phi_y}{\partial Q} + \frac{a^2 H}{t\beta}\frac{dv_0}{dR}.\frac{\partial \phi_y}{\partial Q} - \frac{2aS}{t\beta^2}\frac{\partial^2 w}{\partial R\,\partial Q}.\frac{dv_0}{\partial Q}\right.$$
$$\left. + \frac{a^2 H}{t\beta^2}.\frac{\partial \phi_x}{\partial Q}.\frac{dv_0}{\partial Q} + \frac{a^2 H}{t\beta}.\frac{\partial \phi_y}{\partial R}.\frac{dv_0}{\partial Q}\right)$$

$$+B_{33}\left(-\frac{2aS}{t\beta^2}.\frac{\partial^2 w}{\partial R\,\partial Q}.\frac{du_0}{\partial Q} - \frac{2aS}{t\beta}.\frac{\partial^2 w}{\partial R\,\partial Q}.\frac{dv_0}{dR}\right)$$

$$+B_{33}\left(\frac{a^2 H}{t\beta^2}\frac{du_0}{\partial Q}.\frac{\partial \phi_x}{\partial Q} + \frac{a^2 H}{t\beta}\frac{dv_0}{dR}.\frac{\partial \phi_x}{\partial Q}\right) + B_{33}\left(+\frac{a^2 H}{t\beta}\frac{du_0}{\partial Q}.\frac{\partial \phi_y}{\partial R} + \frac{a^2 H}{t}\frac{dv_0}{dR}.\frac{\partial \phi_y}{\partial R}\right)$$

$$+B_{33}\left(-\frac{2aS}{t\beta^2}\frac{\partial^2 w}{\partial R\,\partial Q}.\frac{du_0}{\partial Q} + \frac{a^2 H}{t\beta^2}.\frac{\partial \phi_x}{\partial Q}.\frac{du_0}{\partial Q} + \frac{a^2 H}{t\beta}\frac{\partial \phi_y}{\partial R}.\frac{du_0}{\partial Q}\right)$$

$$+B_{33}\left(-\frac{2aS}{t\beta}\frac{\partial^2 w}{\partial R\,\partial Q}.\frac{dv_0}{dR} + \frac{a^2 H}{t\beta}.\frac{\partial \phi_x}{\partial Q}.\frac{dv_0}{dR} + \frac{a^2 H}{t}\frac{\partial \phi_y}{\partial R}.\frac{dv_0}{dR}\right)$$

$$+\frac{E_0 t^3}{[1-\mu_{XY}\mu_{YX}]a^4} B_{11}\left(+\frac{a^2}{t^2}.\left(\frac{du_0}{dR}\right)^2\right) + 2B_{12}\left(+\frac{a^2}{t^2\beta}\frac{du_0}{dR}.\frac{dv_0}{\partial Q}\right)$$

$$+2B_{13}\left(\frac{a^2}{t^2}\frac{du_0}{\beta\,\partial Q}.\frac{du_0}{dR} + \frac{a^2}{t^2}.\frac{dv_0}{dR}.\frac{du_0}{dR}\right) + 2B_{22}\left(\frac{a^2}{t^2\beta^2}.\left(\frac{dv_0}{\partial Q}\right)^2\right)$$

$$+2B_{23}\left(\frac{a^2}{t^2\beta^2}\frac{du_0}{\partial Q}\frac{dv_0}{\partial Q} + \frac{a^2}{t^2\beta}\frac{dv_0}{dR}.\frac{dv_0}{\partial Q}\right) + B_{33}\left(\frac{a^2}{t^2\beta^2}.\left(\frac{du_0}{\partial Q}\right)^2 + \frac{a^2}{t^2\beta}\frac{dv_0}{dR}.\frac{du_0}{\partial Q}\right)$$

$$+B_{33}\left(\frac{a^2}{t^2\beta}\frac{du_0}{\partial Q}.\frac{dv_0}{dR}+\frac{a^2}{t^2}\left(\frac{dv_0}{dR}\right)^2\right)$$

$$+B_{44}\frac{a^4}{t^2}\left(\left(\emptyset_x.\frac{\partial H}{\partial S}\right)^2+\emptyset_x.\frac{du_0}{dS}.\frac{\partial H}{\partial S}+\emptyset_x.\frac{\partial H}{\partial S}.\frac{du_0}{dS}+\left(\frac{du_0}{dS}\right)^2\right)$$

$$+B_{55}\frac{a^4}{t^2}\left(\left(\emptyset_y.\frac{\partial H}{\partial S}\right)^2+.\emptyset_y.\frac{dv_0}{dS}.\frac{\partial H}{\partial S}+\emptyset_y.\frac{\partial H}{\partial S}.\frac{dv_0}{dS}+\left(\frac{dv_0}{dS}\right)^2\right) \quad (3.218b)$$

3.2.4.4 Variation Of Total Potential Energy

The various values of the strain energy must be differentiated to obtain the equation of equilibrium of forces. The differentiation was with respect to engineering properties which include $w, u_0\ v_0, \emptyset_y$ and $\emptyset_x$. For thick rectangular anisotropic in pure bending, the Total Potential Energy functional is given as

$$\pi = U + V \quad (3.219)$$

Where V is considered as the external load given as

$$V = qab\int_0^1\int_0^1 W\,dRdQ \quad (3.220)$$

As ealier explained the strain energy equation was categorized under three, namely bending , coupling and axial/membrane stifness. The Formulation of the total potential energy equation for the thick plate was further simplified by considering these three categories of the stinesses. In each stiffness, the strain energy, U was differentiated with respect to $w, u_0\ v_0, \emptyset_y$ and $\emptyset_x$. The external load was later introduced to the derived equations , to give the total potential energy for each of the five parameters.

3.2.4.5 Differentiation of the Strain Energy for the Bending Stiffness

Firstly the Strain Energy for the case of bending stiffness was differentiated with respect to the displacement in z direction, w. The differential value of the external load was introduced later, after summing up the total strain energy from the three stifnesses. This gave the governing equation. The process is as detailed below

$$\frac{dU_b}{dw}=\frac{E_0t^3}{[1-\mu_{XY}\mu_{YX}]}B_{11}\left(2S^2\frac{\partial^4 w}{\partial R^4}-aHS.\frac{\partial^3\phi_x}{\partial R^3}-aHS.\frac{\partial^3\phi_x}{\partial R^3}\right)$$

$$+2B_{12}\left(\frac{2S^2}{\beta^2}.\frac{\partial^4 w}{\partial Q^2\partial R^2}-\frac{aHS}{\beta}.\frac{\partial^3\phi_y}{\partial Q\partial R^2}-\frac{aHS}{\beta^2}\frac{\partial^3\phi_x}{\partial R\partial Q^2}\right)$$

$$+2B_{13}\left(-\frac{2S^2}{\beta}\frac{\partial^4 w}{\partial R^3\partial Q}-\frac{aHS}{\beta}\frac{\partial^3\partial\phi_x}{\partial Q\partial R^2}-aHS\frac{\partial^3\phi_y}{\partial R^3}+\frac{2aHS}{\beta}\frac{\partial^3\phi_x}{\partial R^2\partial Q}\right)$$

$$+\mathrm{B}_{22}\left(\frac{2\mathrm{S}^2}{\beta^4}\frac{\partial^4\mathrm{w}}{\partial\mathrm{Q}^4}-\frac{\mathrm{aHS}}{\beta^3}.\frac{\partial^3\phi_y}{\partial\mathrm{Q}^3}-\frac{\mathrm{aHS}}{\beta^3}\frac{\partial^3\phi_y}{\partial\mathrm{Q}^3}\right)$$

$$+2\mathrm{B}_{23}\left(\frac{4\mathrm{S}^2}{\beta^3}\frac{\partial^4\mathrm{w}}{\partial\mathrm{R}\,\partial\mathrm{Q}^3}-\frac{\mathrm{aSH}}{\beta^3}.\frac{\partial^3\phi_x}{\partial\mathrm{Q}^3}-\frac{\mathrm{aSH}}{\beta^2}.\frac{\partial^3\phi_y}{\partial\mathrm{R}\,\partial\mathrm{Q}^2}-\frac{2\mathrm{aHS}}{\beta^2}\frac{\partial^2\phi_y}{\partial\mathrm{R}\,\partial\mathrm{Q}^2}\right)$$

$$+\mathrm{B}_{33}\left(\frac{8\mathrm{S}^2}{\beta^2}\frac{\partial^4\mathrm{w}}{\partial\mathrm{R}^2\,\partial\mathrm{Q}^2}-\frac{2\mathrm{aHS}}{\beta^2}.\frac{\partial^3\phi_x}{\partial\mathrm{R}\,\partial\mathrm{Q}^2}-\frac{2\mathrm{aHS}}{\beta}.\frac{\partial^3\phi_y}{\partial\mathrm{R}^2\,\partial\mathrm{Q}}\right)+\mathrm{B}_{33}\left(-\frac{2\mathrm{aHS}}{\beta^2}\frac{\partial^3\phi_x}{\partial\mathrm{R}\,\partial\mathrm{Q}^2}\right)$$

$$+\mathrm{B}_{33}\left(-\frac{2\mathrm{aHS}}{\beta}\frac{\partial^2\phi_y}{\partial\mathrm{R}^2\,\partial\mathrm{Q}}\right) \qquad (3.221)$$

Also differentiating the strain energy with respect to middle layer in-plane displacement on x component gives

$$\frac{\mathrm{dU_b}}{\mathrm{du_0}}=0 \qquad (3.222)$$

Similarly differentiating the strain energy with respect to middle layer in-plane displacement y component gives

$$\frac{\mathrm{dU_b}}{\mathrm{dv_0}}=0 \qquad (3.223)$$

But differentiating the strain energy with respect to shear rotation y component gives

$$\frac{\mathrm{dU_b}}{\mathrm{d}\emptyset_y}=\frac{\mathrm{E_0t^3}}{[1-\mu_{XY}\mu_{YX}]}2\mathrm{B}_{12}\left(-\frac{\mathrm{aHS}}{\beta}.\frac{\partial^3\mathrm{w}}{\partial\mathrm{Q}\,\partial\mathrm{R}^2}+\frac{\mathrm{a^2H^2}}{\beta}.\frac{\partial^2\phi_x}{\partial\mathrm{R}\,\partial\mathrm{Q}}\right)$$

$$+2\mathrm{B}_{13}\left(-\mathrm{aHS}\frac{\partial^3\mathrm{w}}{\partial\mathrm{R}^3}+\mathrm{a^2H^2}.\frac{\partial^2\phi_x}{\partial\mathrm{R}^2}\right)+\mathrm{B}_{22}\left(-\frac{\mathrm{aHS}}{\beta^3}.\frac{\partial^3\phi_y}{\partial\mathrm{Q}^3}-\frac{\mathrm{aHS}}{\beta^3}\frac{\partial^3\phi_y}{\partial\mathrm{Q}^3}+\frac{2\mathrm{a^2H^2}}{\beta^2}.\frac{\partial^2\phi_y}{\partial\mathrm{Q}^2}\right)$$

$$+2\mathrm{B}_{23}\left(-\frac{\mathrm{aSH}}{\beta^2}.\frac{\partial^3\mathrm{w}}{\partial\mathrm{R}\,\partial\mathrm{Q}^2}-\frac{2\mathrm{aHS}}{\beta^2}\frac{\partial^3\mathrm{w}}{\partial\mathrm{R}\,\partial\mathrm{Q}^2}+\frac{\mathrm{a^2H^2}}{\beta^2}.\frac{\partial^2\phi_x}{\partial\mathrm{Q}^2}+\frac{\mathrm{a^2H^2}}{\beta}.\frac{\partial^2\phi_y}{\partial\mathrm{Q}\,\partial\mathrm{R}}\right)$$

$$+\mathrm{B}_{33}\left(-\frac{2\mathrm{aHS}}{\beta}.\frac{\partial^3\mathrm{w}}{\partial\mathrm{R}^2\,\partial\mathrm{Q}}\right)+\mathrm{B}_{33}\left(+\frac{\mathrm{a^2H^2}}{\beta}.\frac{\partial\phi_x}{\partial\mathrm{Q}\,\partial\mathrm{R}}\right)$$

$$+B_{33}\left(-\frac{2aHS}{\beta}\frac{\partial^3 w}{\partial R^2\,\partial Q}+\frac{a^2H^2}{\beta}.\frac{\partial^2\phi_x}{\partial R\,\partial Q}+2a^2H^2\frac{\partial^2\phi_y}{\partial R^2}\right)+2B_{55}.\frac{a^4}{t^2}.g_4.\emptyset_y\}dR\,dQ \quad (3.224)$$

Similarly for the case of the shear rotation on x component gives

$$\frac{dU_b}{d\emptyset_x}=\frac{E_0t^3}{[1-\mu_{XY}\mu_{YX}]}B_{11}\left(-aHS.\frac{\partial^3 w}{\partial R^3}-aHS.\frac{\partial^3 w}{\partial R^3}+2a^2S^2.\frac{\partial^2\phi_x}{\partial R^2}\right)$$

$$+2B_{12}\left(-\frac{aHS}{\beta^2}\frac{\partial^3 w}{\partial R\,\partial Q^2}+\frac{a^2H^2}{\beta}.\frac{\partial^2\phi_y}{\partial R\,\partial Q}\right)$$

$$+2B_{13}\left(-\frac{aHS}{\beta}.\frac{\partial^3 w}{\partial Q\,\partial R^2}-\frac{2aHS}{\beta}.\frac{\partial^3 w}{\partial R^2\,\partial Q}+a^2H^2.\frac{\partial^2\phi_y}{\partial R^2}\right)$$

$$+2B_{23}\left(-\frac{aSH}{\beta^3}.\frac{\partial^3 w}{\partial Q^3}+\frac{a^2H^2}{\beta^2}.\frac{\partial^2\phi_y}{\partial Q^2}\right)+B_{33}\left(-\frac{2aHS}{\beta^2}.\frac{\partial^3 w}{\partial R\,\partial Q^2}\right)$$

$$+B_{33}\left(-\frac{2aHS}{\beta^2}\frac{\partial^3 w}{\partial R\,\partial Q^2}+\frac{2a^2H^2}{\beta^2}.\frac{\partial^2\phi_x}{\partial Q^2}+\frac{a^2H^2}{\beta}\frac{\partial^2\phi_y}{\partial Q\,\partial R}\right)$$

$$+B_{33}\left(\frac{a^2H^2}{\beta}.\frac{\partial^2\phi_y}{\partial R\,\partial Q}\right)+2B_{44}.\frac{a^4}{t^2}.g_4.\emptyset_x\}dR\,dQ \qquad (3.225)$$

3.2.4.6 Differentiation of the Strain Energy for the Coupling Stiffness

Secondly the Strain Energy for the case of coupling stiffness was differentiated with respect to the displacement in z direction, w. The differential value of the external load was introduced later, after summing up the total strain energy from the three stifnesses. This gave the governing equation. The process is as detailed below

$$\frac{dU_c}{dw}=\frac{E_0t^3}{[1-\mu_{XY}\mu_{YX}]a^4}B_{11}\left(-\frac{a}{t}S.\frac{\partial^3u_0}{\partial R^3}-\frac{aS}{t}.\frac{\partial^3u_0}{\partial R^3}\right)+2B_{12}\left(-\frac{a}{t\beta}S.\frac{\partial^3v_0}{\partial Q\partial R^2}-\frac{aS}{t\beta^2}\frac{\partial^2u_0}{\partial R\partial Q^2}\right)$$

$$+2\,B_{13}\left(-\frac{a}{t\beta}S\frac{\partial^3u_0}{\partial Q\partial R^2}-\frac{a}{t}S\frac{\partial^3v_0}{\partial R^3}+\frac{2aS}{t\beta}\frac{\partial^3u_0}{\partial R^2\,\partial Q}\right)+B_{22}\left(-\frac{aS}{t\beta^3}.\frac{\partial^3v_0}{\partial Q^3}-\frac{aS}{t\beta^3}\frac{\partial^3v_0}{\partial Q^3}\right)$$

$$+2B_{23}\left(-\frac{aS}{t\beta^3}.\frac{\partial^3u_0}{\partial Q^3}-\frac{aS}{t\beta^2}.\frac{\partial^2v_0}{dR\,\partial Q^2}-\frac{2aS}{t\beta^2}\frac{\partial^3v_0}{\partial R\,\partial Q^2}\right)+B_{33}\left(-\frac{2aS}{t\beta^2}.\frac{\partial^3u_0}{\partial R\partial Q^2}-\frac{2aS}{t\beta}.\frac{\partial^3v_0}{\partial R^2\partial Q}\right)$$

$$+B_{33}\left(-\frac{2aS}{t\beta^2}\frac{\partial^3u_0}{\partial R\,\partial Q^2}\right)+B_{33}\left(-\frac{2aS}{t\beta}\frac{\partial^3v_0}{\partial R^2\,\partial Q}\right) \qquad (3.226)$$

Differentiating the coupling strain energy with respect to middle layer in-plane displacement gives

$$\frac{dU_c}{du_0} = \frac{E_0 t^3}{[1-\mu_{XY}\mu_{YX}]a^4} B_{11}\left(-\frac{a}{t}S.\frac{\partial^3 w}{\partial R^3} + \frac{a^2}{t}H.\frac{\partial^2 \phi_x}{\partial R^2} - \frac{aS}{t}.\frac{\partial^3 w}{\partial R^3} + \frac{a^2}{t}H\frac{\partial^2 \phi_x}{\partial R^2}\right)$$

$$+2B_{12}\left(-\frac{aS}{t\beta^2}\frac{\partial^3 w}{dR\partial Q^2} + \frac{a^2 H}{t\beta}.\frac{\partial^2 \phi_y}{\partial R\partial Q}\right)$$

$$+2B_{13}\left(-\frac{a}{t\beta}S\frac{\partial^3 w}{\partial Q\partial R^2} + \frac{a^2 H}{t\beta}.\frac{\partial^2 \phi_x}{\partial R\partial Q} + \frac{2aS}{t\beta}\frac{\partial^3 w}{\partial R^2\partial Q} + \frac{a^2 H}{t\beta}\frac{\partial^2 \phi_x}{\partial R\partial Q} + \frac{a^2 H}{t\beta}.\frac{\partial^2 \phi_y}{\partial R^2}\right)$$

$$+2B_{23}\left(-\frac{aS}{t\beta^3}.\frac{\partial^3 w}{\partial Q^3} + \frac{a^2 H}{t\beta^2}.\frac{\partial^2 \phi_y}{\partial Q^2}\right) + B_{33}\left(-\frac{2aS}{t\beta^2}.\frac{\partial^3 w}{\partial R\partial\,\partial Q^2}\right) + B_{33}\left(\frac{a^2 H}{t\beta^2}.\frac{\partial^2 \phi_x}{\partial Q^2}\right)$$

$$+B_{33}\left(+\frac{a^2 H}{t\beta}.\frac{\partial^2 \phi_y}{\partial Q\partial R}\right) + B_{33}\left(-\frac{2aS}{t\beta^2}\frac{\partial^3 w}{\partial R\,\partial Q^2} + \frac{a^2 H}{t\beta^2}.\frac{\partial^2 \phi_x}{\partial Q^2} + \frac{a^2 H}{t\beta}\frac{\partial^2 \phi_y}{\partial Q\partial R}\right) \tag{3.227}$$

Similarly differentiating the coupling strain energy with respect to middle layer in-plane displacement v_0 gave

$$\frac{dU_c}{dv_0} = \frac{E_0 t^3}{[1-\mu_{XY}\mu_{YX}]a^4}_{11} + 2B_{12}\left(-\frac{a}{t\beta}S.\frac{\partial^3 w}{\partial Q\,\partial R^2} + \frac{a^2 H}{t\beta}\frac{\partial^2 v_0}{\partial R\,\partial Q}\right)$$

$$+2B_{12}\left(-\frac{a}{t}S\frac{\partial^3 w}{\partial R^3} + \frac{a^2 H}{t}\frac{\partial^2 \phi_x}{\partial R^2}\right) + B_{22}\left(-\frac{aS}{t\beta^3}.\frac{\partial^3 w}{\partial Q^3} + \frac{a^2 H}{t\beta^2}\frac{\partial^2 v_0}{\partial Q^2} - \frac{aS}{t\beta^3}\frac{\partial^2 w}{\partial Q^3} + \frac{a^2 H}{t\beta^2}.\frac{\partial^2 \phi_y}{\partial Q^2}\right)$$

$$+2B_{23}\left(-\frac{aS}{t\beta^2}.\frac{\partial^3 w}{\partial R\,\partial Q^2} + \frac{a^2 H}{t\beta}.\frac{\partial^2 \phi_y}{\partial R\,\partial Q} - \frac{2aS}{t\beta^2}\frac{\partial^3 w}{\partial R\,\partial Q^2} + \frac{a^2 H}{t\beta^2}.\frac{\partial^2 \phi_x}{\partial Q^2} + \frac{a^2 H}{t\beta}.\frac{\partial^2 \phi_y}{\partial Q\,\partial R}\right)$$

$$+B_{33}\left(-\frac{2aS}{t\beta}.\frac{\partial^3 w}{\partial R^2\,\partial Q}\right) + B_{33}\left(\frac{a^2 H}{t\beta}.\frac{\partial^2 \phi_x}{\partial R\,\partial Q}\right) + B_{33}\left(+\frac{a^2 H}{t}.\frac{\partial^2 \phi_y}{\partial R^2}\right)$$

$$+B_{33}\left(-\frac{2aS}{t\beta}\frac{\partial^3 w}{\partial R^2\,\partial Q} + \frac{a^2 H}{t\beta}.\frac{\partial^2 \phi_x}{\partial R\,\partial Q} + \frac{a^2 H}{t}\frac{\partial^2 \phi_y}{\partial R^2}\right) \tag{3.228}$$

Also differentiating coupling strain energy, with respect to shear rotaion on y axis gives

$$\frac{dU_c}{d\emptyset_y} = \frac{E_0 t^3}{[1-\mu_{XY}\mu_{YX}]a^4} + 2B_{12}\left(\frac{a^2 H}{t\beta}.\frac{\partial^2 u_0}{\partial R\,\partial Q}\right) + 2B_{13}\left(+\frac{a^2 H}{t\beta}.\frac{\partial^2 u_0}{\partial R^2}\right)$$

$$+ B_{22}\left(\frac{a^2H}{t\beta^2}\frac{\partial^2 v_0}{\partial Q^2} + \frac{a^2H}{t\beta^2}.\frac{\partial^2 v_0}{\partial Q^2}\right) + 2B_{23}\left(\frac{a^2H}{t\beta^2}\frac{\partial^2 u_0}{\partial Q^2} + \frac{a^2H}{t\beta}\frac{\partial^2 v_0}{\partial Q\,\partial R} + \frac{a^2H}{t\beta}.\frac{\partial^2 v_0}{\partial Q\,\partial R}\right)$$

$$+B_{33}\left(+\frac{a^2H}{t\beta}\frac{\partial^2 u_0}{\partial Q\,\partial R} + \frac{a^2H}{t}\frac{\partial^2 v_0}{\partial R^2}\right) + B_{33}\left(\frac{a^2H}{t\beta}.\frac{\partial^2 u_0}{\partial R^2}\right) + B_{33}\left(\frac{a^2H}{t}.\frac{\partial^2 v_0}{\partial R^2}\right)$$

$$+B_{55}.\frac{a^4}{t^3}.\left[2g_{C3}.\frac{dv_0}{dS}\right] dR\, dQ \qquad (3.229)$$

Similarly also differentiating the coupling strain energy respect to shear rotaion on x axis gives

$$\frac{dU_c}{d\emptyset_x} = \frac{E_0 t^3}{[1-\mu_{XY}\mu_{YX}]a^4} B_{11}\left(\frac{a^2}{t}H.\frac{\partial^2 u_0}{\partial R^2} + \frac{a^2}{t}H.\frac{\partial^2 u_0}{\partial R^2}\right) + 2B_{12}\left(\frac{a^2H}{t\beta}\frac{\partial^2 v_0}{\partial R^2}\right)$$

$$+2B_{13}\left(\frac{a^2H}{t\beta}\frac{\partial^2 u_0}{\partial R\partial Q} + \frac{a^2H}{t}\frac{\partial^2 v_0}{\partial R^2} + \frac{a^2H}{t\beta}.\frac{\partial^2 u_0}{\partial R\partial Q}\right) + 2B_{23}\left(\frac{a^2H}{t\beta^2}.\frac{\partial^2 v_0}{\partial Q^2}\right)$$

$$+B_{33}\left(\frac{a^2H}{t\beta^2}\frac{\partial^2 u_0}{\partial Q^2} + \frac{a^2H}{t\beta}.\frac{\partial^2 v_0}{\partial R\partial Q}\right) + B_{33}\left(\frac{a^2H}{t\beta^2}.\frac{\partial^2 u_0}{\partial Q^2}\right) + B_{33}\left(\frac{a^2H}{t\beta}.\frac{\partial^2 v_0}{\partial Q^2}\right)$$

$$+B_{44}.\frac{a^4}{t^3}\left[2g_{C3}.\frac{du_0}{dS}\right] dR\, dQ \qquad (3.230)$$

3.2.4.7 Differentiation of the Strain Energy for the Axial Stiffness

Thirdly, the Strain Energy for the case of axial/membrane stiffness was differentiated with respect to the displacement in in z direction, w. The differential value of the external load was introduced later, after summing up the total strain energy from the three stifnesses. This gave zero value as shown below.

$$\frac{dU_M}{dw} = 0 \qquad (3.231)$$

Differentiating the membrane strain energy, U_m with respect to the middle layer in-plane displacement gives

$$\frac{dU_M}{du_0} = \frac{E_0 t^3}{[1-\mu_{XY}\mu_{YX}]a^4} B_{11}\left(+\frac{a^2}{t^2}.2\frac{\partial^2 u_0}{\partial R^2}\right) + 2B_{12}\left(+\frac{a^2}{t^2\beta}.\frac{\partial^2 v_0}{dR\,\partial Q}\right)$$

$$+2B_{13}\left(\frac{a^2}{t^2\beta}2\frac{\partial^2 u_0}{\partial R\,\partial Q} + \frac{a^2}{t^2}.\frac{\partial^2 v_0}{\partial R^2}\right) + 2B_{23}\left(\frac{a^2}{t^2\beta^2}.\frac{\partial^2 v_0}{\partial Q^2}\right)$$

$$+B_{33}\left(\frac{a^2}{t^2\beta^2}.2.\frac{\partial^2 u_0}{\partial Q^2} + \frac{a^2}{t^2\beta}\frac{\partial^2 v_0}{\partial Q\,\partial R}\right) + B_{33}\left(\frac{a^2}{t^2\beta}\frac{\partial^2 v_0}{\partial Q\,\partial R}\right)$$

$$+B_{44}.\frac{a^2}{t^2}\left(\emptyset_x\frac{\partial^2 H}{\partial S^2}+\emptyset_x\frac{\partial^2 H}{\partial S^2}+2.\frac{\partial^2 u_0}{\partial S^2}\right)dR\,dQ \qquad (3.232)$$

Similarly, differentiating the membranel strain energy, U_m with respect to middle layer in-plane displacement gives

$$\frac{dU_M}{dv_0}=\frac{E_0t^3}{[1-\mu_{XY}\mu_{YX}]a^4}2B_{12}\left(+\frac{a^2}{t^2\beta}\frac{\partial^2 v_0}{dR\,\partial Q}\right)+2B_{13}\left(\frac{a^2}{t^2}.\frac{\partial^2 u_0}{\partial R^2}.\right)+2B_{22}\left(\frac{a^2}{t^2\beta^2}.2.\frac{\partial^2 v_0}{\partial Q^2}\right)$$

$$+2B_{23}\left(\frac{a^2}{t^2\beta^2}\frac{\partial^2 u_0}{\partial Q^2}+\frac{a^2}{t^2\beta}\frac{\partial^2 v_0}{\partial Q\,\partial R}\right)+B_{33}\left(\frac{a^2}{t^2\beta}\frac{\partial^2 u_0}{\partial Q\,\partial R}\right)+B_{33}\left(\frac{a^2}{t^2\beta}\frac{\partial^2 u_0}{\partial Q\,\partial R}+\frac{a^2}{t^2}2.\frac{\partial^2 v_0}{\partial R^2}\right)$$

$$+B_{55}.\frac{a^2}{t^2}\left(\emptyset_y\frac{\partial^2 H}{\partial S^2}+\emptyset_y\frac{\partial^2 H}{\partial S^2}+2\frac{\partial^2 v_0}{\partial S^2}\right)dR\,dQ \qquad (3.233)$$

But differentiating the same equation (i.e U_m), for the two shear rotations, $\emptyset_y$ *and* $\emptyset_x$ gave zero values as shown Equations (3.234) and (3.235) respectively.

$$\frac{dU_M}{d\emptyset_y}=0 \qquad (3.234)$$

$$\frac{dU_M}{d\emptyset_x}=0 \qquad (3.235)$$

3.2.4.8 Formulation of The Buckling Equations

Considering the external work,V introduced on the laminated thick as

$$\iint\left(0+\frac{N_x}{2}\left(\frac{dw}{dx}\right)^2+0\right)dxdy \qquad (3.236)$$

where

$$V=-\frac{N_x}{2}\int_0^1\int_0^1\left(\frac{dw}{dx}\right)^2 dxdy \qquad (3.237)$$

Expresing V in terms of non dimensional parameters gives

$$V=-N_x 0.5\frac{ab}{a^2}\int_0^1\int_0^1\left(\frac{dw}{dR}\right)^2 dRdQ \qquad (3.238)$$

Where

$$\frac{\partial V}{\partial w}=-\frac{2}{2}N_x a^2 dRdQ=$$
$$-N_x a^2 dRdQ \qquad (3.239a)$$

The total potential energy functional of thick rectangular anisotropic plate is given by the expression in Equation (3.239b)

$$\pi = U + V \tag{3.239b}$$

considering the flexural rigidity as

$$D_O = \frac{E_0 t^3}{[1 - \mu_{XY}\mu_{YX}]a^4} \tag{3.239c}$$

3.2.4.9 Derivation of The Governing Equation

The total strain energy comprises of the strain energy for bending , coupling and axial

Stiffnesses. That is to say

$$U = U_b + U_c + U_m \tag{3.240}$$

$$\text{but } \pi = (U_b + U_c + U_m) + V \tag{3.241}$$

$$\text{or } \pi = U + V \tag{3.242}$$

Differentiating Equation (3.242) with respect to w gives the governing equation

$$\frac{\partial \pi}{\partial w} = \frac{\partial U}{\partial w} + \frac{\partial V}{\partial w} = 0 \tag{3.243}$$

Substituting back Equation (3.221) (3.226) and (3.231) into Equation (3.243) gives

$$\frac{\partial \pi}{\partial w} = \frac{abD_0}{2a^4}\int_0^1\int_0^1 \{B_{11}\left(2S^2\frac{\partial^4 w}{\partial R^4} - aHS.\frac{\partial^3 \phi_x}{\partial R^3} - aHS.\frac{\partial^3 \phi_x}{\partial R^3}\right)$$

$$+2B_{12}\left(\frac{2S^2}{\beta^2}.\frac{\partial^4 w}{\partial Q^2 \partial R^2} - \frac{aHS}{\beta}.\frac{\partial^3 \phi_y}{\partial Q \partial R^2} - \frac{aHS}{\beta^2}\frac{\partial^3 \phi_x}{\partial R \partial Q^2}\right)$$

$$+2B_{13}\left(+\frac{2S^2}{\beta}\frac{\partial^4 w}{\partial R^3 \partial Q} - \frac{aHS}{\beta}\frac{\partial^3 \partial \phi_x}{\partial Q \partial R^2} - aHS\frac{\partial^3 \phi_y}{\partial R^3} - \frac{2aHS}{\beta}\frac{\partial^3 \phi_x}{\partial R^2 \partial Q}\right)$$

$$+B_{22}\left(\frac{2S^2}{\beta^4}\frac{\partial^4 w}{\partial Q^4} - \frac{aHS}{\beta^3}.\frac{\partial^3 \phi_y}{\partial Q^3} - \frac{aHS}{\beta^3}\frac{\partial^3 \phi_y}{\partial Q^3}\right)$$

$$+2B_{23}\left(\frac{4S^2}{\beta^3}\frac{\partial^4 w}{\partial R \partial Q^3} - \frac{aSH}{\beta^3}.\frac{\partial^3 \phi_x}{\partial Q^3} - \frac{aSH}{\beta^2}.\frac{\partial^3 \phi_y}{\partial R \partial Q^2} - \frac{2aHS}{\beta^2}\frac{\partial^2 \phi_y}{\partial R \partial Q^2}\right)$$

$$+B_{33}\left(\frac{8S^2}{\beta^2}\frac{\partial^4 w}{\partial R^2\partial Q^2}-\frac{2aHS}{\beta^2}.\frac{\partial^3\phi_x}{\partial R\partial Q^2}-\frac{2aHS}{\beta}.\frac{\partial^3\phi_y}{\partial R^2\partial Q}\right)+B_{33}\left(-\frac{2aHS}{\beta^2}\frac{\partial^3\phi_x}{\partial R\partial Q^2}\right)$$

$$+B_{33}\left(-\frac{2aHS}{\beta}\frac{\partial^2\phi_y}{\partial R^2\partial Q}\right)+\mathrm{B}_{11}\left(-\frac{\mathrm{a}}{\mathrm{t}}S.\frac{\partial^3\mathrm{u}_0}{\partial\mathrm{R}^3}-\frac{\mathrm{aS}}{\mathrm{t}}.\frac{\partial^3\mathrm{u}_0}{\partial\mathrm{R}^3}\right)$$

$$+2\mathrm{B}_{12}\left(-\frac{a}{t\beta}S.\frac{\partial^3 v_0}{\partial Q\partial R^2}-\frac{aS}{t\beta^2}\frac{\partial^2 u_0}{\partial R\partial Q^2}\right)$$

$$+2\mathrm{B}_{13}\left(-\frac{a}{t\beta}S\frac{\partial^3 u_0}{\partial Q\partial R^2}-\frac{a}{t}S\frac{\partial^3 v_0}{\partial R^3}+\frac{2\mathrm{aS}}{\mathrm{t\beta}}\frac{\partial^3\mathrm{u}_0}{\partial R^2\,\partial\mathrm{Q}}\right)$$

$$+\mathrm{B}_{22}\left(-\frac{\mathrm{aS}}{\mathrm{t\beta^3}}.\frac{\partial^3\mathrm{v}_0}{\partial\mathrm{Q}^3}-\frac{\mathrm{aS}}{\mathrm{t\beta^3}}\frac{\partial^3\mathrm{v}_0}{\partial\mathrm{Q}^3}\right)+2\mathrm{B}_{23}\left(-\frac{\mathrm{aS}}{\mathrm{t\beta^3}}.\frac{\partial^3\mathrm{u}_0}{\partial\mathrm{Q}^3}-\frac{\mathrm{aS}}{\mathrm{t\beta^2}}.\frac{\partial^2\mathrm{v}_0}{\mathrm{dR}\,\partial\mathrm{Q}^2}-\frac{2\mathrm{aS}}{\mathrm{t\beta^2}}\frac{\partial^3\mathrm{v}_0}{\partial\mathrm{R}\,\partial\mathrm{Q}^2}\right)$$

$$+B_{33}\left(-\frac{2aS}{t\beta^2}.\frac{\partial^3 u_0}{\partial R\partial\mathrm{Q}^2}-\frac{2aS}{t\beta}.\frac{\partial^3 v_0}{\partial\mathrm{R}^2\partial Q}\right)+B_{33}\left(-\frac{2aS}{t\beta^2}\frac{\partial^3 u_0}{\partial R\,\partial\mathrm{Q}^2}\right)$$

$$B_{33}\left(-\frac{2aS}{t\beta}\frac{\partial^3 v_0}{\partial\mathrm{R}^2\partial Q}\right)-2\frac{\mathrm{N_x a^2}}{\mathrm{D_0}}\Bigg\}dRdQ\ =0 \tag{3.244}$$

Equation(3.244) was simplified further the to get

$$\frac{\partial\pi}{\partial\mathrm{w}}=\int_0^1\int_0^1 2\,\mathrm{B}_{11}\,\mathrm{S}^2\frac{\partial^4\mathrm{w}}{\partial\mathrm{R}^4}-\mathrm{B}_{11}\mathrm{aHS}\frac{\partial^3\phi_\mathrm{x}}{\partial\mathrm{R}^2}-\mathrm{B}_{11}\mathrm{a}HS.\frac{\partial^3\phi_\mathrm{x}}{\partial\mathrm{R}^3}-\mathrm{B}_{11}\frac{\mathrm{a}}{\mathrm{t}}S.\frac{\partial^3\mathrm{u}_0}{\partial\mathrm{R}^3}$$

$$-\mathrm{B}_{11}\frac{\mathrm{aS}}{\mathrm{t}}.\frac{\partial^3\mathrm{u}_0}{\partial\mathrm{R}^3}+\frac{4\mathrm{B}_{12}\mathrm{S}^2}{\beta^2}.\frac{\partial^4 w}{\partial Q^2\partial R^2}-\frac{2\mathrm{B}_{12}aHS}{\beta}.\frac{\partial^3\phi_y}{\partial Q\partial R^2}-\frac{2\mathrm{B}_{12}aHS}{\beta^2}\frac{\partial^3\phi_x}{\partial R\partial Q^2}$$

$$-\frac{2\mathrm{B}_{12}a}{t\beta}S.\frac{\partial^3 v_0}{\partial Q\partial R^2}-\frac{2\mathrm{B}_{12}aS}{t\beta^2}\frac{\partial^3 u_0}{\partial R\partial Q^2}$$

$$+\frac{4B_{13}S^2}{\beta}\frac{\partial^4 w}{\partial R^3\partial Q}-\frac{2B_{13}aHS}{\beta}\frac{\partial^3\partial\phi_x}{\partial Q\partial R^2}-2B_{13}aHS\frac{\partial^3\phi_y}{\partial R^3}+\frac{4B_{13}aHS}{\beta}\frac{\partial^3\phi_x}{\partial R^2\partial Q}$$

$$-\frac{2\mathrm{B}_{13}a}{t\beta}S\frac{\partial^3 u_0}{\partial Q\partial R^2}-\frac{2\mathrm{B}_{13}a}{t}S\frac{\partial^3 v_0}{\partial R^3}+\frac{4\mathrm{B}_{13}\mathrm{aS}}{\mathrm{t\beta}}\frac{\partial^3\mathrm{u}_0}{\partial R^2\,\partial\mathrm{Q}}$$

$$+\frac{\mathrm{B}_{22}2\mathrm{S}^2}{\beta^4}\frac{\partial^4\mathrm{w}}{\partial\mathrm{Q}^4}-\frac{\mathrm{B}_{22}\mathrm{aHS}}{\beta^3}.\frac{\partial^3\phi_\mathrm{y}}{\partial\mathrm{Q}^3}-\frac{\mathrm{B}_{22}\mathrm{aHS}}{\beta^3}\frac{\partial^3\phi_\mathrm{y}}{\partial\mathrm{Q}^3}-\frac{\mathrm{B}_{22}\mathrm{aS}}{\mathrm{t\beta^3}}.\frac{\partial^3\mathrm{v}_0}{\partial\mathrm{Q}^3}-\frac{\mathrm{B}_{22}\mathrm{aS}}{\mathrm{t\beta^3}}\frac{\partial^3\mathrm{v}_0}{\partial\mathrm{Q}^3}$$

$$+\frac{8B_{23}S^2}{\beta^3}\frac{\partial^4 w}{\partial R\partial Q^3}-\frac{2B_{23}aSH}{\beta^3}.\frac{\partial^3\phi_x}{\partial Q^3}-\frac{2B_{23}aSH}{\beta^2}.\frac{\partial^3\phi_y}{\partial R\partial Q^2}-\frac{4B_{23}aHS}{\beta^2}\frac{\partial^2\phi_y}{\partial R\partial Q^2}$$

$$-\frac{2\mathrm{B}_{23}\mathrm{aS}}{\mathrm{t\beta^3}}.\frac{\partial^3\mathrm{u}_0}{\partial\mathrm{Q}^3}-\frac{2\mathrm{B}_{23}\mathrm{aS}}{\mathrm{t\beta^2}}.\frac{\partial^2\mathrm{v}_0}{\mathrm{dR}\,\partial\mathrm{Q}^2}-\frac{4\mathrm{B}_{23}\mathrm{aS}}{\mathrm{t\beta^2}}\frac{\partial^3\mathrm{v}_0}{\partial\mathrm{R}\,\partial\mathrm{Q}^2}$$

$$+\frac{8B_{33}S^2}{\beta^2}\frac{\partial^4 w}{\partial R^2\,\partial Q^2}-\frac{2B_{33}aHS}{\beta^2}.\frac{\partial^3\phi_x}{\partial R\partial Q^2}-\frac{2B_{33}aHS}{\beta}.\frac{\partial^3\phi_y}{\partial R^2\partial Q}-\frac{2B_{33}aHS}{\beta^2}\frac{\partial^3\phi_x}{\partial R\partial Q^2}-\frac{2B_{33}aHS}{\beta}\frac{\partial^2\phi_y}{\partial R^2\partial Q}$$

$$-\frac{2B_{33}aS}{t\beta^2}.\frac{\partial^3 u_0}{\partial R\partial Q^2}-\frac{2B_{33}aS}{t\beta}.\frac{\partial^3 v_0}{\partial R^2\partial Q}-\frac{2B_{33}aS}{t\beta^2}\frac{\partial^3 u_0}{\partial R\,\partial Q^2}-\frac{2B_{33}aS}{t\beta}\frac{\partial^3 v_0}{\partial R^2\partial Q}$$

$$\left.-2\frac{N_x a^2}{D_0}\right\}dRdQ=0 \tag{3.245}$$

That is:

$$\int_0^1\int_0^1\{2\,B_{11}S^2\frac{\partial^4 w}{\partial R^4}+\frac{4B_{12}S^2}{\beta^2}.\frac{\partial^4 w}{\partial Q^2\partial R^2}+\frac{4B_{13}S^2}{\beta}\frac{\partial^4 w}{\partial R^3\partial Q}+\frac{B_{22}2S^2}{\beta^4}\frac{\partial^4 w}{\partial Q^4}$$

$$+\frac{8B_{23}S^2}{\beta^3}\frac{\partial^4 w}{\partial R\partial Q^3}+\frac{8B_{33}S^2}{\beta^2}\frac{\partial^4 w}{\partial R^2\partial Q^2}-B_{11}aHS.\frac{\partial^3\phi_x}{\partial R^3}-B_{11}aHS.\frac{\partial^3\phi_x}{\partial R^3}$$

$$-\frac{2B_{12}aHS}{\beta^2}\frac{\partial^3\phi_x}{\partial R\partial Q^2}-\frac{2B_{13}aHS}{\beta}\frac{\partial^3\partial\phi_x}{\partial Q\partial R^2}+\frac{4B_{13}aHS}{\beta}\frac{\partial^3\phi_x}{\partial R^2\partial Q}-\frac{2B_{23}aSH}{\beta^3}.\frac{\partial^3\phi_x}{\partial Q^3}-\frac{2B_{33}aHS}{\beta^2}.\frac{\partial^3\phi_x}{\partial R\partial Q^2}-\frac{2B_{33}aHS}{\beta^2}\frac{\partial^3\phi_x}{\partial R\partial Q^2}-\frac{2B_{12}aHS}{\beta}.\frac{\partial^3\phi_y}{\partial Q\partial R^2}-2B_{13}aHS\frac{\partial^3\phi_y}{\partial R^3}-\frac{B_{22}aHS}{\beta^3}.\frac{\partial^3\phi_y}{\partial Q^3}-\frac{B_{22}aHS}{\beta^3}\frac{\partial^3\phi_y}{\partial Q^3}$$

$$-\frac{2B_{23}aSH}{\beta^2}\frac{\partial^3\phi_y}{\partial R\partial Q^2}-\frac{4B_{23}aHS}{\beta^2}\frac{\partial^2\phi_y}{\partial R\partial Q^2}-\frac{2B_{33}aHS}{\beta}.\frac{\partial^3\phi_y}{\partial R^2\partial Q}-\frac{2B_{33}aHS}{\beta}\frac{\partial^2\phi_y}{\partial R^2\partial Q}$$

$$-B_{11}\frac{a}{t}S.\frac{\partial^3 u_0}{\partial R^3}-B_{11}\frac{aS}{t}.\frac{\partial^3 u_0}{\partial R^3}-\frac{2B_{12}aS}{t\beta^2}\frac{\partial^3 u_0}{\partial R\,\partial Q^2}-\frac{2B_{13}a}{t\beta}S\frac{\partial^3 u_0}{\partial Q\,\partial R^2}+\frac{4B_{13}aS}{t\beta}\frac{\partial^3 u_0}{\partial R^2\,\partial Q}-\frac{2B_{23}aS}{t\beta^3}.\frac{\partial^3 u_0}{\partial Q^3}-\frac{2B_{33}aS}{t\beta^2}.\frac{\partial^3 u_0}{\partial R\,\partial Q^2}-\frac{2B_{33}aS}{t\beta^2}\frac{\partial^3 u_0}{\partial R\,\partial Q^2}$$

$$-\frac{2B_{12}a}{t\beta}S.\frac{\partial^3 v_0}{\partial Q\partial R^2}-\frac{2B_{13}a}{t}S\frac{\partial^3 v_0}{\partial R^3}-\frac{B_{22}aS}{t\beta^3}.\frac{\partial^3 v_0}{\partial Q^3}-\frac{B_{22}aS}{t\beta^3}\frac{\partial^3 v_0}{\partial Q^3}-\frac{2B_{23}aS}{t\beta^2}$$

$$.\frac{\partial^2 v_0}{dR\,\partial Q^2}-\frac{4B_{23}aS}{t\beta^2}\frac{\partial^3 v_0}{\partial R\,\partial Q^2}-\frac{2B_{33}aS}{t\beta}.\frac{\partial^3 v_0}{\partial R^2\partial Q}-\frac{2B_{33}aS}{t\beta}$$

$$\frac{\partial^3 v_0}{\partial \mathrm{R}^2 \partial Q} - \frac{\mathrm{N_x a^2}}{\mathrm{D_0}}\Bigg\} \, dRdQ \qquad (3.246)$$

Further factorizing Equation (3.246) gives:

$$\int_0^1 \int_0^1 \{2\,\mathrm{B_{11}S^2}\frac{\partial^4 \mathrm{w}}{\partial \mathrm{R}^4} + \frac{2\mathrm{S}^2}{\beta^2}[\mathrm{B_{12}} + 2B_{33}]\frac{\partial^4 w}{\partial R^2 \partial Q^2} + \frac{4B_{13}S^2}{\beta}\frac{\partial^4 w}{\partial R^3 \partial Q} + \frac{8B_{23}S^2}{\beta^3}\frac{\partial^4 w}{\partial R \partial Q^3} + \frac{\mathrm{B_{22}2S^2}}{\beta^4}\frac{\partial^4 \mathrm{w}}{\partial \mathrm{Q}^4}$$

$$-2\mathrm{B_{11}a}HS.\frac{\partial^3 \phi_x}{\partial \mathrm{R}^3} - \frac{2aHS}{\beta^2}[\mathrm{B_{12}} + 2B_{33}]\frac{\partial^3 \phi_x}{\partial R \partial Q^2} + \frac{2B_{13}aHS}{\beta}\frac{\partial^3 \phi_x}{\partial R^2 \partial Q} - \frac{2B_{23}aSH}{\beta^3}.\frac{\partial^3 \phi_x}{\partial Q^3}$$

$$-2B_{13}aHS\frac{\partial^3 \phi_y}{\partial R^3} - \frac{2aHS}{\beta}[\mathrm{B_{12}} + 2B_{33}]\frac{\partial^3 \phi_y}{\partial R^2 \partial Q} - \frac{2aSH}{\beta^2}[B_{23} + 2B_{23}]\frac{\partial^2 \phi_y}{\partial R \partial Q^2} - \frac{2\mathrm{B_{22}aHS}}{\beta^3}.\frac{\partial^3 \phi_y}{\partial \mathrm{Q}^3}$$

$$-2\mathrm{B_{11}}\frac{\mathrm{aS}}{\mathrm{t}}.\frac{\partial^3 \mathrm{u_0}}{\partial \mathrm{R}^3} + \frac{2\mathrm{B_{13}aS}}{\mathrm{t}\beta}\frac{\partial^3 \mathrm{u_0}}{\partial R^2\,\partial \mathrm{Q}} - \frac{2aS}{t\beta^2}[\mathrm{B_{12}} + 4B_{33}]\frac{\partial^3 u_0}{\partial R\,\partial \mathrm{Q}^2} - \frac{2\mathrm{B_{23}aS}}{\mathrm{t}\beta^3}.\frac{\partial^3 \mathrm{u_0}}{\partial \mathrm{Q}^3}$$

$$-\frac{2\mathrm{B_{13}}a}{t}S\frac{\partial^3 v_0}{\partial R^3} - \frac{2aS}{t\beta}[\mathrm{B_{12}} + 2B_{33}]\frac{\partial^3 v_0}{\partial \mathrm{R}^2 \partial Q} - \frac{6\mathrm{B_{23}aS}}{\mathrm{t}\beta^2}\frac{\partial^3 \mathrm{v_0}}{\partial \mathrm{R}\,\partial \mathrm{Q}^2} - \frac{2\mathrm{B_{22}aS}}{\mathrm{t}\beta^3}.\frac{\partial^3 \mathrm{v_0}}{\partial \mathrm{Q}^3} - \frac{\mathrm{N_x a^2}}{\mathrm{D_0}}\Bigg\} \, dRdQ \qquad (3.247a)$$

Considering the x-y plane elastic modulus parameter as

$$\mathrm{B}_{xy} = \mathrm{B_{12}} + 2B_{33} \qquad (3.247b)$$

and putting it back into Equation $(3.247a)$ gives

$$\frac{\partial \pi}{\partial \mathrm{w}} = \int_0^1 \int_0^1 \{2\,\mathrm{B_{11}S^2}\frac{\partial^4 \mathrm{w}}{\partial \mathrm{R}^4} + \frac{2\mathrm{S}^2}{\beta^2}[\mathrm{B}_{xy}]\frac{\partial^4 w}{\partial R^2 \partial Q^2} + \frac{4B_{13}S^2}{\beta}\frac{\partial^4 w}{\partial R^3 \partial Q} + \frac{8B_{23}S^2}{\beta^3}\frac{\partial^4 w}{\partial R \partial Q^3} + \frac{\mathrm{B_{22}2S^2}}{\beta^4}\frac{\partial^4 \mathrm{w}}{\partial \mathrm{Q}^4} - 2\mathrm{B_{11}a}HS.\frac{\partial^3 \phi_x}{\partial \mathrm{R}^3}$$

$$-\frac{2aHS}{\beta^2}[\mathrm{B}_{xy}]\frac{\partial^3 \phi_x}{\partial R \partial Q^2} + \frac{2B_{13}aHS}{\beta}\frac{\partial^3 \phi_x}{\partial R^2 \partial Q} - \frac{2B_{23}aSH}{\beta^3}.\frac{\partial^3 \phi_x}{\partial Q^3} - 2B_{13}aHS$$

$$\frac{\partial^3 \phi_y}{\partial R^3} - \frac{2aHS}{\beta}[\mathrm{B}_{xy}]\frac{\partial^3 \phi_y}{\partial R^2 \partial Q} - \frac{2aSH}{\beta^2}[B_{23} + 2B_{23}]\frac{\partial^2 \phi_y}{\partial R \partial Q^2} - \frac{2\mathrm{B_{22}aHS}}{\beta^3}.\frac{\partial^3 \phi_y}{\partial \mathrm{Q}^3}$$

$$-2B_{11}\frac{aS}{t}.\frac{\partial^3 u_0}{\partial R^3}+\frac{2B_{13}aS}{t\beta}\frac{\partial^3 u_0}{\partial R^2\,\partial Q}-\frac{aS}{t\beta^2}[B_{xy}]\frac{\partial^3 u_0}{\partial R\,\partial Q^2}-\frac{2B_{23}aS}{t\beta^3}.\frac{\partial^3 u_0}{\partial Q^3}$$

$$-\frac{2B_{13}a}{t}S\frac{\partial^3 v_0}{\partial R^3}-\frac{2aS}{t\beta}[B_{xy}]\frac{\partial^3 v_0}{\partial R^2\partial Q}-\frac{6B_{23}aS}{t\beta^2}\frac{\partial^3 v_0}{\partial R\,\partial Q^2}-\frac{2B_{22}aS}{t\beta^3}.\frac{\partial^3 v_0}{\partial Q^3}$$

$$\left.-\frac{N_x a^2}{D_0}\right\} dRdQ=0 \tag{3.248}$$

Substitutitng g_i for S as already stated in Equations (3.205) to (3.211) in to Equation (3.248) gives:

$$\frac{\partial \pi}{\partial w}=\int_0^1\int_0^1\{2\,B_{11}g_1\frac{\partial^4 w}{\partial R^4}+\frac{2g_1}{\beta^2}[B_{xy}]\frac{\partial^4 w}{\partial R^2\partial Q^2}-\frac{4B_{13}g_1}{\beta}\frac{\partial^4 w}{\partial R^3\partial Q}+\frac{8B_{23}g_1}{\beta^3}\frac{\partial^4 w}{\partial R\partial Q^3}$$

$$+\frac{2B_{22}g_1}{\beta^4}\frac{\partial^4 w}{\partial Q^4}-2B_{11}g_2.\frac{\partial^4 w}{\partial R^4}-\frac{2g_2}{\beta^2}[B_{xy}]\frac{\partial^4 w}{\partial R^2\partial Q^2}$$

$$+\frac{2B_{13}g_2}{\beta}\frac{\partial^4 w}{\partial R^3\partial Q}-\frac{2B_{23}g_2}{\beta^3}.\frac{\partial^4 w}{\partial R\partial Q^3}-\frac{2B_{13}}{\beta}g_2\frac{\partial^4 w}{\partial Q\partial R^3}-\frac{2g_2}{\beta^2}[B_{xy}]\frac{\partial^4 w}{\partial R^2\partial Q^2}$$

$$-\frac{6B_{23}g_2}{\beta^3}\frac{\partial^4 w}{\partial R\partial Q^3}-\frac{2B_{22}g_2}{\beta^4}.\frac{\partial^4 w}{\partial Q^4}-2B_{11}\frac{g_{c1}}{t}.\frac{\partial^4 w}{\partial R^4}+\frac{2B_{13}g_{c1}}{t\beta}\frac{\partial^4 w}{\partial R^3\,\partial Q}$$

$$-\frac{2g_{c1}}{t\beta^2}[B_{xy}]\frac{\partial^4 w}{\partial R^2\,\partial Q^2}-\frac{2B_{23}g_{c1}}{t\beta^3}.\frac{\partial^4 w}{\partial Q^4}-\frac{2B_{13}g_{c1}}{t\beta}.\frac{\partial^4 w}{\partial R^3\partial Q}$$

$$-\frac{2g_{c1}}{t\beta^2}[B_{xy}]\frac{\partial^4 w}{\partial R^2\,\partial Q^2}-\frac{6B_{23}g_{c1}}{t\beta^3}\frac{\partial^4 w}{\partial R\,\partial Q^3}-\frac{2B_{22}g_{c1}}{t\beta^4}$$

$$\left..\frac{\partial^4 w}{\partial Q^4}-\frac{N_x a^2}{D_0}\right\} dRdQ=0 \tag{3.249}$$

Further rearranging Equation (3.249) gives

$$\int_0^1\int_0^1\{.\left[2B_{11}g_1-2B_{11}g_2-2B_{11}\frac{g_{c1}}{t}\right]\frac{\partial^4 w}{\partial R^4}$$

$$+\left[\frac{2g_1}{\beta^2}[B_{xy}]-\frac{2g_2}{\beta^2}[B_{xy}]+\frac{2g_1}{\beta^2}[B_{xy}]-\frac{2g_2}{\beta^2}[B_{xy}]-\frac{2g_{c1}}{t\beta^2}[B_{xy}]-\frac{2g_{c1}}{t\beta^2}[B_{xy}]\right]$$

$$\frac{\partial^4 w}{\partial R^2\,\partial Q^2}+\left[\frac{2B_{22}g_1}{\beta^4}-\frac{2B_{22}g_2}{\beta^4}-\frac{2B_{23}g_{c1}}{t\beta^3}-\frac{2B_{22}g_{c1}}{t\beta^4}\right]\frac{\partial^4 w}{\partial Q^4}$$

$$+\left[-\frac{4B_{13}g_1}{\beta}+\frac{2B_{13}g_2}{\beta}+\frac{2\mathrm{B}_{13}g_{c1}}{\mathrm{t}\beta}-\frac{2\mathrm{B}_{13}g_{c1}}{t\beta}\right]\frac{\partial^4 w}{\partial R^3 \partial Q}$$

$$+\left[\frac{8B_{23}g_1}{\beta^3}-\frac{2B_{23}g_2}{\beta^3}-\frac{6B_{23}g_2}{\beta^3}-\frac{6\mathrm{B}_{23}g_{c1}}{\mathrm{t}\beta^3}\right]\frac{\partial^4 \mathrm{w}}{\partial \mathrm{R}\,\partial \mathrm{Q}^3}-\frac{\mathrm{N_x a^2}}{\mathrm{D_0}}\Bigg\}\, dRdQ=0 \qquad (3.250)$$

Further factorization of Equation (3.250) gives:

$$\int_0^1\int_0^1\{.\left[2\mathrm{B}_{11}g_1-2\mathrm{B}_{11}g_2-2\mathrm{B}_{11}\frac{g_{c1}}{\mathrm{t}}\right]\frac{\partial^4 \mathrm{w}}{\partial \mathrm{R}^4}$$

$$+\left[\frac{4g_1}{\beta^2}[\mathrm{B}_{xy}]-\frac{4g_2}{\beta^2}[\mathrm{B}_{xy}]-\frac{3g_{c1}}{t\beta^2}[\mathrm{B}_{xy}]\right]\frac{\partial^4 w}{\partial \mathrm{R}^2\,\partial \mathrm{Q}^2}$$

$$+\left[\frac{2\mathrm{B}_{22}g_1}{\beta^4}-\frac{2\mathrm{B}_{22}g_2}{\beta^4}-\frac{2\mathrm{B}_{23}g_{c1}}{\mathrm{t}\beta^3}-\frac{2\mathrm{B}_{22}g_{c1}}{\mathrm{t}\beta^4}\right]\frac{\partial^4 \mathrm{w}}{\partial \mathrm{Q}^4}+\left[-\frac{4B_{13}g_1}{\beta}+\frac{2B_{13}g_2}{\beta}\right]$$

$$\frac{\partial^4 w}{\partial R^3 \partial Q}+\left[\frac{8B_{23}g_1}{\beta^3}-\frac{8B_{23}g_2}{\beta^3}-\frac{6\mathrm{B}_{23}g_{c1}}{\mathrm{t}\beta^3}\right]\frac{\partial^4 \mathrm{w}}{\partial \mathrm{R}\,\partial \mathrm{Q}^3}-\frac{\mathrm{N_x a^2}}{\mathrm{D_0}}\Bigg\}\, dRdQ=0 \qquad (3.251)$$

Each of the terms in Equation (3.251), was divided by the coefficient of $\frac{\partial^4 \mathrm{w}}{\partial \mathrm{R}^4}$ and that gives:

$$\int_0^1\int_0^1\{.\,\frac{\partial^4 \mathrm{w}}{\partial \mathrm{R}^4}+\frac{\frac{2}{\beta^2}\left[4g_1[\mathrm{B}_{xy}]-4g_2[\mathrm{B}_{xy}]-\frac{4g_{c1}}{t}[\mathrm{B}_{xy}]\right]}{\left[\mathrm{B}_{11}g_1-\mathrm{B}_{11}g_2-\mathrm{B}_{11}\frac{g_{c1}}{\mathrm{t}}\right]}\frac{\partial^4 w}{\partial \mathrm{R}^2\,\partial \mathrm{Q}^2}$$

$$+\frac{\frac{1}{\beta^4}\left[\mathrm{B}_{22}\mathrm{g}_1-\mathrm{B}_{22}\mathrm{g}_2-\frac{\mathrm{B}_{23}\beta \mathrm{g}_{c1}}{\mathrm{t}}-\mathrm{B}_{22}\mathrm{g}_{c1}\right]}{\left[\mathrm{B}_{11}\mathrm{g}_1-\mathrm{B}_{11}\mathrm{g}_2-\mathrm{B}_{11}\frac{\mathrm{g}_{c1}}{\mathrm{t}}\right]}\frac{\partial^4 \mathrm{w}}{\partial \mathrm{Q}^4}+\frac{\frac{\mathrm{B}_{13}}{\beta}[-2\mathrm{g}_1+\mathrm{B}_{13}\mathrm{g}_2]}{\left[\mathrm{B}_{11}\mathrm{g}_1-\mathrm{B}_{11}\mathrm{g}_2-\mathrm{B}_{11}\frac{\mathrm{g}_{c1}}{\mathrm{t}}\right]}\frac{\partial^4 \mathrm{w}}{\partial \mathrm{R}^3\,\partial \mathrm{Q}}$$

$$+\frac{\frac{1}{\beta^3}\left[4\mathrm{B}_{23}\mathrm{g}_1-4\mathrm{B}_{23}\mathrm{g}_2-\frac{3\mathrm{B}_{23}\beta \mathrm{g}_{c1}}{\mathrm{t}}\right]}{\left[\mathrm{B}_{11}\mathrm{g}_1-\mathrm{B}_{11}\mathrm{g}_2-\mathrm{B}_{11}\frac{\mathrm{g}_{c1}}{\mathrm{t}}\right]}\frac{\partial^4 \mathrm{w}}{\partial \mathrm{R}\,\partial \mathrm{Q}^3}-\frac{\mathrm{qa}^4}{\mathrm{D}_0\left[\mathrm{B}_{11}\mathrm{g}_1-\mathrm{B}_{11}\mathrm{g}_2-\mathrm{B}_{11}\frac{\mathrm{g}_{c1}}{\mathrm{t}}\right]}\Bigg\}\,\mathrm{dRd}$$

$$=0 \qquad (3.252)$$

Equation (3.252) can be further reduced to

$$\int_0^1\int_0^1.\left\{\frac{\partial^4 \mathrm{w}}{\partial \mathrm{R}^4}+G_2.\frac{1}{\beta^2}.\frac{\partial^4 w}{\partial \mathrm{R}^2\,\partial \mathrm{Q}^2}+G_3.\frac{2}{\beta^4}.\frac{\partial^4 \mathrm{w}}{\partial \mathrm{Q}^4}+G_4.\frac{1}{\beta}.\frac{\partial^4 w}{\partial R^3 \partial Q}+G_5.\frac{1}{\beta^3}.\frac{\partial^4 \mathrm{w}}{\partial \mathrm{R}\,\partial \mathrm{Q}^3}-G_6\,\frac{\mathrm{N_x a^2}}{\mathrm{D_0}}\right\}$$

$$dRdQ\ =0 \qquad (3.253)$$

where

$$G_1 = \frac{\left[B_{11}g_1 - B_{11}g_2 - B_{11}\frac{g_{c1}}{t}\right]}{\left[B_{11}g_1 - B_{11}g_2 - B_{11}\frac{g_{c1}}{t}\right]} \qquad (3.254)$$

$$G_2 = \frac{\left[4g_1[B_{xy}] - 4g_2[B_{xy}] - \frac{3g_{c1}}{t}[B_{xy}]\right]}{\left[B_{11}g_1 - B_{11}g_2 - B_{11}\frac{g_{c1}}{t}\right]} \qquad (3.355)$$

$$G_3 = \frac{\left[B_{22}g_1 - B_{22}g_2 - \frac{B_{23}\beta g_{c1}}{t} - B_{22}g_{c1}\right]}{\left[B_{11}g_1 - B_{11}g_2 - B_{11}\frac{g_{c1}}{t}\right]} \qquad (3.356)$$

$$G_4 = \frac{[-2g_1 + B_{13}g_2]}{\left[B_{11}g_1 - B_{11}g_2 - B_{11}\frac{g_{c1}}{t}\right]} \qquad (3.357)$$

$$G_5 = \frac{\left[4B_{23}g_1 - 4B_{23}g_2 - \frac{3B_{23}\beta g_{c1}}{t}\right]}{\left[B_{11}g_1 - B_{11}g_2 - B_{11}\frac{g_{c1}}{t}\right]} \qquad (3.358)$$

$$G_6 = \frac{1}{\left[B_{11}g_1 - B_{11}g_2 - B_{11}\frac{g_{c1}}{t}\right]} \qquad (3.359)$$

Equation (3.253) can be expressed as

$$\int_0^1 \int_0^1 \{[A] + [B]\} dR dQ = 0 \qquad (3.260)$$

where $$[A] = \left[\frac{\partial^4 w}{\partial R^4} + G_2.\frac{2}{\beta^2}.\frac{\partial^4 w}{\partial R^2\,\partial Q^2} + G_3.\frac{1}{\beta^4}.\frac{\partial^4 w}{\partial Q^4} - G_6\,\frac{N_x a^2}{D_0}\right] \qquad (3.261)$$

$$[B] = \frac{1}{\beta}.\frac{\partial^2 w}{\partial R\,\partial Q}\left[G_4.\frac{\partial^2 w}{\partial R^2} + \frac{G_5}{\beta^2}.\frac{\partial^2 w}{\partial Q^2}\right] \qquad (3.261)$$

3.2.5 Derivation of The Compatibilty Equaions

The first compatibility equation was obtained by differenting the total potential equation by the middle layer in-plane displacement in the direction of x. That gives

$$\frac{\partial\pi}{\partial u_0} = \frac{\partial U}{\partial u_0} + \frac{\partial V}{\partial u_0} = \frac{\partial U}{\partial u_0} + 0 = 0 \qquad (3.262)$$

Introducing the real values into Equation (3.262) gives

$$\frac{\partial\pi}{\partial u_0} = \frac{abD_0}{2a^4}\int_0^1\int_0^1 \{B_{11}\left(-\frac{a}{t}S.\frac{\partial^3 w}{\partial R^3} + \frac{a^2}{t}H.\frac{\partial^2\phi_x}{\partial R^2} - \frac{aS}{t}.\frac{\partial^3 w}{\partial R^3} + \frac{a^2}{t}H\frac{\partial^2\phi_x}{\partial R^2}8\right)$$
$$+ 2B_{12}\left(-\frac{aS}{t\beta^2}\frac{\partial^3 w}{dR\,\partial Q^2} + \frac{a^2H}{t\beta}.\frac{\partial^2\phi_y}{\partial R\,\partial Q}\right) + B_{13}\left(-\frac{a}{t\beta}S\frac{\partial^3 w}{\partial Q\,\partial R^2} + \frac{a^2H}{t\beta}.\frac{\partial^2\phi_x}{\partial R\,\partial Q}\right.$$
$$\left. - \frac{2aS}{t\beta}\frac{\partial^3 w}{\partial R^2\,\partial Q} + \frac{a^2H}{t\beta}\frac{\partial^2\phi_x}{\partial R\,\partial Q} + \frac{a^2H}{t\beta}.\frac{\partial^2\phi_y}{\partial R^2}\right) + 2B_{23}\left(-\frac{aS}{t\beta^3}.\frac{\partial^3 w}{\partial Q^3} + \frac{a^2H}{t\beta^2}.\frac{\partial^2\phi_y}{\partial Q^2}\right)$$
$$+B_{33}\left(-\frac{2aS}{t\beta^2}.\frac{\partial^3 w}{\partial R\,\partial\,\partial Q^2}\right) + B_{33}\left(\frac{a^2H}{t\beta^2}.\frac{\partial^2\phi_x}{\partial Q^2}\right) + B_{33}\left(+\frac{a^2H}{t\beta}.\frac{\partial^2\phi_y}{\partial Q\,\partial R}\right) + B_{33}$$
$$\left(-\frac{2aS}{t\beta^2}\frac{\partial^3 w}{\partial R\,\partial Q^2} + \frac{a^2H}{t\beta^2}.\frac{\partial^2\phi_x}{\partial Q^2} + \frac{a^2H}{t\beta}\frac{\partial^2\phi_y}{\partial Q\,\partial R}\right) + B_{11}\left(+\frac{a^2}{t^2}.2\frac{\partial^2 u_0}{\partial R^2}\right) + 2B_{12}$$
$$\left(+\frac{a^2}{t^2\beta}.\frac{\partial^2 v_0}{dR\,\partial Q}\right) + 2B_{13}\left(\frac{a^2}{t^2\beta}2\frac{\partial^2 u_0}{\partial R\,\partial Q} + \frac{a^2}{t^2}.\frac{\partial^2 v_0}{\partial R^2}\right) + 2B_{23}\left(\frac{a^2}{t^2\beta^2}.\frac{\partial^2 v_0}{\partial Q^2}\right) + B_{33}$$
$$\left(2\frac{a^2}{t^2\beta^2}.\frac{\partial^2 u_0}{\partial Q^2} + \frac{a^2}{t^2\beta}\frac{\partial^2 v_0}{\partial Q\,\partial R}\right) + B_{33}\left(\frac{a^2}{t^2\beta}\frac{\partial^2 v_0}{\partial Q\,\partial R}\right) + 2B_{44}.\frac{a^2}{t^2}.\frac{\partial^2 u_0}{\partial S^2}.\}dR\,dQ \qquad (3.263)$$

when the brackets are removed, Eqaution (3.263) becomes

$$\frac{\partial\pi}{\partial u_0} = \int_0^1\int_0^1\{.-B_{11}\frac{a}{t}S.\frac{\partial^3 w}{\partial R^3} + B_{11}\frac{a^2}{t}H.\frac{\partial^2\phi_x}{\partial R^2} - B_{11}\frac{aS}{t}.\frac{\partial^3 w}{\partial R^3}$$
$$+B_{11}\frac{a^2}{t}H\frac{\partial^2\phi_x}{\partial R^2} - \frac{2B_{12}aS}{t\beta^2}\frac{\partial^3 w}{dR\,\partial Q^2} + \frac{2B_{12}a^2H}{t\beta}.\frac{\partial^2\phi_y}{\partial R\,\partial Q} - \frac{B_{13}a}{t\beta}S\frac{\partial^3 w}{\partial Q\,\partial R^2}$$
$$+\frac{B_{13}a^2H}{t\beta}.\frac{\partial^2\phi_x}{\partial R\,\partial Q} - \frac{2B_{13}aS}{t\beta}\frac{\partial^3 w}{\partial R^2\,\partial Q} + \frac{B_{13}a^2H}{t\beta}\frac{\partial^2\phi_x}{\partial R\,\partial Q} + \frac{B_{13}a^2H}{t\beta}.\frac{\partial^2\phi_y}{\partial R^2} - \frac{2B_{23}aS}{t\beta^3}.$$
$$\frac{\partial^3 w}{\partial Q^3} + \frac{2B_{23}a^2H}{t\beta^2}.\frac{\partial^2\phi_y}{\partial Q^2} - \frac{2B_{33}aS}{t\beta^2}.\frac{\partial^3 w}{\partial R\,\partial\,\partial Q^2} + \frac{B_{33}a^2H}{t\beta^2}.\frac{\partial^2\phi_x}{\partial Q^2} + \frac{B_{33}a^2H}{t\beta}.\frac{\partial^2\phi_y}{\partial Q\,\partial R}$$
$$-\frac{2B_{33}aS}{t\beta^2}\frac{\partial^3 w}{\partial R\,\partial Q^2} + \frac{B_{33}a^2H}{t\beta^2}.\frac{\partial^2\phi_x}{\partial Q^2} + \frac{B_{33}a^2H}{t\beta}\frac{\partial^2\phi_y}{\partial Q\,\partial R} + \frac{2B_{11}a^2}{t^2}.\frac{\partial^2 u_0}{\partial R^2}$$

$$+\frac{2B_{12}a^2}{t^2\beta}.\frac{\partial^2 v_0}{dR\,\partial Q}+\frac{4B_{13}a^2}{t^2\beta}.\frac{\partial^2 u_0}{\partial R\,\partial Q}+\frac{2B_{13}a^2}{t^2}.\frac{\partial^2 v_0}{\partial R^2}+\frac{2B_{23}a^2}{t^2\beta^2}.\frac{\partial^2 v_0}{\partial Q^2}$$

$$+2\frac{B_{33}a^2}{t^2\beta^2}.\frac{\partial^2 u_0}{\partial Q^2}+\frac{B_{33}a^2}{t^2\beta}\frac{\partial^2 v_0}{\partial Q\,\partial R}+\frac{B_{33}a^2}{t^2\beta}\frac{\partial^2 v_0}{\partial Q\,\partial R}+2B_{44}.\frac{a^2}{t^2}.\frac{\partial^2 u_0}{\partial S^2}.\}dR\,dQ \qquad (3.264)$$

Collecting the like terms together gives

$$\frac{\partial \pi}{\partial u_0}=-B_{11}\frac{aS}{t}.\frac{\partial^3 w}{\partial R^3}+B_{11}\frac{a^2}{t}H.\frac{\partial^2 \phi_x}{\partial R^2}-B_{11}\frac{aS}{t}.\frac{\partial^3 w}{\partial R^3}$$

$$+B_{11}\frac{a^2}{t}H\frac{\partial^2 \phi_x}{\partial R^2}+\frac{2B_{11}a^2}{t^2}.\frac{\partial^2 u_0}{\partial R^2}-\frac{2B_{12}aS}{t\beta^2}\frac{\partial^3 w}{dR\,\partial Q^2}+\frac{2B_{12}a^2H}{t\beta}.\frac{\partial^2 \phi_y}{\partial R\,\partial Q}$$

$$+\frac{2B_{12}a^2}{t^2\beta}.\frac{\partial^2 v_0}{dR\,\partial Q}-\frac{B_{13}aS}{t\beta}.\frac{\partial^3 w}{\partial Q\,\partial R^2}+\frac{B_{13}a^2H}{t\beta}.\frac{\partial^2 \phi_x}{\partial R\,\partial Q}-\frac{2B_{13}aS}{t\beta}\frac{\partial^3 w}{\partial R^2\,\partial Q}$$

$$+\frac{B_{13}a^2H}{t\beta}\frac{\partial^2 \phi_x}{\partial R\,\partial Q}+\frac{B_{13}a^2H}{t\beta}.\frac{\partial^2 \phi_y}{\partial R^2}+\frac{4B_{13}a^2}{t^2\beta}.\frac{\partial^2 u_0}{\partial R\,\partial Q}+\frac{2B_{13}a^2}{t^2}.\frac{\partial^2 v_0}{\partial R^2}-\frac{2B_{23}aS}{t\beta^3}$$

$$.\frac{\partial^3 w}{\partial Q^3}+\frac{2B_{23}a^2H}{t\beta^2}.\frac{\partial^2 \phi_y}{\partial Q^2}+\frac{2B_{23}a^2}{t^2\beta^2}.\frac{\partial^2 v_0}{\partial Q^2}-\frac{2B_{33}aS}{t\beta^2}.\frac{\partial^3 w}{\partial R\,\partial\,\partial Q^2}+\frac{B_{33}a^2H}{t\beta^2}.\frac{\partial^2 \phi_x}{\partial Q^2}$$

$$+\frac{B_{33}a^2H}{t\beta}.\frac{\partial^2 \phi_y}{\partial Q\,\partial R}-\frac{2B_{33}aS}{t\beta^2}\frac{\partial^3 w}{\partial R\,\partial Q^2}+\frac{B_{33}a^2H}{t\beta^2}.\frac{\partial^2 \phi_x}{\partial Q^2}+\frac{B_{33}a^2H}{t\beta}\frac{\partial^2 \phi_y}{\partial Q\,\partial R}$$

$$+2\frac{B_{33}a^2}{t^2\beta^2}.\frac{\partial^2 u_0}{\partial Q^2}+\frac{B_{33}a^2}{t^2\beta}\frac{\partial^2 v_0}{\partial Q\,\partial R}+\frac{B_{33}a^2}{t^2\beta}\frac{\partial^2 v_0}{\partial Q\,\partial R}+2B_{44}.\frac{a^2}{t^2}.\frac{\partial^2 u_0}{\partial S^2}.\}dR\,dQ \qquad (3.265)$$

Further factorization of Equation (3.265) with respect to B gives

$$B_{11}\left[-\frac{2aS}{t}.\frac{\partial^3 w}{\partial R^3}+\frac{2a^2}{t^2}.\frac{\partial^2 u_0}{\partial R^2}+\frac{2a^2H}{t}.\frac{\partial^2 \phi_x}{\partial R^2}\right]$$

$$\frac{B_{12}}{\beta^2}\left[-\frac{2aS}{t}\frac{\partial^3 w}{dR\,\partial Q^2}+\frac{2a^2\beta}{t^2}.\frac{\partial^2 v_0}{dR\,\partial Q}+\frac{2a^2H\beta}{t}.\frac{\partial^2 \phi_y}{\partial R\,\partial Q}\right]$$

$$+\frac{B_{13}}{\beta}\left[-\frac{3aS}{t}.\frac{\partial^3 w}{\partial R^2\,\partial Q}+\frac{2a^2}{t^2}\left[4.\frac{\partial^2 u_0}{\partial R\,\partial Q}+\beta.\frac{\partial^2 v_0}{\partial R^2}\right]+\frac{a^2H}{t}\left[2.\frac{\partial^2 \phi_x}{\partial R\,\partial Q}+\frac{\partial^2 \phi_y}{\partial R^2}\right]\right]$$

$$+\frac{B_{23}}{\beta^3}\left[-\frac{2B_{23}aS}{t}.\frac{\partial^3 w}{\partial Q^3}+\frac{2B_{23}a^2\beta}{t^2}.\frac{\partial^2 v_0}{\partial Q^2}+\frac{2B_{23}a^2H\beta}{t}.\frac{\partial^2 \phi_y}{\partial Q^2}\right]$$

$$+\frac{B_{33}}{\beta^2}\left[-\frac{4aS}{t}.\frac{\partial^3 w}{\partial R\,\partial\,\partial Q^2}+\frac{2a^2}{t^2}.\left[\frac{\partial^2 u_0}{\partial Q^2}+\beta\frac{\partial^2 v_0}{\partial Q\,\partial R}\right]+\frac{2a^2 H}{t}.\left[\frac{\partial^2\phi_x}{\partial Q^2}+\beta.\frac{\partial^2\phi_y}{\partial Q\,\partial R}\right]\right] \qquad (3.266)$$

The second compatibility equation was similarly gotten by differenting the total potential equation by the middle layer in-plane displacement but in the direction of y. That gives

$$\frac{\partial\pi}{\partial v_0}=\frac{\partial U}{\partial v_0}+\frac{\partial V}{\partial v_0}=\frac{\partial U}{\partial v_0}+0=0 \qquad (3.267)$$

Substituting the real values in to Equations (3.569) gives

$$\frac{\partial\pi}{\partial v_0}=\frac{abD_0}{2a^4}\int_0^1\int_0^1\{2B_{12}\left(-\frac{a}{t\beta}S.\frac{\partial^3 w}{\partial Q\,\partial R^2}+\frac{a^2H}{t\beta}\frac{\partial^2 v_0}{\partial R\,\partial Q}\right)+2B_{13}$$

$$\left(-\frac{a}{t}S\frac{\partial^3 w}{\partial R^3}+\frac{a^2H}{t}\frac{\partial^2\phi_x}{\partial R^2}\right)+B_{22}\left(-\frac{aS}{t\beta^3}.\frac{\partial^3 w}{\partial Q^3}+\frac{a^2H}{t\beta^2}\frac{\partial^2 v_0}{\partial Q^2}-\frac{aS}{t\beta^3}\frac{\partial^2 w}{\partial Q^3}+\frac{a^2H}{t\beta^2}.\frac{\partial^2\phi_y}{\partial Q^2}\right)$$

$$+2B_{23}\left(-\frac{aS}{t\beta^2}.\frac{\partial^3 w}{\partial R\partial Q^2}+\frac{a^2H}{t\beta}.\frac{\partial^2\phi_y}{\partial R\partial Q}-\frac{2aS}{t\beta^2}\frac{\partial^3 w}{\partial R\,\partial Q^2}+\frac{a^2H}{t\beta^2}.\frac{\partial^2\phi_x}{\partial Q^2}+\frac{a^2H}{t\beta}.\frac{\partial^2\phi_y}{\partial Q\partial R}\right)$$

$$+B_{33}\left(-\frac{2aS}{t\beta}.\frac{\partial^3 w}{\partial R^2\,\partial Q}\right)+B_{33}\left(\frac{a^2H}{t\beta}.\frac{\partial^2\phi_x}{\partial R\,\partial Q}\right)+B_{33}\left(+\frac{a^2H}{t}.\frac{\partial^2\phi_y}{\partial R^2}\right)$$

$$+B_{33}\left(-\frac{2aS}{t\beta}\frac{\partial^3 w}{\partial R^2\,\partial Q}+\frac{a^2H}{t\beta}.\frac{\partial^2\phi_x}{\partial R\,\partial Q}+\frac{a^2H}{t}\frac{\partial^2\phi_y}{\partial R^2}\right)+2B_{12}\left(+\frac{a^2}{t^2\beta}\frac{\partial^2 v_0}{dR\,\partial Q}\right)$$

$$+2B_{13}\left(\frac{a^2}{t^2}.\frac{\partial^2 u_0}{\partial R^2}.\right)+2B_{22}\left(\frac{a^2}{t^2\beta^2}.2.\frac{\partial^2 v_0}{\partial Q^2}\right)+2B_{23}\left(\frac{a^2}{t^2\beta^2}\frac{\partial^2 u_0}{\partial Q^2}+\frac{a^2}{t^2\beta}\frac{\partial^2 v_0}{\partial Q\,\partial R}\right)$$

$$+2B_{13}\left(\frac{a^2}{t^2}.\frac{\partial^2 u_0}{\partial R^2}.\right)+2B_{22}\left(\frac{a^2}{t^2\beta^2}.2.\frac{\partial^2 v_0}{\partial Q^2}\right)+2B_{23}\left(\frac{a^2}{t^2\beta^2}\frac{\partial^2 u_0}{\partial Q^2}+\frac{a^2}{t^2\beta}\frac{\partial^2 v_0}{\partial Q\,\partial R}\right)$$

$$+B_{33}\left(\frac{a^2}{t^2\beta}\frac{\partial^2 u_0}{\partial Q\,\partial R}\right)+B_{33}\left(\frac{a^2}{t^2\beta}\frac{\partial^2 u_0}{\partial Q\,\partial R}+\frac{a^2}{t^2}2.\frac{\partial^2 v_0}{\partial R^2}\right)+B_{55}.\frac{a^2}{t^2}.2\frac{\partial^2 v_0}{\partial S^2}.\}dR\,dQ \qquad (3.268)$$

Opening the brackets gives

$$\frac{\partial\pi}{\partial v_0}=-\frac{2B_{12}aS}{t\beta}.\frac{\partial^3 w}{\partial Q\,\partial R^2}+\frac{2B_{12}a^2H}{t\beta}\frac{\partial^2\phi_x}{\partial R\,\partial Q}-\frac{2B_{13}aS}{t}.\frac{\partial^3 w}{\partial R^3}$$

$$+\frac{2B_{13}a^2H}{t}\frac{\partial^2\phi_x}{\partial R^2}-\frac{B_{22}aS}{t\beta^3}.\frac{\partial^3 w}{\partial Q^3}+\frac{B_{22}a^2H}{t\beta^2}\frac{\partial^2\phi_y}{\partial Q^2}-\frac{B_{22}aS}{t\beta^3}\frac{\partial^2 w}{\partial Q^3}$$

$$+\frac{B_{22}a^2H}{t\beta^2}.\frac{\partial^2\phi_y}{\partial Q^2}-\frac{2B_{23}aS}{t\beta^2}.\frac{\partial^3 w}{\partial R\,\partial Q^2}+\frac{2B_{23}a^2H}{t\beta}.\frac{\partial^2\phi_y}{\partial R\,\partial Q}-\frac{4B_{23}aS}{t\beta^2}$$

$$\frac{\partial^3 w}{\partial R\,\partial Q^2}+\frac{2B_{23}a^2H}{t\beta^2}.\frac{\partial^2\phi_x}{\partial Q^2}+\frac{2B_{23}a^2H}{t\beta}.\frac{\partial^2\phi_y}{\partial Q\,\partial R}-\frac{2B_{33}aS}{t\beta}.\frac{\partial^3 w}{\partial R^2\,\partial Q}$$

$$+\frac{B_{33}a^2H}{t\beta}.\frac{\partial^2\phi_x}{\partial R\,\partial Q}+\frac{B_{33}a^2H}{t}.\frac{\partial^2\phi_y}{\partial R^2}-\frac{2B_{33}aS}{t\beta}\frac{\partial^3 w}{\partial R^2\,\partial Q}+\frac{B_{33}a^2H}{t\beta}.\frac{\partial^2\phi_x}{\partial R\,\partial Q}$$

$$+\frac{B_{33}a^2H}{t}\frac{\partial^2\phi_y}{\partial R^2}+\frac{2B_{12}a^2}{t^2\beta}\frac{\partial^2 v_0}{dR\,\partial Q}+\frac{2B_{13}a^2}{t^2}.\frac{\partial^2 u_0}{\partial R^2}+\frac{4B_{22}a^2}{t^2\beta^2}.\frac{\partial^2 v_0}{\partial Q^2}$$

$$+\frac{2B_{23}a^2}{t^2\beta^2}\frac{\partial^2 u_0}{\partial Q^2}+\frac{2B_{23}a^2}{t^2\beta}\frac{\partial^2 v_0}{\partial Q\,\partial R}+\frac{2B_{13}a^2}{t^2}.\frac{\partial^2 u_0}{\partial R^2}+\frac{2B_{23}a^2}{t^2\beta^2}\frac{\partial^2 u_0}{\partial Q^2}$$

$$+\frac{2B_{23}a^2}{t^2\beta}\frac{\partial^2 v_0}{\partial Q\,\partial R}+\frac{B_{33}a^2}{t^2\beta}\frac{\partial^2 u_0}{\partial Q\,\partial R}+\frac{B_{33}a^2}{t^2\beta}\frac{\partial^2 u_0}{\partial Q\,\partial R}+\frac{2B_{33}a^2}{t^2}.\frac{\partial^2 v_0}{\partial R^2}$$

$$+B_{55}.\frac{a^2}{t^2}.2\frac{\partial^2 v_0}{\partial S^2}.\}dR\,dQ \qquad (3.269)$$

Factorizing Equation (3.269) the like terms with respect to B gives

$$\frac{\partial\pi}{\partial v_0}=\frac{B_{12}}{\beta}\left[-\frac{2aS}{t}.\frac{\partial^3 w}{\partial Q\,\partial R^2}+\frac{2a^2}{t}\left[\frac{H\,\partial^2\phi_x}{\partial R\,\partial Q}+\frac{\partial^2 v_0}{tdR\,\partial Q}\right]\right]$$

$$+\frac{B_{13}}{\beta}\left[-\frac{2aS\beta}{t}.\frac{\partial^3 w}{\partial R^3}+\frac{2a^2\beta}{t}.\left[\frac{\partial^2 u_0}{t\,\partial R^2}+H\frac{\partial^2\phi_x}{\partial R^2}\right]\right]$$

$$+\frac{B_{22}}{\beta^3}\left[-\frac{2aS}{t}.\frac{\partial^3 w}{\partial Q^3}+\frac{4a^2\beta}{t^2}.\frac{\partial^2 v_0}{\partial Q^2}+\frac{2a^2\beta H}{t}.\frac{\partial^2\phi_y}{\partial Q^2}\right]$$

$$+\frac{B_{23}}{\beta^2}\left[-\frac{6aS}{t}.\frac{\partial^3 w}{\partial R\,\partial Q^2}+\frac{4a^2}{t^2}\left[\frac{\partial^2 u_0}{\partial Q^2}+\beta\frac{\partial^2 v_0}{\partial Q\,\partial R}\right]+\frac{2a^2H}{t}.\left[\frac{2\beta\,\partial^2\phi_y}{\partial R\,\partial Q}+\frac{\partial^2\phi_x}{\partial Q^2}\right]\right]$$

$$+\frac{B_{33}}{\beta}\left[-\frac{4aS}{t}\frac{\partial^3 w}{\partial R^2\,\partial Q}+\frac{2a^2}{t^2}\frac{\partial^2 u_0}{\partial Q\,\partial R}+\frac{2a^2\beta}{t^2}.\frac{\partial^2 v_0}{\partial R^2}+\frac{2a^2H\beta}{t}.\frac{\partial^2\phi_y}{\partial R^2}+\frac{2a^2H}{t}.\frac{\partial^2\phi_x}{\partial R\,\partial Q}\right]$$

$$+B_{55}.\frac{a^2}{t^2}.2\frac{\partial^2 v_0}{\partial S^2}.\}dR\,dQ \qquad (3.270)$$

Further factorization of Equation (3.270) with respect to B gives

$$\frac{\partial\pi}{\partial v_0} = \frac{B_{12}}{\beta}\left[-\frac{2aS}{t}.\frac{\partial^3 w}{\partial Q\,\partial R^2} + \frac{2a^2}{t^2}\frac{\partial^2 v_0}{dR\,\partial Q} + \frac{2a^2H}{t}\frac{\partial^2\phi_x}{\partial R\,\partial Q}\right]$$

$$+\frac{B_{13}}{\beta}\left[-\frac{2aS\beta}{t}.\frac{\partial^3 w}{\partial R^3} + \frac{2a^2\beta}{t^2}.\frac{\partial^2 u_0}{\partial R^2} + \frac{2a^2H\beta}{t}\frac{\partial^2\phi_x}{\partial R^2}\right]$$

$$+\frac{B_{22}}{\beta^3}\left[-\frac{2aS}{t}.\frac{\partial^3 w}{\partial Q^3} + \frac{8a^2\beta}{t^2}.\frac{\partial^2 v_0}{\partial Q^2} + \frac{2a^2H\beta}{t}\frac{\partial^2\phi_y}{\partial Q^2}\right]$$

$$+\frac{B_{23}}{\beta^2}\left[-\frac{6aS}{t}.\frac{\partial^3 w}{\partial R\,\partial Q^2} + \frac{2a^2}{t^2}\left[\frac{\partial^2 u_0}{\partial Q^2} + \beta.\frac{\partial^2 v_0}{\partial Q\,\partial R}\right] + \frac{2a^2H}{t}\left[2\beta\frac{\partial^2\phi_y}{\partial R\,\partial Q} +.\frac{\partial^2\phi_x}{\partial Q^2}\right]\right]$$

$$+\frac{B_{33}}{\beta}\left[-\frac{4aS}{t}.\frac{\partial^3 w}{\partial R^2\,\partial Q} + \frac{2a^2}{t^2}\left[\frac{\partial^2 u_0}{\partial Q\,\partial R} + \beta.\frac{\partial^2 v_0}{\partial R^2}\right] + \frac{2a^2H}{t}.\left[\frac{\partial^2\phi_x}{\partial R\,\partial Q} + \beta.\frac{\partial^2\phi_y}{\partial R^2}\right]\right]$$

$$+B_{55}.\frac{a^2}{t^2}.2\frac{\partial^2 v_0}{\partial S^2}.\}dR\,dQ \qquad (3.272)$$

Considering the differential values for the case of the shear rotation on y-z plane, i. e ϕ_y gives

$$\frac{\partial\pi}{\partial\phi_y} = \frac{\partial U}{\partial\phi_y} + \frac{\partial V}{\partial\phi_y} = \frac{\partial U}{\partial\phi_y} + 0 = 0 \qquad (3.273)$$

Also introducing the real values into Equation(3.273) gives

$$\frac{\partial\pi}{\partial\phi_y} = \frac{abD_0}{2a^4}.\int_0^1\int_0^1\{2B_{12}\left(-\frac{aHS}{\beta}.\frac{\partial^3 w}{\partial Q\,\partial R^2} + \frac{a^2H^2}{\beta}.\frac{\partial^2\phi_x}{\partial R\,\partial Q}\right) + 2B_{13}$$

$$\left(-aHS\frac{\partial^3 w}{\partial R^3} + a^2H^2.\frac{\partial^2\phi_x}{\partial R^2}\right) + B_{22}\left(-\frac{aHS}{\beta^3}.\frac{\partial^3\phi_y}{\partial Q^3} - \frac{aHS}{\beta^3}\frac{\partial^3\phi_y}{\partial Q^3} + \frac{2a^2H^2}{\beta^2}.\frac{\partial^2\phi_y}{\partial Q^2}\right)$$

$$+2B_{23}\left(-\frac{aSH}{\beta^2}.\frac{\partial^3 w}{\partial R\,\partial Q^2} - \frac{2aHS}{\beta^2}\frac{\partial^3 w}{\partial R\,\partial Q^2} + \frac{a^2H^2}{\beta^2}.\frac{\partial^2\phi_x}{\partial Q^2} + \frac{a^2H^2}{\beta}.\frac{\partial^2\phi_y}{\partial Q\,\partial R}\right) + B_{33}$$

$$\left(+\frac{a^2H^2}{\beta}.\frac{\partial\phi_x}{\partial Q\,\partial R}\right) + B_{33}\left(-\frac{2aHS}{\beta}\frac{\partial^3 w}{\partial R^2\,\partial Q} + \frac{a^2H^2}{\beta}.\frac{\partial^2\phi_x}{\partial R\,\partial Q} + 2a^2H^2\frac{\partial^2\phi_y}{\partial R^2}\right)$$

$$+2B_{55}.\frac{a^4}{t^2}.g_4.\emptyset_y\}dR\,dQ \qquad (3.274)$$

That means:

$$\frac{\partial\pi}{\partial\phi_y} = +2B_{12}\left(\frac{a^2H}{t\beta}.\frac{\partial^2u_0}{\partial R\,\partial Q}\right) + 2B_{13}\left(+\frac{a^2H}{t\beta}.\frac{\partial^2u_0}{\partial R^2}\right) + B_{22}\left(+\frac{a^2H}{t\beta^2}\frac{\partial^2v_0}{\partial Q^2} + \frac{a^2H}{t\beta^2}.\frac{\partial^2v_0}{\partial Q^2}\right)$$

$$+2B_{23}\left(\frac{a^2H}{t\beta^2}\frac{\partial^2u_0}{\partial Q^2} + \frac{a^2H}{t\beta}\frac{\partial^2v_0}{\partial Q\partial R} + \frac{a^2H}{t\beta}.\frac{\partial^2v_0}{\partial Q\partial R}\right) + B_{33}\left(+\frac{a^2H}{t\beta}\frac{\partial^2u_0}{\partial Q\partial R} + \frac{a^2H}{t}\frac{\partial^2v_0}{\partial R^2}\right)$$

$$+ B_{33}\left(\frac{a^2H}{t\beta}.\frac{\partial^2u_0}{\partial R^2}\right) + B_{33}\left(\frac{a^2H}{t}.\frac{\partial^2v_0}{\partial R^2}\right) + B_{55}.\frac{a^4}{t^3}.\left[2g_{C3}.\frac{dv_0}{dS}\right].\}dR\,dQ \qquad (3.275)$$

Opening the bracket gives

$$\frac{\partial\pi}{\partial\phi_y} = \int_0^1\int_0^1\{-\frac{2B_{12}aHS}{\beta}.\frac{\partial^3w}{\partial Q\,\partial R^2} + \frac{2B_{12}a^2H^2}{\beta}.\frac{\partial^2\phi_x}{\partial R\,\partial Q} - 2B_{13}aHS\frac{\partial^3w}{\partial R^3} + 2B_{13}a^2H^2.$$

$$\frac{\partial^2\phi_x}{\partial R^2} - \frac{B_{22}aHS}{\beta^3}.\frac{\partial^3w}{\partial Q^3} - \frac{B_{22}aHS}{\beta^3}\frac{\partial^3w}{\partial Q^3} + \frac{2\,B_{22}a^2H^2}{\beta^2}.\frac{\partial^2\phi_y}{\partial Q^2} - \frac{2B_{23}aSH}{\beta^2}.\frac{\partial^3w}{\partial R\partial Q^2} -$$

$$\frac{4B_{23}aHS}{\beta^2}\frac{\partial^3w}{\partial R\partial Q^2} + \frac{2B_{23}a^2H^2}{\beta^2}.\frac{\partial^2\phi_x}{\partial Q^2} + \frac{2B_{23}H^2}{\beta}.\frac{\partial^2\phi_y}{\partial Q\partial R} - \frac{2B_{33}aHS}{\beta}.\frac{\partial^3w}{\partial R^2\partial Q} + \frac{B_{33}a^2H^2}{\beta}$$

$$.\frac{\partial\phi_x}{\partial Q\partial R} - \frac{2B_{33}aHS}{\beta}\frac{\partial^3w}{\partial R^2\partial Q} + \frac{B_{33}a^2H^2}{\beta}.\frac{\partial^2\phi_x}{\partial R\partial Q} + 2B_{33}a^2H^2\frac{\partial^2\phi_y}{\partial R^2} + 2B_{55}.\frac{a^4}{t^2}.g_4.$$

$$\emptyset_y\}dR\,dQ + \frac{2B_{12}a^2H}{t\beta}.\frac{\partial^2u_0}{\partial R\partial Q} + \frac{2B_{13}a^2H}{t\beta}.\frac{\partial^2u_0}{\partial R^2} + \frac{B_{22}a^2H}{t\beta^2}\frac{\partial^2v_0}{\partial Q^2} + \frac{B_{22}a^2H}{t\beta^2}.\frac{\partial^2v_0}{\partial Q^2}$$

$$+\frac{2B_{23}a^2H}{t\beta^2}\frac{\partial^2u_0}{\partial Q^2} + \frac{2B_{23}a^2H}{t\beta}\frac{\partial^2v_0}{\partial Q\partial R} + \frac{2B_{23}a^2H}{t\beta}.\frac{\partial^2v_0}{\partial Q\partial R} + \frac{B_{33}a^2H}{t\beta}\frac{\partial^2u_0}{\partial Q\partial R}$$

$$+\frac{B_{33}a^2H}{t}\frac{\partial^2v_0}{\partial R^2} + \frac{B_{33}a^2H}{t\beta}.\frac{\partial^2u_0}{\partial R\partial Q} + \frac{B_{33}a^2H}{t}.\frac{\partial^2v_0}{\partial R^2} + B_{55}.\frac{a^4}{t^3}\left[2g_{C3}.\frac{dv_0}{dS}\right].\}dR\,dQ \qquad (3.278)$$

.

Collecting the like terms together with respect to B gives

$$\frac{\partial\pi}{\partial\phi_y} = \frac{B_{12}}{\beta}\left[-2aHS.\frac{\partial^3w}{\partial Q\,\partial R^2} + \frac{2a^2H}{t}.\frac{\partial^2u_0}{\partial R\,\partial Q} + 2a^2H^2.\frac{\partial^2\phi_x}{\partial R\,\partial Q}\right]$$

$$+\frac{B_{13}}{\beta}\left[-2aHS\beta\frac{\partial^3w}{\partial R^3} + \frac{2a^2H}{t}.\frac{\partial^2u_0}{\partial R^2} + 2a^2H^2\beta.\frac{\partial^2\phi_x}{\partial R^2}\right]$$

$$+\frac{B_{22}}{\beta^3}\left[-2aHS.\frac{\partial^3w}{\partial Q^3} + 2a^2H^2\beta.\frac{\partial^2\phi_y}{\partial Q^2} + \frac{2a^2H\beta}{t}\frac{\partial^2v_0}{\partial Q^2}\right] + \frac{B_{23}}{\beta^2}$$

$$\left[-6aHS.\frac{\partial^3 w}{\partial R\partial Q^2}+\frac{2a^2H}{t}.\frac{\partial^2 u_0}{\partial Q^2}+\frac{4a^2H\beta}{t}\frac{\partial^2 \mathrm{v}_0}{\partial Q\partial R}+2a^2H^2.\frac{\partial^2\phi_x}{\partial Q^2}+2H^2\beta.\frac{\partial^2\phi_y}{\partial Q\partial R}\right]+\frac{B_{33}}{\beta}$$

$$\left[-4aHS.\frac{\partial^3 w}{\partial R^2\partial Q}+\frac{2a^2H}{t}.\frac{\partial^2 \mathrm{u}_0}{\partial R\partial Q}+\frac{2a^2H\beta}{t}.\frac{\partial^2 \mathrm{v}_0}{\partial R^2}+2a^2H^2.\frac{\partial\phi_x}{\partial Q\partial R}+2a^2H^2\beta\frac{\partial^2\phi_y}{\partial R^2}\right]$$

$$+2\mathrm{B}_{55}.\frac{a^4}{\mathrm{t}^2}.g_4.\emptyset_y+\mathrm{B}_{55}.\frac{a^4}{\mathrm{t}^3}.\left[2g_{C3}.\frac{dv_0}{dS}\right].\}\mathrm{dR\,dQ} \qquad (3.279)$$

And finally for the case of the shear rotation on x-z plane, the fourth compatibility equation was formulated as shown below

$$\frac{\partial\pi}{\partial\phi_x}=\frac{\partial \mathrm{U}}{\partial\phi_x}+\frac{\partial \mathrm{V}}{\partial\phi_x}=\frac{\partial \mathrm{U}}{\partial\phi_x}+0=0 \qquad (3.280)$$

Putting the actual parameters into Equation (3.280) gives

$$\frac{\partial\pi}{\partial\phi_x}=\frac{\mathrm{abD}_0}{2\mathrm{a}^4}\int_0^1\int_0^1\Big\{\mathrm{B}_{11}\left(-\mathrm{aHS}.\frac{\partial^3\mathrm{w}}{\partial\mathrm{R}^3}-\mathrm{aHS}.\frac{\partial^3\mathrm{w}}{\partial\mathrm{R}^3}+2\mathrm{a}^2\mathrm{S}^2.\frac{\partial^2\phi_x}{\partial\mathrm{R}^2}\right)$$

$$+2\mathrm{B}_{12}\left(-\frac{\mathrm{aHS}}{\beta^2}\frac{\partial^3\mathrm{w}}{\partial\mathrm{Q}^2}.\frac{\partial\phi_x}{\partial\mathrm{R}}+\frac{\mathrm{a}^2\mathrm{H}^2}{\beta}.\frac{\partial^2\phi_y}{\partial\mathrm{R}\,\partial\mathrm{Q}}\right)+2\mathrm{B}_{13}$$

$$\left(-\frac{\mathrm{aHS}}{\beta}.\frac{\partial^3\mathrm{w}}{\partial\mathrm{Q}\,\partial\mathrm{R}^2}-\frac{2\mathrm{aHS}}{\beta}.\frac{\partial^3\mathrm{w}}{\partial\mathrm{R}^2\,\partial\mathrm{Q}}+\mathrm{a}^2\mathrm{H}^2.\frac{\partial^2\phi_y}{\partial\mathrm{R}^2}\right)+2\mathrm{B}_{23}\left(-\frac{\mathrm{aSH}}{\beta^3}.\frac{\partial^3\mathrm{w}}{\partial\mathrm{Q}^3}+\frac{\mathrm{a}^2\mathrm{H}^2}{\beta^2}.\frac{\partial^2\phi_y}{\partial\mathrm{Q}^2}\right)$$

$$+\mathrm{B}_{33}\left(-\frac{2\mathrm{aHS}}{\beta^2}.\frac{\partial^3\mathrm{w}}{\partial\mathrm{R}\,\partial\mathrm{Q}^2}\right)+\mathrm{B}_{33}\left(-\frac{2\mathrm{aHS}}{\beta^2}\frac{\partial^3\mathrm{w}}{\partial\mathrm{R}\,\partial\mathrm{Q}^2}+\frac{2\mathrm{a}^2\mathrm{H}^2}{\beta^2}.\frac{\partial^2\phi_x}{\partial\mathrm{Q}^2}+\frac{\mathrm{a}^2\mathrm{H}^2}{\beta}\frac{\partial^2\phi_y}{\partial\mathrm{Q}\,\partial\mathrm{R}}\right)$$

$$+\mathrm{B}_{33}\left(\frac{\mathrm{a}^2\mathrm{H}^2}{\beta}.\frac{\partial^2\phi_y}{\partial\mathrm{R}\,\partial\mathrm{Q}}\right)+2\mathrm{B}_{44}.\frac{\mathrm{a}^4}{\mathrm{t}^2}.\mathrm{g}_4.\emptyset_\mathrm{x}\}\mathrm{dR\,dQ}+\mathrm{B}_{11}\left(\frac{\mathrm{a}^2}{\mathrm{t}}\mathrm{H}.\frac{\partial^2\mathrm{u}_0}{\partial\mathrm{R}^2}+\frac{\mathrm{a}^2}{\mathrm{t}}\mathrm{H}.\frac{\partial^2\mathrm{u}_0}{\partial\mathrm{R}^2}\right)$$

$$+2\mathrm{B}_{12}\left(\frac{\mathrm{a}^2\mathrm{H}}{\mathrm{t}\beta}\frac{\partial^2\mathrm{v}_0}{\partial\mathrm{R}^2}\right)+2\mathrm{B}_{13}\left(+\frac{\mathrm{a}^2\mathrm{H}}{\mathrm{t}\beta}\frac{\partial^2\mathrm{u}_0}{\partial\mathrm{R}\,\partial\mathrm{Q}}+\frac{\mathrm{a}^2\mathrm{H}}{\mathrm{t}}\frac{\partial^2\mathrm{v}_0}{\partial\mathrm{R}^2}+\frac{\mathrm{a}^2\mathrm{H}}{\mathrm{t}\beta}.\frac{\partial^2\mathrm{u}_0}{\partial\mathrm{R}\,\partial\mathrm{Q}}\right)$$

$$+2\mathrm{B}_{23}\left(\frac{\mathrm{a}^2\mathrm{H}}{\mathrm{t}\beta^2}.\frac{\partial^2\mathrm{v}_0}{\partial\mathrm{Q}^2}\right)+\mathrm{B}_{33}\left(\frac{\mathrm{a}^2\mathrm{H}}{\mathrm{t}\beta^2}\frac{\partial^2\mathrm{u}_0}{\partial\mathrm{Q}^2}+\frac{\mathrm{a}^2\mathrm{H}}{\mathrm{t}\beta}.\frac{\partial^2\mathrm{v}_0}{\partial\mathrm{R}\,\partial\mathrm{Q}}\right)+\mathrm{B}_{33}\left(\frac{\mathrm{a}^2\mathrm{H}}{\mathrm{t}\beta^2}.\frac{\partial^2\mathrm{u}_0}{\partial\mathrm{Q}^2}\right)$$

$$+\mathrm{B}_{33}\left(\frac{\mathrm{a}^2\mathrm{H}}{\mathrm{t}\beta}.\frac{\partial^2\mathrm{v}_0}{\partial\mathrm{Q}^2}\right)+\mathrm{B}_{44}.\frac{\mathrm{a}^4}{\mathrm{t}^3}\left[2\mathrm{g}_{C3}.\frac{\mathrm{du}_0}{\mathrm{dS}}\right]\Big\}\mathrm{dR\,dQ} \qquad (3.281)$$

Opening the bracket gives

$$\frac{\partial\pi}{\partial\phi_x} = -B_{11}aHS.\frac{\partial^3 w}{\partial R^3} - B_{11}aHS.\frac{\partial^3 w}{\partial R^3} + 2B_{11}a^2S^2.\frac{\partial^2\phi_x}{\partial R^2} - \frac{2B_{12}aHS}{\beta^2}\frac{\partial^3 w}{\partial R\partial Q^2}$$

$$+\frac{2B_{12}a^2H^2}{\beta}.\frac{\partial^2\phi_y}{\partial R\partial Q} \quad -\frac{2B_{13}aHS}{\beta}.\frac{\partial^3 w}{\partial Q\,\partial R^2} - \frac{4aHS}{\beta}.\frac{\partial^3 w}{\partial R^2\,\partial Q}$$

$$+2B_{13}a^2H^2.\frac{\partial^2\phi_y}{\partial R^2} - \frac{2B_{23}aSH}{\beta^3}.\frac{\partial^3 w}{\partial Q^3} + \frac{2B_{23}a^2H^2}{\beta^2}.\frac{\partial^2\phi_y}{\partial Q^2} - \frac{2B_{33}aHS}{\beta^2}$$

$$.\frac{\partial^3 w}{\partial R\,\partial Q^2} - \frac{2B_{33}aHS}{\beta^2}\frac{\partial^3 w}{\partial R\,\partial Q^2} + \frac{2B_{33}a^2H^2}{\beta^2}.\frac{\partial^2\phi_x}{\partial Q^2} + \frac{B_{33}a^2H^2}{\beta}\frac{\partial^2\phi_y}{\partial Q\,\partial R}$$

$$+\frac{B_{33}a^2H^2}{\beta}.\frac{\partial^2\phi_y}{\partial R\,\partial Q} + \frac{B_{11}a^2H}{t}.\frac{\partial^2 u_0}{\partial R^2} + \frac{B_{11}a^2H}{t}.\frac{\partial^2 u_0}{\partial R^2} + \frac{2B_{12}a^2H}{t\beta}\frac{\partial^2 v_0}{\partial R\,\partial Q}$$

$$+\frac{2B_{13}a^2H}{t\beta}\frac{\partial^2 u_0}{\partial R\,\partial Q} + \frac{2B_{13}a^2H}{t}\frac{\partial^2 v_0}{\partial R^2} + \frac{2B_{13}a^2H}{t\beta}.\frac{\partial^2 u_0}{\partial R\,\partial Q} + \frac{2B_{23}a^2H}{t\beta^2}.\frac{\partial^2 v_0}{\partial Q^2}$$

$$+\frac{B_{33}a^2H}{t\beta^2}\frac{\partial^2 u_0}{\partial Q^2} + \frac{B_{33}a^2H}{t\beta}.\frac{\partial^2 v_0}{\partial R\,\partial Q} + \frac{B_{33}a^2H}{t\beta^2}.\frac{\partial^2 u_0}{\partial Q^2} + \frac{B_{33}a^2H}{t\beta}.\frac{\partial^2 v_0}{\partial R\,\partial Q}$$

$$+ B_{44}.\frac{a^4}{t^3}\left[2g_{C3}\,.\frac{du_0}{dS}\right] + 2B_{44}.\frac{a^4}{t^2}.g_4.\emptyset_x\}dR\,dQ \qquad (3.282)$$

Upon factorization of the like terms together with respect to B, Equation (3.282) gives

$$\frac{\partial\pi}{\partial\phi_x} = B_{11}\left[-2aHS.\frac{\partial^3 w}{\partial R^3} + \frac{2a^2H}{t}.\frac{\partial^2 u_0}{\partial R^2} + 2a^2S^2.\frac{\partial^2\phi_x}{\partial R^2}\right]$$

$$+\frac{B_{12}}{\beta^2}\left[-2aHS\frac{\partial^3 w}{\partial R\,\partial Q^2} + \frac{2a^2H\beta}{t}\frac{\partial^2 v_0}{\partial R\,\partial Q} + 2a^2H^2\beta.\frac{\partial^2\phi_y}{\partial R\,\partial Q}\right]$$

$$+\frac{B_{13}}{\beta}\left[-5aHS.\frac{\partial^3 w}{\partial Q\,\partial R^2} + \frac{4a^2H}{t}.\frac{\partial^2 u_0}{\partial R\,\partial Q} + \frac{2a^2H\beta}{t}\frac{\partial^2 v_0}{\partial R^2} + 2a^2H^2\beta.\frac{\partial^2\phi_y}{\partial R^2}\right]$$

$$+\frac{B_{23}}{\beta^3}\left[-2aSH.\frac{\partial^3 w}{\partial Q^3} + \frac{2a^2H\beta}{t}.\frac{\partial^2 v_0}{\partial Q^2} + 2a^2H^2\beta.\frac{\partial^2\phi_y}{\partial Q^2}\right] + \frac{B_{33}}{\beta^2}$$

$$\left[-4aHS.\frac{\partial^3 w}{\partial R\,\partial Q^2} + \frac{2a^2H}{t}.\left[\frac{\partial^2 u_0}{\partial Q^2} + \beta.\frac{\partial^2 v_0}{\partial R\,\partial Q}\right] + 2a^2H^2\left[.\frac{\partial^2\phi_x}{\partial Q^2} + \beta.\frac{\partial^2\phi_y}{\partial R\,\partial Q}\right]\right]$$

$$+B_{44}.\frac{a^4}{t^3}\left[2g_{C3}\,.\frac{du_0}{dS}\right] + 2B_{44}.\frac{a^4}{t^2}.g_4.\emptyset_x\}dR\,dQ \qquad (3.283)$$

Bringing the the four compatibility equations together gives

$$\frac{\partial \pi}{\partial u_0}+\frac{\partial \pi}{\partial v_0}+\frac{\partial \pi}{\partial \phi_y}+\frac{\partial \pi}{\partial \phi_x}$$

That is

$$B_{11}\left[-\frac{2aS}{t}.\frac{\partial^3 w}{\partial R^3}+\frac{2a^2}{t^2}.\frac{\partial^2 u_0}{\partial R^2}+\frac{2a^2H}{t}.\frac{\partial^2 \phi_x}{\partial R^2}\right]$$

$$\frac{B_{12}}{\beta^2}\left[-\frac{2aS}{t}\frac{\partial^3 w}{dR\,\partial Q^2}+\frac{2a^2\beta}{t^2}.\frac{\partial^2 v_0}{dR\,\partial Q}+\frac{2a^2H\beta}{t}.\frac{\partial^2 \phi_y}{\partial R\,\partial Q}\right]$$

$$+\frac{B_{13}}{\beta}\left[-\frac{3aS}{t}.\frac{\partial^3 w}{\partial R^2\,\partial Q}+\frac{2\alpha^2}{t^2}\left[4.\frac{\partial^2 u_0}{\partial R\,\partial Q}+\beta.\frac{\partial^2 v_0}{\partial R^2}\right]+\frac{a^2H}{t}\left[2.\frac{\partial^2 \phi_x}{\partial R\,\partial Q}+\frac{\partial^2 \phi_y}{\partial R^2}\right]\right]$$

$$+\frac{B_{23}}{\beta^3}\left[-\frac{2B_{23}aS}{t}.\frac{\partial^3 w}{\partial Q^3}+\frac{2B_{23}a^2\beta}{t^2}.\frac{\partial^2 v_0}{\partial Q^2}+\frac{2B_{23}a^2H\beta}{t}.\frac{\partial^2 \phi_y}{\partial Q^2}\right]$$

$$+\frac{B_{33}}{\beta^2}\left[-\frac{4aS}{t}.\frac{\partial^3 w}{\partial R\,\partial\,\partial Q^2}+\frac{2a^2}{t^2}.\left[\frac{\partial^2 u_0}{\partial Q^2}+\beta\frac{\partial^2 v_0}{\partial Q\,\partial R}\right]+\frac{2a^2H}{t}.\left[\frac{\partial^2 \phi_x}{\partial Q^2}+\beta.\frac{\partial^2 \phi_y}{\partial Q\,\partial R}\right]\right]\quad +$$

$$\frac{B_{12}}{\beta}\left[-\frac{2aS}{t}.\frac{\partial^3 w}{\partial Q\,\partial R^2}+\frac{2a^2}{t^2}\frac{\partial^2 v_0}{dR\,\partial Q}+\frac{2a^2H}{t}\frac{\partial^2 \phi_x}{\partial R\,\partial Q}\right]$$

$$+\frac{B_{13}}{\beta}\left[-\frac{2aS\beta}{t}.\frac{\partial^3 w}{\partial R^3}+\frac{2a^2\beta}{t^2}.\frac{\partial^2 u_0}{\partial R^2}+\frac{2a^2H\beta}{t}\frac{\partial^2 \phi_x}{\partial R^2}\right]$$

$$\frac{B_{22}}{\beta^3}\left[-\frac{2aS}{t}.\frac{\partial^3 w}{\partial Q^3}+\frac{8a^2\beta}{t^2}.\frac{\partial^2 v_0}{\partial Q^2}+\frac{2a^2H\beta}{t}\frac{\partial^2 \phi_y}{\partial Q^2}\right]$$

$$+\frac{B_{23}}{\beta^2}\left[-\frac{6aS}{t}.\frac{\partial^3 w}{\partial R\,\partial Q^2}+\frac{2a^2}{t^2}\left[\frac{\partial^2 u_0}{\partial Q^2}+\beta.\frac{\partial^2 v_0}{\partial Q\,\partial R}\right]+\frac{2a^2H}{t}\left[2\beta\frac{\partial^2 \phi_y}{\partial R\,\partial Q}+.\frac{\partial^2 \phi_x}{\partial Q^2}\right]\right]$$

$$\frac{B_{33}}{\beta}\left[-\frac{4aS}{t}.\frac{\partial^3 w}{\partial R^2\,\partial Q}+\frac{2a^2}{t^2}\left[\frac{\partial^2 u_0}{\partial Q\,\partial R}+\beta.\frac{\partial^2 v_0}{\partial R^2}\right]+\frac{2a^2H}{t}.\left[\frac{\partial^2 \phi_x}{\partial R\,\partial Q}+\beta.\frac{\partial^2 \phi_y}{\partial R^2}\right]\right]$$

$$+B_{55}.\frac{a^2}{t^2}.2\frac{\partial^2 v_0}{\partial S^2}.\}dR\,dQ\qquad +$$

$$\frac{B_{12}}{\beta}\left[-2aHS.\frac{\partial^3 w}{\partial Q\,\partial R^2}+\frac{2a^2H}{t}.\frac{\partial^2 u_0}{\partial R\partial Q}+2a^2H^2.\frac{\partial^2 \phi_x}{\partial R\,\partial Q}\right]$$

$$+\frac{B_{13}}{\beta}\left[-2aHS\beta\frac{\partial^3 w}{\partial R^3}+\frac{2a^2H}{t}.\frac{\partial^2 u_0}{\partial R^2}+2a^2H^2\beta.\frac{\partial^2 \phi_x}{\partial R^2}\right]$$

$$+\frac{B_{22}}{\beta^3}\left[-2aHS.\frac{\partial^3 w}{\partial Q^3}+2a^2H^2\beta.\frac{\partial^2 \phi_y}{\partial Q^2}+\frac{2a^2H\beta}{t}\frac{\partial^2 v_0}{\partial Q^2}\right]$$

$$+\frac{B_{23}}{\beta^2}\left[-6aHS.\frac{\partial^3 w}{\partial R\,\partial Q^2}+\frac{2a^2H}{t}.\frac{\partial^2 u_0}{\partial Q^2}+\frac{4a^2H\beta}{t}\frac{\partial^2 v_0}{\partial Q\,\partial R}+2a^2H^2.\frac{\partial^2 \phi_x}{\partial Q^2}+2H^2\beta.\frac{\partial^2 \phi_y}{\partial Q\,\partial R}\right]$$

$$\frac{B_{33}}{\beta}\left[-4aHS.\frac{\partial^3 w}{\partial R^2\,\partial Q}+\frac{2a^2H}{t}.\frac{\partial^2 u_0}{\partial R\,\partial Q}+\frac{2a^2H\beta}{t}.\frac{\partial^2 v_0}{\partial R^2}+2a^2H^2.\frac{\partial \phi_x}{\partial Q\,\partial R}+2a^2H^2\beta\frac{\partial^2 \phi_y}{\partial R^2}\right]$$

$$+2B_{55}.\frac{a^4}{t^2}.g_4.\emptyset_y+B_{55}.\frac{a^4}{t^3}.\left[2g_{C3}.\frac{dv_0}{dS}\right].\}dR\,dQ \quad +$$

$$B_{11}\left[-2aHS.\frac{\partial^3 w}{\partial R^3}+\frac{2a^2H}{t}.\frac{\partial^2 u_0}{\partial R^2}+2a^2S^2.\frac{\partial^2 \phi_x}{\partial R^2}\right]$$

$$+\frac{B_{12}}{\beta^2}\left[-2aHS\frac{\partial^3 w}{\partial R\partial Q^2}+\frac{2a^2H\beta}{t}\frac{\partial^2 v_0}{\partial R\partial Q}+2a^2H^2\beta.\frac{\partial^2 \phi_y}{\partial R\partial Q}\right]$$

$$+\frac{B_{13}}{\beta}\left[-5aHS.\frac{\partial^3 w}{\partial Q\partial R^2}+\frac{4a^2H}{t}.\frac{\partial^2 u_0}{\partial R\partial Q}+\frac{2a^2H\beta}{t}\frac{\partial^2 v_0}{\partial R^2}+2a^2H^2\beta.\frac{\partial^2 \phi_y}{\partial R^2}\right]$$

$$+\frac{B_{23}}{\beta^3}\left[-2aSH.\frac{\partial^3 w}{\partial Q^3}+\frac{2a^2H\beta}{t}.\frac{\partial^2 v_0}{\partial Q^2}+2a^2H^2\beta.\frac{\partial^2 \phi_y}{\partial Q^2}\right]$$

$$+\frac{B_{33}}{\beta^2}\left[-4aHS.\frac{\partial^3 w}{\partial R\,\partial Q^2}+\frac{2a^2H}{t}.\left[\frac{\partial^2 u_0}{\partial Q^2}+\beta.\frac{\partial^2 v_0}{\partial R\,\partial Q}\right]+2a^2H^2\left[.\frac{\partial^2 \phi_x}{\partial Q^2}+\beta.\frac{\partial^2 \phi_y}{\partial R\,\partial Q}\right]\right]$$

$$+B_{44}.\frac{a^4}{t^3}\left[2g_{C3}.\frac{du_0}{dS}\right]+2B_{44}.\frac{a^4}{t^2}.g_4.\emptyset_x=0 \qquad (3.283)$$

Equation (3.283) was further expanded by opening of the brackets and that gave

$$B_{11}\left[-\frac{2aS}{t}.\frac{\partial^3 w}{\partial R^3}-2aHS.\frac{\partial^3 w}{\partial R^3}+\frac{2a^2}{t^2}.\frac{\partial^2 u_0}{\partial R^2}+\frac{2a^2H}{t}.\frac{\partial^2 u_0}{\partial R^2}+\frac{2a^2H}{t}.\frac{\partial^2 \phi_x}{\partial R^2}++2a^2S^2.\frac{\partial^2 \phi_x}{\partial R^2}\right]$$

$$\frac{B_{12}}{\beta^2}\Bigg[-\frac{2aS}{t}\frac{\partial^3 w}{dR\,\partial Q^2}-2aH\beta S.\frac{\partial^3 w}{\partial Q\,\partial R^2}-2aHS\frac{\partial^3 w}{\partial R\,\partial Q^2}-\frac{2aS\beta}{t}.\frac{\partial^3 w}{\partial Q\,\partial R^2}+\frac{2a^2H\beta}{t}.\frac{\partial^2 u_0}{\partial R\,\partial Q}$$
$$+\frac{2a^2\beta}{t^2}.\frac{\partial^2 v_0}{dR\,\partial Q}+\frac{2a^2H\beta}{t}\frac{\partial^2 v_0}{\partial R\,\partial Q}+\frac{2a^2\beta}{t^2}\frac{\partial^2 v_0}{dR\,\partial Q}+2a^2H^2\beta.\frac{\partial^2 \phi_x}{\partial R\,\partial Q}$$
$$+2a^2H^2\beta.\frac{\partial^2 \phi_y}{\partial R\,\partial Q}+\frac{2a^2H\beta}{t}.\frac{\partial^2 \phi_y}{\partial R\,\partial Q}+\frac{2a^2H\beta}{t}\frac{\partial^2 \phi_x}{\partial R\,\partial Q}\Bigg]$$

$$+\frac{B_{13}}{\beta}\Bigg[-\frac{3aS}{t}.\frac{\partial^3 w}{\partial R^2\,\partial Q}-4aHS.\frac{\partial^3 w}{\partial Q\,\partial R^2}-2aHS\beta\frac{\partial^3 w}{\partial R^3}-\frac{2aS\beta}{t}.\frac{\partial^3 w}{\partial R^3}+\frac{8a^2}{t^2}.\frac{\partial^2 u_0}{\partial R\,\partial Q}$$
$$+\frac{4a^2H}{t}.\frac{\partial^2 u_0}{\partial R\,\partial Q}+\frac{2a^2\beta}{t^2}.\frac{\partial^2 u_0}{\partial R^2}+\frac{2a^2H}{t}.\frac{\partial^2 u_0}{\partial R^2}+\frac{2a^2\beta}{t^2}.\frac{\partial^2 v_0}{\partial R^2}+\frac{2a^2H\beta}{t}\frac{\partial^2 v_0}{\partial R^2}$$
$$+\frac{2a^2H}{t}.\frac{\partial^2 \phi_x}{\partial R\partial Q}+2a^2H^2\beta.\frac{\partial^2 \phi_x}{\partial R^2}+\frac{2a^2H\beta}{t}\frac{\partial^2 \phi_x}{\partial R^2}+2a^2H^2\beta.\frac{\partial^2 \phi_y}{\partial R^2}$$
$$+\frac{a^2H}{t}.\frac{\partial^2 \phi_y}{\partial R^2}\Bigg]$$

$$+\frac{B_{22}}{\beta^3}\Bigg[-2aHS.\frac{\partial^3 w}{\partial Q^3}-\frac{2aS}{t}.\frac{\partial^3 w}{\partial Q^3}+2a^2H^2\beta.\frac{\partial^2 \phi_y}{\partial Q^2}+\frac{8a^2\beta}{t^2}.\frac{\partial^2 v_0}{\partial Q^2}+\frac{2a^2H\beta}{t}\frac{\partial^2 v_0}{\partial Q^2}$$
$$+\frac{2a^2H\beta}{t}\frac{\partial^2 \phi_y}{\partial Q^2}\Bigg]$$

$$+\frac{B_{23}}{\beta^3}\Bigg[-2aSH.\frac{\partial^3 w}{\partial Q^3}-\frac{2aS}{t}.\frac{\partial^3 w}{\partial Q^3}-6aH\beta S.\frac{\partial^3 w}{\partial R\partial Q^2}-\frac{6aS\beta}{t}.\frac{\partial^3 w}{\partial R\partial Q^2}+\frac{2a^2H\beta}{t}.\frac{\partial^2 u_0}{\partial Q^2}$$
$$+\frac{2a^2\beta}{t^2}.\frac{\partial^2 u_0}{\partial Q^2}+\frac{2a^2H\beta}{t}.\frac{\partial^2 v_0}{\partial Q^2}+\frac{4a^2H\beta^2}{t}\frac{\partial^2 v_0}{\partial Q\partial R}+\frac{2a^2\beta^2}{t^2}.\frac{\partial^2 v_0}{\partial Q\partial R}$$
$$+2H^2\beta^2.\frac{\partial^2 \phi_y}{\partial Q\partial R}+\frac{4a^2H\beta^2}{t}\frac{\partial^2 \phi_y}{\partial R\partial Q}\frac{2a^2\beta}{t^2}.\frac{\partial^2 v_0}{\partial Q^2}+2a^2H^2\beta.\frac{\partial^2 \phi_x}{\partial Q^2}$$
$$+\frac{2a^2H\beta}{t}.\frac{\partial^2 \phi_y}{\partial Q^2}+2a^2H^2\beta.\frac{\partial^2 \phi_y}{\partial Q^2}\Bigg]$$

$$\frac{B_{33}}{\beta^2}\Bigg[-\frac{4aS\beta}{t}.\frac{\partial^3 w}{\partial R^2\,\partial Q}-4aHS.\frac{\partial^3 w}{\partial R\,\partial Q^2}-4aHS\beta.\frac{\partial^3 w}{\partial R^2\,\partial Q}+\frac{2a^2\beta}{t^2}.\frac{\partial^2 u_0}{\partial Q\,\partial R}+\frac{2a^2H\beta}{t}.\frac{\partial^2 u_0}{\partial R\,\partial Q}$$
$$+\frac{2a^2\beta^2}{t^2}.\frac{\partial^2 v_0}{\partial R^2}+\frac{2a^2H\beta}{t}.\frac{\partial^2 v_0}{\partial R\partial Q}+\frac{2a^2H\beta^2}{t}.\frac{\partial^2 v_0}{\partial R^2}+\frac{2a^2H\beta}{t}.\frac{\partial^2 \phi_x}{\partial R\partial Q}$$
$$+2a^2H^2\beta.\frac{\partial \phi_x}{\partial Q\partial R}+2a^2H^2.\frac{\partial^2 \phi_x}{\partial Q^2}+\frac{2a^2H}{t}.\frac{\partial^2 u_0}{\partial Q^2}+2a^2H^2\beta^2\frac{\partial^2 \phi_y}{\partial R^2}$$
$$+2a^2H^2\beta.\frac{\partial^2 \phi_y}{\partial R\partial Q}+\frac{2a^2H\beta^2}{t}.\frac{\partial^2 \phi_y}{\partial R^2}\Bigg]$$

$$+\frac{a^2}{t^2}\Bigg[2B_{55}.\frac{\partial^2 v_0}{\partial S^2}+2B_{55}.a^2.g_4.\emptyset_y+2B_{55}.\frac{a^2}{t}.g_{C3}.\frac{dv_0}{dS}+2B_{44}.\frac{a^2}{t}.g_{C3}.\frac{du_0}{dS}$$
$$+2B_{44}.a^2.g_4.\emptyset_x\Bigg]=0 \qquad (3.284)$$

Simplifying Equation (3.284) further gives

$$B_{11}\Bigg[-\frac{2aS}{t}.\frac{\partial^3 w}{\partial R^3}-2aHS.\frac{\partial^3 w}{\partial R^3}+\frac{2a^2}{t^2}.\frac{\partial^2 u_0}{\partial R^2}+\frac{2a^2H}{t}.\frac{\partial^2 u_0}{\partial R^2}+\frac{2a^2H}{t}.\frac{\partial^2 \phi_x}{\partial R^2}++2a^2S^2.\frac{\partial^2 \phi_x}{\partial R^2}\Bigg]$$

$$\frac{B_{12}}{\beta^2}\Bigg[-\frac{2aS}{t}\frac{\partial^3 w}{dR\partial Q^2}-2aH\beta S.\frac{\partial^3 w}{\partial Q\,\partial R^2}-2aHS\frac{\partial^3 w}{\partial R\partial Q^2}-\frac{2aS\beta}{t}.\frac{\partial^3 w}{\partial Q\partial R^2}+\frac{2a^2H\beta}{t}.\frac{\partial^2 u_0}{\partial R\partial Q}$$
$$+\frac{2a^2\beta}{t^2}.\frac{\partial^2 v_0}{dR\partial Q}+\frac{2a^2H\beta}{t}\frac{\partial^2 v_0}{\partial R\partial Q}+\frac{2a^2\beta}{t^2}\frac{\partial^2 v_0}{dR\partial Q}+2a^2H^2\beta.\frac{\partial^2 \phi_x}{\partial R\,\partial Q}$$
$$+2a^2H^2\beta.\frac{\partial^2 \phi_y}{\partial R\partial Q}+\frac{2a^2H\beta}{t}.\frac{\partial^2 \phi_y}{\partial R\partial Q}+\frac{2a^2H\beta}{t}\frac{\partial^2 \phi_x}{\partial R\partial Q}\Bigg]$$

$$+\frac{B_{13}}{\beta}\Bigg[\frac{aS}{t}.\frac{\partial^3 w}{\partial R^2\partial Q}-5aHS.\frac{\partial^3 w}{\partial Q\partial R^2}-2aHS\beta\frac{\partial^3 w}{\partial R^3}-\frac{2aS\beta}{t}.\frac{\partial^3 w}{\partial R^3}+\frac{8a^2}{t^2}.\frac{\partial^2 u_0}{\partial R\partial Q}$$
$$+\frac{4a^2H}{t}.\frac{\partial^2 u_0}{\partial R\partial Q}+\frac{2a^2\beta}{t^2}.\frac{\partial^2 u_0}{\partial R^2}+\frac{2a^2H}{t}.\frac{\partial^2 u_0}{\partial R^2}+\frac{2a^2\beta}{t^2}.\frac{\partial^2 v_0}{\partial R^2}+\frac{2a^2H\beta}{t}\frac{\partial^2 v_0}{\partial R^2}$$
$$+\frac{2a^2H}{t}.\frac{\partial^2 \phi_x}{\partial R\,\partial Q}+2a^2H^2\beta.\frac{\partial^2 \phi_x}{\partial R^2}+\frac{2a^2H\beta}{t}\frac{\partial^2 \phi_x}{\partial R^2}+2a^2H^2\beta.\frac{\partial^2 \phi_y}{\partial R^2}$$
$$+\frac{a^2H}{t}.\frac{\partial^2 \phi_y}{\partial R^2}\Bigg]$$

$$+\frac{B_{22}}{\beta^3}\left[-2aHS.\frac{\partial^3 w}{\partial Q^3}-\frac{2aS}{t}.\frac{\partial^3 w}{\partial Q^3}+2a^2H^2\beta.\frac{\partial^2\phi_y}{\partial Q^2}+\frac{8a^2\beta}{t^2}.\frac{\partial^2 v_0}{\partial Q^2}+\frac{2a^2H\beta}{t}\frac{\partial^2 v_0}{\partial Q^2}\right.$$
$$\left.+\frac{2a^2H\beta}{t}\frac{\partial^2\phi_y}{\partial Q^2}\right]$$

$$+\frac{B_{23}}{\beta^3}\left[-2aSH.\frac{\partial^3 w}{\partial Q^3}-\frac{2aS}{t}.\frac{\partial^3 w}{\partial Q^3}-6aH\beta S.\frac{\partial^3 w}{\partial R\partial Q^2}-\frac{6aS\beta}{t}.\frac{\partial^3 w}{\partial R\partial Q^2}+\frac{2a^2H\beta}{t}.\frac{\partial^2 u_0}{\partial Q^2}\right.$$
$$+\frac{2a^2\beta}{t^2}.\frac{\partial^2 u_0}{\partial Q^2}+\frac{2a^2H\beta}{t}.\frac{\partial^2 v_0}{\partial Q^2}+\frac{4a^2H\beta^2}{t}\frac{\partial^2 v_0}{\partial Q\partial R}+\frac{2a^2\beta^2}{t^2}.\frac{\partial^2 v_0}{\partial Q\partial R}$$
$$+2H^2\beta^2.\frac{\partial^2\phi_y}{\partial Q\partial R}+\frac{4a^2H\beta^2}{t}\frac{\partial^2\phi_y}{\partial R\partial Q}\frac{2a^2\beta}{t^2}.\frac{\partial^2 v_0}{\partial Q^2}+2a^2H^2\beta.\frac{\partial^2\phi_x}{\partial Q^2}$$
$$\left.+\frac{2a^2H\beta}{t}.\frac{\partial^2\phi_y}{\partial Q^2}+2a^2H^2\beta.\frac{\partial^2\phi_y}{\partial Q^2}\right]$$

$$\frac{B_{33}}{\beta^2}\left[-\frac{4aS\beta}{t}.\frac{\partial^3 w}{\partial R^2\partial Q}-4aHS.\frac{\partial^3 w}{\partial R\partial Q^2}-4aHS\beta.\frac{\partial^3 w}{\partial R^2\partial Q}+\frac{2a^2\beta}{t^2}.\frac{\partial^2 u_0}{\partial Q\partial R}+\frac{2a^2H\beta}{t}.\frac{\partial^2 u_0}{\partial R\partial Q}\right.$$
$$+\frac{2a^2\beta^2}{t^2}.\frac{\partial^2 v_0}{\partial R^2}+\frac{2a^2H\beta}{t}.\frac{\partial^2 v_0}{\partial R\partial Q}+\frac{2a^2H\beta^2}{t}.\frac{\partial^2 v_0}{\partial R^2}+\frac{2a^2H\beta}{t}.\frac{\partial^2\phi_x}{\partial R\partial Q}$$
$$+2a^2H^2\beta.\frac{\partial\phi_x}{\partial Q\partial R}+2a^2H^2.\frac{\partial^2\phi_x}{\partial Q^2}+\frac{2a^2H}{t}.\frac{\partial^2 u_0}{\partial Q^2}+2a^2H^2\beta^2\frac{\partial^2\phi_y}{\partial R^2}$$
$$\left.+2a^2H^2\beta.\frac{\partial^2\phi_y}{\partial R\partial Q}+\frac{2a^2H\beta^2}{t}.\frac{\partial^2\phi_y}{\partial R^2}\right]+\frac{a^2}{t^2}$$

$$\left[2B_{55}\left[\frac{\partial^2 v_0}{\partial S^2}+a^2.g_4.\emptyset_y+\frac{a^2}{t}.g_{C3}.\frac{dv_0}{dS}\right]+2B_{44}\left[\frac{a^2}{t}.g_{C3}.\frac{du_0}{dS}+a^2.g_4.\emptyset_x\right]\right]$$
$$=0 \qquad (3.285)$$

Collecting the like terms together and regrouping Equation (3.285) gives

$$-2B_{13}aHS\frac{\partial^3 w}{\partial R^3}-\frac{2aHS}{\beta}[B_{12}+2B_{33}]\frac{\partial^3 w}{\partial R^2\partial Q}-\frac{6B_{23}aSH}{\beta^2}.\frac{\partial^3 w}{\partial R\partial Q^2}-\frac{2B_{22}aHS}{\beta^3}\frac{\partial^3 w}{\partial Q^3}$$

$$+\frac{2B_{13}a^2H}{t\beta}.\frac{\partial^2 u_0}{\partial R^2}+\frac{2a^2H}{t\beta}[B_{12}+B_{33}].\frac{\partial^2 u_0}{\partial R\partial Q}+\frac{2B_{23}a^2H}{t\beta^2}\frac{\partial^2 u_0}{\partial Q^2}$$

$$+\frac{2B_{33}a^2H}{t}\frac{\partial^2 v_0}{\partial R^2}+\frac{4B_{23}a^2H}{t\beta}.\frac{\partial^2 v_0}{\partial Q\partial R}+\frac{2\,B_{22}a^2H}{t\beta^2}.\frac{\partial^2 v_0}{\partial Q^2}$$

$$+2B_{33}a^2H^2\frac{\partial^2\phi_y}{\partial R^2}+\frac{2B_{23}H^2}{\beta}.\frac{\partial^2\phi_y}{\partial Q\partial R}+\frac{2\,B_{22}a^2H^2}{\beta^2}.\frac{\partial^2\phi_y}{\partial Q^2}$$

$$+2B_{13}a^2H^2.\frac{\partial^2\phi_x}{\partial R^2}+\frac{2a^2H^2}{\beta}[B_{12}+B_{33}]\frac{\partial^2\phi_x}{\partial R\partial Q}+\frac{2B_{23}a^2H^2}{\beta^2}.\frac{\partial^2\phi_x}{\partial Q^2}$$

$$+2B_{55}.\frac{a^4}{t^2}.g_4.\emptyset_y\}dR\,dQ+B_{55}.\frac{a^4}{t^3}.\left[2g_{C3}.\frac{dv_0}{dS}\right].\}dR\,dQ\;=0 \tag{3.286}$$

The Summation of (3.266), (3.272), (3.279) and (3.283) gives

$$B_{11}\left[-\frac{2aS}{t}.\frac{\partial^3 w}{\partial R^3}+\frac{2a^2}{t^2}.\frac{\partial^2 u_0}{\partial R^2}+\frac{2a^2H}{t}.\frac{\partial^2\phi_x}{\partial R^2}\right]$$

$$\frac{B_{12}}{\beta^2}\left[-\frac{2aS}{t}\frac{\partial^3 w}{dR\partial Q^2}+\frac{2a^2\beta}{t^2}.\frac{\partial^2 v_0}{dR\partial Q}+\frac{2a^2H\beta}{t}.\frac{\partial^2\phi_y}{\partial R\partial Q}\right]+\frac{B_{13}}{\beta}$$

$$\left[\frac{aS}{t}.\frac{\partial^3 w}{\partial R^2\,\partial Q}+\frac{2a^2}{t^2}\left[4.\frac{\partial^2 u_0}{\partial R\,\partial Q}+\beta.\frac{\partial^2 v_0}{\partial R^2}\right]+\frac{a^2H}{t}\left[2.\frac{\partial^2\phi_x}{\partial R\,\partial Q}+\frac{\partial^2\phi_y}{\partial R^2}\right]\right]$$

$$+\frac{B_{23}}{\beta^3}\left[-\frac{2B_{23}aS}{t}.\frac{\partial^3 w}{\partial Q^3}+\frac{2B_{23}a^2\beta}{t^2}.\frac{\partial^2 v_0}{\partial Q^2}+\frac{2B_{23}a^2H\beta}{t}.\frac{\partial^2\phi_y}{\partial Q^2}\right]+\frac{B_{33}}{\beta^2}$$

$$\left[-\frac{4aS}{t}.\frac{\partial^3 w}{\partial R\partial\,\partial Q^2}+\frac{2a^2}{t^2}.\left[\frac{\partial^2 u_0}{\partial Q^2}+\beta\frac{\partial^2 v_0}{\partial Q\partial R}\right]+\frac{2a^2H}{t}.\left[\frac{\partial^2\phi_x}{\partial Q^2}+\beta.\frac{\partial^2\phi_y}{\partial Q\partial R}\right]\right]$$

$$+2B_{44}\frac{a^4}{t^2}\left(\frac{\partial^2 H}{\partial S^2}.\emptyset_x+\frac{\partial^2 u_0}{\partial S^2}\right)$$

$$\frac{B_{12}}{\beta}\left[-\frac{2aS}{t}.\frac{\partial^3 w}{\partial Q\partial R^2}+\frac{2a^2}{t^2}\frac{\partial^2 v_0}{dR\partial Q}+\frac{2a^2H}{t}\frac{\partial^2\phi_x}{\partial R\partial Q}\right]$$

$$+\frac{B_{13}}{\beta}\left[-\frac{2aS\beta}{t}.\frac{\partial^3 w}{\partial R^3}+\frac{2a^2\beta}{t^2}.\frac{\partial^2 u_0}{\partial R^2}+\frac{2a^2H\beta}{t}\frac{\partial^2\phi_x}{\partial R^2}\right]$$

$$\frac{B_{22}}{\beta^3}\left[-\frac{2aS}{t}.\frac{\partial^3 w}{\partial Q^3}+\frac{8a^2\beta}{t^2}.\frac{\partial^2 v_0}{\partial Q^2}+\frac{2a^2H\beta}{t}\frac{\partial^2\phi_y}{\partial Q^2}\right]$$

$$+\frac{B_{23}}{\beta^2}\left[-\frac{6aS}{t}.\frac{\partial^3 w}{\partial R\partial Q^2}+\frac{2a^2}{t^2}\left[\frac{\partial^2 u_0}{\partial Q^2}+\beta.\frac{\partial^2 v_0}{\partial Q\partial R}\right]+\frac{2a^2 H}{t}\left[2\beta\frac{\partial^2\phi_y}{\partial R\partial Q}+.\frac{\partial^2\phi_x}{\partial Q^2}\right]\right]$$

$$\frac{B_{33}}{\beta}\left[-\frac{4aS}{t}.\frac{\partial^3 w}{\partial R^2\partial Q}+\frac{2a^2}{t^2}\left[\frac{\partial^2 u_0}{\partial Q\partial R}+\beta.\frac{\partial^2 v_0}{\partial R^2}\right]+\frac{2a^2 H}{t}.\left[\frac{\partial^2\phi_x}{\partial R\partial Q}+\beta.\frac{\partial^2\phi_y}{\partial R^2}\right]\right]$$

$$+2B_{55}\frac{a^4}{t^2}\left(\frac{\partial^2 H}{\partial S^2}.\emptyset_y+\frac{\partial^2 v_0}{\partial S^2}\right).\}dR\,dQ \quad +$$

$$\frac{B_{12}}{\beta}\left[-\frac{2aHS}{t}.\frac{\partial^3 w}{\partial Q\,\partial R^2}+\frac{2a^2 H}{t^2}.\frac{\partial^2 u_0}{\partial R\partial Q}+\frac{2a^2H^2}{t}.\frac{\partial^2\phi_x}{\partial R\,\partial Q}\right]$$

$$+\frac{B_{13}}{\beta}\left[-\frac{2aHS\beta}{t}\frac{\partial^3 w}{\partial R^3}+\frac{2a^2 H}{t^2}.\frac{\partial^2 u_0}{\partial R^2}+\frac{2a^2H^2\beta}{t}.\frac{\partial^2\phi_x}{\partial R^2}\right]$$

$$+\frac{B_{22}}{\beta^3}\left[-\frac{2aHS}{t}.\frac{\partial^3 w}{\partial Q^3}+\frac{2a^2H^2\beta.}{t}\frac{\partial^2\phi_y}{\partial Q^2}+\frac{2a^2H\beta}{t^2}\frac{\partial^2 v_0}{\partial Q^2}\right]+\frac{B_{23}}{\beta^2}$$

$$\left[-\frac{6aHS}{t}.\frac{\partial^3 w}{\partial R\partial Q^2}+\frac{2a^2H}{t^2}.\frac{\partial^2 u_0}{\partial Q^2}+\frac{4a^2H\beta}{t^2}\frac{\partial^2 v_0}{\partial Q\partial R}+\frac{2a^2H^2}{t^2}.\frac{\partial^2\phi_x}{\partial Q^2}+\frac{2H^2\beta}{t}.\frac{\partial^2\phi_y}{\partial Q\partial R}\right]\frac{B_{33}}{\beta}$$

$$\left[-\frac{4aHS}{t}.\frac{\partial^3 w}{\partial R^2\partial Q}+\frac{2a^2H}{t^2}.\frac{\partial^2 u_0}{\partial R\partial Q}+\frac{2a^2H\beta}{t^2}.\frac{\partial^2 v_0}{\partial R^2}+\frac{2a^2H^2}{t}.\frac{\partial\phi_x}{\partial Q\partial R}+\frac{2a^2H^2\beta}{t}\frac{\partial^2\phi_y}{\partial R^2}\right]$$

$$+2B_{55}\frac{a^4}{t^3}\left(\emptyset_y.\left(\frac{dH}{dS}\right)^2+\frac{dv_0}{dS}.\frac{\partial H}{\partial S}\right).\}dR\,dQ+B_{11}$$

$$\left[-\frac{2aHS}{t}.\frac{\partial^3 w}{\partial R^3}+\frac{2a^2H}{t^2}.\frac{\partial^2 u_0}{\partial R^2}+\frac{2a^2S^2}{t}.\frac{\partial^2\phi_x}{\partial R^2}\right]+\frac{B_{12}}{\beta^2}$$

$$\left[-\frac{2aHS}{t}\frac{\partial^3 w}{\partial R\partial Q^2}+\frac{2a^2H\beta}{t^2}\frac{\partial^2 v_0}{\partial R\partial Q}+\frac{2a^2H^2\beta}{t}.\frac{\partial^2\phi_y}{\partial R\partial Q}\right]+\frac{B_{13}}{\beta}$$

$$\left[-\frac{4aHS}{t}.\frac{\partial^3 w}{\partial Q\partial R^2}+\frac{4a^2H}{t^2}.\frac{\partial^2 u_0}{\partial R\partial Q}+\frac{2a^2H\beta}{t^2}\frac{\partial^2 v_0}{\partial R^2}+\frac{2a^2H^2\beta}{t}.\frac{\partial^2\phi_y}{\partial R^2}\right]$$

$$+\frac{B_{23}}{\beta^3}\left[-\frac{2aSH}{t}.\frac{\partial^3 w}{\partial Q^3}+\frac{2a^2H\beta}{t^2}.\frac{\partial^2 v_0}{\partial Q^2}+\frac{2a^2H^2\beta}{t}.\frac{\partial^2\phi_y}{\partial Q^2}\right]+\frac{B_{33}}{\beta^2}$$

$$\left[-\frac{4aHS}{t}.\frac{\partial^3 w}{\partial R\partial Q^2}+\frac{2a^2H}{t^2}.\left[\frac{\partial^2 u_0}{\partial Q^2}+\beta.\frac{\partial^2 v_0}{\partial R\partial Q}\right]+\frac{2a^2H^2}{t}\left[.\frac{\partial^2 \phi_x}{\partial Q^2}+\beta.\frac{\partial^2 \phi_y}{\partial R\partial Q}\right]\right]$$

$$+2B_{44}\frac{a^4}{t^3}\left(\emptyset_x.\left(\frac{dH}{dS}\right)^2+\frac{du_0}{dS}.\frac{\partial H}{\partial S}\right) \tag{3.287}$$

That is

$$B_{11}\ \left[-\frac{2aS}{t}.\frac{\partial^3 w}{\partial R^3}-2aHS.\frac{\partial^3 w}{\partial R^3}+\frac{2a^2}{t^2}.\frac{\partial^2 u_0}{\partial R^2}+\frac{2a^2H}{t}.\frac{\partial^2 u_0}{\partial R^2}+\frac{2a^2H}{t}.\frac{\partial^2 \phi_x}{\partial R^2}++2a^2S^2.\frac{\partial^2 \phi_x}{\partial R^2}\right]$$

$$\frac{B_{12}}{\beta^2}\left[-\frac{2aS}{t}\frac{\partial^3 w}{dR\partial Q^2}-2aH\beta S.\frac{\partial^3 w}{\partial Q\,\partial R^2}-2aHS\frac{\partial^3 w}{\partial R\partial Q^2}-\frac{2aS\beta}{t}.\frac{\partial^3 w}{\partial Q\partial R^2}+\frac{2a^2H\beta}{t}.\frac{\partial^2 u_0}{\partial R\partial Q}\right.$$

$$+\frac{2a^2\beta}{t^2}.\frac{\partial^2 v_0}{dR\partial Q}+\frac{2a^2H\beta}{t}\frac{\partial^2 v_0}{\partial R\partial Q}+\frac{2a^2\beta}{t^2}\frac{\partial^2 v_0}{dR\partial Q}+2a^2H^2\beta.\frac{\partial^2 \phi_x}{\partial R\,\partial Q}$$

$$\left.+2a^2H^2\beta.\frac{\partial^2 \phi_y}{\partial R\partial Q}+\frac{2a^2H\beta}{t}.\frac{\partial^2 \phi_y}{\partial R\partial Q}+\frac{2a^2H\beta}{t}\frac{\partial^2 \phi_x}{\partial R\partial Q}\right]$$

$$+\frac{B_{13}}{\beta}\left[\frac{aS}{t}.\frac{\partial^3 w}{\partial R^2\,\partial Q}-4aHS.\frac{\partial^3 w}{\partial Q\partial R^2}-2aHS\beta\frac{\partial^3 w}{\partial R^3}-\frac{2aS\beta}{t}.\frac{\partial^3 w}{\partial R^3}+\frac{8a^2}{t^2}.\frac{\partial^2 u_0}{\partial R\,\partial Q}\right.$$

$$+\frac{4a^2H}{t}.\frac{\partial^2 u_0}{\partial R\partial Q}+\frac{2a^2\beta}{t^2}.\frac{\partial^2 u_0}{\partial R^2}+\frac{2a^2H}{t}.\frac{\partial^2 u_0}{\partial R^2}+\frac{2a^2\beta}{t^2}.\frac{\partial^2 v_0}{\partial R^2}+\frac{2a^2H\beta}{t}\frac{\partial^2 v_0}{\partial R^2}$$

$$+\frac{2a^2H}{t}.\frac{\partial^2 \phi_x}{\partial R\,\partial Q}+2a^2H^2\beta.\frac{\partial^2 \phi_x}{\partial R^2}+\frac{2a^2H\beta}{t}\frac{\partial^2 \phi_x}{\partial R^2}+2a^2H^2\beta.\frac{\partial^2 \phi_y}{\partial R^2}$$

$$\left.+\frac{a^2H}{t}.\frac{\partial^2 \phi_y}{\partial R^2}\right]$$

$$+\frac{B_{22}}{\beta^3}\left[-2aHS.\frac{\partial^3 w}{\partial Q^3}-\frac{2aS}{t}.\frac{\partial^3 w}{\partial Q^3}+2a^2H^2\beta.\frac{\partial^2 \phi_y}{\partial Q^2}+\frac{8a^2\beta}{t^2}.\frac{\partial^2 v_0}{\partial Q^2}+\frac{2a^2H\beta}{t}\frac{\partial^2 v_0}{\partial Q^2}\right.$$

$$\left.++\frac{2a^2H\beta}{t}\frac{\partial^2 \phi_y}{\partial Q^2}\right]$$

$$+\frac{B_{23}}{\beta^3}\Bigg[-2aSH.\frac{\partial^3 w}{\partial Q^3}-\frac{2aS}{t}.\frac{\partial^3 w}{\partial Q^3}- 6aH\beta S.\frac{\partial^3 w}{\partial R\partial Q^2}-\frac{6aS\beta}{t}.\frac{\partial^3 w}{\partial R\partial Q^2}+\frac{2a^2H\beta}{t}.\frac{\partial^2 u_0}{\partial Q^2}$$

$$+\frac{2a^2\beta}{t^2}.\frac{\partial^2 u_0}{\partial Q^2}+\frac{2a^2H\beta}{t}.\frac{\partial^2 v_0}{\partial Q^2}+\frac{4a^2H\beta^2}{t}\frac{\partial^2 v_0}{\partial Q\partial R}+\frac{2a^2\beta^2}{t^2}.\frac{\partial^2 v_0}{\partial Q\partial R}$$

$$+2H^2\beta^2.\frac{\partial^2\phi_y}{\partial Q\partial R}+\frac{4a^2H\beta^2}{t}\frac{\partial^2\phi_y}{\partial R\partial Q}\frac{2a^2\beta}{t^2}.\frac{\partial^2 v_0}{\partial Q^2}+2a^2H^2\beta.\frac{\partial^2\phi_x}{\partial Q^2}$$

$$+\frac{2a^2H\beta}{t}.\frac{\partial^2\phi_y}{\partial Q^2}+2a^2H^2\beta.\frac{\partial^2\phi_y}{\partial Q^2}\Bigg]$$

$$\frac{B_{33}}{\beta^2}\Bigg[-\frac{4aS\beta}{t}.\frac{\partial^3 w}{\partial R^2\partial Q}-4aHS.\frac{\partial^3 w}{\partial R\partial Q^2}-4aHS\beta.\frac{\partial^3 w}{\partial R^2\partial Q}+\frac{2a^2\beta}{t^2}.\frac{\partial^2 u_0}{\partial Q\partial R}+\frac{2a^2H\beta}{t}.\frac{\partial^2 u_0}{\partial R\partial Q}$$

$$+\frac{2a^2\beta^2}{t^2}.\frac{\partial^2 v_0}{\partial R^2}+\frac{2a^2H\beta}{t}.\frac{\partial^2 v_0}{\partial R\partial Q}+\frac{2a^2H\beta^2}{t}.\frac{\partial^2 v_0}{\partial R^2}+\frac{2a^2H\beta}{t}.\frac{\partial^2\phi_x}{\partial R\partial Q}$$

$$+2a^2H^2\beta.\frac{\partial\phi_x}{\partial Q\partial R}+2a^2H^2.\frac{\partial^2\phi_x}{\partial Q^2}+\frac{2a^2H}{t}.\frac{\partial^2 u_0}{\partial Q^2}+2a^2H^2\beta^2\frac{\partial^2\phi_y}{\partial R^2}$$

$$+2a^2H^2\beta.\frac{\partial^2\phi_y}{\partial R\partial Q}+\frac{2a^2H\beta^2}{t}.\frac{\partial^2\phi_y}{\partial R^2}\Bigg]$$

$$+2B_{44}\frac{a^4}{t^2}\left(\frac{\partial^2 H}{\partial S^2}.\emptyset_x+\frac{\partial^2 u_0}{\partial S^2}\right)+B_{55}\frac{a^4}{t^2}\left(\frac{\partial^2 H}{\partial S^2}.\emptyset_y+\frac{\partial^2 v_0}{\partial S^2}\right)$$

$$+2B_{55}\frac{a^4}{t^2}\left(\emptyset_y.\left(\frac{dH}{dS}\right)^2+\frac{dv_0}{dS}.\frac{\partial H}{\partial S}\right)$$

$$+2B_{44}\frac{a^4}{t^2}\left(\emptyset_x.\left(\frac{dH}{dS}\right)^2+\frac{du_0}{dS}.\frac{\partial H}{\partial S}\right)=0 \qquad (3.288)$$

The values of S in Equation (3.288) were replaced by J_i. Recalling that for the case of Bending stiffness, the expressions were as follows

$$J_1=\sum_{m=1}^{m=n}\int S^2\,dS=\sum_{m=1}^{m=n}\frac{1}{3}(S_m^3-S_{m-1}^3)=\frac{1}{3}\sum_{m=1}^{m=n}(S_m^3-S_{m-1}^3) \qquad (3.289)$$

where m stands for number of laminas in the plate and n is considered as the total

number of laminas

$$J_2=\sum_{m=1}^{m=n}\int SH\,dS=\sum_{m=1}^{m=n}\int\left[S^2-\frac{4}{3}S^4\right]dS$$

That is:

$$J_2 = \frac{1}{3}\sum_{m=1}^{m=n}\left[\left(S_m^3 - \frac{4}{5}S_m^5\right) - \left(S_{m-1}^3 - \frac{4}{5}S_{m-1}^5\right)\right] \tag{3.290}$$

$$J_3 = \sum_{m=1}^{m=n}\int H^2\,dS = \sum_{m=1}^{m=n}\int\left[S^2 - \frac{8}{3}S^4 + \frac{16}{9}S^6\right]dS = \sum_{m=1}^{m=n}\left[\frac{S^3}{3} - \frac{8}{15}S^5 + \frac{16}{63}S^7\right]$$

That is:

$$J_3 = \frac{1}{3}\sum_{m=1}^{m=n}\left[\left(S_m^3 - \frac{8}{5}S_m^5 + \frac{16}{21}S_m^7\right) - \left(S_{m-1}^3 - \frac{8}{5}S_{m-1}^5 + \frac{16}{21}S_{m-1}^7\right)\right] \tag{3.291}$$

$$J_4 = \sum_{m=1}^{m=n}\int\left[\frac{\partial H}{\partial S}\right]^2 dS = \sum_{m=1}^{m=n}\int[1 - 8S^2 + 16S^4]\,dS = \sum_{m=1}^{m=n}\left[S - 8\frac{S^3}{3} + 16\frac{S^5}{5}\right]$$

$$J_4 = \frac{1}{3}\sum_{m=1}^{m=n}\left[\left(3S_m^1 - 8S_m^3 + \frac{48}{5}S_m^5\right) - \left(3S_{m-1}^1 - 8S_{m-1}^3 + \frac{48}{5}S_{m-1}^5\right)\right] \tag{3.292}$$

Similarly, for the case of coupling stiffness, the expressions were as follows also for the case of coupling stiffness,

$$J_{C1} = \sum_{m=1}^{m=n}\int S\,dS = \sum_{m=1}^{m=n}\frac{1}{2}(S_m^2 - S_{m-1}^2) = \frac{1}{2} \tag{3.293}$$

$$J_{C2} = \sum_{m=1}^{m=n}\int H\,dS = \sum_{m=1}^{m=n}\int\left[S - \frac{4}{3}S^3\right]dS = \sum_{m=1}^{m=n}\left[\frac{S^2}{2} - \frac{S^4}{3}\right] \tag{3.294}$$

That is:

$$J_{C2} = \frac{1}{2}\sum_{m=1}^{m=n}\left[\left(S_m^2 - \frac{2}{3}S_m^4\right) - \left(S_{m-1}^2 - \frac{2}{3}S_{m-1}^4\right)\right] \tag{3.295}$$

$$J_{C3} = \sum_{m=1}^{m=n}\int\frac{\partial H}{\partial S}dS = \sum_{m=1}^{m=n}\int[1 - 4S^2]\,dS = \sum_{m=1}^{m=n}\left[S - \frac{4}{3}S^3\right] \tag{3.296}$$

That is:

$$\mathrm{J_{C3}} = \sum_{\mathrm{m=1}}^{\mathrm{m=n}} \left[\left(\mathrm{S_m^1} - \frac{4}{3}\mathrm{S_m^3} \right) - \left(\mathrm{S_{m-1}^1} - \frac{4}{3}\mathrm{S_{m-1}^3} \right) \right] \tag{3.297}$$

and finally the axial or membrane stiffness is given as

$$\mathrm{J_M} = \sum_{\mathrm{m=1}}^{\mathrm{m=n}} \int 1\,\mathrm{dS} = \sum_{\mathrm{m=1}}^{\mathrm{m=n}} \int [1]\,\mathrm{dS} = \sum_{\mathrm{m=1}}^{\mathrm{m=n}} [\mathrm{S}] \tag{3.298}$$

That is:

$$\mathrm{J_M} = \sum_{\mathrm{m=1}}^{\mathrm{m=n}} [(\mathrm{S_m^1}) - (\mathrm{S_{m-1}^1})] \tag{3.299}$$

Introducing the J_{ij} values in to the Equations (3.598)) gives

$$B_{11}\left[-\frac{2\mathrm{aJ_{C1}}}{\mathrm{t}}.\frac{\partial^3 \mathrm{w}}{\partial \mathrm{R}^3} - 2\mathrm{aJ_2}.\frac{\partial^3 \mathrm{w}}{\partial \mathrm{R}^3} + \frac{2\mathrm{a^2 J_M}}{\mathrm{t}^2}.\frac{\partial^2 \mathrm{u_0}}{\partial R^2} + \frac{2a^2 \mathrm{J_{C2}}}{\mathrm{t}}.\frac{\partial^2 \mathrm{u_0}}{\partial R^2} + \frac{2a^2 \mathrm{J_{C2}}}{\mathrm{t}}.\frac{\partial^2 \phi_x}{\partial \mathrm{R}^2} + 2a^2 \mathrm{J_1}.\frac{\partial^2 \phi_x}{\partial R^2}\right]$$

$$+\frac{\mathrm{B_{12}}}{\beta^2}\left[-\frac{2a\mathrm{J_{C1}}}{t}\frac{\partial^3 w}{dR\partial Q^2} - 2\mathrm{aJ_2}\upbeta.\frac{\partial^3 \mathrm{w}}{\partial \mathrm{Q}\,\partial \mathrm{R}^2} - 2a\mathrm{J_2}\frac{\partial^3 w}{\partial R\partial Q^2} - \frac{2a\mathrm{J_{C1}}\beta}{t}.\frac{\partial^3 w}{\partial Q\partial R^2} + \frac{2a^2\mathrm{J_{C2}}\upbeta}{t}.\frac{\partial^2 \mathrm{u_0}}{\partial R\partial Q} + \frac{2a^2\beta \mathrm{J_M}}{t^2}.\frac{\partial^2 v_0}{dR\partial Q} + \frac{2a^2\mathrm{J_{C2}}\beta}{t}\frac{\partial^2 \mathrm{v_0}}{\partial R\partial Q} + \frac{2a^2\mathrm{J_M}\beta}{t^2}\frac{\partial^2 v_0}{dR\partial Q} + 2\mathrm{a^2 J_3}\upbeta.\frac{\partial^2 \upphi_\mathrm{x}}{\partial \mathrm{R}\,\partial \mathrm{Q}} + 2a^2\mathrm{J_3}\beta.\frac{\partial^2 \phi_y}{\partial R\partial Q} + \frac{2a^2\mathrm{J_{C2}}\beta}{t}.\frac{\partial^2 \phi_y}{\partial R\partial Q} + \frac{2a^2\mathrm{J_{C2}}\beta}{t}\frac{\partial^2 \phi_x}{\partial R\partial Q}\right]$$

$$+\frac{\mathrm{B_{13}}}{\beta}\left[\frac{\mathrm{aJ_{C1}}}{\mathrm{t}}.\frac{\partial^3 \mathrm{w}}{\partial R^2\,\partial \mathrm{Q}} - 4a\mathrm{J_2}.\frac{\partial^3 w}{\partial Q\partial R^2} - 2a\mathrm{J_2}\beta\frac{\partial^3 w}{\partial R^3} - \frac{2a\mathrm{J_{C1}}\beta}{t}.\frac{\partial^3 w}{\partial R^3} + \frac{8a^2\mathrm{J_M}}{t^2}.\frac{\partial^2 \mathrm{u_0}}{\partial R\,\partial \mathrm{Q}} + \frac{4a^2\mathrm{J_{C2}}}{t}.\frac{\partial^2 \mathrm{u_0}}{\partial R\partial Q} + \frac{2a^2\beta \mathrm{J_M}}{t^2}.\frac{\partial^2 u_0}{\partial R^2} + \frac{2a^2\mathrm{J_{C2}}}{t}.\frac{\partial^2 \mathrm{u_0}}{\partial R^2} + \frac{2a^2\beta \mathrm{J_M}}{t^2}.\frac{\partial^2 v_0}{\partial R^2} + \frac{2a^2\mathrm{J_{C2}}\beta}{t}\frac{\partial^2 \mathrm{v_0}}{\partial R^2} + \frac{2a^2\mathrm{J_{C2}}}{t}.\frac{\partial^2 \upphi_\mathrm{x}}{\partial R\,\partial \mathrm{Q}} + 2a^2\mathrm{J_3}\beta.\frac{\partial^2 \phi_x}{\partial R^2} + \frac{2a^2\mathrm{J_{C2}}\beta}{t}\frac{\partial^2 \phi_x}{\partial R^2} + 2a^2\mathrm{J_3}\beta.\frac{\partial^2 \phi_y}{\partial R^2} + \frac{a^2\mathrm{J_{C2}}}{t}.\frac{\partial^2 \upphi_\mathrm{y}}{\partial R^2}\right]$$

$$+\frac{B_{22}}{\beta^3}\left[-2aJ_2.\frac{\partial^3 w}{\partial Q^3}-\frac{2aJ_{C1}}{t}.\frac{\partial^3 w}{\partial Q^3}+2a^2J_3\beta.\frac{\partial^2\phi_y}{\partial Q^2}+\frac{8a^2\beta J_M}{t^2}.\frac{\partial^2 v_0}{\partial Q^2}+\frac{2a^2J_{C2}\beta}{t}\frac{\partial^2 v_0}{\partial Q^2}\right.$$
$$\left.+\frac{2a^2\beta J_{C2}}{t}\frac{\partial^2\phi_y}{\partial Q^2}\right]$$

$$+\frac{B_{23}}{\beta^3}\left[-2aJ_2.\frac{\partial^3 w}{\partial Q^3}-\frac{2aS}{t}.\frac{\partial^3 w}{\partial Q^3}-6a\beta J_2.\frac{\partial^3 w}{\partial R\partial Q^2}-\frac{6a\beta J_{C1}}{t}.\frac{\partial^3 w}{\partial R\partial Q^2}+\frac{2a^2\beta J_{C2}}{t}.\frac{\partial^2 u_0}{\partial Q^2}\right.$$
$$+\frac{2a^2\beta J_M}{t^2}.\frac{\partial^2 u_0}{\partial Q^2}+\frac{2a^2\beta J_{C2}}{t}.\frac{\partial^2 v_0}{\partial Q^2}+\frac{4a^2\beta^2 J_{C2}}{t}\frac{\partial^2 v_0}{\partial Q\partial R}+\frac{2a^2\beta^2 J_M}{t^2}.\frac{\partial^2 v_0}{\partial Q\partial R}$$
$$+2\beta^2J_3.\frac{\partial^2\phi_y}{\partial Q\partial R}+\frac{4a^2\beta^2 J_{C2}}{t}\frac{\partial^2\phi_y}{\partial R\partial Q}+\frac{2a^2\beta J_M}{t^2}.\frac{\partial^2 v_0}{\partial Q^2}+2a^2\beta J_3.\frac{\partial^2\phi_x}{\partial Q^2}$$
$$\left.+\frac{2a^2\beta J_{C2}}{t}.\frac{\partial^2\phi_y}{\partial Q^2}+2a^2H^2\beta J_3.\frac{\partial^2\phi_y}{\partial Q^2}\right]$$

$$+\frac{B_{33}}{\beta^2}\left[-\frac{4a\beta J_{C1}}{t}.\frac{\partial^3 w}{\partial R^2\partial Q}-4aJ_2.\frac{\partial^3 w}{\partial R\partial Q^2}-4aJ_2\beta.\frac{\partial^3 w}{\partial R^2\partial Q}+\frac{2a^2\beta J_M}{t^2}.\frac{\partial^2 u_0}{\partial Q\partial R}\right.$$
$$+\frac{2a^2\beta J_{C2}}{t}.\frac{\partial^2 u_0}{\partial R\partial Q}+\frac{2a^2\beta^2 J_M}{t^2}.\frac{\partial^2 v_0}{\partial R^2}+\frac{2a^2\beta J_{C2}}{t}.\frac{\partial^2 v_0}{\partial R\partial Q}+\frac{2a^2\beta^2 J_{C2}}{t}.\frac{\partial^2 v_0}{\partial R^2}$$
$$+\frac{2a^2\beta J_{C2}}{t}.\frac{\partial^2\phi_x}{\partial R\partial Q}+2a^2\beta J_3.\frac{\partial\phi_x}{\partial Q\partial R}+2a^2J_3.\frac{\partial^2\phi_x}{\partial Q^2}+\frac{2a^2J_{C2}}{t}.\frac{\partial^2 u_0}{\partial Q^2}$$
$$\left.+2a^2\beta^2J_3\frac{\partial^2\phi_y}{\partial R^2}+2a^2\beta J_3.\frac{\partial^2\phi_y}{\partial R\partial Q}+\frac{2a^2\beta^2 J_{C2}}{t}.\frac{\partial^2\phi_y}{\partial R^2}\right]$$

$$+2B_{44}\frac{a^4}{t^2}\left(\frac{\partial^2 H}{\partial S^2}.\emptyset_x+\frac{\partial^2 u_0}{\partial S^2}\right)+B_{55}\frac{a^4}{t^2}\left(\frac{\partial^2 H}{\partial S^2}.\emptyset_y+\frac{\partial^2 v_0}{\partial S^2}\right)$$
$$+2B_{55}\frac{a^4}{t^2}\left(\emptyset_y.\left(\frac{dH}{dS}\right)^2+\frac{dv_0}{dS}.\frac{\partial H}{\partial S}\right)$$

$$+2B_{44}\frac{a^4}{t^2}\left(\emptyset_x.\left(\frac{dH}{dS}\right)^2+\frac{du_0}{dS}.\frac{\partial H}{\partial S}\right)=0 \qquad (3.300)$$

*T*he quantities, J_i was then expressed in terms of g_i by multiply each by 12. That is:

$$g_1=12J_1=12\times\tfrac{1}{3}\textstyle\sum_{m=1}^{m=n}(S_m^3-S_{m-1}^3)=4\sum_{m=1}^{m=n}(S_m^3-S_{m-1}^3) \qquad (3.301\ )$$

$$g_2=12J_2=12\times\frac{1}{3}\sum_{m=1}^{m=n}\left[\left(S_m^3-\frac{4}{5}S_m^5\right)-\left(S_{m-1}^3-\frac{4}{5}S_{m-1}^5\right)\right]$$

$$= 4\sum_{m=1}^{m=n}\left[\left(S_m^3 - \frac{4}{5}S_m^5\right) - \left(S_{m-1}^3 - \frac{4}{5}S_{m-1}^5\right)\right] \tag{3.302}$$

$$\begin{aligned} g_3 = 12J_3 = 12 \times \frac{1}{3}\sum_{m=1}^{m=n}\left[\left(S_m^3 - \frac{8}{5}S_m^5 + \frac{16}{21}S_m^7\right) - \left(S_{m-1}^3 - \frac{8}{5}S_{m-1}^5 + \frac{16}{21}S_{m-1}^7\right)\right] \\ = 4\sum_{m=1}^{m=n}\left[\left(S_m^3 - \frac{8}{5}S_m^5 + \frac{16}{21}S_m^7\right)\right. \\ \left. - \left(S_{m-1}^3 - \frac{8}{5}S_{m-1}^5 + \frac{16}{21}S_{m-1}^7\right)\right] \end{aligned} \tag{3.303}$$

$$\begin{aligned} g_4 = 12J_4 = 12 \times \frac{1}{3}\sum_{m=1}^{m=n}\left[\left(3S_m^1 - 8S_m^3 + \frac{48}{5}S_m^5\right) - \left(3S_{m-1}^1 - 8S_{m-1}^3 + \frac{48}{5}S_{m-1}^5\right)\right] \\ = 4\sum_{m=1}^{m=n}\left[\left(3S_m^1 - 8S_m^3 + \frac{48}{5}S_m^5\right)\right. \\ \left. - \left(3S_{m-1}^1 - 8S_{m-1}^3 + \frac{48}{5}S_{m-1}^5\right)\right] \end{aligned} \tag{3.304}$$

$$g_{C1} = 12J_{C1} = 12 \times \frac{1}{2}\sum_{m=1}^{m=n}(S_m^2 - S_{m-1}^2) = 6\sum_{m=1}^{m=n}(S_m^2 - S_{m-1}^2) \tag{3.305}$$

$$\begin{aligned} g_{C2} = 12J_{C2} = 12 \times \frac{1}{2}\sum_{m=1}^{m=n}\left[\left(S_m^2 - \frac{2}{3}S_m^4\right) - \left(S_{m-1}^2 - \frac{2}{3}S_{m-1}^4\right)\right] \\ = 6\sum_{m=1}^{m=n}\left[\left(S_m^2 - \frac{2}{3}S_m^4\right) - \left(S_{m-1}^2 - \frac{2}{3}S_{m-1}^4\right)\right] \end{aligned} \tag{3.306}$$

$$g_{C3} = 12J_{C3} = 12\sum_{m=1}^{m=n}\left[\left(S_m^1 - \frac{4}{3}S_m^3\right) - \left(S_{m-1}^1 - \frac{4}{3}S_{m-1}^3\right)\right] \tag{3.307}$$

$$g_M = 12J_{C4} = 12\sum_{m=1}^{m=n}[(S_m^1) - (S_{m-1}^1)] \tag{3.308}$$

and substituting These values of g_{ij} back into Eqaiton (3.308) that gives

$$B_{11}\left[-\frac{2ag_{C1}}{t}.\frac{\partial^3 w}{\partial R^3}-2ag_2.\frac{\partial^3 w}{\partial R^3}+\frac{2a^2g_M}{t^2}.\frac{\partial^2 u_0}{\partial R^2}+\frac{2a^2g_{C2}}{t}.\frac{\partial^2 u_0}{\partial R^2}+\frac{2a^2g_{C2}}{t}.\frac{\partial^2 \phi_x}{\partial R^2}\right.$$
$$\left.+2a^2g_1.\frac{\partial^2 \phi_x}{\partial R^2}\right]$$

$$+\frac{B_{12}}{\beta^2}\left[-\frac{2ag_{C1}}{t}\frac{\partial^3 w}{dR\partial Q^2}-2ag_2\beta.\frac{\partial^3 w}{\partial Q\,\partial R^2}-2ag_2\frac{\partial^3 w}{\partial R\partial Q^2}-\frac{2ag_{C1}\beta}{t}.\frac{\partial^3 w}{\partial Q\partial R^2}\right.$$
$$+\frac{2a^2g_{C2}\beta}{t}.\frac{\partial^2 u_0}{\partial R\partial Q}+\frac{2a^2\beta g_M}{t^2}.\frac{\partial^2 v_0}{dR\partial Q}+\frac{2a^2g_{C2}\beta}{t}\frac{\partial^2 v_0}{\partial R\partial Q}+\frac{2a^2g_M\beta}{t^2}\frac{\partial^2 v_0}{dR\partial Q}$$
$$\left.+2a^2g_3\beta.\frac{\partial^2 \phi_x}{\partial R\,\partial Q}+2a^2g_3\beta.\frac{\partial^2 \phi_y}{\partial R\partial Q}+\frac{2a^2g_{C2}\beta}{t}.\frac{\partial^2 \phi_y}{\partial R\partial Q}+\frac{2a^2g_{C2}\beta}{t}\frac{\partial^2 \phi_x}{\partial R\partial Q}\right]$$

$$+\frac{B_{13}}{\beta}\left[\frac{ag_{C1}}{t}.\frac{\partial^3 w}{\partial R^2\,\partial Q}-4ag_2.\frac{\partial^3 w}{\partial Q\partial R^2}-2ag_2\beta\frac{\partial^3 w}{\partial R^3}-\frac{2ag_{C1}\beta}{t}.\frac{\partial^3 w}{\partial R^3}+\frac{8a^2g_M}{t^2}.\frac{\partial^2 u_0}{\partial R\,\partial Q}\right.$$
$$+\frac{4a^2g_{C2}}{t}.\frac{\partial^2 u_0}{\partial R\partial Q}+\frac{2a^2\beta g_M}{t^2}.\frac{\partial^2 u_0}{\partial R^2}+\frac{2a^2g_{C2}}{t}.\frac{\partial^2 u_0}{\partial R^2}+\frac{2a^2\beta g_M}{t^2}.\frac{\partial^2 v_0}{\partial R^2}$$
$$+\frac{2a^2g_{C2}\beta}{t}\frac{\partial^2 v_0}{\partial R^2}+\frac{2a^2g_{C2}}{t}.\frac{\partial^2 \phi_x}{\partial R\,\partial Q}+2a^2g_3\beta.\frac{\partial^2 \phi_x}{\partial R^2}+\frac{2a^2g_{C2}\beta}{t}\frac{\partial^2 \phi_x}{\partial R^2}$$
$$\left.+2a^2g_3\beta.\frac{\partial^2 \phi_y}{\partial R^2}+\frac{a^2g_{C2}}{t}.\frac{\partial^2 \phi_y}{\partial R^2}\right]$$

$$+\frac{B_{22}}{\beta^3}\left[-2ag_2.\frac{\partial^3 w}{\partial Q^3}-\frac{2ag_{C1}}{t}.\frac{\partial^3 w}{\partial Q^3}+2a^2g_3\beta.\frac{\partial^2 \phi_y}{\partial Q^2}+\frac{8a^2\beta g_M}{t^2}.\frac{\partial^2 v_0}{\partial Q^2}+\frac{2a^2g_{C2}\beta}{t}\frac{\partial^2 v_0}{\partial Q^2}\right.$$
$$\left.+\frac{2a^2\beta g_{C2}}{t}\frac{\partial^2 \phi_y}{\partial Q^2}\right]$$

$$+\frac{B_{23}}{\beta^3}\left[-2ag_2.\frac{\partial^3 w}{\partial Q^3}-\frac{2aS}{t}.\frac{\partial^3 w}{\partial Q^3}-6a\beta g_2.\frac{\partial^3 w}{\partial R\partial Q^2}-\frac{6a\beta g_{C1}}{t}.\frac{\partial^3 w}{\partial R\partial Q^2}+\frac{2a^2\beta g_{C2}}{t}.\frac{\partial^2 u_0}{\partial Q^2}\right.$$
$$+\frac{2a^2\beta g_M}{t^2}.\frac{\partial^2 u_0}{\partial Q^2}+\frac{2a^2\beta g_{C2}}{t}.\frac{\partial^2 v_0}{\partial Q^2}+\frac{4a^2\beta^2g_{C2}}{t}\frac{\partial^2 v_0}{\partial Q\partial R}+\frac{2a^2\beta^2g_M}{t^2}.\frac{\partial^2 v_0}{\partial Q\partial R}$$
$$+2\beta^2g_3.\frac{\partial^2 \phi_y}{\partial Q\partial R}+\frac{4a^2\beta^2g_{C2}}{t}\frac{\partial^2 \phi_y}{\partial R\partial Q}+\frac{2a^2\beta g}{t^2}.\frac{\partial^2 v_0}{\partial Q^2}+2a^2\beta g_3.\frac{\partial^2 \phi_x}{\partial Q^2}$$
$$\left.+\frac{2a^2\beta g_{C2}}{t}.\frac{\partial^2 \phi_y}{\partial Q^2}+2a^2H^2\beta g_3.\frac{\partial^2 \phi_y}{\partial Q^2}\right]$$

$$+\frac{B_{33}}{\beta^2}\left[-\frac{4a\beta g_{C1}}{t}.\frac{\partial^3 w}{\partial R^2\partial Q}-4ag_2.\frac{\partial^3 w}{\partial R\partial Q^2}-4ag_2\beta.\frac{\partial^3 w}{\partial R^2\partial Q}+\frac{2a^2\beta g_M}{t^2}.\frac{\partial^2 u_0}{\partial Q\partial R}\right.$$

$$+\frac{2a^2\beta g_{C2}}{t}.\frac{\partial^2 u_0}{\partial R\partial Q}+\frac{2a^2\beta^2 g_M}{t^2}.\frac{\partial^2 v_0}{\partial R^2}+\frac{2a^2\beta g_{C2}}{t}.\frac{\partial^2 v_0}{\partial R\partial Q}+\frac{2a^2\beta^2 g_{C2}}{t}.\frac{\partial^2 v_0}{\partial R^2}$$

$$+\frac{2a^2\beta g_{C2}}{t}.\frac{\partial^2 \phi_x}{\partial R\partial Q}+2a^2\beta g_3.\frac{\partial \phi_x}{\partial Q\partial R}+2a^2 g_3.\frac{\partial^2 \phi_x}{\partial Q^2}+\frac{2a^2 g_{C2}}{t}.\frac{\partial^2 u_0}{\partial Q^2}$$

$$\left.+2a^2\beta^2 g_3\frac{\partial^2 \phi_y}{\partial R^2}+2a^2\beta g_3.\frac{\partial^2 \phi_y}{\partial R\partial Q}+\frac{2a^2\beta^2 g_{C2}}{t}.\frac{\partial^2 \phi_y}{\partial R^2}\right]$$

$$+2B_{44}\frac{a^4}{t^2}\left(\frac{\partial^2 H}{\partial S^2}.\emptyset_x+\frac{\partial^2 u_0}{\partial S^2}\right)+B_{55}\frac{a^4}{t^2}\left(\frac{\partial^2 H}{\partial S^2}.\emptyset_y+\frac{\partial^2 v_0}{\partial S^2}\right)$$

$$+2B_{55}\frac{a^4}{t^2}\left(\emptyset_y.\left(\frac{dH}{dS}\right)^2+\frac{dv_0}{dS}.\frac{\partial H}{\partial S}\right)+\ 2B_{44}\frac{a^4}{t^2}\left(\emptyset_x.\left(\frac{dH}{dS}\right)^2+\frac{du_0}{dS}.\frac{\partial H}{\partial S}\right)=0 \qquad (3.309)$$

One of the conditions of having the Equation (3.309) as zero is when the individual terms in the equation are considered as zero, and so considering the expression interms of coefficient of B gives.

$$B_{11}\left[-\frac{2ag_{C1}}{t}.\frac{\partial^3 w}{\partial R^3}-\frac{2ag_2}{t}.\frac{\partial^3 w}{\partial R^3}+\frac{2a^2 g_M}{t^2}.\frac{\partial^2 u_0}{\partial R^2}+\frac{2a^2 g_{C2}}{t}.\frac{\partial^2 u_0}{\partial R^2}+\frac{2a^2 g_{C2}}{t}.\frac{\partial^2 \phi_x}{\partial R^2}\right.$$

$$\left.+\frac{2a^2 g_1}{t}.\frac{\partial^2 \phi_x}{\partial R^2}\right]=0 \qquad (3.310)$$

$$+\frac{B_{12}}{\beta^2}\left[-\frac{2ag_{C1}}{t}\frac{\partial^3 w}{dR\partial Q^2}-\frac{2ag_2}{t}\beta.\frac{\partial^3 w}{\partial Q\,\partial R^2}-\frac{2ag_2}{t}\frac{\partial^3 w}{\partial R\partial Q^2}-\frac{2ag_{C1}\beta}{t}.\frac{\partial^3 w}{\partial Q\partial R^2}\right.$$

$$+\frac{2a^2 g_{C2}\beta}{t}.\frac{\partial^2 u_0}{\partial R\partial Q}+\frac{2a^2\beta g_M}{t^2}.\frac{\partial^2 v_0}{dR\partial Q}+\frac{2a^2 g_{C2}\beta}{t}\frac{\partial^2 v_0}{\partial R\partial Q}+\frac{2a^2 g_M\beta}{t^2}\frac{\partial^2 v_0}{dR\partial Q}$$

$$\left.+2a^2 g_3\beta.\frac{\partial^2 \phi_x}{\partial R\,\partial Q}+2a^2 g_3\beta.\frac{\partial^2 \phi_y}{\partial R\partial Q}+\frac{2a^2 g_{C2}\beta}{t}.\frac{\partial^2 \phi_y}{\partial R\partial Q}+\frac{2a^2 g_{C2}\beta}{t}\frac{\partial^2 \phi_x}{\partial R\partial Q}\right]$$

$$=0 \qquad (3.311)$$

For the purpose of the establishing the relationship between the deflection, middle-layer , displacement and shear rotation as demenotrated in Sub-section **3.2.5.1,** Equation (3.311) was broken down as shown in Equations (3.312 to (3.324)).That gives:

$$-\frac{2ag_{C1}}{t}\frac{\partial^3 w}{dR\partial Q^2}-2ag_2\beta.\frac{\partial^3 w}{\partial Q\,\partial R^2}-2ag_2\frac{\partial^3 w}{\partial R\partial Q^2}-\frac{2ag_{C1}\beta}{t}.\frac{\partial^3 w}{\partial Q\partial R^2}$$

$$+\frac{2a^2 g_{C2}\beta}{t}.\frac{\partial^2 u_0}{\partial R\partial Q}=0. \qquad (3.312)$$

Factorizing Equation (3.312) gives

$$\frac{dw g_{C1}}{dR}\left[-\frac{2a}{t}\frac{\partial^2 w}{\partial Q^2}-\frac{2a\beta}{t}\cdot\frac{\partial^2 w}{\partial Q\partial R}\right]=\left[\frac{2a^2 g_{C2}\beta}{t}\cdot\frac{\partial^2}{\partial R\partial Q}\right]u_0 \quad (3.313)$$

$$\frac{dw}{dR}\left[-2ag_2\beta.\frac{\partial^2 w}{\partial Q\partial R}-2ag_2\frac{\partial^2 w}{\partial Q^2}\right]=\left[\frac{2a^2 g_{C2}\beta}{t}\cdot\frac{\partial^2}{\partial R\partial Q}\right]u_0 \quad (3.314)$$

$$-\frac{2ag_{C1}}{t}\frac{\partial^3 w}{dR\partial Q^2}-2ag_2\beta.\frac{\partial^3 w}{\partial Q\,\partial R^2}-2ag_2\frac{\partial^3 w}{\partial R\partial Q^2}-\frac{2ag_{C1}\beta}{t}\cdot\frac{\partial^3 w}{\partial Q\partial R^2}$$

$$+\frac{2a^2\beta g_M}{t^2}\cdot\frac{\partial^2 v_0}{dR\partial Q}+\frac{2a^2 g_{C2}\beta}{t}\frac{\partial^2 v_0}{\partial R\partial Q}+\frac{2a^2 g_M\beta}{t^2}\frac{\partial^2 v_0}{dR\partial Q}=0 \quad (3.315)$$

$$-\frac{2ag_{C1}}{t}\frac{\partial^3 w}{dR\partial Q^2}-2ag_2\beta.\frac{\partial^3 w}{\partial Q\,\partial R^2}-2ag_2\frac{\partial^3 w}{\partial R\partial Q^2}-\frac{2ag_{C1}\beta}{t}.$$

$$\frac{\partial^3 w}{\partial Q\partial R^2}2a^2 g_3\beta.\frac{\partial^2\phi_x}{\partial R\,\partial Q}+\frac{2a^2 g_{C2}\beta}{t}\frac{\partial^2\phi_x}{\partial R\partial Q}=0 \quad (3.316)$$

$$-\frac{2ag_{C1}}{t}\frac{\partial^3 w}{dR\partial Q^2}-2ag_2\beta.\frac{\partial^3 w}{\partial Q\,\partial R^2}-2ag_2\frac{\partial^3 w}{\partial R\partial Q^2}-\frac{2ag_{C1}\beta}{t}\cdot\frac{\partial^3 w}{\partial Q\partial R^2}$$

$$+2a^2 g_3\beta.\frac{\partial^2\phi_y}{\partial R\partial Q}+\frac{2a^2 g_{C2}\beta}{t}\cdot\frac{\partial^2\phi_y}{\partial R\partial Q}=0 \quad (3.317)$$

$$\frac{2a^2 g_{C2}\beta}{t}\cdot\frac{\partial^2\phi_y}{\partial R\,\partial Q}+\frac{2a^2 g_{C2}\beta}{t}\cdot\frac{\partial^2 u_0}{\partial R\,\partial Q}+\frac{2a^2 g_{C2}\beta}{t}\frac{\partial^2\phi_x}{\partial R\,\partial Q}+\frac{2a^2 g_{C2}\beta}{t}\cdot\frac{\partial^2 u_0}{\partial R\,\partial Q}=$$

$$\frac{2ag_{C1}\beta}{t}\cdot\frac{\partial^3 w}{\partial Q\,\partial R^2}+\frac{2ag_{C1}\beta}{t}\cdot\frac{\partial^3 w}{\partial Q\,\partial R^2} \quad (3.318)$$

$$\left[\frac{2a^2\beta}{t}\cdot\frac{\partial^2\phi_y}{\partial R\partial Q}+\frac{2a^2\beta}{t}\cdot\frac{\partial^2 u_0}{\partial R\partial Q}+\frac{2a^2\beta}{t}\frac{\partial^2\phi_x}{\partial R\partial Q}+\frac{2a^2\beta}{t}\cdot\frac{\partial^2 u_0}{\partial R\partial Q}\right]g_{C2}=$$

$$\left[\frac{2a\beta}{t}\cdot\frac{\partial^3 w}{\partial Q\partial R^2}+\frac{2a\beta}{t}\cdot\frac{\partial^3 w}{\partial Q\partial R^2}\right]g_{C1} \quad (3.319)$$

$$\frac{B_{13}}{\beta}\left[\frac{ag_{C1}}{t}.\frac{\partial^3 w}{\partial R^2\,\partial Q}-4ag_2.\frac{\partial^3 w}{\partial Q\,\partial R^2}-2ag_2\beta\frac{\partial^3 w}{\partial R^3}-\frac{2ag_{C1}\beta}{t}.\frac{\partial^3 w}{\partial R^3}+\frac{8a^2g_M}{t^2}.\frac{\partial^2 u_0}{\partial R\,\partial Q}\right.$$
$$+\frac{4a^2g_{C2}}{t}.\frac{\partial^2 u_0}{\partial R\,\partial Q}+\frac{2a^2\beta g_M}{t^2}.\frac{\partial^2 u_0}{\partial R^2}+\frac{2a^2g_{C2}}{t}.\frac{\partial^2 u_0}{\partial R^2}+\frac{2a^2\beta g_M}{t^2}.\frac{\partial^2 v_0}{\partial R^2}$$
$$+\frac{2a^2g_{C2}\beta}{t}\frac{\partial^2 v_0}{\partial R^2}+\frac{2a^2g_{C2}}{t}.\frac{\partial^2 \phi_x}{\partial R\,\partial Q}+2a^2g_3\beta.\frac{\partial^2 \phi_x}{\partial R^2}+\frac{2a^2g_{C2}\beta}{t}\frac{\partial^2 \phi_x}{\partial R^2}$$
$$\left.+2a^2g_3\beta.\frac{\partial^2 \phi_y}{\partial R^2}+\frac{a^2g_{C2}}{t}.\frac{\partial^2 \phi_y}{\partial R^2}\right]=\frac{g_{C1}}{g_{C2}}=0 \qquad (3.320)$$

$$\frac{B_{22}}{\beta^3}\left[-2ag_2.\frac{\partial^3 w}{\partial Q^3}-\frac{2ag_{C1}}{t}.\frac{\partial^3 w}{\partial Q^3}+2a^2g_3\beta.\frac{\partial^2 \phi_y}{\partial Q^2}+\frac{8a^2\beta g_M}{t^2}.\frac{\partial^2 v_0}{\partial Q^2}+\frac{2a^2g_{C2}\beta}{t}\frac{\partial^2 v_0}{\partial Q^2}\right.$$
$$\left.+\frac{2a^2\beta g_{C2}}{t}\frac{\partial^2 \phi_y}{\partial Q^2}\right]=0 \qquad (3.321)$$

$$+\frac{B_{23}}{\beta^3}\left[-2ag_2.\frac{\partial^3 w}{\partial Q^3}-\frac{2aS}{t}.\frac{\partial^3 w}{\partial Q^3}-6a\beta g_2.\frac{\partial^3 w}{\partial R\partial Q^2}-\frac{6a\beta g_{C1}}{t}.\frac{\partial^3 w}{\partial R\partial Q^2}+\frac{2a^2\beta g_{C2}}{t}.\frac{\partial^2 u_0}{\partial Q^2}\right.$$
$$+\frac{2a^2\beta g_M}{t^2}.\frac{\partial^2 u_0}{\partial Q^2}+\frac{2a^2\beta g_{C2}}{t}.\frac{\partial^2 v_0}{\partial Q^2}+\frac{4a^2\beta^2 g_{C2}}{t}\frac{\partial^2 v_0}{\partial Q\partial R}+\frac{2a^2\beta^2 g_M}{t^2}.\frac{\partial^2 v_0}{\partial Q\partial R}$$
$$+2\beta^2g_3.\frac{\partial^2 \phi_y}{\partial Q\partial R}+\frac{4a^2\beta^2 g_{C2}}{t}\frac{\partial^2 \phi_y}{\partial R\partial Q}+\frac{2a^2\beta g}{t^2}.\frac{\partial^2 v_0}{\partial Q^2}+2a^2\beta g_3.\frac{\partial^2 \phi_x}{\partial Q^2}$$
$$\left.+\frac{2a^2\beta g_{C2}}{t}.\frac{\partial^2 \phi_y}{\partial Q^2}+2a^2H^2\beta g_3.\frac{\partial^2 \phi_y}{\partial Q^2}\right]=0 \qquad (3.322)$$

$$+\frac{B_{33}}{\beta^2}\left[-\frac{4a\beta g_{C1}}{t}.\frac{\partial^3 w}{\partial R^2\partial Q}-4ag_2.\frac{\partial^3 w}{\partial R\partial Q^2}-4ag_2\beta.\frac{\partial^3 w}{\partial R^2\partial Q}+\frac{2a^2\beta g_M}{t^2}.\frac{\partial^2 u_0}{\partial Q\partial R}\right.$$
$$+\frac{2a^2\beta g_{C2}}{t}.\frac{\partial^2 u_0}{\partial R\partial Q}+\frac{2a^2\beta^2 g_M}{t^2}.\frac{\partial^2 v_0}{\partial R^2}+\frac{2a^2\beta g_{C2}}{t}.\frac{\partial^2 v_0}{\partial R\partial Q}+\frac{2a^2\beta^2 g_{C2}}{t}.\frac{\partial^2 v_0}{\partial R^2}$$
$$+\frac{2a^2\beta g_{C2}}{t}.\frac{\partial^2 \phi_x}{\partial R\partial Q}+2a^2\beta g_3.\frac{\partial \phi_x}{\partial Q\partial R}+2a^2g_3.\frac{\partial^2 \phi_x}{\partial Q^2}+\frac{2a^2g_{C2}}{t}.\frac{\partial^2 u_0}{\partial Q^2}$$
$$\left.+2a^2\beta^2g_3\frac{\partial^2 \phi_y}{\partial R^2}+2a^2\beta g_3.\frac{\partial^2 \phi_y}{\partial R\partial Q}+\frac{2a^2\beta^2 g_{C2}}{t}.\frac{\partial^2 \phi_y}{\partial R^2}\right]=0 \qquad (3.323)$$

$$+4\frac{a^4}{t^2}\left[B_{44}\left(g_4{}^2.\emptyset_x+\frac{\partial^2 u_0}{\partial S^2}\right)+B_{55}\left(g_4{}^2.\emptyset_y+\frac{\partial^2 v_0}{\partial S^2}\right)\right]=0 \qquad (3.324)$$

3.2.5.1 Establishment of the Relationship Between the Deflection, Middle-Layer Displacements and Shear Rotations

The compatibility equations were further factorize to derive the relationship between deflection, w *and* $u_0, v_0, \emptyset_x, \emptyset_y$The values obtained shall be substituted into the governing equation . That is the equilibrium equation of forces, which shows that the sum of action and reaction is zero. That gives

From Equations (3.319) and (3.318), it was gotten that $u_0 = \emptyset_x$,

also from Equations (3.319), (3.321), (3.322), (3.323), it was confirmed that $v_0 = \emptyset_y$

From Equations (3.320) and (3.324) $\emptyset_x = -\frac{B_{55}\emptyset_y.}{B_{44}}$ also $1 = -\frac{\phi_y}{\phi_x}$ from Equation (3.320)

From Equations (3.617), (3.618) it is obtained that:

$$u_0 = \frac{g_{C1}t}{ag_{C2}} \cdot \frac{\partial w}{\partial R} = \frac{r_1}{a} \cdot \frac{\partial w}{\partial R} \tag{3.325}$$

Similarly in Equations (3.319) , (3.321), (3.322) and (3.323) it is obtained that:

$$v_0 = \frac{g_{C1}t}{ag_{C2}\beta} \cdot \frac{\partial w}{\partial Q} = \frac{r_2}{a\beta} \cdot \frac{\partial w}{\partial Q} \tag{3.326}$$

Also in Equations (3.321) , (3.322), (3.319) and (3.323) it is obtained that:

$$\phi_y = \frac{g_{C1}}{ag_{C2}\beta} \cdot \frac{\partial w}{\partial Q} = \frac{r_3}{a\beta} \cdot \frac{\partial w}{\partial R} \tag{3.327}$$

And finally in Equations (3.318) and (3.319) it is obtained that:

$$\phi_x = \frac{g_{c1}}{ag_{C2}} \cdot \frac{\partial w}{\partial R} = \frac{r_4}{a} \cdot \frac{\partial w}{\partial R} \tag{3.328}$$

From Equations (3.320), (3.321), (3.322) it is obtained that:

$$\phi_x = -1 * \phi_y \tag{3.329}$$

Also from the same Equation (3.324) it can be shown that:

$$\emptyset_x = -\frac{B_{55}}{B_{44}} * \emptyset_y \tag{3.330}$$

where $m = -\frac{B_{55}}{B_{44}}$ (3.331)

and also

$$u_0 = -\frac{B_{55}}{2B_{44}} * v_0 \tag{3.332}$$

Equation (3.324) also gives :

$$u_0 = \frac{g_{C3}}{J_M} \frac{B_{55}}{B_{44}} . v_0 \tag{3.333}$$

Similarly let the coefficient be m_{11} and that gives:

$$u_0 = -m_{11} * v_0 \tag{3.334}$$

$$\int_0^1 \int_0^1 \{[A] + [B]\} dRdQ = 0 \tag{3.335}$$

where $[A] = \left[\frac{\partial^4 w}{\partial R^4} + G_2 . \frac{2}{\beta^2} . \frac{\partial^4 w}{\partial R^2 \partial Q^2} + G_3 . \frac{1}{\beta^4} . \frac{\partial^4 w}{\partial Q^4} - G_6 \frac{N_x a^2}{D_0}\right]$ (3.336)

$$[B] = \frac{1}{\beta} . \frac{\partial^2 w}{\partial R \partial Q} \left[G_4 . \frac{\partial^2 w}{\partial R^2} + \frac{G_5}{\beta^2} . \frac{\partial^2 w}{\partial Q^2}\right] \tag{3.337}$$

and one of the true conditions of Equation (3.327) is when

$$\int_0^1 \int_0^1 \left[\frac{\partial^4 w}{\partial R^4} + G_2 . \frac{2}{\beta^2} . \frac{\partial^4 w}{\partial R^2 \partial Q^2} + G_3 . \frac{1}{\beta^4} . \frac{\partial^4 w}{\partial Q^4} - G_6 \frac{N_x a^2}{D_0}\right] dRdQ = 0 \tag{3.338}$$

$$\int_0^1 \int_0^1 \left[G_4 . \frac{\partial^2 w}{\partial R^2} + \frac{G_5}{\beta^2} . \frac{\partial^2 w}{\partial Q^2}\right] dRdQ = 0 \tag{3.339}$$

From Equation (3.638) comes $\frac{\partial^2 w}{\partial R^2} = -\frac{G_5}{G_4 \beta^2} . \frac{\partial^2 w}{\partial Q^2} = 0$ (3.340)

Considering a constant "1" as the summation of three constant values; q_1, q_2 and q_3 in an equation given as: $q_1 + q_2 + q_3 = 1$ (3.341)

and defining orthogonal deflection in terms of split deflection as

$$w = w_x * w_y \tag{3.342}$$

bringing deflection on both x and y components gives

$$\int_0^1\int_0^1\Big[w_y * \frac{\partial^4 w_x}{\partial R^4} + G_2 * \frac{2}{\beta^2} * \frac{\partial^2 w_x}{\partial R^2} * \frac{\partial^2 w_y}{\partial Q^2} + w_x * G_3 * \frac{1}{\beta^4} * \frac{\partial^4 w_y}{\partial Q^4} - G_6\,\frac{N_x a^2}{D_0}(q_1 + q_2 + q_3)\Big]\,dRdQ = 0\,. \quad (3.343)$$

and breaking this further gives

$$\int_0^1\int_0^1\left[\left(w_y * \frac{\partial^4 w_x}{\partial R^4} - G_6\,\frac{N_x a^2}{D_0} q_1\right) + \left(G_2 * \frac{2}{\beta^2} * \frac{\partial^2 w_x}{\partial R^2} * \frac{\partial^2 w_y}{\partial Q^2} - G_6\,\frac{N_x a^2}{D_0}\,q_2\right) + \left(w_x * G_3 * \frac{1}{\beta^4} * \frac{\partial^4 w_y}{\partial Q^4} - G_6\,\frac{N_x a^2}{D_0}\right) q_3\right] dRdQ = 0 \quad (3.344)$$

Just like in the previous case, for Equation (3.344)to be zero, is simply for each term in the equation to be considered as zero. That is:

$$\int_0^1\int_0^1\left(w_y * \frac{\partial^4 w_x}{\partial R^4} - G_6\,\frac{N_x a^2}{D_0} q_1\right) dRdQ = 0 \quad (3.345)$$

$$\int_0^1\int_0^1\left(G_2 * \frac{2}{\beta^2} * \frac{\partial^2 w_x}{\partial R^2} * \frac{\partial^2 w_y}{\partial Q^2} - G_6\,\frac{N_x a^2}{D_0}\,q_2\right) dRdQ = 0 \quad (3.346)$$

$$\int_0^1\int_0^1\left(w_x * G_3 * \frac{1}{\beta^4} * \frac{\partial^4 w_y}{\partial Q^4} - G_6\,\frac{N_x a^2}{D_0} q_3\right) dRdQ = 0 \quad (3.347)$$

Conducting closed domain integration in equation (3.345) and (3.346) for the case of Q and R respectively gives

$$\int_0^1\left(w_3 * \frac{\partial^4 w_x}{\partial R^4} - G_6\,\frac{N_x a^2}{D_0} q_1\right) dR = 0 \quad (3.348)$$

and

$$\int_0^1\int_0^1\left(w_1 * G_3 * \frac{1}{\beta^4} * \frac{\partial^4 w_y}{\partial Q^4} - G_6\,\frac{N_x a^2}{D_0} q_3\right) dR = 0 \quad (3.349)$$

The integrands in equations (3.348)and (3.349) must be zero for it to be true, with the w_3 and w_1 as the constants, expressed as

$$w_3 = \int_0^1 w_y * dQ \tag{3.350}$$

$$w_1 = \int_0^1 w_x * dR \tag{3.351}$$

For Equation (3.348), the integrands is expressed as

$$\frac{\partial^4 w_x}{\partial R^4} = G_6 \frac{N_x a^2}{D_0} * \frac{q_1}{w_3} \tag{3.352}$$

For Equation (3.351), the integrands is expressed as

$$G_6 \frac{N_x a^2}{D_0} * \frac{\beta^4 q_3}{G_3 w_1} \tag{3.353}$$

The ready solution for Equation (3.352) can be derived as shown below as

$$\frac{\partial^4 w}{\partial R^4} = \frac{N_x a^2}{D_0} * \frac{G_6\, q_1}{w_3} * R^0 \tag{3.354}$$

$$\frac{\partial^3 w}{\partial R^3} = \frac{N_x a^2}{D_0} * \frac{G_6\, q_1}{w_3} * R + a_3 \tag{3.355}$$

$$\frac{\partial^2 w}{\partial R^2} = \frac{N_x a^2}{D_0} * \frac{G_6\, q_1}{w_3} * \frac{R^2}{2} + Ra_3 + R^0 a_2 \tag{3.356}$$

$$\frac{\partial w}{\partial R} = \frac{N_x a^2}{D_0} * \frac{G_6\, q_1}{w_3} * \frac{R^3}{6} + \frac{R^2 a_3}{2} + Ra_2 + R^0 a_1 \tag{3.357}$$

and then the deflection equation is given as

$$w_x = \frac{N_x a^2}{D_0} * \frac{G_6\, q_1}{w_3} * \frac{R^4}{24} + \frac{R^3 a_3}{6} + \frac{R^2 a_2}{2} + Ra_1 + R^0 a_0 \tag{3.358}$$

Similarly

$$w_y = \frac{N_x a^2}{D_0} * \frac{G_6\, q_1}{w_1} * \frac{Q^4}{24} + \frac{Q^3 b_3}{6} + \frac{Q^2 b_2}{2} + Qb_1 + Q^0 b_0 \tag{3.359}$$

Expressing the Equations (3.358) and (3.359) in matrix form gives

$$w_x = [R^0\ R^1\ R^2\ R^3\ R^4] * \begin{bmatrix} a_0 \\ a_1 \\ \frac{a_2}{2} \\ \frac{a_3}{6} \\ \frac{N_x a^2}{24D_{11}}.\frac{q_1}{w_3} \end{bmatrix} \tag{3.360}$$

Similarly,

$$w_y = [Q^0\ Q^1\ Q^2\ Q^3\ Q^4] * \begin{bmatrix} b_0 \\ b_1 \\ \frac{b_2}{2} \\ \frac{b_3}{6} \\ \frac{N_x a^2}{24D_{22}}.\frac{q_3}{w_1} \end{bmatrix} \tag{3.361}$$

Recall that deflection, w is the product of the shape function, h and the Amplitude, A

That is: $w = [h] * [A]$ (3.362)

and so $w = [h_x][A_{1x}] * [h_y][A_{1y}]$ (3.363)

and finally $w = h * A_1$ (3.364)

Putting these values back into the Equations (3.624), (3.625), (3.626) gives

$$u_0 = \frac{r_2}{a}.\frac{\partial h * P_1}{\partial R} = \frac{r_2}{a}.\frac{\partial h * A_1}{\partial R} \tag{3.365}$$

$$= \frac{A_1 * r_2}{a}.\frac{\partial h}{\partial R} \tag{3.366}$$

$$\text{and } v_0 = \frac{r_3}{a\beta}.\frac{\partial w}{\partial Q} = \frac{r_3}{a\beta}.\frac{\partial h * A_2}{\partial Q}$$

$$= \frac{A_1 * r_3}{a\beta}.\frac{\partial h}{\partial Q} \tag{3.367}$$

$$also\ \ \phi_y = \frac{r_4}{a\beta}.\frac{\partial w}{\partial R} = \frac{r_4}{a\beta}.\frac{\partial h * A_1}{\partial R}$$

$$= \frac{A_1 * r_4}{\alpha\beta} . \frac{\partial h}{\partial Q} \tag{3.368}$$

and finally $\phi_x = \frac{r_5}{a} . \frac{\partial h * A_1}{\partial R}$

$$= \frac{A_1 * r_5}{a} . \frac{\partial h}{\partial R} \tag{3.369}$$

3.2.5.2 Direct Variation Of Total Potential Energy

Recalling that the total potential energy is expressed as

$$\pi = (U_B + U_C + U_M) + V \tag{3.370}$$

where the U_B, U_C and U_M represents the strain energy for the case of bending , coupling and axial stiffnesses respectively. Alternatively , the total strain energy can be expressed as

$$U = \frac{abt}{2} \int_0^1 \int_0^1 \int_{-0.5}^{0.5} [(\sigma.\varepsilon)_B + (\sigma.\varepsilon)_C + (\sigma.\varepsilon)_M] \, dR \, dQ \, dS \tag{3.371}$$

Representing the Equation (3.371) in terms of g values gives

$$\pi = \frac{abD_o}{2a^4} \int_0^1 \int_0^1 \{B_{11} \left(g_1 \left(\frac{\partial^2 w}{\partial R^2} \right)^2 - ag_2 . \frac{\partial^2 w}{\partial R^2} . \frac{\partial \phi_x}{\partial R} - \frac{a}{t} g_{c1} . \frac{\partial^2 w}{\partial R^2} . \frac{du_0}{dR} - ag_2 . \frac{\partial^2 w}{\partial R^2} . \frac{\partial \phi_x}{\partial R} \right.$$

$$+ a^2 g_1 . \left(\frac{\partial \phi_x}{\partial R} \right)^2 + \frac{a^2}{t} g_{c2} . \frac{du_0}{dR} . \frac{\partial \phi_x}{\partial R} - \frac{ag_{c1}}{t} . \frac{\partial^2 w}{\partial R^2} . \frac{du_0}{\partial R} + \frac{a^2}{t} g_{c2} \frac{\partial \phi_x}{\partial R} . \frac{du_0}{dR}$$

$$\left. + \frac{a^2}{t^2} . \left(\frac{du_0}{dR} \right)^2 \right) +$$

$$2B_{12} \left(\frac{g_1}{\beta^2} . \frac{\partial^2 w}{\partial R^2} . \frac{\partial^2 w}{\partial Q^2} - \frac{ag_2}{\beta} . \frac{\partial^2 w}{\partial R^2} . \frac{\partial \phi_y}{\partial Q} - \frac{a}{t\beta} g_{c1} . \frac{\partial^2 w}{\partial R^2} . \frac{dv_0}{\partial Q} - \frac{ag_2}{\beta^2} \frac{\partial^2 w}{\partial Q^2} . \frac{\partial \phi_x}{\partial R} \right.$$

$$+ \frac{a^2 g_3}{\beta} . \frac{\partial \phi_y}{\partial Q} . \frac{\partial \phi_x}{\partial R} + \frac{a^2 g_{c2}}{t\beta} \frac{dv_0}{\partial Q} . \frac{\partial \phi_x}{\partial R} - \frac{ag_{c1}}{t\beta^2} \frac{\partial^2 w}{\partial Q^2} . \frac{du_0}{dR} + \frac{a^2 g_{c2}}{t\beta} . \frac{\partial \phi_y}{\partial Q} . \frac{du_0}{dR}$$

$$\left. + \frac{a^2}{t^2 \beta} \frac{du_0}{dR} . \frac{dv_0}{\partial Q} \right) + 2B_{13}$$

$$\left(-\frac{2g_1}{\beta}\frac{\partial^2 w}{\partial R\,\partial Q}.\frac{\partial^2 w}{\partial R^2}-\frac{ag_2}{\beta}\frac{\partial^2 w}{v\,\partial R^2}.\frac{\partial \phi_x}{\partial Q}-ag_2\frac{\partial^2 w}{\partial R^2}.\frac{\partial \phi_y}{\partial R}-\frac{a}{t\beta}g_{c1}\frac{\partial^2 w}{\partial R^2}.\frac{du_0}{\partial Q}-\frac{a}{t}g_{c1}\frac{\partial^2 w}{\partial R^2}.\frac{dv_0}{dR}\right)$$
$$+\left(\frac{2ag_2}{\beta}\frac{\partial^2 w}{\partial R\,\partial Q}.\frac{\partial \phi_x}{\partial R}+\frac{a^2g_3}{\beta}.\frac{\partial \phi_x}{\partial R}.\frac{\partial \phi_x}{\partial Q}+a^2g_3.\frac{\partial \phi_y}{\partial R}.\frac{\partial \phi_x}{\partial R}+\frac{a^2g_{c2}}{t}\frac{du_0}{\beta\,\partial Q}.\frac{\partial \phi_x}{\partial R}\right.$$
$$\left.+\frac{a^2g_{c2}}{t}\frac{dv_0}{dR}.\frac{\partial \phi_x}{\partial R}\right)$$

$$+\left(\frac{2ag_{c1}}{t\beta}\frac{\partial^2 w}{\partial R\,\partial Q}.\frac{du_0}{dR}+\frac{a^2g_{c2}}{t\beta}\frac{\partial \phi_x}{\partial Q}.\frac{du_0}{dR}+\frac{a^2g_{c2}}{t\beta}.\frac{\partial \phi_y}{\partial R}.\frac{du_0}{dR}+\frac{a^2}{t^2}\frac{du_0}{\beta\,\partial Q}.\frac{du_0}{dR}+\frac{a^2}{t^2}.\frac{dv_0}{dR}.\frac{du_0}{dR}\right)$$

$$+B_{22}\left(\frac{g_1}{\beta^4}\left(\frac{\partial^2 w}{\partial Q^2}\right)^2-\frac{ag_2}{\beta^3}.\frac{\partial^2 w}{\partial Q^2}.\frac{\partial \phi_y}{\partial Q}-\frac{ag_{c1}}{t\beta^3}.\frac{\partial^2 w}{\partial Q^2}.\frac{dv_0}{\partial Q}-\frac{ag_2}{\beta^3}\frac{\partial^2 w}{\partial Q^2}.\frac{\partial \phi_y}{\partial Q}+\frac{a^2g_3}{\beta^2}.\left(\frac{\partial \phi_y}{\partial Q}\right)^2\right.$$
$$\left.+\ \frac{a^2g_{c2}}{t\beta^2}\frac{dv_0}{\partial Q}.\frac{\partial \phi_y}{\partial Q}-\frac{ag_{c1}}{t\beta^3}\frac{\partial^2 w}{\partial Q^2}.\frac{dv_0}{\partial Q}+\frac{a^2g_{c2}}{t\beta^2}.\frac{\partial \phi_y}{\partial Q}.\frac{dv_0}{\partial Q}+\frac{a^2}{t^2\beta^2}.\left(\frac{dv_0}{\partial Q}\right)^2\right)$$

$$+2B_{23}\left(\frac{2g_1}{\beta^3}\frac{\partial^2 w}{\partial R\,\partial Q}.\frac{\partial^2 w}{\partial Q^2}-\frac{ag_2}{\beta^3}.\frac{\partial \phi_x}{\partial Q}.\frac{\partial^2 w}{\partial Q^2}-\frac{ag_2}{\beta^2}.\frac{\partial \phi_y}{\partial R}.\frac{\partial^2 w}{\partial Q^2}-\frac{ag_{c1}}{t\beta^3}\frac{du_0}{\partial Q}.\frac{\partial^2 w}{\partial Q^2}\right.$$
$$-\frac{ag_{c1}}{t\beta^2}\frac{dv_0}{dR}.\frac{\partial^2 w}{\partial Q^2}-\ \frac{2ag_2}{\beta^2}\frac{\partial^2 w}{\partial R\,\partial Q}.\frac{\partial \phi_y}{\partial Q}+\frac{a^2g_3}{\beta^2}.\frac{\partial \phi_x}{\partial Q}.\frac{\partial \phi_y}{\partial Q}+\frac{a^2g_3}{\beta}.\frac{\partial \phi_y}{\partial R}.\frac{\partial \phi_y}{\partial Q}$$
$$+\frac{a^2g_{c2}}{t\beta^2}\frac{du_0}{\partial Q}.\frac{\partial \phi_y}{\partial Q}+\frac{a^2g_{c2}}{t\beta}\frac{dv_0}{dR}.\frac{\partial \phi_y}{\partial Q}-\ \frac{2ag_{c1}}{t\beta^2}\frac{\partial^2 w}{\partial R\,\partial Q}.\frac{dv_0}{\partial Q}+\frac{a^2g_{c2}}{t\beta^2}.\frac{\partial \phi_x}{\partial Q}.\frac{dv_0}{\partial Q}$$
$$\left.+\frac{a^2g_{c2}}{t\beta}.\frac{\partial \phi_y}{\partial R}.\frac{dv_0}{\partial Q}+\frac{a^2}{t^2\beta^2}\frac{du_0}{\partial Q}\frac{dv_0}{\partial Q}+\frac{a^2}{t^2\beta}\frac{dv_0}{dR}.\frac{dv_0}{\partial Q}\right)$$

$$+B_{33}\left(\frac{4g_1}{\beta^2}\left(\frac{\partial^2 w}{\partial R\,\partial Q}\right)^2-\frac{2ag_2}{\beta^2}.\frac{\partial^2 w}{\partial R\,\partial Q}.\frac{\partial \phi_x}{\partial Q}-\frac{2ag_2}{\beta}.\frac{\partial^2 w}{\partial R\,\partial Q}.\frac{\partial \phi_y}{\partial R}-\frac{2ag_{c1}}{t\beta^2}.\frac{\partial^2 w}{\partial R\,\partial Q}.\frac{du_0}{\partial Q}\right.$$
$$\left.-\frac{2ag_{c1}}{t\beta}.\frac{\partial^2 w}{\partial R\,\partial Q}.\frac{dv_0}{dR}\right)+B_{33}$$

$$\left(-\frac{2ag_2}{\beta^2}\frac{\partial^2 w}{\partial R\,\partial Q}.\frac{\partial \phi_x}{\partial Q}+\frac{a^2g_3}{\beta^2}.\left(\frac{\partial \phi_x}{\partial Q}\right)^2+\frac{a^2g_3}{\beta}\frac{\partial \phi_y}{\partial R}.\frac{\partial \phi_x}{\partial Q}+\frac{a^2g_{c2}}{t\beta^2}\frac{du_0}{\partial Q}.\frac{\partial \phi_x}{\partial Q}+\frac{a^2g_{c2}}{t\beta}\frac{dv_0}{dR}.\frac{\partial \phi_x}{\partial Q}\right)$$

$$+B_{33}\left(-\frac{2ag_2}{\beta}\frac{\partial^2 w}{\partial R\,\partial Q}.\frac{\partial \phi_y}{\partial R}+\frac{a^2g_3}{\beta}.\frac{\partial \phi_x}{\partial Q}.\frac{\partial \phi_y}{\partial R}+a^2g_3\left(\frac{\partial \phi_y}{\partial R}\right)^2+\frac{a^2g_{c2}}{t\beta}\frac{du_0}{\partial Q}.\frac{\partial \phi_y}{\partial R}+\frac{a^2g_{c2}}{t}\frac{dv_0}{dR}.\frac{\partial \phi_y}{\partial R}\right)$$

$$+ B_{33}\left(-\frac{2ag_{c1}}{t\beta^2}\frac{\partial^2 w}{\partial R\,\partial Q}.\frac{du_0}{\partial Q}+\frac{a^2 g_{c2}}{t\beta^2}.\frac{\partial\phi_x}{\partial Q}.\frac{du_0}{\partial Q}+\frac{a^2 g_{c2}}{t\beta}\frac{\partial\phi_y}{\partial R}.\frac{du_0}{\partial Q}+\frac{a^2}{t^2\beta^2}.\left(\frac{du_0}{\partial Q}\right)^2 + \frac{a^2}{t^2\beta}\frac{dv_0}{dR}.\frac{du_0}{\partial Q}\right)+$$

$$B_{33}\left(-\frac{2ag_{c1}}{t\beta}\frac{\partial^2 w}{\partial R\partial Q}.\frac{dv_0}{dR}+\frac{a^2 g_{c2}}{t\beta}.\frac{\partial\phi_x}{\partial Q}.\frac{dv_0}{dR}+\frac{a^2 g_{c2}}{t}\frac{\partial\phi_y}{\partial R}.\frac{dv_0}{dR}+\frac{a^2}{t^2\beta}\frac{du_0}{\partial Q}.\frac{dv_0}{dR}+\frac{a^2}{t^2}\left(\frac{dv_0}{dR}\right)^2\right)$$

$$+B_{44}\frac{a^4}{t^2}\left((\emptyset_x.g_4)^2+\emptyset_x.\frac{du_0}{dS}.g_4+\emptyset_x.g_4.\frac{du_0}{dS}+\left(\frac{du_0}{dS}\right)^2\right) + B_{55}\frac{a^4}{t^2}\left(\left(\emptyset_y.g_4\right)^2+.\emptyset_y.\frac{dv_0}{dS}.g_4+\emptyset_y.g_4.\frac{dv_0}{dS}+\left(\frac{dv_0}{dS}\right)^2\right) + 2\frac{N_x a^2}{D_0}w\,\}\,dRdQ \qquad (3.372)$$

Considering the expressions A_1r_2, A_1r_3, A_1r_4 and A_1r_5 as A_1, A_2, A_3, A_4 and A_5 respectively and substituting them into the Total Potential Energy equations gives

$$\pi=\frac{abD_o}{2a^4}\int_0^1\int_0^1\{B_{11}\left(g_1\left(\frac{\partial^2 A_1 h}{\partial R^2}\right)^2-ag_2.\frac{\partial^2 A_1 h}{\partial R^2}.\frac{\partial}{\partial R}\left(\frac{A_5}{a}\frac{\partial h}{\partial R}\right) - \frac{a}{t}g_{c1}.\frac{\partial^2 A_1 h}{\partial R^2}.\frac{d}{dR}\left(\frac{A_2}{a}.\frac{\partial h}{\partial R}\right)-ag_2.\frac{\partial^2 A_1 h}{\partial R^2}.\frac{\partial}{\partial R}\left(\frac{A_5}{a}\frac{\partial h}{\partial R}\right) + a^2 g_1.\left(\frac{\partial}{\partial R}\frac{A_5}{a}\frac{\partial h}{\partial R}\right)^2+\frac{a^2}{t}g_{c2}.\frac{d}{dR}\left(\frac{A_2}{a}.\frac{\partial h}{\partial R}\right).\frac{\partial}{\partial R}\left(\frac{A_5}{a}\frac{\partial h}{\partial R}\right) - \frac{ag_{c1}}{t}.\frac{\partial^2 A_1 h}{\partial R^2}.\frac{d}{\partial R}\left(\frac{A_2}{a}.\frac{\partial h}{\partial R}\right)+\frac{a^2}{t}g_{c2}\frac{\partial}{\partial R}\left(\frac{A_5}{a}\frac{\partial h}{\partial R}\right).\frac{d}{dR}\left(\frac{A_2}{a}.\frac{\partial h}{\partial R}\right) + \frac{a^2}{t^2}.\left(\frac{du_0}{dR}\left(\frac{A_2}{a}.\frac{\partial h}{\partial R}\right)\right)^2\right)+$$

$$2B_{12}\left(\frac{g_1}{\beta^2}.\frac{\partial^2(A_1h)}{\partial R^2}.\frac{\partial^2(A_1h)}{\partial Q^2}-\frac{ag_2}{\beta}.\frac{\partial^2(A_1h)}{\partial R^2}.\frac{\partial}{\partial Q}\left(\frac{A_4}{a\beta}.\frac{\partial h}{\partial Q}\right)\right.$$

$$-\frac{a}{t\beta}g_{c1}.\frac{\partial^2(A_1h)}{\partial R^2}.\frac{d}{\partial Q}\left(\frac{A_3}{a\beta}.\frac{\partial h}{\partial R}\right)-\frac{ag_2}{\beta^2}\frac{\partial^2(A_1h)}{\partial Q^2}.\frac{\partial}{\partial R}\left(\frac{A_5}{a}\frac{\partial h}{\partial R}\right)$$

$$+\frac{a^2g_3}{\beta}.\frac{\partial}{\partial Q}\left(\frac{A_4}{a\beta}.\frac{\partial h}{\partial Q}\right).\frac{\partial}{\partial R}\left(\frac{A_5}{a}\frac{\partial h}{\partial R}\right)+\frac{a^2g_{c2}}{t\beta}\frac{d}{\partial Q}\left(\frac{A_3}{a\beta}.\frac{\partial h}{\partial R}\right).\frac{\partial}{\partial R}\left(\frac{A_5}{a}\frac{\partial h}{\partial R}\right)$$

$$-\frac{ag_{c1}}{t\beta^2}\frac{\partial^2(A_1h)}{\partial Q^2}.\frac{d}{dR}\left(\frac{A_2}{a}.\frac{\partial h}{\partial R}\right)+\frac{a^2g_{c2}}{t\beta}.\frac{\partial}{\partial Q}\left(\frac{A_4}{a\beta}.\frac{\partial h}{\partial Q}\right).\frac{d}{dR}\left(\frac{A_2}{a}.\frac{\partial h}{\partial R}\right)$$

$$\left.+\frac{a^2}{t^2\beta}\frac{d}{dR}\left(\frac{A_2}{a}.\frac{\partial h}{\partial R}\right).\frac{d}{\partial Q}\left(\frac{A_3}{a\beta}.\frac{\partial h}{\partial R}\right)\right)+2B_{13}$$

$$(\left(-\frac{2g_1}{\beta}\frac{\partial^2(A_1h)}{\partial R\,\partial Q}.\frac{\partial^2(A_1h)}{\partial R^2}-\frac{ag_2}{\beta}\frac{\partial^2 A_1h}{\partial R^2}.\frac{\partial}{\partial Q}\left(\frac{A_5}{a}.\frac{\partial h}{\partial R}\right)-ag_2\frac{\partial^2(A_1h)}{\partial R^2}.\frac{\partial}{\partial R}\left(\frac{A_4}{a\beta}.\frac{\partial h}{\partial Q}\right)\right.$$

$$\left.-\frac{a}{t\beta}g_{c1}\frac{\partial^2(A_1h)}{\partial R^2}.\frac{d}{\partial Q}\left(\frac{A_2}{a}.\frac{\partial h}{\partial R}\right)-\frac{a}{t}g_{c1}\frac{\partial^2(A_1h)}{\partial R^2}.\frac{d}{dR}\left(\frac{A_3}{a\beta}.\frac{\partial h}{\partial Q}\right)\right)$$

$$+\left(\frac{2ag_2}{\beta}\frac{\partial^2(A_1h)}{\partial R\,\partial Q}.\frac{\partial}{\partial R}\left(\frac{A_5}{a}.\frac{\partial h}{\partial R}\right)+\frac{a^2g_3}{\beta}.\frac{\partial}{\partial R}\left(\frac{A_5}{a}.\frac{\partial h}{\partial R}\right).\frac{\partial}{\partial Q}\left(\frac{A_5}{a}.\frac{\partial h}{\partial R}\right)\right.$$

$$+a^2g_3.\frac{\partial}{\partial R}\left(\frac{A_4}{a\beta}.\frac{\partial h}{\partial Q}\right).\frac{\partial}{\partial R}\left(\frac{A_5}{a}.\frac{\partial h}{\partial R}\right)+\frac{a^2g_{c2}}{t}\frac{d}{\beta\,\partial Q}\left(\frac{A_2}{a}.\frac{\partial h}{\partial R}\right).\frac{\partial}{\partial R}\left(\frac{A_5}{a}.\frac{\partial h}{\partial R}\right)$$

$$\left.+\frac{a^2g_{c2}}{t}\frac{d}{dR}\left(\frac{A_3}{a\beta}.\frac{\partial h}{\partial Q}\right).\frac{\partial}{\partial R}\left(\frac{A_5}{a}.\frac{\partial h}{\partial R}\right)\right)$$

$$+\left(\frac{2ag_{c1}}{t\beta}\frac{\partial^2(A_1h)}{\partial R\,\partial Q}.\frac{d}{dR}\left(\frac{A_2}{a}.\frac{\partial h}{\partial R}\right)+\frac{a^2g_{c2}}{t\beta}\frac{\partial}{\partial Q}\left(\frac{A_5}{a}.\frac{\partial h}{\partial R}\right).\frac{d}{dR}\left(\frac{A_2}{a}.\frac{\partial h}{\partial R}\right)\right.$$

$$+\frac{a^2g_{c2}}{t\beta}.\frac{\partial}{\partial R}\left(\frac{A_4}{a\beta}.\frac{\partial h}{\partial Q}\right).\frac{d}{dR}\left(\frac{A_2}{a}.\frac{\partial h}{\partial R}\right)+\frac{a^2}{t^2}\frac{d}{\beta\,\partial Q}\left(\frac{A_2}{a}.\frac{\partial h}{\partial R}\right).\frac{d}{dR}\left(\frac{A_2}{a}.\frac{\partial h}{\partial R}\right)$$

$$\left.+\frac{a^2}{t^2}.\frac{d}{dR}\left(\frac{A_3}{a\beta}.\frac{\partial h}{\partial Q}\right).\frac{d}{dR}\left(\frac{A_2}{a}.\frac{\partial h}{\partial R}\right)\right))$$

$$+\mathrm{B}_{22}\left(\frac{\mathrm{g}_1}{\beta^4}\left(\frac{\partial^2(A_1h)}{\partial \mathrm{Q}^2}\right)^2-\frac{\mathrm{ag}_2}{\beta^3}.\frac{\partial^2(A_1h)}{\partial \mathrm{Q}^2}.\frac{\partial}{\partial \mathrm{Q}}\left(\frac{\mathrm{A}_4}{a\beta}.\frac{\partial \mathrm{h}}{\partial Q}\right)-\frac{\mathrm{ag}_{c1}}{\mathrm{t}\beta^3}.\frac{\partial^2(A_1h)}{\partial \mathrm{Q}^2}.\frac{\mathrm{d}}{\partial \mathrm{Q}}\left(\frac{\mathrm{A}_3}{a\beta}.\frac{\partial \mathrm{h}}{\partial \mathrm{Q}}\right)\right.$$

$$-\frac{\mathrm{ag}_2}{\beta^3}\frac{\partial^2(A_1h)}{\partial \mathrm{Q}^2}.\frac{\partial}{\partial \mathrm{Q}}\left(\frac{\mathrm{A}_4}{a\beta}.\frac{\partial \mathrm{h}}{\partial Q}\right)+\frac{\mathrm{a}^2\mathrm{g}_3}{\beta^2}.\left(\frac{\partial}{\partial \mathrm{Q}}\left(\frac{\mathrm{A}_4}{a\beta}.\frac{\partial \mathrm{h}}{\partial Q}\right)\right)^2$$

$$+\frac{\mathrm{a}^2\mathrm{g}_{c2}}{\mathrm{t}\beta^2}\frac{\mathrm{d}}{\partial \mathrm{Q}}\left(\frac{\mathrm{A}_3}{a\beta}.\frac{\partial \mathrm{h}}{\partial \mathrm{Q}}\right).\frac{\partial}{\partial \mathrm{Q}}\left(\frac{\mathrm{A}_4}{a\beta}.\frac{\partial \mathrm{h}}{\partial Q}\right)-\frac{\mathrm{ag}_{c1}}{\mathrm{t}\beta^3}\frac{\partial^2(A_1h)}{\partial \mathrm{Q}^2}.\frac{\mathrm{d}}{\partial \mathrm{Q}}\left(\frac{\mathrm{A}_3}{a\beta}.\frac{\partial \mathrm{h}}{\partial \mathrm{Q}}\right)$$

$$\left.+\frac{\mathrm{a}^2\mathrm{g}_{c2}}{\mathrm{t}\beta^2}.\frac{\partial}{\partial \mathrm{Q}}\left(\frac{\mathrm{A}_4}{a\beta}.\frac{\partial \mathrm{h}}{\partial Q}\right).\frac{\mathrm{d}}{\partial \mathrm{Q}}\left(\frac{\mathrm{A}_3}{a\beta}.\frac{\partial \mathrm{h}}{\partial \mathrm{Q}}\right)+\frac{\mathrm{a}^2}{\mathrm{t}^2\beta^2}.\left(\frac{\mathrm{d}}{\partial \mathrm{Q}}\left(\frac{\mathrm{A}_3}{a\beta}.\frac{\partial \mathrm{h}}{\partial \mathrm{Q}}\right)\right)^2\right)$$

$$\left(\frac{4\mathrm{g}_1}{\beta^2}\left(\frac{\partial^2(A_1h)}{\partial R\partial Q}\right)^2-\frac{2a\mathrm{g}_2}{\beta^2}.\frac{\partial^2(A_1h)}{\partial R\partial Q}.\frac{\partial}{\partial Q}\left(\frac{\mathrm{A}_5}{a}.\frac{\partial \mathrm{h}}{\partial R}\right)-\frac{2a\mathrm{g}_2}{\beta}.\frac{\partial^2(A_1h)}{\partial R\partial Q}.\frac{\partial}{\partial R}\left(\frac{\mathrm{A}_4}{a\beta}.\frac{\partial \mathrm{h}}{\partial Q}\right)\right.$$

$$\left.-\frac{2a\mathrm{g}_{c1}}{t\beta^2}.\frac{\partial^2(A_1h)}{\partial R\partial Q}.\frac{d}{\partial Q}\left(\frac{\mathrm{A}_2}{a}.\frac{\partial \mathrm{h}}{\partial R}\right)-\frac{2a\mathrm{g}_{c1}}{t\beta}.\frac{\partial^2(A_1h)}{\partial R\partial Q}.\frac{d}{dR}\left(\frac{\mathrm{A}_3}{a\beta}.\frac{\partial \mathrm{h}}{\partial Q}\right)\right)+$$

$$B_{33}\left(-\frac{2a\mathrm{g}_2}{\beta^2}\frac{\partial^2(A_1h)}{\partial R\partial Q}.\frac{\partial}{\partial Q}\left(\frac{\mathrm{A}_5}{a}.\frac{\partial \mathrm{h}}{\partial R}\right)+\frac{a^2\mathrm{g}_3}{\beta^2}.\left(\frac{\partial}{\partial Q}\left(\frac{\mathrm{A}_5}{a}.\frac{\partial \mathrm{h}}{\partial R}\right)\right)^2+\frac{a^2\mathrm{g}_3}{\beta}\frac{\partial}{\partial R}\left(\frac{\mathrm{A}_4}{a\beta}.\frac{\partial \mathrm{h}}{\partial Q}\right).\frac{\partial}{\partial Q}\left(\frac{\mathrm{A}_5}{a}.\frac{\partial \mathrm{h}}{\partial R}\right)\right.$$

$$\left.+\frac{a^2\mathrm{g}_{c2}}{t\beta^2}\frac{d}{\partial Q}\left(\frac{\mathrm{A}_2}{a}.\frac{\partial \mathrm{h}}{\partial R}\right).\frac{\partial}{\partial Q}\left(\frac{\mathrm{A}_5}{a}.\frac{\partial \mathrm{h}}{\partial R}\right)+\frac{a^2\mathrm{g}_{c2}}{t\beta}\frac{d}{dR}\left(\frac{\mathrm{A}_3}{a\beta}.\frac{\partial \mathrm{h}}{\partial Q}\right).\frac{\partial}{\partial Q}\left(\frac{\mathrm{A}_5}{a}.\frac{\partial \mathrm{h}}{\partial R}\right)\right)+$$

$$B_{33}\left(-\frac{2a\mathrm{g}_2}{\beta}\frac{\partial^2(A_1h)}{\partial R\partial Q}.\frac{\partial}{\partial R}\left(\frac{\mathrm{A}_4}{a\beta}.\frac{\partial \mathrm{h}}{\partial Q}\right)+\frac{a^2\mathrm{g}_3}{\beta}.\frac{\partial}{\partial Q}\left(\frac{\mathrm{A}_5}{a}.\frac{\partial \mathrm{h}}{\partial R}\right).\frac{\partial}{\partial R}\left(\frac{\mathrm{A}_4}{a\beta}.\frac{\partial \mathrm{h}}{\partial Q}\right)+a^2\mathrm{g}_3\left(\frac{\partial}{\partial R}\left(\frac{\mathrm{A}_4}{a\beta}.\frac{\partial \mathrm{h}}{\partial Q}\right)\right)^2\right.$$

$$\left.+\frac{a^2\mathrm{g}_{c2}}{t\beta}\frac{d}{\partial Q}\left(\frac{\mathrm{A}_2}{a}.\frac{\partial \mathrm{h}}{\partial R}\right).\frac{\partial}{\partial R}\left(\frac{\mathrm{A}_4}{a\beta}.\frac{\partial \mathrm{h}}{\partial Q}\right)+\frac{a^2\mathrm{g}_{c2}}{t}\frac{d}{dR}\left(\frac{\mathrm{A}_3}{a\beta}.\frac{\partial \mathrm{h}}{\partial Q}\right).\frac{\partial}{\partial R}\left(\frac{\mathrm{A}_4}{a\beta}.\frac{\partial \mathrm{h}}{\partial Q}\right)\right)+$$

$$B_{33}\left(-\frac{2a\mathrm{g}_{c1}}{t\beta^2}\frac{\partial^2(A_1h)}{\partial R\partial Q}.\frac{d}{\partial Q}\left(\frac{\mathrm{A}_2}{a}.\frac{\partial \mathrm{h}}{\partial R}\right)+\frac{a^2\mathrm{g}_{c2}}{t\beta^2}.\frac{\partial}{\partial Q}\left(\frac{\mathrm{A}_5}{a}.\frac{\partial \mathrm{h}}{\partial R}\right).\frac{d}{\partial Q}\left(\frac{\mathrm{A}_2}{a}.\frac{\partial \mathrm{h}}{\partial R}\right)\right.$$

$$+\frac{a^2\mathrm{g}_{c2}}{t\beta}\frac{\partial}{\partial R}\left(\frac{\mathrm{A}_4}{a\beta}.\frac{\partial \mathrm{h}}{\partial Q}\right).\frac{d}{\partial Q}\left(\frac{\mathrm{A}_2}{a}.\frac{\partial \mathrm{h}}{\partial R}\right)+\frac{a^2}{t^2\beta^2}.\left(\frac{d}{\partial Q}\left(\frac{\mathrm{A}_2}{a}.\frac{\partial \mathrm{h}}{\partial R}\right)\right)^2$$

$$\left.+\frac{a^2}{t^2\beta}\frac{d}{dR}\left(\frac{\mathrm{A}_3}{a\beta}.\frac{\partial \mathrm{h}}{\partial Q}\right).\frac{d}{\partial Q}\left(\frac{\mathrm{A}_2}{a}.\frac{\partial \mathrm{h}}{\partial R}\right)\right)$$

$$+B_{33}\left(-\frac{2a\mathrm{g}_{c1}}{t\beta}\frac{\partial^2(A_1h)}{\partial R\partial Q}.\frac{d}{dR}\left(\frac{\mathrm{A}_3}{a\beta}.\frac{\partial \mathrm{h}}{\partial Q}\right)+\frac{a^2\mathrm{g}_{c2}}{t\beta}.\frac{\partial}{\partial Q}\left(\frac{\mathrm{A}_5}{a}.\frac{\partial \mathrm{h}}{\partial R}\right).\frac{d}{dR}\left(\frac{\mathrm{A}_3}{a\beta}.\frac{\partial \mathrm{h}}{\partial Q}\right)\right.$$

$$+\frac{a^2\mathrm{g}_{c2}}{t}\frac{\partial}{\partial R}\left(\frac{\mathrm{A}_4}{a\beta}.\frac{\partial \mathrm{h}}{\partial Q}\right).\frac{d}{dR}\left(\frac{\mathrm{A}_3}{a\beta}.\frac{\partial \mathrm{h}}{\partial Q}\right)+\frac{a^2}{t^2\beta}\left(\frac{\mathrm{A}_2}{a}.\frac{\partial \mathrm{h}}{\partial R}\right).\frac{d}{dR}\left(\frac{\mathrm{A}_3}{a\beta}.\frac{\partial \mathrm{h}}{\partial Q}\right)$$

$$\left.+\frac{a^2}{t^2}\left(\frac{d}{dR}\left(\frac{\mathrm{A}_3}{a\beta}.\frac{\partial \mathrm{h}}{\partial Q}\right)\right)^2\right)$$

$$+B_{44}\frac{a^4}{t^2}\left(\left(\mathrm{g}_4\left(\frac{\mathrm{A}_5}{a}.\frac{\partial \mathrm{h}}{\partial R}\right)\right)^2+\frac{2\mathrm{g}_4}{\mathrm{J_M}}.\left(\frac{\mathrm{A}_5}{a}.\frac{\partial \mathrm{h}}{\partial R}\right).\left(\frac{\mathrm{A}_2}{a}.\frac{\partial^2 \mathrm{h}}{\partial R^2}\right)+\left(\frac{1}{\mathrm{J_M}}\left(\frac{\mathrm{A}_2}{a}.\frac{\partial^2 \mathrm{h}}{\partial R^2}\right)\right)^2\right)$$

$$+B_{55}\frac{a^4}{t^2}\left(\left(\mathrm{g}_4\left(\frac{\mathrm{A}_4}{a\beta}.\frac{\partial \mathrm{h}}{\partial Q}\right).\right)^2+\frac{2\mathrm{g}_4}{\mathrm{J_M}}\left(\frac{\mathrm{A}_4}{a\beta}.\frac{\partial \mathrm{h}}{\partial Q}\right)\left(\frac{\mathrm{A}_3}{a\beta}.\frac{\partial^2 \mathrm{h}}{\partial Q^2}\right)+\left(\frac{1}{\mathrm{J_M}}\left(\frac{\mathrm{A}_3}{a\beta}.\frac{\partial^2 \mathrm{h}}{\partial Q^2}\right)\right)^2\right)$$

$$+\ 2\frac{\mathrm{N_x a^2}}{\mathrm{D_0}}\mathrm{A}_1\left(\frac{\partial \mathrm{h}}{\partial \mathrm{R}}\right)^2\ \}\,dRdQ \qquad (3.373)$$

Further reduction of Equation (3.373) gives

$$\pi=\frac{\mathrm{abD_o}}{2\mathrm{a}^4}\int_0^1\int_0^1\{\mathrm{B}_{11}\left(\mathrm{g_1A_1A_1}-\mathrm{g_2A_1A_5}-\frac{\mathrm{g}_{c1}}{\mathrm{t}}A_1A_2-\mathrm{g}_2A_1A_5+a^2\mathrm{g_1A_5A_5}+\frac{\mathrm{g}_{c2}}{\mathrm{t}^2}\mathrm{A_2A_5}\right.$$

$$\left.-\frac{\mathrm{g}_{c1}}{\mathrm{t}}\mathrm{A_1A_2}+\frac{\mathrm{g}_{c2}}{\mathrm{t}}\mathrm{A_2A_5}+\frac{\mathrm{A_2A_2}}{\mathrm{t}^2}\right)\frac{\partial^4\mathrm{h}}{\partial R^4}$$

$$+\ 2\mathrm{B}_{12}\left(\frac{\mathrm{g}_1}{\beta^2}\mathrm{A_1A_1}-\frac{\mathrm{g}_2}{\beta^2}\mathrm{A_1A_4}-\frac{\mathrm{g}_{c1}}{\mathrm{t}\beta^2}A_1A_3-\frac{\mathrm{g}_2}{\beta^2}A_1A_5+\frac{\mathrm{g}_3}{\beta^2}\mathrm{A_4A_5}\right.$$

$$\left.+\frac{\mathrm{g}_{c2}}{\mathrm{t}\beta^2}\mathrm{A_3A_5}.-\frac{\mathrm{g}_{c1}}{\mathrm{t}\beta^2}\mathrm{A_1A_2}.+\frac{\mathrm{g}_{c2}}{\mathrm{t}\beta^2}\mathrm{A_2A_4}+\frac{\mathrm{A_2A_3}}{\mathrm{t^2\beta^2}}\right)\frac{\partial^4\mathrm{h}}{\partial R^2\partial Q^2}$$

$$+2B_{13}\left(-\frac{2g_1}{\beta}A_1A_1-\frac{g_2}{\beta}A_1A_5-\frac{g_2}{\beta}A_1A_4-\frac{g_{c1}}{t\beta}A_1A_2-\frac{g_{c1}}{t\beta}A_1A_3+\frac{2g_2}{\beta}A_1A_5+\frac{g_3}{\beta}A_5A_5\right.$$

$$+\frac{g_3}{\beta}A_4A_5+\frac{g_{c2}}{t\beta}A_2A_5+\frac{g_{c2}}{t\beta}A_3A_5+\frac{2g_{c1}}{t\beta}A_1A_2+\frac{g_{c2}}{t\beta}A_2A_5+\frac{g_{c2}}{t\beta^2}A_2A_4$$

$$\left.+\frac{A_2A_2}{t^2\beta}+\frac{A_2A_3}{t^2\beta}\right)\frac{\partial^4 h}{\partial R^3\partial Q}$$

$$+B_{22}\left(\frac{g_1}{\beta^4}A_1A_1-\frac{g_2}{\beta^4}A_1A_4-\frac{g_{c1}}{t\beta^4}A_1A_3-\frac{g_2}{\beta^4}A_1A_4+\frac{g_3}{\beta^4}A_4A_4+\frac{g_{c2}}{t\beta^4}A_3A_4-\frac{g_{c1}}{t\beta^4}A_1A_3\right.$$

$$\left.+\frac{g_{c2}}{t\beta^4}A_3A_4+\frac{A_3A_3}{t^2\beta^4}\right)\frac{\partial^4 h}{\partial Q^4}$$

$$+2B_{23}\left(\frac{2g_1}{\beta^3}A_1A_1-\frac{g_2}{\beta^3}A_1A_5-\frac{g_2}{\beta^3}A_1A_4-\frac{g_{c1}}{t\beta^3}A_1A_2-\frac{g_{c1}}{t\beta^3}A_1A_3-\frac{2g_2}{\beta^3}A_1A_4+\frac{g_3}{\beta^3}A_4A_5\right.$$

$$+\frac{g_3}{\beta^3}A_4A_4+\frac{g_{c2}}{t\beta^3}A_2A_4+\frac{g_{c2}}{t\beta^3}A_3A_4-\frac{2g_{c1}}{t\beta^3}A_1A_3+\frac{g_{c2}}{t\beta^3}A_3A_5+\frac{g_{c2}}{t\beta^3}A_3A_4$$

$$\left.+\frac{A_2A_3}{t^2\beta^3}+\frac{A_3A_3}{t^2\beta^3}\right)\frac{\partial^4 h}{\partial R\partial Q^3}$$

$$+B_{33}\left(\frac{4g_1}{\beta^2}A_1A_1-\frac{2g_2}{\beta^2}A_1A_5-\frac{2g_2}{\beta^2}A_1A_4-\frac{2g_{c1}}{t\beta^2}A_1A_2-\frac{2g_{c1}}{t\beta^2}A_1A_3-\frac{2g_2}{\beta^2}A_1A_5\right.$$

$$\left.+\frac{g_3}{\beta^2}A_5A_5\right)\frac{\partial^4 h}{\partial R^2\partial Q^2}$$

$$+B_{33}\left(\frac{g_3}{\beta^2}A_4A_5+\frac{g_{c2}}{t\beta^2}A_2A_5+\frac{g_{c2}}{t\beta^2}A_3A_5-\frac{2g_2}{\beta^2}A_1A_4+\frac{g_3}{\beta^2}A_4A_5+\frac{g_3}{\beta^2}A_4A_4+\frac{g_{c2}}{t\beta^2}A_2A_4\right)\frac{\partial^4 h}{\partial R^2\partial Q^2}$$

$$+B_{33}\left(\frac{g_{c2}}{t\beta^2}A_3A_4-\frac{2g_{c1}}{t\beta^2}A_1A_2+\frac{g_{c2}}{t\beta^2}A_2A_5+\frac{g_{c2}}{t\beta^2}A_2A_4+\frac{A_2A_2}{t^2\beta^2}+\frac{A_2A_3}{t^2\beta^2}\right)\frac{\partial^4 h}{\partial R^2\partial Q^2}$$

$$+B_{33}\left(-\frac{2g_{c1}}{t\beta^2}A_1A_3+\frac{g_{c2}}{t\beta^2}A_3A_5+\frac{g_{c2}}{t\beta^2}A_3A_4+\frac{A_2A_3}{t^2\beta^2}+\frac{A_3A_3}{t^2\beta^2}\right)\frac{\partial^4 h}{\partial R^2\partial Q^2}$$

$$+B_{44}\frac{a^4}{t^2}\left(\left(g_4\left(\frac{A_5}{a}.\frac{\partial h}{\partial R}\right)\right)^2+\frac{2g_4}{J_M}.\left(\frac{A_5}{a}.\frac{\partial h}{\partial R}\right).\left(\frac{A_2}{a}.\frac{\partial^2 h}{\partial R^2}\right)+\left(\frac{1}{J_M}\left(\frac{A_2}{a}.\frac{\partial^2 h}{\partial R^2}\right)\right)^2\right)$$

$$+ B_{55}\frac{a^4}{t^2}\left(\left(\mathrm{g}_4\left(\frac{\mathrm{A}_4}{a\beta}.\frac{\partial \mathrm{h}}{\partial Q}\right)\right)^2 + \frac{2\mathrm{g}_4}{\mathrm{J_M}}\left(\frac{\mathrm{A}_4}{a\beta}.\frac{\partial \mathrm{h}}{\partial Q}\right)\left(\frac{\mathrm{A}_3}{a\beta}.\frac{\partial^2 \mathrm{h}}{\partial Q^2}\right) + \left(\frac{1}{\mathrm{J_M}}\left(\frac{\mathrm{A}_3}{a\beta}.\frac{\partial^2 \mathrm{h}}{\partial Q^2}\right)\right)^2\right)$$

$$+ \ 2\frac{\mathrm{N_x a^2}}{\mathrm{D_0}}\mathrm{A}_1\left(\frac{\partial \mathrm{h}}{\partial \mathrm{R}}\right)^2 \ \} \, dRdQ \qquad (3.374)$$

and finally from Equation (3.374) comes

$$\pi = \frac{\mathrm{abD_o}}{2\mathrm{a}^4}\int_0^1\int_0^1 \{\mathrm{B}_{11}\left(\mathrm{g}_1\mathrm{A}_1\mathrm{A}_1 - 2\mathrm{g}_2\mathrm{A}_1\mathrm{A}_5 - \frac{\mathrm{g_{c1}}}{\mathrm{t}}A_1A_2 + a^2\mathrm{g}_1\mathrm{A}_5\mathrm{A}_5 + \frac{\mathrm{g_{c2}}}{\mathrm{t}^2}\mathrm{A}_2\mathrm{A}_5 - \frac{\mathrm{g_{c1}}}{\mathrm{t}}\mathrm{A}_1\mathrm{A}_2 + \frac{\mathrm{g_{c2}}}{\mathrm{t}}\mathrm{A}_2\mathrm{A}_5 + \frac{\mathrm{A}_2\mathrm{A}_2}{\mathrm{t}^2}\right)\frac{\partial^4 \mathrm{h}}{\partial R^4}$$

$$+2\mathrm{B}_{12}\left(\frac{\mathrm{g}_1}{\beta^2}\mathrm{A}_1\mathrm{A}_1 - \frac{\mathrm{g}_2}{\beta^2}\mathrm{A}_1\mathrm{A}_4 - \frac{\mathrm{g_{c1}}}{\mathrm{t}\beta^2}A_1A_3 - \frac{\mathrm{g}_2}{\beta^2}A_1A_5 + \frac{\mathrm{g}_3}{\beta^2}\mathrm{A}_4\mathrm{A}_5 + \frac{\mathrm{g_{c2}}}{\mathrm{t}\beta^2}\mathrm{A}_3\mathrm{A}_5. - \frac{\mathrm{g_{c1}}}{\mathrm{t}\beta^2}\mathrm{A}_1\mathrm{A}_2. + \frac{\mathrm{g_{c2}}}{\mathrm{t}\beta^2}\mathrm{A}_2\mathrm{A}_4 + \frac{\mathrm{A}_2\mathrm{A}_3}{\mathrm{t}^2\beta^2}\right)\frac{\partial^4 \mathrm{h}}{\partial R^2\partial Q^2}$$

$$+2\mathrm{B}_{13}\left(-\frac{2\mathrm{g}_1}{\beta}\mathrm{A}_1\mathrm{A}_1 + \frac{\mathrm{g}_2}{\beta}A_1A_5 - \frac{\mathrm{g}_2}{\beta}\mathrm{A}_1\mathrm{A}_4 + \frac{\mathrm{g_{c1}}}{\mathrm{t}\beta}A_1A_2 - \frac{\mathrm{g_{c1}}}{\mathrm{t}\beta}A_1A_3 + \frac{\mathrm{g}_3}{\beta}A_5A_5 + \frac{\mathrm{g}_3}{\beta}A_4A_5 + \frac{2\mathrm{g_{c2}}}{\mathrm{t}\beta}A_2\mathrm{A}_5 + \frac{\mathrm{g_{c2}}}{\mathrm{t}\beta}\mathrm{A}_3\mathrm{A}_5 + \frac{\mathrm{g_{c2}}}{\mathrm{t}\beta^2}A_2\mathrm{A}_4 + \frac{A_2A_2}{\mathrm{t}^2\beta} + \frac{\mathrm{A}_2\mathrm{A}_3}{\mathrm{t}^2\beta}\right)\frac{\partial^4 \mathrm{h}}{\partial R^3\partial Q}$$

$$+\mathrm{B}_{22}\left(\frac{\mathrm{g}_1}{\beta^4}\mathrm{A}_1\mathrm{A}_1 - \frac{2\mathrm{g}_2}{\beta^4}\mathrm{A}_1\mathrm{A}_4 - \frac{2\mathrm{g_{c1}}}{\mathrm{t}\beta^4}A_1A_3 + \frac{\mathrm{g}_3}{\beta^4}\mathrm{A}_4\mathrm{A}_4 + \frac{2\mathrm{g_{c2}}}{\mathrm{t}\beta^4}\mathrm{A}_3\mathrm{A}_4 + \frac{\mathrm{A}_3\mathrm{A}_3}{\mathrm{t}^2\beta^4}\right)\frac{\partial^4 \mathrm{h}}{\partial Q^4}$$

$$+2\mathrm{B}_{23}\left(\frac{2\mathrm{g}_1}{\beta^3}\mathrm{A}_1\mathrm{A}_1 - \frac{\mathrm{g}_2}{\beta^3}A_1A_5 - \frac{3\mathrm{g}_2}{\beta^3}\mathrm{A}_1\mathrm{A}_4 - \frac{\mathrm{g_{c1}}}{\mathrm{t}\beta^3}A_1A_2 - \frac{3\mathrm{g_{c1}}}{\mathrm{t}\beta^3}A_1A_3 + \frac{\mathrm{g}_3}{\beta^3}\mathrm{A}_4A_5 + \frac{\mathrm{g}_3}{\beta^3}A_4\mathrm{A}_4 + \frac{\mathrm{g_{c2}}}{\mathrm{t}\beta^3}A_2\mathrm{A}_4 + \frac{2\mathrm{g_{c2}}}{\mathrm{t}\beta^3}A_3\mathrm{A}_4 + \frac{\mathrm{g_{c2}}}{\mathrm{t}\beta^3}A_3\mathrm{A}_5 + \frac{A_2A_3}{\mathrm{t}^2\beta^3} + \frac{\mathrm{A}_3\mathrm{A}_3}{\mathrm{t}^2\beta^3}\right)\frac{\partial^4 \mathrm{h}}{\partial R\partial Q^3}$$

$$+\mathrm{B}_{33}\left(\frac{4\mathrm{g}_1}{\beta^2}\mathrm{A}_1\mathrm{A}_1 - \frac{4\mathrm{g}_2}{\beta^2}A_1A_5 - \frac{4\mathrm{g}_2}{\beta^2}\mathrm{A}_1\mathrm{A}_4 - \frac{4\mathrm{g_{c1}}}{\mathrm{t}\beta^2}A_1A_3 + \frac{\mathrm{g}_3}{\beta^2}\mathrm{A}_5\mathrm{A}_5\right)\frac{\partial^4 \mathrm{h}}{\partial \mathrm{R}^2\partial \mathrm{Q}^2}$$

$$+\mathrm{B}_{33}\left(\frac{2\mathrm{g}_3}{\beta^2}\mathrm{A}_4A_5 + \frac{2\mathrm{g_{c2}}}{\mathrm{t}\beta^2}A_2A_5 + \frac{2\mathrm{g_{c2}}}{\mathrm{t}\beta^2}A_3\mathrm{A}_5 + \frac{\mathrm{g}_3}{\beta^2}\mathrm{A}_4\mathrm{A}_4 + \frac{2\mathrm{g_{c2}}}{\mathrm{t}\beta^2}A_2\mathrm{A}_4\right)\frac{\partial^4 \mathrm{h}}{\partial \mathrm{R}^2\partial \mathrm{Q}^2}$$

$$+\mathrm{B}_{33}\left(\frac{2\mathrm{g_{c2}}}{\mathrm{t}\beta^2}A_3\mathrm{A}_4 + \frac{A_3A_3}{\mathrm{t}^2\beta^2} - \frac{4\mathrm{g_{c1}}}{\mathrm{t}\beta^2}\mathrm{A}_1A_2 + \frac{A_2A_2}{\mathrm{t}^2\beta^2} + \frac{2A_2A_3}{\mathrm{t}^2\beta^2}\right)\frac{\partial^4 \mathrm{h}}{\partial \mathrm{R}^2\partial \mathrm{Q}^2}$$

$$+B_{44}\frac{a^2}{t^2}\left(\left(g_4{A_5}^2\left(\frac{\partial h}{\partial R}\right)\right)^2+\frac{2g_4A_5A_2}{J_M}.\left(\frac{\partial h}{\partial R}\right).\left(\frac{\partial^2 h}{\partial R^2}\right)+\frac{{A_2}^2}{{J_M}^2}\left(\frac{\partial^2 h}{\partial R^2}\right)^2\right)$$

$$+B_{55}\frac{a^2}{t^2}\left(\left(\frac{g_4{A_4}^2}{\beta^2}\left(\frac{\partial h}{\partial Q}\right).\right)^2+\frac{2g_4A_4A_3}{J_M\beta^2}\left(\frac{\partial h}{\partial Q}\right)\left(\frac{\partial^2 h}{\partial Q^2}\right)+\frac{{A_3}^2}{{J_M}^2\beta^2}\left(\frac{\partial^2 h}{\partial Q^2}\right)\right)$$

$$+2\frac{N_x a^2}{D_0}A_1\left(\frac{\partial h}{\partial R}\right)^2\ \}\,dRdQ \tag{3.375}$$

3.2.5.3 Differential Values of The Shape Functions

Equation (3.375) was further expressed in terms of stiffness coefficients,

Recalling that

$$\overline{K}_1 = \left(\frac{d^3h}{dR^3}\right)\left(\frac{dh}{dR}\right) \tag{3.375}$$

$$\overline{K}_2 = \left(\frac{d^3h}{dR^2dQ}\right)\left(\frac{dh}{dQ}\right) \tag{3.376}$$

$$\overline{K}_3 = \left(\frac{d^3h}{dQ^3}\right)\left(\frac{dh}{dQ}\right) \tag{3.377}$$

$$\overline{K}_4 == \left(\frac{d^3h}{dR^3}\right)\left(\frac{dh}{dQ}\right) \tag{3.378}$$

$$\overline{K}_5 = \left(\frac{d^3h}{dQ^3}\right)\left(\frac{dh}{dR}\right) \tag{3.379}$$

$$\overline{K}_6 = \left(\frac{dh}{dR}\right)\left(\frac{dh}{dR}\right) \tag{3.380}$$

$$\overline{K}_7 = \left(\frac{dh}{dQ}\right)\left(\frac{dh}{dQ}\right) \tag{3.381}$$

$$\overline{K}_8 = (h) \tag{3.382}$$

$$\overline{K}_{11} = \left(\frac{\partial^3 h}{\partial R^3}\right) \tag{3.383}$$

$$\overline{K}_{12} = \left(\frac{\partial^4 h}{\partial Q^3}\right) \tag{3.384}$$

Where the integral values of $\overline{K}_1, \overline{K}_2\ \overline{K}_3\ \overline{K}_4\ \overline{K}_5\ \overline{K}_6\ \overline{K}_7\ \overline{K}_8$ and $\overline{K}_9$ represents $k_1, k_2, k_3, k_4, k_5,$ k_6, k_7, k_8 *and* k_9 respectively. Substituting them into the Total Potential equation gives

$$\pi = \frac{abD_o}{2a^4}\int_0^1\int_0^1 \{B_{11}\left(g_1A_1A_1 - 2g_2A_1A_5 - \frac{g_{c1}}{t}A_1A_2 + g_1A_5A_5 + \frac{g_{c2}}{t^2}A_2A_5 - \frac{g_{c1}}{t}A_1A_2 + \frac{g_{c2}}{t}A_2A_5 + \frac{A_2A_2}{t^2}\right)k_1$$

$$+2B_{12}\left(\frac{g_1}{\beta^2}A_1A_1 - \frac{g_2}{\beta^2}A_1A_4 - \frac{g_{c1}}{t\beta^2}A_1A_3 - \frac{g_2}{\beta^2}A_1A_5 + \frac{g_3}{\beta^2}A_4A_5 + \frac{g_{c2}}{t\beta^2}A_3A_5. - \frac{g_{c1}}{t\beta^2}A_1A_2. + \frac{g_{c2}}{t\beta^2}A_2A_4 + \frac{A_2A_3}{t^2\beta^2}\right)k_2$$

$$+2B_{13}\left(-\frac{2g_1}{\beta}A_1A_1 + \frac{g_2}{\beta}A_1A_5 - \frac{g_2}{\beta}A_1A_4 + \frac{g_{c1}}{t\beta}A_1A_2 - \frac{g_{c1}}{t\beta}A_1A_3 + \frac{g_3}{\beta}A_5A_5 + \frac{g_3}{\beta}A_4A_5 + \frac{2g_{c2}}{t\beta}A_2A_5 + \frac{g_{c2}}{t\beta}A_3A_5 + \frac{g_{c2}}{t\beta^2}A_2A_4 + \frac{A_2A_2}{t^2\beta} + \frac{A_2A_3}{t^2\beta}\right)k_4$$

$$+ B_{22}\left(\frac{g_1}{\beta^4}A_1A_1 - \frac{2g_2}{\beta^4}A_1A_4 - \frac{2g_{c1}}{t\beta^4}A_1A_3 + \frac{g_3}{\beta^4}A_4A_4 + \frac{2g_{c2}}{t\beta^4}A_3A_4 + \frac{A_3A_3}{t^2\beta^4}\right)k_3$$

$$+ 2B_{23}\left(\frac{2g_1}{\beta^3}A_1A_1 - \frac{g_2}{\beta^3}A_1A_5 - \frac{3g_2}{\beta^3}A_1A_4 - \frac{g_{c1}}{t\beta^3}A_1A_2 - \frac{3g_{c1}}{t\beta^3}A_1A_3 + \frac{g_3}{\beta^3}A_4A_5 + \frac{g_3}{\beta^3}A_4A_4 + \frac{g_{c2}}{t\beta^3}A_2A_4 + \frac{2g_{c2}}{t\beta^3}A_3A_4 + \frac{g_{c2}}{t\beta^3}A_3A_5 + \frac{A_2A_3}{t^2\beta^3} + \frac{A_3A_3}{t^2\beta^3}\right)k_5$$

$$+ B_{33}\left(\frac{4g_1}{\beta^2}A_1A_1 - \frac{4g_2}{\beta^2}A_1A_5 - \frac{4g_2}{\beta^2}A_1A_4 - \frac{4g_{c1}}{t\beta^2}A_1A_3 + \frac{g_3}{\beta^2}A_5A_5\right)k_2$$

$$+ B_{33}\left(\frac{2g_3}{\beta^2}A_4A_5 + \frac{2g_{c2}}{t\beta^2}A_2A_5 + \frac{2g_{c2}}{t\beta^2}A_3A_5 + \frac{g_3}{\beta^2}A_4A_4 + \frac{2g_{c2}}{t\beta^2}A_2A_4\right)k_2$$

$$+ B_{33}\left(\frac{2g_{c2}}{t\beta^2}A_3A\ ;;;;_4 + \frac{A_3A_3}{t^2\beta^2} - \frac{4g_{c1}}{t\beta^2}A_1A_2 + \frac{A_2A_2}{t^2\beta^2} + \frac{2A_2A_3}{t^2\beta^2}\right)k_2$$

$$+ B_{44}\frac{a^2}{t^2}\left({g_4}^2{A_5}^2k_6 + \frac{2g_4A_5A_2}{J_M}k_{11} + \frac{{A_2}^2}{{J_M}^2}k_1\right)$$

$$+ B_{55}\frac{a^2}{t^2}\left(\frac{{g_4}^2{A_4}^2}{\beta^2}k_7 + \frac{2g_4A_4A_3}{J_M\beta^2}k_{12} + \frac{{A_3}^2}{{J_M}^2\beta^2}k_3\right)\}dRdQ + \frac{N_x}{2a^2}{A_1}^2k_6\,dRdQ \qquad (3.385)$$

Differentiating the total potential energy with respect to the Amplitude gives

$$\frac{d\pi}{dA_1} = \frac{abD_o}{2a^4}\int_0^1\int_0^1 \{B_{11}\left(2g_1A_1 - 2g_2A_5 - \frac{2g_{c1}}{t}A_2\right)k_1$$

$$+2B_{12}\left(\frac{2g_1}{\beta^2}A_1-\frac{g_2}{\beta^2}A_4-\frac{g_{c1}}{t\beta^2}A_3-\frac{g_2}{\beta^2}A_5-\frac{g_{c1}}{t\beta^2}A_2\right)k_2$$

$$+2B_{13}\left(-\frac{4g_1}{\beta}A_1+\frac{g_2}{\beta}A_5-\frac{g_2}{\beta}A_4+\frac{g_{c1}}{t\beta}A_2-\frac{g_{c1}}{t\beta}A_3\right)k_4+B_{22}$$

$$\left(\frac{2g_1}{\beta^4}A_1-\frac{2g_2}{\beta^4}A_4-\frac{2g_{c1}}{t\beta^4}A_3\right)k_3+B_{33}\left(\frac{8g_1}{\beta^2}A_1-\frac{4g_2}{\beta^2}A_5-\frac{4g_2}{\beta^2}A_4-\frac{4g_{c1}}{t\beta^2}A_3\right)$$

$$k_2+B_{33}\left(-\frac{4g_{c1}}{t\beta^2}A_2\right)k_2+\frac{N_xa^2}{D_0}A_1k_6\ \}\,dRdQ \qquad (3.386)$$

Simplifying it further gives

$$\{B_{11}\left(2g_1A_1-2g_2A_5-\frac{2g_{c1}}{t}A_2\right)k_1+\ \frac{2B_{12}}{\beta^2}\left(2g_1A_1-g_2[A_4+A_5]-\frac{g_{c1}}{t}[A_3+A_2]\right)k_2$$

$$+\frac{2B_{13}}{\beta}\left(-4g_1A_1+g_2[A_5-A_4]+\frac{g_{c1}}{t}[A_2-A_3]\right)k_4+\frac{B_{22}}{\beta^4}\left(2g_1A_1-2g_2A_4-\frac{2g_{c1}}{t}A_3\right)$$

$$k_3+\frac{2B_{23}}{\beta^3}\left(4g_1A_1-g_2[A_5-3A_4]-\frac{g_{c1}}{t}[A_2+3A_3]\right)k_5$$

$$+\frac{4B_{33}}{\beta^2}\left(2g_1A_1-g_2[A_5+A_4]-\frac{g_{c1}}{t}[A_3+4A_2]\right)k_2\ \}\ +\ 2\frac{N_xa^2}{D_0}k_6\ =0 \qquad (3.387)$$

The differential value of the total potential energy in Equation (3.387) was derived, for the case of second Amplitude, giving

$$\frac{d\pi}{dA_2}=\frac{abD_o}{2a^4}\int_0^1\int_0^1\{B_{11}\left(-\frac{2g_{c1}}{t}A_1+\frac{g_{c2}}{t^2}A_2+\frac{g_{c2}}{t}A_5+\frac{2A_2}{t^2}\right)k_1$$

$$+2B_{12}\left(-\frac{g_{c1}}{t\beta^2}A_1+\frac{g_{c2}}{t\beta^2}A_4+\frac{A_3}{t^2\beta^2}\right)k_2$$

$$+2B_{13}\left(\frac{g_{c1}}{t\beta}A_1+\frac{2g_{c2}}{t\beta}A_5+\frac{g_{c2}}{t\beta^2}A_4+\frac{2A_2}{t^2\beta}+\frac{A_3}{t^2\beta}\right)k_4$$

$$+2B_{23}\left(-\frac{g_{c1}}{t\beta^3}A_1+\frac{g_{c2}}{t\beta^3}A_4+\frac{A_3}{t^2\beta^3}\right)k_5$$

$$+B_{33}\left(\frac{2g_{c2}}{t\beta^2}A_5+\frac{2g_{c2}}{t\beta^2}A_4-\frac{4g_{c1}}{t\beta^2}A_1+\frac{2A_2}{t^2\beta^2}+\frac{2A_3}{t^2\beta^2}\right)k_2$$

$$+B_{44}\frac{a^2}{t^2}\left(+\frac{2g_4A_5}{J_M}k_{11}+\frac{2A_2}{{J_M}^2}k_1\right)\ \}\ dRdQ \qquad (3.388)$$

Further factorizing gives

$$\{\left(-\frac{2B_{11}g_{c1}}{t}A_1+\frac{B_{11}g_{c2}}{t^2}A_2+\frac{B_{11}g_{c2}}{t}A_5+\frac{2B_{11}A_2}{t^2}\right)k_1 + \left(-\frac{2B_{12}g_{c1}}{t\beta^2}A_1+\frac{2B_{12}g_{c2}}{t\beta^2}A_4+\frac{2B_{12}A_3}{t^2\beta^2}\right)k_2$$

$$\left(\frac{2B_{13}g_{c1}}{t\beta}A_1+\frac{4B_{13}g_{c2}}{t\beta}A_5+\frac{2B_{13}g_{c2}}{t\beta^2}A_4+\frac{4B_{13}A_2}{t^2\beta}+\frac{2B_{13}A_3}{t^2\beta}\right)k_4$$

$$+\left(-\frac{2B_{23}g_{c1}}{t\beta^3}A_1+\frac{2B_{23}g_{c2}}{t\beta^3}A_4+\frac{2B_{23}A_3}{t^2\beta^3}\right)k_5$$

$$+\left(\frac{2B_{33}g_{c2}}{t\beta^2}A_5+\frac{2B_{33}g_{c2}}{t\beta^2}A_4-\frac{4B_{33}g_{c1}}{t\beta^2}A_1+\frac{2B_{33}A_2}{t^2\beta^2}+\frac{2B_{33}A_3}{t^2\beta^2}\right)k_2$$

$$+\left(B_{44}\frac{a^2}{t^2}\frac{2g_4A_5}{J_M}k_{11}+B_{44}\frac{a^2}{t^2}\frac{2A_2}{{J_M}^2}k_1\right)\} = 0 \qquad (3.388)$$

Expressing Equation (3.388) in terms of the Amplitude gives

$$A_1\left[-\frac{2B_{11}g_{c1}}{t}k_1-\frac{2B_{12}g_{c1}}{t\beta^2}k_2+\frac{2B_{13}g_{c1}}{t\beta}k_4-\frac{2B_{23}g_{c1}}{t\beta^3}k_5-\frac{4B_{33}g_{c1}}{t\beta^2}k_2\right]$$

$$A_2\left[\frac{B_{11}g_{c2}}{t^2}k_1+\frac{2B_{11}}{t^2}k_1+\frac{4B_{13}}{t^2\beta}k_4+\frac{2B_{33}}{t^2\beta^2}k_2+B_{44}\frac{a^2}{t^2}\frac{2}{{J_M}^2}k_1\right]+$$

$$A_3\left[\frac{2B_{12}}{t^2\beta^2}k_2+\frac{2B_{13}}{t^2\beta}k_4+\frac{2B_{23}}{t^2\beta^3}k_5+\frac{2B_{33}}{t^2\beta^2}k_2\right]$$

$$+A_4\left[\frac{2B_{12}g_{c2}}{t\beta^2}k_2+\frac{2B_{13}g_{c2}}{t\beta^2}k_4+\frac{2B_{23}g_{c2}}{t\beta^3}k_5+\frac{2B_{33}g_{c2}}{t\beta^2}k_2\right]$$

$$+A_5\left[\frac{B_{11}g_{c2}}{t}k_1+\frac{4B_{13}g_{c2}}{t\beta}k_4+\frac{2B_{33}g_{c2}}{t\beta^2}k_2+B_{44}\frac{a^2}{t^2}\frac{2g_4}{J_M}k_{11}\right]=0 \qquad (3.389)$$

For the case the third Amplitude gives

$$\frac{d\pi}{dA_3}=\frac{abD_o}{2a^4}\int_0^1\int_0^1\Big\{2B_{12}\left(-\frac{g_{c1}}{t\beta^2}A_1+\frac{g_{c2}}{t\beta^2}A_5+\frac{A_2}{t^2\beta^2}\right)k_2$$

$$+2B_{13}\left(-\frac{g_{c1}}{t\beta}A_1+\frac{g_{c2}}{t\beta}A_5+\frac{A_3}{t^2\beta}\right)k_4+B_{22}\left(-\frac{2g_{c1}}{t\beta^4}A_1+\frac{2g_{c2}}{t\beta^4}A_4+\frac{2A_3}{t^2\beta^4}\right)k_3$$

$$+2B_{23}\left(-\frac{3g_{c1}}{t\beta^3}A_1+\frac{2g_{c2}}{t\beta^3}A_4+\frac{g_{c2}}{t\beta^3}A_5+\frac{A_2}{t^2\beta^3}+\frac{2A_3}{t^2\beta^3}\right)k_5$$

$$+B_{33}\left(-\frac{4g_{c1}}{t\beta^2}A_1\right)k_2 + B_{33}\left(\frac{2g_{c2}}{t\beta^2}A_5\right)k_2 + B_{33}\left(\frac{2g_{c2}}{t\beta^2}A_4 + \frac{2A_3}{t^2\beta^2} + \frac{2A_2}{t^2\beta^2}\right)k_2$$

$$+ B_{55}\frac{a^2}{t^2}\left(\frac{2g_4A_4}{J_M\beta^2}k_{12} + \frac{2A_3}{{J_M}^2\beta^2}k_3\right)\} \qquad (3.390)$$

Further factorization gives

$$\{\left(-\frac{2B_{12}g_{c1}}{t\beta^2}A_1 + \frac{2B_{12}g_{c2}}{t\beta^2}A_5 + \frac{2B_{12}A_2}{t^2\beta^2}\right)k_2$$

$$+\left(-\frac{2B_{13}g_{c1}}{t\beta}A_1 + \frac{2B_{13}g_{c2}}{t\beta}A_5 + \frac{2B_{13}A_3}{t^2\beta}\right)k_4 + \left(-\frac{2B_{22}g_{c1}}{t\beta^4}A_1 + \frac{2B_{22}g_{c2}}{t\beta^4}A_4 + \frac{2B_{22}A_3}{t^2\beta^4}\right)$$

$$k_3 + \left(-\frac{6B_{23}g_{c1}}{t\beta^3}A_1 + \frac{4B_{23}g_{c2}}{t\beta^3}A_4 + \frac{2B_{23}g_{c2}}{t\beta^3}A_5 + \frac{2B_{23}A_2}{t^2\beta^3} + \frac{4B_{23}A_3}{t^2\beta^3}\right)k_5$$

$$+\left(-\frac{4B_{33}g_{c1}}{t\beta^2}A_1\right)k_2 + \left(\frac{2B_{33}g_{c2}}{t\beta^2}A_5\right)k_2 + \left(\frac{2B_{33}g_{c2}}{t\beta^2}A_4 + \frac{2B_{33}A_3}{t^2\beta^2} + \frac{2B_{33}A_2}{t^2\beta^2}\right)k_2$$

$$+ \left(B_{55}\frac{a^2}{t^2}\frac{2g_4A_4}{J_M\beta^2}k_{12} + B_{55}\frac{a^2}{t^2}\frac{2A_3}{{J_M}^2\beta^2}k_3\right)\} = 0 \qquad (3.391)$$

Expressing it in terms of the Amplitude gives

$$A_1\left[-\frac{2B_{12}g_{c1}}{t\beta^2}k_2 - \frac{2B_{13}g_{c1}}{t\beta}k_4 - \frac{2B_{22}g_{c1}}{t\beta^4}k_3 - \frac{6B_{23}g_{c1}}{t\beta^3}k_5 - \frac{4B_{33}g_{c1}}{t\beta^2}k_2\right]$$

$$A_2\left[\frac{2B_{12}}{t^2\beta^2}k_2 + \frac{2B_{23}}{t^2\beta^3}k_5 + \frac{2B_{33}}{t^2\beta^2}k_2\right]$$

$$+A_3\left[\frac{2B_{13}}{t^2\beta}k_4 + \frac{2B_{22}}{t^2\beta^4}k_3 + \frac{4B_{23}}{t^2\beta^3}k_5 + \frac{2B_{33}}{t^2\beta^2}k_2 + B_{55}\frac{a^2}{t^2}\frac{2}{{J_M}^2\beta^2}k_3\right]$$

$$+A_4\left[\frac{2B_{22}g_{c2}}{t\beta^4}k_3 + \frac{4B_{23}g_{c2}}{t\beta^3}k_5 + \frac{2B_{33}g_{c2}}{t\beta^2}k_2 + B_{55}\frac{a^2}{t^2}\frac{2g_4}{J_M\beta^2}k_{12}\right]$$

$$+A_5\left[\frac{2B_{12}g_{c2}}{t\beta^2}k_2 + \frac{2B_{13}g_{c2}}{t\beta}k_4 + \frac{2B_{23}g_{c2}}{t\beta^3}k_5 + \frac{2B_{33}g_{c2}}{t\beta^2}k_2\right] = 0 \qquad (3.392)$$

For the case of the Fourth Amplitude gives

$$\frac{d\pi}{dA_4} = \frac{abD_o}{2a^4}\int_0^1\int_0^1 \{. + 2B_{12}\left(-\frac{g_2}{\beta^2}A_1 + \frac{g_3}{\beta^2}A_5 + \frac{g_{c2}}{t\beta^2}A_2\right)k_2 + 2B_{13}$$

$$\left(-\frac{g_2}{\beta}A_1 + \frac{g_3}{\beta}A_5 + \frac{g_{c2}}{t\beta^2}A_2\right)k_4 + B_{22}\left(-\frac{2g_2}{\beta^4}A_1 + \frac{2g_3}{\beta^4}A_4 + \frac{2g_{c2}}{t\beta^4}A_3\right)k_3$$

$$+2B_{23}\left(-\frac{3g_2}{\beta^3}A_1 + \frac{g_3}{\beta^3}A_5 + \frac{2g_3}{\beta^3}A_4 + \frac{g_{c2}}{t\beta^3}A_2 + \frac{2g_{c2}}{t\beta^3}A_3\right)k_5$$

$$+B_{33}\left(-\frac{4g_2}{\beta^2}A_1\right)k_2 + B_{33}\left(\frac{2g_3}{\beta^2}A_5 + \frac{2g_3}{\beta^2}A_4 + \frac{2g_{c2}}{t\beta^2}A_2\right)k_2$$

$$+B_{33}\left(\frac{2g_{c2}}{t\beta^2}A_3\right)k_2 + B_{55}\frac{a^2}{t^2}\left(\frac{2{g_4}^2A_4}{\beta^2}k_7 + \frac{2g_4A_3}{J_M\beta^2}k_{12}\right)\} dRdQ \qquad (3.393)$$

Factorizing Equation (3.682)further gives

$$\frac{d\pi}{dA_4} = \left(-\frac{2B_{12}g_2}{\beta^2}A_1 + \frac{2B_{12}g_3}{\beta^2}A_5 + \frac{2B_{12}g_{c2}}{t\beta^2}A_2\right)k_2$$

$$+\left(-\frac{2B_{13}g_2}{\beta}A_1 + \frac{2B_{13}g_3}{\beta}A_5 + \frac{2B_{13}g_{c2}}{t\beta^2}A_2\right)k_4$$

$$+B_{22}\left(-\frac{2B_{22}g_2}{\beta^4}A_1 + \frac{2B_{22}g_3}{\beta^4}A_4 + \frac{2B_{22}g_{c2}}{t\beta^4}A_3\right)k_3 + 2B_{23}$$

$$\left(-\frac{6B_{23}g_2}{\beta^3}A_1 + \frac{2B_{23}g_3}{\beta^3}A_5 + \frac{4B_{23}g_3}{\beta^3}A_4 + \frac{2B_{23}g_{c2}}{t\beta^3}A_2 + \frac{4B_{23}g_{c2}}{t\beta^3}A_3\right)k_5$$

$$+\left(-\frac{4B_{33}g_2}{\beta^2}A_1\right)k_2 + \left(\frac{2B_{33}g_3}{\beta^2}A_5 + \frac{2B_{33}g_3}{\beta^2}A_4 + \frac{2B_{33}g_{c2}}{t\beta^2}A_2\right)k_2$$

$$+\left(\frac{2B_{33}g_{c2}}{t\beta^2}A_3\right)k_2 + B_{55}\frac{a^2}{t^2}\left(B_{55}\frac{a^2}{t^2}\frac{2{g_4}^2A_4}{\beta^2}k_7 + B_{55}\frac{a^2}{t^2}\frac{2g_4A_3}{J_M\beta^2}k_{12}\right)\} = 0 \qquad (3.394)$$

Also expressing it terms of the Amplitude gives

$$+A_2\left[\frac{2B_{12}g_{c2}}{t\beta^2}k_2 + \frac{2B_{13}g_{c2}}{t\beta^2}k_4 + \frac{2B_{23}g_{c2}}{t\beta^3}k_5 + \frac{2B_{33}g_{c2}}{t\beta^2}k_2\right]$$

$$+A_3\left[\frac{2B_{22}g_{c2}}{t\beta^4}k_3 + \frac{4B_{23}g_{c2}}{t\beta^3}k_5 + \frac{2B_{33}g_{c2}}{t\beta^2}k_2 + B_{55}\frac{a^2}{t^2}\frac{2g_4}{J_M\beta^2}k_{12}\right]$$

$$+A_4\left[\frac{2B_{22}g_3}{\beta^4}k_3+\frac{4B_{23}g_3}{\beta^3}k_5+\frac{2B_{33}g_3}{\beta^2}k_2+B_{55}\frac{a^2}{t^2}\frac{2{g_4}^2}{\beta^2}k_7\right]$$

$$+A_5\left[\frac{2B_{12}g_3}{\beta^2}k_2+\frac{2B_{33}g_3}{\beta^2}k_2+\frac{2B_{13}g_3}{\beta}k_4+\frac{2B_{23}g_3}{\beta^3}k_5\right]\ =$$

$$-A_1\left[-\frac{2B_{12}g_2}{\beta^2}k_2-\frac{2B_{13}g_2}{\beta}k_4-\frac{2B_{22}g_2}{\beta^4}k_3-\frac{6B_{23}g_2}{\beta^3}k_5-\frac{4B_{33}g_2}{\beta^2}k_2\right] \tag{3.395}$$

and finally for the fifth Amplitude gives

$$\frac{d\pi}{dA_5}=\frac{abD_o}{2a^4}\int_0^1\int_0^1\{B_{11}\left(-2g_2A_1+2g_1A_5+\frac{g_{c2}}{t^2}A_2+\frac{g_{c2}}{t}A_2\right)k_1$$

$$+\ 2B_{12}\left(-\frac{g_2}{\beta^2}A_1+\frac{g_3}{\beta^2}A_4+\frac{g_{c2}}{t\beta^2}A_3\right)k_2$$

$$+2B_{13}\left(-\frac{g_2}{\beta}A_1+\frac{2g_3}{\beta}A_5+\frac{g_3}{\beta}A_4+\frac{2g_{c2}}{t\beta}A_2+\frac{g_{c2}}{t\beta}A_3\right)k_4$$

$$+2B_{23}\left(-\frac{g_2}{\beta^3}A_1+\frac{g_3}{\beta^3}A_4+\frac{g_{c2}}{t\beta^3}A_3\right)k_5$$

$$+B_{33}\left(-\frac{4g_2}{\beta^2}A_1+\frac{2g_3}{\beta^2}A_5\right)k_2+B_{33}\left(\frac{2g_3}{\beta^2}A_4+\frac{2g_{c2}}{t\beta^2}A_2+\frac{2g_{c2}}{t\beta^2}A_3\right)k_2$$

$$+B_{44}\frac{a^2}{t^2}\left(2{g_4}^2A_5k_6\ +\frac{2g_4A_2}{J_M}k_{11}\right)\ \}\ dRdQ \tag{3.396}$$

Further factorization gives

$$\frac{d\pi}{dA_5}=\left(-2B_{11}g_2A_1+2B_{11}g_1A_5+\frac{B_{11}g_{c2}}{t^2}A_2+\frac{B_{11}g_{c2}}{t}A_2\right)k_1$$

$$+\left(-\frac{2B_{12}g_2}{\beta^2}A_1+\frac{2B_{12}g_3}{\beta^2}A_4+\frac{2B_{12}g_{c2}}{t\beta^2}A_3\right)k_2$$

$$+\left(-\frac{2B_{13}g_2}{\beta}A_1+\frac{4B_{13}g_3}{\beta}A_5+\frac{2B_{13}g_3}{\beta}A_4+\frac{4B_{13}g_{c2}}{t\beta}A_2+\frac{2B_{13}g_{c2}}{t\beta}A_3\right)k_4$$

$$+\left(-\frac{2B_{23}g_2}{\beta^3}A_1+\frac{2B_{23}g_3}{\beta^3}A_4+\frac{2B_{23}g_{c2}}{t\beta^3}A_3\right)k_5+\left(-\frac{4B_{33}g_2}{\beta^2}A_1+\frac{2B_{33}g_3}{\beta^2}A_5\right)$$

$$k_2 + B_{33}\left(\frac{2B_{33}g_3}{\beta^2}A_4 + \frac{2B_{33}g_{c2}}{t\beta^2}A_2 + \frac{2B_{33}g_{c2}}{t\beta^2}A_3\right)k_2$$

$$+\left(2B_{44}\frac{a^2}{t^2}g_4{}^2 A_5 k_6 + B_{44}\frac{a^2}{t^2}\frac{2g_4A_2}{J_M}k_{11}\right)\} = 0 \qquad (3.397)$$

and That is

$$+A_2\left[\frac{B_{11}g_{c2}}{t^2}k_1 + \frac{B_{11}g_{c2}}{t}k_1 + \frac{4B_{13}g_{c2}}{t\beta}k_4 + \frac{2B_{33}g_{c2}}{t\beta^2}k_2 + B_{44}\frac{a^2}{t^2}\frac{2g_4}{J_M}k_{11}\right]$$

$$+A_3\left[\frac{2B_{12}g_{c2}}{t\beta^2}k_2 + \frac{2B_{13}g_{c2}}{t\beta}k_4 + \frac{2B_{23}g_{c2}}{t\beta^3}k_5 + \frac{2B_{33}g_{c2}}{t\beta^2}k_2\right]$$

$$+A_4\left[\frac{2B_{12}g_3}{\beta^2}k_2 + \frac{2B_{13}g_3}{\beta}k_4 + \frac{2B_{23}g_3}{\beta^3}k_5\frac{2B_{33}g_3}{\beta^2}k_2\right]$$

$$+A_5\left[2B_{11}g_1k_1 + \frac{4B_{13}g_3}{\beta}k_4 + \frac{2B_{33}g_3}{\beta^2}k_2 + 2B_{44}\frac{a^2}{t^2}g_4{}^2k_6\right] =$$

$$-A_1\left[-2B_{11}g_2k_1 - \frac{2B_{12}g_2}{\beta^2}k_2 - \frac{2B_{13}g_2}{\beta}k_4 - \frac{2B_{23}g_2}{\beta^3}k_5 - \frac{4B_{33}g_2}{\beta^2}k_2\right] \qquad (3.398)$$

3.2.5.4 The Compatibility Equations and Buckling Load Equation

For the sake of simplicity, Equations (3.389), (3.392), (3.395) and (3.398) were reduced to

$$Z_{11}A_2 + Z_{12}A_3 + Z_{13}A_4 + Z_{14}A_5 = Z_{15}A_1 \qquad (3.399)$$

$$Z_{21}A_2 + Z_{22}A_3 + Z_{23}A_4 + Z_{24}A_5 = Z_{25}A_1 \qquad (3.400)$$

$$Z_{31}A_2 + Z_{32}A_3 + Z_{33}A_4 + Z_{34}A_5 = Z_{35}A_1 \qquad (3.401)$$

$$Z_{41}A_2 + Z_{42}A_3 + Z_{43}A_4 + Z_{44}A_5 = Z_{45}A_1 \qquad (3.402)$$

respectively, were

$$Z_{11} = \left[-\frac{2B_{11}g_{c1}}{t}k_1 - \frac{2B_{12}g_{c1}}{t\beta^2}k_2 + \frac{2B_{13}g_{c1}}{t\beta}k_4 - \frac{2B_{23}g_{c1}}{t\beta^3}k_5 - \frac{4B_{33}g_{c1}}{t\beta^2}k_2\right] \qquad (3.403)$$

$$Z_{12} = \left[\frac{B_{11}g_{c2}}{t^2}k_1 + \frac{2B_{11}}{t^2}k_1 + \frac{4B_{13}}{t^2\beta}k_4 + \frac{2B_{33}}{t^2\beta^2}k_2 + B_{44}\frac{a^2}{t^2}\frac{2}{J_M{}^2}k_1\right] \qquad (3.403b)$$

$$Z_{13} = \left[\frac{2B_{12}}{t^2\beta^2}k_2 + \frac{2B_{13}}{t^2\beta}k_4 + \frac{2B_{23}}{t^2\beta^3}k_5 + \frac{2B_{33}}{t^2\beta^2}k_2\right] \qquad (3.403c)$$

$$Z_{14} = \left[\frac{2B_{12}g_{c2}}{t\beta^2}k_2 + \frac{2B_{13}g_{c2}}{t\beta^2}k_4 + \frac{2B_{23}g_{c2}}{t\beta^3}k_5 + \frac{2B_{33}g_{c2}}{t\beta^2}k_2\right] \quad (3.403d)$$

$$Z_{15} = \left[\frac{B_{11}g_{c2}}{t}k_1 + \frac{4B_{13}g_{c2}}{t\beta}k_4 + \frac{2B_{33}g_{c2}}{t\beta^2}k_2 + B_{44}\frac{a^2}{t^2}\frac{2g_4}{J_M}k_{11}\right] \quad (3.403e)$$

in terms of the second amplitude and also

$$Z_{21} = \left[-\frac{2B_{12}g_{c1}}{t\beta^2}k_2 - \frac{2B_{13}g_{c1}}{t\beta}k_4 - \frac{2B_{22}g_{c1}}{t\beta^4}k_3 - \frac{6B_{23}g_{c1}}{t\beta^3}k_5 - \frac{4B_{33}g_{c1}}{t\beta^2}k_2\right] \quad (3.403f)$$

$$Z_{22} = \left[\frac{2B_{12}}{t^2\beta^2}k_2 + \frac{2B_{23}}{t^2\beta^3}k_5 + \frac{2B_{33}}{t^2\beta^2}k_2\right] \quad (3.403g)$$

$$Z_{23} = \left[\frac{2B_{13}}{t^2\beta}k_4 + \frac{2B_{22}}{t^2\beta^4}k_3 + \frac{4B_{23}}{t^2\beta^3}k_5 + \frac{2B_{33}}{t^2\beta^2}k_2 + B_{55}\frac{a^2}{t^2}\frac{2}{J_M{}^2\beta^2}k_3\right] \quad (3.403h)$$

$$Z_{24} = \left[\frac{2B_{22}g_{c2}}{t\beta^4}k_3 + \frac{4B_{23}g_{c2}}{t\beta^3}k_5 + \frac{2B_{33}g_{c2}}{t\beta^2}k_2 + B_{55}\frac{a^2}{t^2}\frac{2g_4}{J_M\beta^2}k_{12}\right] \quad (3.403i)$$

$$Z_{25} = \left[\frac{2B_{12}g_{c2}}{t\beta^2}k_2 + \frac{2B_{13}g_{c2}}{t\beta}k_4 + \frac{2B_{23}g_{c2}}{t\beta^3}k_5 + \frac{2B_{33}g_{c2}}{t\beta^2}k_2\right] \quad (3.403j)$$

in terms of the third amplitude, while in terms of the fourth amplitude

$$Z_{31} = \left[-\frac{2B_{12}g_2}{\beta^2}k_2 - \frac{2B_{13}g_2}{\beta}k_4 - \frac{2B_{22}g_2}{\beta^4}k_3 - \frac{6B_{23}g_2}{\beta^3}k_5 - \frac{4B_{33}g_2}{\beta^2}k_2\right] \quad (3.403k)$$

$$Z_{32} = \left[\frac{2B_{12}g_{c2}}{t\beta^2}k_2 + \frac{2B_{13}g_{c2}}{t\beta^2}k_4 + \frac{2B_{23}g_{c2}}{t\beta^3}k_5 + \frac{2B_{33}g_{c2}}{t\beta^2}k_2\right] \quad (3.403l)$$

$$Z_{33} = \left[\frac{2B_{22}g_{c2}}{t\beta^4}k_3 + \frac{4B_{23}g_{c2}}{t\beta^3}k_5 + \frac{2B_{33}g_{c2}}{t\beta^2}k_2 + B_{55}\frac{a^2}{t^2}\frac{2g_4}{J_M\beta^2}k_{12}\right] \quad (3.403m)$$

$$Z_{34} = \left[\frac{2B_{22}g_3}{\beta^4}k_3 + \frac{4B_{23}g_3}{\beta^3}k_5 + \frac{2B_{33}g_3}{\beta^2}k_2 + B_{55}\frac{a^2}{t^2}\frac{2g_4{}^2}{\beta^2}k_7\right] \quad (3.403n)$$

$$Z_{35} = \left[\frac{2B_{12}g_3}{\beta^2}k_2 + \frac{2B_{33}g_3}{\beta^2}k_2 + \frac{2B_{13}g_3}{\beta}k_4 + \frac{2B_{23}g_3}{\beta^3}k_5\right] \quad (3.403o)$$

For the fifth case,

$$Z_{41} = \left[-2B_{11}g_2k_1 - \frac{2B_{12}g_2}{\beta^2}k_2 - \frac{2B_{13}g_2}{\beta}k_4 - \frac{2B_{23}g_2}{\beta^3}k_5 - \frac{4B_{33}g_2}{\beta^2}k_2\right] \quad (3.403p)$$

$$Z_{42} = \left[\frac{B_{11}g_{c2}}{t^2}k_1 + \frac{B_{11}g_{c2}}{t}k_1 + \frac{4B_{13}g_{c2}}{t\beta}k_4 + \frac{2B_{33}g_{c2}}{t\beta^2}k_2 + B_{44}\frac{a^2}{t^2}\frac{2g_4}{J_M}k_{11}\right] \quad (3.403q)$$

$$Z_{43} = \left[\frac{2B_{12}g_{c2}}{t\beta^2}k_2 + \frac{2B_{13}g_{c2}}{t\beta}k_4 + \frac{2B_{23}g_{c2}}{t\beta^3}k_5 + \frac{2B_{33}g_{c2}}{t\beta^2}k_2\right] \tag{3.403r}$$

$$Z_{44} = \left[\frac{2B_{12}g_3}{\beta^2}k_2 + \frac{2B_{13}g_3}{\beta}k_4 + \frac{2B_{23}g_3}{\beta^3}k_5 \frac{2B_{33}g_3}{\beta^2}k_2\right] \tag{3.403s}$$

$$Z_{45} = \left[2B_{11}g_1k_1 + \frac{4B_{13}g_3}{\beta}k_4 + \frac{2B_{33}g_3}{\beta^2}k_2 + 2B_{44}\frac{a^2}{t^2}g_4{}^2k_6\right] \tag{3.403t}$$

Applying the method of Gauss elimination, Equations (3.399), (3.400), (3.401) and (3.402) were resolved and that gave:

$$A_5 = P_5A_1, \quad A_4 = P_4A_1, \quad A_3 = P_3A_1, \quad A_2 = P_2A_1$$

and substituting them into the governing Equation gives

$$B_{11}\left(2g_1A_1 - 2g_2P_5A_1 - \frac{2g_{c1}}{t}P_2A_1\right)k_1$$

$$+ \frac{2B_{12}}{\beta^2}\left(2g_1A_1 - g_2[P_4A_1 + P_5A_1] - \frac{g_{c1}}{t}[P_3A_1 + P_2A_1]\right)k_2$$

$$+ \frac{2B_{13}}{\beta}\left(-4g_1A_1 + g_2[P_5A_1 - P_4A_1] + \frac{g_{c1}}{t}[P_2A_1 - P_3A_1]\right)k_4$$

$$+ \frac{B_{22}}{\beta^4}\left(2g_1A_1 - 2g_2P_4A_1 - \frac{2g_{c1}}{t}P_3A_1\right)k_3$$

$$+ \frac{2B_{23}}{\beta^3}\left(4g_1A_1 - g_2P_4A_1 - \frac{g_{c1}}{t}[P_2A_1 + 3P_3A_1]\right)k_5$$

$$+ \frac{4B_{33}}{\beta^2}\left(2g_1A_1 - g_2[P_5A_1 + P_4A_1] - \frac{g_{c1}}{t}[P_3A_1 + 4P_2A_1]\right)k_2 = \frac{N_xa^2}{D_0}A_1k_6 \tag{3.404}$$

Dividing Equation (3.404) by A_1 gives

$$B_{11}\left(2g_1 - 2g_2P_5 - \frac{2g_{c1}}{t}P_2\right)k_1$$

$$+\frac{2}{\beta^2}\Big[2g_1(B_{12} + 2B_{33}) - g_2(B_{12} + 2B_{33})[P_5 + P_4]$$
$$- \frac{g_{c1}}{t}(B_{12}[P_3 + P_2] + 2B_{33}[P_3 + 4P_2])\Big]k_2$$

$$+\left(\frac{4g_1A_1}{\beta^2}[B_{12} + 2B_{33}] - \frac{2g_2}{\beta^2}[B_{12} + 2B_{33}][P_5 + P_4] - \frac{2g_{c1}}{t\beta^2}\Big[P_3[B_{12} + 2B_{33}] + P_2[B_{12} + 8B_{33}]\Big]\right)k_2$$

$$+\frac{B_{22}}{\beta^4}\left(2g_1 - 2g_2P_4 - \frac{2g_{c1}}{t}P_3\right)k_3 + \frac{2B_{13}}{\beta}\left(-4g_1 + g_2[P_5 - P_4] + \frac{g_{c1}}{t}[P_2 - P_3]\right)k_4$$

$$+\frac{2B_{23}}{\beta^3}\left(4g_1 - g_2P_4 - \frac{g_{c1}}{t}[P_2 + 3P_3]\right)k_5 \; = \; \frac{N_xa^2}{D_0}k_6 \tag{3.405}$$

Rearranging Equation (3.405) gives

$$\frac{N_xa^2}{D_0} = \frac{1}{k_6}\{B_{11}\left(2g_1 - 2g_2P_5 - \frac{2g_{c1}}{t}P_2\right)k_1$$

$$+\frac{2}{\beta^2}\frac{1}{k_6}\Big[2g_1(B_{12} + 2B_{33}) - g_2(B_{12} + 2B_{33})[P_5 + P_4]$$

$$-\frac{g_{c1}}{t}(B_{12}[P_3 + P_2] + 2B_{33}[P_3 + 4P_2])\Big]k_2$$

$$+\frac{B_{22}}{\beta^4}\frac{1}{k_6}\left(2g_1 - 2g_2P_4 - \frac{2g_{c1}}{t}P_3\right)k_3$$

$$+\frac{2B_{13}}{\beta}\frac{1}{k_6}\left(-4g_1 + g_2[P_5 - P_4] + \frac{g_{c1}}{t}[P_2 - P_3]\right)k_4$$

$$+\frac{2B_{23}}{\beta^3}\frac{1}{k_6}\left(4g_1 - g_2P_4 - \frac{g_{c1}}{t}[P_2 + 3P_3]\right)k_5\,\} \tag{3.406}$$

Where $K_{T1} = B_{11}\left(g_1 - g_2P_5 - \frac{g_{c1}}{t}P_2\right)k_1$ (3.407)

$$K_{T2} = \frac{1}{\beta^2k_6}\Big[2g_1(B_{12} + 2B_{33}) - g_2(B_{12} + 2B_{33})[P_5 + P_4] - \frac{g_{c1}}{t}(B_{12}[P_3 + P_2] + 2B_{33}[P_3 + 4P_2])\Big]k_2 \tag{3.408}$$

$$K_{T3} = \frac{B_{22}}{\beta^4k_6}\left(g_1 - g_2P_4 - \frac{g_{c1}}{t}P_3\right)k_3 \tag{3.409}$$

$$K_{T4} = \frac{B_{13}}{\beta k_6}\left(-4g_1 + g_2[P_5 - P_4] + \frac{g_{c1}}{t}[P_2 - P_3]\right)k_4 \tag{3.410}$$

$$K_{T5} = \frac{B_{23}}{\beta^3k_6}\left(4g_1 - g_2P_4 - \frac{g_{c1}}{t}[P_2 + 3P_3]\right)k_5 \tag{3.411}$$

$$\frac{N_xa^2}{D_0} = \frac{E_0}{D_0}\left(\frac{K_{T1} + K_{T2} + K_{T3} + K_{T4} + K_{T5}}{k_6}\right) \tag{3.412}$$

3.2.6 Numerical Analysis of Thick Laminated Anisotropic Plate for Different Edge / Boundary Conditions

The analysis of laminated thick plate, consisting of 3 different plate laminas were considered differently. In the first case, 2 laminated plate with two different laminas. Also the cases of three, four, five and more laminated conditions can be conducted using the same numerical procedures. It was observed that as the values of the lamina, m changes, the corresponding values of J_i. and g_i changes both in Bending, Coupling and Axial membrane. These processes were repeated for all the different plate conditions as shown in the sections below.

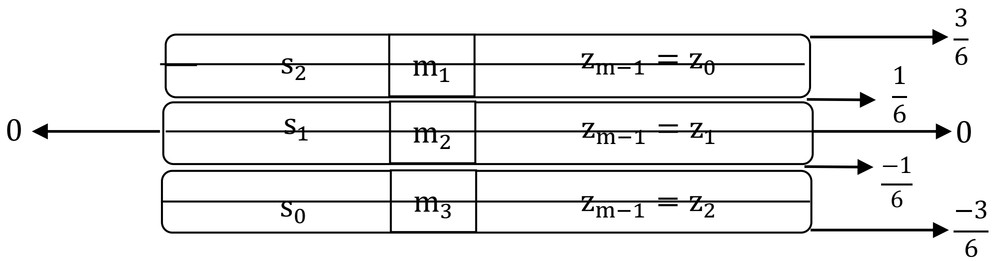


Figure 3.4 Three Laminated Thick Plate

with m = lamina , z = thichness, s = non dimensional coordinate on z direction and

n = total number of laminas

the mid position is considered as zero.

Two major angles were considered in the orientation of the plate, namely 0^0 and 90^0 and the values obtained were considered as follows:

For $\emptyset = 0, \cos 0 = 1, \sin 0 = 0$

But when $\emptyset = 90, \cos 90 = 0, \sin 90 = 1$

And considering m as $\cos \emptyset$ and n as the $\sin\emptyset$

$\mathrm{G}_{12} = 0.5$, $\mathrm{G}_{13} = 0.5$, $\mathrm{G}_{23} = 0.2$, $\mu_{12} = \mu_2 = \frac{E_2}{E_1}\mu_{12} = 0.01$,

$E_{44} = \mathrm{d}_{11} = \frac{\mathrm{E}_{11}}{\mathrm{E}_0} = 25$, $\mathrm{d}_{12} = \frac{\mathrm{E}_{12}}{\mathrm{E}_0} = 0.25$, $\mathrm{d}_{21} = \frac{\mathrm{E}_{21}}{\mathrm{E}_0} = 0.25$, $\mathrm{d}_{22} = \frac{\mathrm{E}_{22}}{\mathrm{E}_0} = 1$

$d_{33} = \frac{E_{33}}{E_0} = 0.49875$, but $E_{33} = G_{12}(1 - \mu_{12}\mu_{21})$

$d_{44} = \frac{E_{44}}{E_0} = 0.49875$ but $E_{44} = G_{13}(1 - \mu_{12}\mu_{21})$

$d_{55} = \frac{E_{55}}{E_0} = 0.1995$ but $E_{55} = G_{23}(1 - \mu_{12}\mu_{21})$

The case of an angle of 0^o was considered. That is $\emptyset = 0$ and the results were as detailed below Taking $m = \cos 0 = 1$ *and* $n = \sin 0 = 0$ and also considering B_{ij} expressions as

$$B_{11} = m^4 d_{11} + 2m^2n^2(d_{12} + 2d_{33}) + n^4 d_{22} \quad (3.413)$$

$$B_{12} = d_{12}(n^4 + m^4) + m^2n^2(d_{11} + d_{22} - 4d_{33}) \quad (3414)$$

$$B_{13} = m^3n(d_{11} - d_{12} - 2d_{33}) + mn^3(d_{12} - d_{22} + 2d_{33}) \quad (3.415)$$

$$B_{22} = n^4 d_{11} + 2m^2n^2(d_{12} + 2d_{33}) + m^4 \quad (3.416)$$

$$B_{23} = mn^3 d_{11} - m^3 n d_{22} + (m^3n - mn^3)(d_{12} + 2d_{33}) \quad (3.417)$$

$$B_{33} = m^2n^2(d_{11} - 2d_{12} + d_{22} - 2d_{33}) + d_{33}(m^4 + n^4) \quad (3.418)$$

and $B_{21} = B_{12}$, $B_{31} = B_{13}$ *and* $B_{32} = B_{23}$

$B_{44} = d_{44}$

$B_{55} = d_{55}$

Putting the real values of m_{ij} and d_{ij} gives

$B_{11} = 1^4 * 25 + 2 * 1^2 0^2(0.25 + 2 * 0.49875) + 0^4 * 1$

$B_{11} = 25$

$B_{12} = 0.25(0^4 + 1^4) + 1^2 * 0^2(25 + 1 - 4 * 0.49875)$

$B_{12} = 0.25$

$B_{13} = 1^3 * 0(d_{11} - d_{12} - 2d_{33}) + mn^3(d_{12} - d_{22} + 2d_{33})$

$B_{13} = 0$

$B_{23} = m0^3 d_{11} - m^3 0 d_{22} + (m^3 0 - m0^3)(d_{12} + 2d_{33})$

$B_{23} = 0$

$B_{33} = 1^2 0^2(d_{11} - 2d_{12} + d_{22} - 2d_{33}) + 0.49875(1^4 + 0^4)$

$B_{33} = 0.49875$

$B_{44} = 0.49875$

Similarly the case of an angle of 90° was considered. That is $\emptyset$ = 90° and the results were as detailed below. Taking $m = \cos 0 = 1$ *and* $n = \sin 0 = 0$ and substitutitng B_{ij}expressions as gives

$$B_{11} = m^4 d_{11} + 2m^2 n^2 (d_{12} + 2d_{33}) + n^4 d_{22} \quad (3.419)$$

$$B_{11} = 1$$

$$B_{12} = d_{12}(n^4 + m^4) + m^2 n^2 (d_{11} + d_{22} - 4d_{33}) \quad (3.420)$$

$$B_{12} = 0.25$$

$$B_{13} = m^3 n(d_{11} - d_{12} - 2d_{33}) + mn^3 (d_{12} - d_{22} + 2d_{33}) \quad (3.421)$$

$$B_{13} = 0$$

$$B_{22} = n^4 d_{11} + 2m^2 n^2 (d_{12} + 2d_{33}) + m^4 d_{22} \quad (3.422)$$

$$B_{22} = 25$$

$$B_{23} = mn^3 d_{11} - m^3 n d_{22} + (m^3 n - mn^3)(d_{12} + 2d_{33}) \quad (3.423)$$

$$B_{23} = 0$$

$$B_{33} = m^2 n^2 (d_{11} - 2d_{12} + d_{22} - 2d_{33}) + d_{33}(m^4 + n^4) \quad (3.424)$$

$$B_{33} = 0.49875$$

and $B_{21} = B_{12}$, $B_{31} = B_{13}$ and $B_{32} = B_{23}$

$Z -$ values for different lamina arrangement

These explains the outcome of the various stiffnesses when different values of thickness were substituted at different layer arrangement. As earlier explained, this is traceable to the changes in lamina, m values as it increases.

3.2.6.1 Analysis of SSSS Thick Laminated Anisotropic Plates

The analysis for the conditions of two and three SSSS plate were considered and the following observations below were made. Various cases of SSSS plate were considered depending on the number of laminas in each laminated plate. For example two laminas were considered in 2-laminated plate. Three laminas were considered for the case of 3-laminated plate and the entire procedures followed the same trend.

3.2.6.2 Two Laminated SSSS Thick Anisotropic Plate Condition

Here, the total of two laminas were considered. For the case of m = 1 and for the case of m = 2 . The values obtained, were queued in to the appropriate equation for the derivations of the of the J_i. and g_i values needed for the next stage of the analysis. Three stiffness conditions were considered. For the Bending stiffness, the values of J_{ij} when the total number of laminas, n is two, were derived as follows

The bending stiffness for for the case of 2–Laminates, when m = 1 is given as

$$J_1 = \left[\frac{(s_m^3 - s_{m-1}^3)}{3}\right] * 2 \tag{3.425}$$

Since this is the first lamina, that means m = 1. From Figure (3.5), all the possible values of s were clearly derived and substituting 1 for m in Equation (3.425) leaves it as

$$J_1 = \left[\frac{(s_1^3 - s_0^3)}{3}\right] * 2 = \left[\frac{((-0.3333)^3 - (-0.5)^3)}{3}\right] * 2 \tag{3.426}$$

Also

$$J_2 = \left[\frac{\left(s_m^3 - \frac{4}{5}s_m^5\right) - \left(s_{m-1}^3 - \frac{4}{5}s_{m-1}^5\right)}{3}\right] * 2 \tag{3.427}$$

Substituting the values of the unknowns gives

$$J_2 = \left[\frac{\left((-0.3333)^3 - \frac{4}{5}(-0.3333)^5\right) - \left((-0.5)^3 - \frac{4}{5}(-0.5)^3\right)}{3}\right] * 2 \tag{3.428}$$

while the

$$J_3 = \left[\frac{\left(s_m^3 - \frac{8}{5}s_m^5 + \frac{16}{21}s_m^7\right) - \left(s_{m-1}^3 - \frac{8}{5}s_{m-1}^5 + \frac{16}{21}s_{m-1}^7\right)}{3}\right] * 2 \tag{3.429}$$

From Equation (3.429)comes

$$J_3 = \left[\frac{\left(s_1^3 - \frac{8}{5}s_1^5 + \frac{16}{21}s_1^7\right) - \left(s_0^3 - \frac{8}{5}s_0^5 + \frac{16}{21}s_0^7\right)}{3}\right] * 2 \tag{3.430}$$

Putting the values of the non dimensional coordinates into Equation (3.430) gives

$$J_3 = \left[\frac{\left((-0.3333)^3 - \frac{8}{5}(-0.3333)^5 + \frac{16}{21}(-0.3333)^7\right) - \left((-0.5)^3 - \frac{8}{5}(-0.5)^5 + \frac{16}{21}(-0.5)^7\right)}{3}\right] * 2 \qquad (3.431)$$

and finally for the bending stiffness

$$J_4 = \left[\frac{\left(3s_m^1 - 8s_m^3 + \frac{48}{5}s_m^5\right) - \left(3s_{m-1}^1 - 8s_{m-1}^3 + \frac{48}{5}s_{m-1}^5\right)}{3}\right] * 2 \qquad (3.432)$$

That is

$$J_4 = \left[\frac{\left(3s_1^1 - 8s_1^3 + \frac{48}{5}s_1^5\right) - \left(3s_0^1 - 8s_0^3 + \frac{48}{5}s_0^5\right)}{3}\right] * 2 \qquad (3.433)$$

Substituting the values of s into Equation (3.433) gives

$J_4 =$

$$\left[\frac{J_{4r} - J_{4l}}{3}\right] * 2 \qquad (3.434)$$

where

$$J_{4r} = \left(3(-0.3333)^1 - 8(-0.3333)^3 + \frac{48}{5}(-0.3333)^5\right) \qquad (3.434b)$$

and

$$J_{4l} = \left(3(-0.5)^1 - 8(-0.5)^3 + \frac{48}{5}(-0.5)^5\right) \qquad (3.434c)$$

Also the coupling stiffness for the case of 2-Laminates, when m =1 and the total number of laminas, n is two, was derived as follows:

$$J_{c1} = \left[\frac{(s_m^2 - s_{m-1}^2)}{2}\right]*2 \qquad (3.435)$$

That is $J_{c1} = \left[\frac{(s_1^2 - s_0^2)}{2}\right]*2$ (3.436)

Putting the values of unknown gives

$$J_{c1} = \left[\frac{\left((-0.3333)^2-(-0.5)^2\right)}{2}\right]*2 \tag{3.437}$$

Also for the

$$J_{c2} = \left[\frac{\left(s_m^2 - \frac{2}{3}s_m^4\right) - \left(s_{m-1}^2 - \frac{2}{3}s_{m-1}^4\right)}{3}\right] * 2 \tag{3.438}$$

That is

$$J_{c2} = \left[\frac{\left(s_1^2 - \frac{2}{3}s_1^4\right) - \left(s_0^2 - \frac{2}{3}s_0^4\right)}{3}\right] * 2 \tag{3.439}$$

Substituting the values of s gives

$$J_{c2} = \left[\frac{\left((-0.3333)^2 - \frac{2}{3}(-0.3333)^4\right) - \left((-0.5)^2 - \frac{2}{3}(-0.5)^4\right)}{3}\right] * 2 \tag{3.440}$$

and finally for

$$J_{c3} = \left[\left(\mathrm{s}_\mathrm{m}^1 - \frac{4}{3}\mathrm{s}_\mathrm{m}^3\right) - \left(\mathrm{s}_{\mathrm{m}-1}^1 - \frac{4}{3}\mathrm{s}_{\mathrm{m}-1}^3\right)\right] * 2 \tag{3.441}$$

That is

$$J_{c3} = \left(\mathrm{s}_1^1 - \frac{4}{3}\mathrm{s}_1^3\right) - \left(\mathrm{s}_0^1 - \frac{4}{3}\mathrm{s}_0^3\right) * 2 \tag{3.442}$$

Substituting the values of s gives

$$J_{c3} = \left[\left((-0.3333)^1 - \frac{4}{3}(-0.3333)^3\right) - \left((-0.5)^1 - \frac{4}{3}(-0.5)^3\right)\right] * 2 \tag{3.443}$$

Similarly the Membrane Stiffness for the case of 2-Laminates, when m = 1 and the total number of laminas, n is two, was derived as follows

$$J_\mathrm{M} = [(s_m^1) - (s_{m-1}^1)] * 2 \tag{3.445}$$

That is

$$J_M = [(s_1^1) - (s_0^1)] * 2 \tag{3.446}$$

Putting the values of s gives

$$J_M = [(-0.3333)^1 - (-0.5)^1] * 2 \tag{3.447}$$

Bending Stiffness for 2–Laminates when m = 2, three stiffness conditions were considered. The values of J_{ij} when the total number of laminas, n is two, were derived as follows

$$J_1 = \left[\frac{(s_2^3 - s_1^3)}{3}\right] * 2 \tag{3.448}$$

Since this is the second lamina, that means m = 2. From Figure 3.5, all the possible values of s were clearly derived and substituting 1 for m in Equation (3.448), leaves it as

$$J_1 = \left[\frac{(s_2^3 - s_1^3)}{3}\right] * 2 = \left[\frac{((-0.16667)^3 - (-0.333)^3)}{3}\right] * 2 \tag{3.449}$$

Also

$$J_2 = \left[\frac{\left(s_m^3 - \frac{4}{5}s_m^5\right) - \left(s_{m-1}^3 - \frac{4}{5}s_{m-1}^5\right)}{3}\right] * 2 \tag{3.450}$$

Substituting the values of the unknowns gives

$$J_2 = \left[\frac{\left((-0.16667)^3 - \frac{4}{5}(-0.16667)^5\right) - \left((-0.3333)^3 - \frac{4}{5}(-0.3333)^5\right)}{3}\right] * 2 \tag{3.451}$$

while the

$$J_3 = \left[\frac{\left(s_m^3 - \frac{8}{5}s_m^5 + \frac{16}{21}s_m^7\right) - \left(s_{m-1}^3 - \frac{8}{5}s_{m-1}^5 + \frac{16}{21}s_{m-1}^7\right)}{3}\right] * 2 \tag{3.452}$$

From Equation (3.452) comes

$$J_3 = \left[\frac{\left(s_2^3 - \frac{8}{5}s_2^5 + \frac{16}{21}s_2^7\right) - \left(s_1^3 - \frac{8}{5}s_1^5 + \frac{16}{21}s_1^7\right)}{3}\right] * 2 \tag{3.453}$$

Putting the values of the non dimensional coordinates into Equation (3.453) gives

$$J_3 = \left[\frac{\left((-0.16667)^3 - \frac{8}{5}(-0.16667)^5 + \frac{16}{21}(-0.16667)^7\right) - \left((-0.3333)^3 - \frac{8}{5}(-0.3333)^5 + \frac{16}{21}(-0.3333)^7\right)}{3}\right] * 2 \tag{3.454}$$

and finally for the bending stiffness

$$J_4 = \left[\frac{\left(3s_m^1 - 8s_m^3 + \frac{48}{5}s_m^5\right) - \left(3s_{m-1}^1 - 8s_{m-1}^3 + \frac{48}{5}s_{m-1}^5\right)}{3}\right] * 2 \quad (3.455)$$

That is

$$J_4 = \left[\frac{\left(3s_2^1 - 8s_2^3 + \frac{48}{5}s_2^5\right) - \left(3s_1^1 - 8s_1^3 + \frac{48}{5}s_1^5\right)}{3}\right] * 2 \quad (3.456)$$

Substituting the values of s into Equation (3.456) gives

$$J_4 = \left[\frac{\left(3(-0.16667)^1 - 8(-0.16667)^3 + \frac{48}{5}(-0.16667)^5\right) - \left(3(-0.333)^1 - 8(-0.333)^3 + \frac{48}{5}(-0.333)^5\right)}{3}\right] * 2 \quad (3.457)$$

For the Coupling Stiffness for 2-Laminates when m = 2 , the values of J_{ij} when the total number of laminas, n is two, were derived as follows:

For

$$J_{c1} = \left[\frac{(s_m^2 - s_{m-1}^2)}{2}\right] * 2 \quad (3.458)$$

That is $J_{c1} = \left[\frac{(s_2^2 - s_1^2)}{2}\right] * 2$ (3.459)

Putting the values of unknown gives

$$J_{c1} = \left[\frac{((-0.16667)^2 - (-0.333)^2)}{2}\right] * 2 \quad (3.460)$$

Also for the

$$J_{c2} = \left[\frac{\left(s_m^2 - \frac{2}{3}s_m^4\right) - \left(s_{m-1}^2 - \frac{2}{3}s_{m-1}^4\right)}{3}\right] * 2 \quad (3.461)$$

That is

$$J_{c2} = \left[\frac{\left(s_2^2 - \frac{2}{3}s_2^4\right) - \left(s_1^2 - \frac{2}{3}s_1^4\right)}{3}\right] * 2 \qquad (3.462)$$

Substituting the values of s gives

$$J_{c2} = \left[\frac{\left((-0.16667)^2 - \frac{2}{3}(-0.16667)^4\right) - \left((-0.333)^2 - \frac{2}{3}(-0.333)^4\right)}{3}\right] * 2 \qquad (3.463)$$

and finally for

$$J_{c3} = \left[\left(s_m^1 - \frac{4}{3}s_m^3\right) - \left(s_{m-1}^1 - \frac{4}{3}s_{m-1}^3\right)\right] * 2 \qquad (3.464)$$

That is

$$J_{c3} = \left(s_2^1 - \frac{4}{3}s_2^3\right) - \left(s_1^1 - \frac{4}{3}s_1^3\right) * 2 \qquad (3.465)$$

Substituting the values of s gives

$$J_{c3} = \left[\left((-0.16667)^1 - \frac{4}{3}(-0.16667)^3\right) - \left((-0.3333)^1 - \frac{4}{3}(-0.3333)^3\right)\right] * 2 \qquad (3.467)$$

Membrane Stiffness for 2–Laminates when m = 2

$$J_M = [(s_m^1) - (s_{m-1}^1)] * 2 \qquad (3.468)$$

That is

$$J_M = [(s_2^1) - (s_1^1)] * 2 \qquad (3.469)$$

Putting the values of s gives

$$J_M = [(-0.16667)^1 - (-0.3333)^1] * 2 \qquad (3.470)$$

3.2.6.3 Two Laminated SSSS Plate for 0^0 0^0 Arrangement

The various values of Z for SSSS plate were derived as shown in Equation (3.787) to

$$Z_{11} = [(B_{11}g_{c2}k_1)/t^2 + (2B_{11}k_1)/t^2 + (4B_{13}k_4)/\beta t^2 + (2B_{33}k_2)/\beta^2 t^2 + (2a^2B_{44}k_1)/J_M^2] \quad (3.471)$$

$$Z_{12} = \left[\frac{(2B_{12}k_2)}{\beta^2 t^2} + \frac{(2B_{13}k_4)}{\beta t^2} + \frac{(2B_{23}k_5)}{\beta^3 t^2} + \frac{(2B_{33}k_2)}{\beta^2 t^2}\right] \quad (3.472)$$

$$Z_{13} = [(2B_{12}g_{c2}k_2)/\beta^2 t + (2B_{13}g_{c2}k_1)/t^2 + (2B_{23}g_{c2}k_5)/\beta^3 t + (2B_{33}g_{c2}k_2)/t\beta^2] \quad (3.473a)$$

$$Z_{14} = [(B_{11}g_{c2}k_1)/t + (4B_{13}g_{c2}k_4)/t\beta + (2B_{33}g_{c2}k_2)/\beta t^2 + (2a^2g_4B_{44}k_{11})/J_M t^2] \quad (3.473b)$$

$$Z_{15} = \left[-\frac{(2B_{11}g_{c1}k_1)}{t} + \frac{(2B_{12}g_{c1}k_2)}{\beta t^2} + \frac{(2B_{13}g_{c1}k_4)}{t\beta} - (2B_{23}g_{c1}k_5)/t\beta^3 - (4B_{33}g_{c1}k_2)/t\beta^2\right] \quad (3.474a)$$

$$Z_{21} = [(2B_{12}k_2)/\beta^2 t^2 + (2B_{23}k_5)/\beta^3 t^2 + (2B_{33}k_2)/\beta^2 t^2] \quad (3.474b)$$

$$Z_{22} = \left[\frac{(2B_{13}k_4)}{\beta t^2} + \frac{(2B_{22}k_3)}{\beta^4 t^2} + \frac{(4B_{23}k_5)}{\beta^3 t^2} + \frac{(2B_{33}k_2)}{\beta^2 t^2} + \frac{2a^2B_{55}k_3}{t^2\beta^2 J_M^2}\right] \quad (3.475)$$

$$Z_{23} = [(2B_{22}g_{c2}k_3)/\beta^4 t + (4B_{23}g_{c2}k_5)/\beta^3 t + (2B_{33}g_{c2}k_2)/\beta^2 t + (2a^2B_{55}k_{12})/\beta^2 t^2 J_M^2] \quad (3.476)$$

$$Z_{24} = [(2B_{12}g_{c2}k_2)/t\beta^2 + (2B_{13}g_{c2}k_4)/t\beta + (2B_{23}g_{c2}k_5)/\beta t^2 + (2B_{33}g_{c2}k_2)/t\beta^2] \quad (3.477)$$

$$Z_{25} = [(-2B_{12}g_{c1}k_2)/t\beta^2 - (2B_{13}g_{c1}k_4)/t\beta - (2B_{22}g_{c1}k_3)/t\beta^4 - (6B_{23}g_{c1}k_5)/t\beta^3 - (4B_{33}g_{c1}k_2)/t\beta^2] \quad (3.478)$$

$$Z_{31} = [(2B_{12}g_{c2}k_2)/t\beta^2 + (2B_{13}g_{c2}k_4)/t\beta^2 + (2B_{23}g_{c2}k_5)/t\beta^3 + (2B_{33}g_{c2}k_2)/t\beta^2] \quad (3.479)$$

$$Z_{32} = [(2B_{22}g_{c2}k_3)/\beta^4 t + (4B_{23}g_{c2}k_5)/\beta^3 t + (2B_{33}g_{c2}k_2)/t\beta^2 + (2a^2g_4B_{55}k_{12})/\beta^2 J_M] \quad (3.480)$$

$$Z_{33} = [(2B_{22}g_3k_3)/\beta^4 + (4B_{23}g_3k_5)/\beta^3 + (2B_{33}g_3k_2)/\beta^2 + (2a^2B_{55}{g_4}^2k_7)/\beta^2 t^2] \quad (3.481)$$

$$Z_{34} = [(2B_{12}g_3k_2)/\beta^2 + (2B_{33}g_3k_2)/\beta^2 + (2B_{13}g_3k_4)/\beta + (2B_{23}g_3k_5)/\beta^3] \quad (3.482)$$

$$Z_{35} = \left[\frac{(-2B_{12}g_2k_2)}{\beta^2} - \frac{(2B_{13}g_2k_4)}{\beta} - \frac{(2B_{22}g_2k_3)}{\beta^4} - (6B_{23}g_2k_5)/\beta^3 - (4B_{33} * g_2k_2)/\beta^2\right] \quad (3.483)$$

$$Z_{41}[(B_{11}g_{c2}k_1)/t^2 + (B_{11}g_{c2}k_1)/t + (4B_{13}g_{c2}k_4)/t\beta + (2B_{33}g_{c2}k_2)/t\beta^2 + (2a^2B_{44}g_4k_{11})/t^2J_M] \quad (3.484)$$

$$Z_{42} = [(2B_{12}g_{c2}k_2)/t\beta^2 + (2B_{13}g_{c2}k_4)/t\beta + (2B_{23}g_{c2}k_5)/t\beta^3 + (2B_{33}g_{c2}k_2)/t\beta^2] \quad (3.485)$$

$$Z_{43} = [(2B_{12}g_3k_2)/\beta^2 + (2B_{13}g_3k_4)/\beta + (2B_{23}g_3k_5)/\beta^3 + (2B_{33}g_3k_2)/\beta^2] \quad (3.486)$$

$$Z_{44} = [(2B_{11}g_1k_1) + (4B_{13}g_3k_4)/\beta + (2B_{33}g_3k_2)/\beta^2 + (2{g_4}^2a^2B_{44}k_6)/t^2] \quad (3.487)$$

$$Z_{45} = [-2 * (B_{11}g_2k_1) - (2B_{12}g_2k_2)/\beta^2 - (2B_{13}g_2k_4)/\beta - (2B_{23}g_2k_5)/\beta^3 - (4B_{33}g_2k_2)/\beta^2] \quad (3.488)$$

Derivation of L-values for SSSS plate

$$L_{11} = [(B_{11}g_{c2}k_1)/t^2 + (2B_{11}k_1)/t^2 + (4B_{13}k_4)/\beta t^2 + (2B_{33}k_2)/\beta^2t^2 + (2a^2B_{44}k_1)/J_M^2] \quad (3.489)$$

$$L_{12} = [(2B_{12}k_2)/\beta^2t^2 + (2B_{13}k_4)/\beta t^2 + (2B_{23}k_5)/\beta^3t^2 + (2B_{33}k_2)/\beta^2t^2] \quad (3.490)$$

$$L_{13} = [(2B_{12}g_{c2}k_2)/\beta^2t + (2B_{13}g_{c2}k_1)/t^2 + (2B_{23}g_{c2}k_5)/\beta^3t + (2B_{33}g_{c2}k_2)/t\beta^2] \quad (3.491)$$

$$L_{14} = [(B_{11}g_{c2}k_1)/t + (4 * B_{13}g_{c2}k_4)/t\beta + (2B_{33}g_{c2}k_2)/\beta t^2 + (2a^2g_4B_{44}k_{11})/J_M t^2] \quad (3.500)$$

$$L_{15} = [(2B_{11}g_{c1}k_1)/t + (2B_{12}g_{c1}k_2)/\beta t^2 + (2B_{13}g_{c1}k_4)/t\beta + (2B_{23}g_{c1}k_5)/t\beta^3 - (4 * B_{33}g_{c1}k_2)/t\beta^2] \quad (3.501)$$

$$L_{21} = Z_{21}$$

$$L_{22} = \left(Z_{21} * \frac{Z_{12}}{Z_{11}} - Z_{22}\right) \quad (3.502)`$$

$$L_{23} = \left(Z_{21} * \frac{Z_{13}}{Z_{11}} - Z_{23}\right) \quad (3.503)$$

$$L_{24} = \left(Z_{21} * \frac{Z_{14}}{Z_{11}} - Z_{24}\right) \quad (3.504)$$

$$L_{33} = (Z_{21}(Z_{12}Z_{33} - Z_{32}Z_{13}) + Z_{31}(Z_{22}Z_{13} - Z_{23}Z_{12}) + Z_{11}(Z_{32}Z_{23} - Z_{32}Z_{22}))/(Z_{21} * Z_{12} - Z_{11}Z_{22})) \quad (3.505)$$

$$L_{34} = (Z_{14}(Z_{22}Z_{31} - Z_{32}Z_{21}) + Z_{12}(Z_{34}Z_{21} - Z_{24}Z_{31}) + Z_{11}(Z_{32}Z_{24} - Z_{22}Z_{34}))/(Z_{21}Z_{12} - Z_{11}Z_{22})) * A_5 \quad (3.506)$$

$$L_{35} = (Z_{15}(Z_{31}Z_{22} - Z_{32}Z_{21}) + Z_{12}(Z_{21}Z_{35} - Z_{31}Z_{25}) + Z_{11}(Z_{32}Z_{25} - Z_{22}Z_{35})/(Z_{21}Z_{12} - Z_{11}Z_{22})) \quad (3.507)$$

$$L_{43} = (Z_{13}(Z_{41}Z_{22} - Z_{42}Z_{21}) + Z_{12}(Z_{21}Z_{43} - Z_{41}Z_{23}) + Z_{11}(Z_{42}Z_{23} - Z_{43}Z_{22})/(Z_{21}Z_{12} - Z_{11}Z_{22})) \quad (3.508)$$

$$L_{44} = (Z_{14}(Z_{41}Z_{22} - Z_{42}Z_{21}) + Z_{12}(Z_{21}Z_{44} - Z_{24}Z_{41}) + Z_{11}(Z_{42}Z_{24} - Z_{22}Z_{44})/(Z_{21}Z_{12} - Z_{11}Z_{22})) \quad (3.509)$$

$$L_{45} = (Z_{15}(Z_{22}Z_{41} - Z_{42}Z_{21}) + Z_{12}(Z_{45}Z_{21} - Z_{41}Z_{25}) + Z_{11}(Z_{42}Z_{25} - Z_{22}Z_{45})/(Z_{21}Z_{12} - Z_{11}Z_{22})) \quad (3.510)$$

Derivation of E and P-values for SSSS plate

The values of E were all determined before the generation of P values. This was achieved by putting the E values in the various Equations of P as demonstrated below. The excel program gave account of the calculation process.

$E_1 = L_{43}L_{35}$, (3.511)

$E_2 = L_{33}L_{45}$ (3.512)

$E_3 = L_{43}L_{34}$, (3.513)

$E_4 = L_{33}L_{44}$ (3.514)

$P_5 = (E_1 - E_2)/(E_3 - E_4)$ (3.515)

$P_4 = ((L_{35} - L_{34}P_5)/L_{33})$ (3.516)

$P_3 = ((L_{35} - L_{23}P_4 - L_{24}P_5)/L_{22})$ (3.517)

$P_2 = ((L_{15} - L_{12}P_3 + L_{13}P_4 + L_{14}P_5)/L_{11})$ (3.518)

Derivation of K_T Values for SSSS plate

For the first case of K,

$B_{11}(g_1 - g_2P_5 - (g_{c1}/t)P_2)k_1 = K_{T1}$ (3.519)

also the second case of K can be expressed as

$$\frac{1}{\beta^2 * k_6}\left[2g_1(B_{12} + 2B_{33}) - g_2(B_{12} + 2B_{33})(P_5 + P_4) - \left(\frac{g_{c1}}{t}\right)(B_{12}(P_3 + P_2) + 2B_{33*}(P_3 + 4P_2))\right]k_2 = K_{T2} \quad (3.520)$$

Further more

$$\frac{B_{22}}{\beta^4 k_6}\left(g_1 - g_2P_4 - \frac{g_{c1}}{t}P_3\right)k_3 = K_{T3} \quad (3.521)$$

and finally,

$$B_{13}/(\beta k_6)\left(-4g_1 + g_2\frac{g_{c1}}{t}(P_2 - P_3)\right)k_4 = K_{T4} \quad (3.522)$$

and

$$B_{23}/(\beta^3 k_6)\left(4g_1 - g_2P_4 - \frac{g_{c1}}{t}(P_2 + 3P_3)\right)k_5 = K_{T5} \quad (3.523)$$

From the various values of K_T the buckling Equation can be given as

$$\frac{N_x a^2}{D_0} = E_0 * (\frac{K_{T1}+K_{T2}+K_{T3}+K_{T4}+K_{T5}}{k_6})D_0 \quad (3.524)$$

3.2.6.4 Two Laminated SSSS Plates for 0^0 90^0 Arrangement

In this case, the derivation of Z-values for SSSS plate was conducted by considering to angles, the 0^0 and 90^0

$Z_{11} =$

$$[(B_{11}g_{c2}k_1)/t^2 + (2B_{11}k_1)/t^2 + (4B_{13}k_4)/\beta t^2 + (2B_{33}k_2)/\beta^2 t^2 + (2a^2 B_{44}k_1)/J_M^2] \quad (3.525)$$

$Z_{12} =$

$$\left[\frac{(2B_{12}k_2)}{\beta^2 t^2} + \frac{(2B_{13}k_4)}{\beta t^2} + \frac{(2B_{23}k_5)}{\beta^3 t^2} + (2B_{33}k_2)/\beta^2 t^2\right] \quad (3.526)$$

$Z_{13} =$

$$[(2B_{12}g_{c2}k_2)/\beta^2 t + (2B_{13}g_{c2}k_1)/t^2 + (2B_{23}g_{c2}k_5)/\beta^3 t(2B_{33}g_{c2}k_2)/t\beta^2] \quad (3.527a)$$

$Z_{14} =$

$$[(B_{11}g_{c2}k_1)/t + (4B_{13}g_{c2}k_4)/t\beta + (2B_{33}g_{c2}k_2)/\beta t^2 + (2a^2 g_4 B_{44}k_{11})/J_M * t^2] \quad (3.527b)$$

$Z_{15} =$

$$\left[-\frac{(2B_{11}g_{c1}k_1)}{t} + \frac{(2B_{12}g_{c1}k_2)}{\beta t^2} + \frac{(2B_{13}g_{c1}k_4)}{t\beta} - \frac{(2B_{23}g_{c1}k_5)}{t\beta^3} - (4B_{33}g_{c1}k_2)/t\beta^2\right] \quad (3.528)$$

$Z_{21} =$

$$[(2B_{12}k_2)/\beta^2 t^2 + (2B_{23}k_5)/\beta^3 t^2 + (2B_{33}k_2)/\beta^2 * t^2] \quad (3.520)$$

$Z_{22} =$

$$[(2B_{13}k_4)/\beta t^2 + (2B_{22}k_3)/\beta^4 t^2 + (4B_{23}k_5)/\beta^3 t^2 + (2B_{33}k_2)/\beta^2 t^2 + (2a^2 B_{55}k_3)/t^2\beta^2 J_M^2] \quad (3.521)$$

$Z_{23} =$

$$[(2B_{22}g_{c2}k_3)/\beta^4 t + (4B_{23}g_{c2}k_5)/\beta^3 t + (2B_{33}g_{c2}k_2)/\beta^2 t + (2a^2 B_{55}k_{12})/\beta^2 t^2 J_M^2] \quad (3.522)$$

$Z_{24} =$

$$[(2B_{12}g_{c2}k_2)/t\beta^2 + (2B_{13}g_{c2}k_4)/t\beta + (2B_{23}g_{c2}k_5)/\beta t^2 + (2B_{33}g_{c2}k_2)/t * \beta^2] \quad (3.523)$$

$Z_{25} =$

$$[(-2B_{12}g_{c1}k_2)/t\beta^2 - (2B_{13}g_{c1}k_4)/t\beta - (2B_{22}g_{c1}k_3)/t\beta^4 - (6B_{23}g_{c1}k_5)/t\beta^3 - (4B_{33}g_{c1}k_2)/t\beta^2] \quad (3.524)$$

$Z_{31} =$

$$[(2B_{12}g_{c2}k_2)/t\beta^2 + (2B_{13}g_{c2}k_4)/t\beta^2 + (2B_{23}g_{c2}k_5)/t\beta^3 + (2B_{33}g_{c2}k_2)/t\beta^2] \quad (3.525)$$

$Z_{32} =$

$$[(2B_{22}g_{c2}k_3)/\beta^4 t + (4B_{23}g_{c2}k_5)/\beta^3 t + (2B_{33}g_{c2}k_2)/t\beta^2 + (2a^2 g_4 B_{55}k_{12})/\beta^2 J_M] \quad (3.526)$$

$Z_{33} =$

$$[(2B_{22}g_3k_3)/\beta^4 + (4B_{23}g_3k_5)/\beta^3 + (2B_{33}g_3k_2)/\beta^2 + (2a^2 B_{55}g_4{}^2k_7)/\beta^2 t^2] \quad (3.527)$$

$Z_{34} =$

$$[(2B_{12}g_3k_2)/\beta^2 + (2B_{33}g_3k_2)/\beta^2 + (2B_{13}g_3k_4)/\beta + (2B_{23}g_3k_5)/\beta^3] \quad (3.528)$$

$Z_{35} =$

$$[(-2B_{12}g_2k_2)/\beta^2 - (2B_{13}g_2k_4)/\beta - (2B_{22}g_2k_3)/\beta^4 - (6 * B_{23} * g_2 * k_5)/\beta^3 - (4 * B_{33}g_2k_2)/\beta^2] \quad (3529)$$

$Z_{41} =$

$$[(B_{11}g_{c2}k_1)/t^2 + (B_{11} * g_{c2} * k_1)/t + (4B_{13}g_{c2}k_4)/t\beta + (2B_{33}g_{c2}k_2)/t\beta^2 + (2a^2 B_{44}g_4k_{11})/t^2 J_M] \quad (3.530)$$

$Z_{42} =$

$$[(2B_{12}g_{c2}k_2)/t\beta^2 + (2B_{13}g_{c2}k_4)/t\beta + (2B_{23}g_{c2}k_5)/t\beta^3 + (2B_{33}g_{c2}k_2)/t\beta^2] \quad (3.531)$$

$Z_{43} =$

$[(2B_{12}g_3k_2)/\beta^2 + (2B_{13}g_3k_4)/\beta + (2B_{23}g_3k_5)/\beta^3 + (2B_{33}g_3k_2)/\beta^2]$ (3.532)

$Z_{44} =$

$[(2B_{11}g_1k_1) + (4B_{13}g_3k_4)/\beta + (2B_{33}g_3k_2)/\beta^2k_6)/t^2]$
$+(2{g_4}^2a^2B_{44}$ (3.533)

$Z_{45} =$

$[-2(B_{11}g_2k_1) - (2B_{12}g_2k_2)/\beta^2 - (2B_{13}g_2k_4)/\beta - (2B_{23}g_2k_5)/\beta^3 - (4B_{33}g_2k_2)/\beta^2]$ (3.534)

Derivation of L-values for SSSS plate

$L_{11} =$

$[(B_{11}g_{c2}k_1)/t^2 + (2B_{11}k_1)/t^2 + (4B_{13}k_4)/\beta t^2 + (2B_{33}k_2)/\beta^2t^2 + (2a^2B_{44}k_1)/J_M^2]$ (3.536)

$L_{12} =$

$[(2B_{12}k_2)/\beta^2t^2 + (2B_{13}k_4)/\beta t^2 + (2 * B_{23}k_5)/\beta^3t^2 + (2B_{33}k_2)/\beta^2t^2]$ (3.537)

$L_{13} =$

$\left[(2B_{12}g_{c2}k_2)/\beta^2t + (2B_{13}g_{c2}k_1)/t^2 + (2B_{23}g_{c2}k_5)/\beta^3t + \frac{(2B_{33}g_{c2}k_2)}{t\beta^2}\right]$ (3.538)

$L_{14} = [(B_{11}g_{c2}k_1)/t + (4B_{13}g_{c2}k_4)/t\beta + (2B_{33}g_{c2}k_2)/\beta t^2 + (2 * a^2 * g_4 * B_{44} * k_{11})/J_Mt^2]$ (3.539)

$L_{15} = [(2B_{11}g_{c1}k_1)/t + (2B_{12}g_{c1}k_2)/\beta t^2 + (2B_{13}g_{c1}k_4)/t\beta + (2B_{23}g_{c1}k_5)/t\beta^3 - (4B_{33}g_{c1}k_2)/t\beta^2]$ (3.540)

$= Z_{21}$

$L_{22} = (Z_{21}Z_{12}/Z_{11} - Z_{22})$` (3.541)

$$L_{23} = \left(Z_{21}\frac{Z_{13}}{Z_{11}} - Z_{23}\right) \quad (3.542)$$

$$L_{24} = (Z_{21}Z_{14}/Z_{11} - Z_{24}) \quad (3.543)$$

$$L_{33} = (Z_{21}(Z_{12}Z_{33} - Z_{32}Z_{13}) + Z_{31}(Z_{22}Z_{13} - Z_{23}Z_{12}) + Z_{11}(Z_{32}Z_{23} - Z_{32}Z_{22}))/(Z_{21}Z_{12} - Z_{11}Z_{22})) \quad (3.544)$$

$$L_{34} = (Z_{14}(Z_{22}Z_{31} - Z_{32}Z_{21}) + Z_{12}(Z_{34}Z_{21} - Z_{24}Z_{31}) + Z_{11}(Z_{32}Z_{24} - Z_{22} * Z_{34}))/(Z_{21} * Z_{12} - Z_{11} * Z_{22})) * A_5 \quad (3.545)$$

$$L_{35} = (Z_{15}(Z_{31}Z_{22} - Z_{32}Z_{21}) + Z_{12}(Z_{21}Z_{35} - Z_{31}Z_{25}) + Z_{11}(Z_{32}Z_{25} - Z_{22}Z_{35})/(Z_{21}Z_{12} - Z_{11}Z_{22})) \quad (3.546)$$

$$L_{43} = (Z_{13}(Z_{41}Z_{22} - Z_{42}Z_{21}) + Z_{12}(Z_{21}Z_{43} - Z_{41}Z_{23}) + Z_{11}(Z_{42}Z_{23} - Z_{43}Z_{22})/(Z_{21}Z_{12} - Z_{11}Z_{22})) \quad (3.547)$$

$$L_{44} = (Z_{14}(Z_{41}Z_{22} - Z_{42}Z_{21}) + Z_{12}(Z_{21}Z_{44} - Z_{24}Z_{41}) + Z_{11}(Z_{42}Z_{24} - Z_{22}Z_{44})/(Z_{21}Z_{12} - Z_{11}Z_{22})) \quad (3.548)$$

$$L_{45} = (Z_{15}(Z_{22}Z_{41} - Z_{42}Z_{21}) + Z_{12}(Z_{45}Z_{21} - Z_{41}Z_{25}) + Z_{11}(Z_{42}Z_{25} - Z_{22}Z_{45})/(Z_{21}Z_{12} - Z_{11} * Z_{22})) \quad (3.549)$$

Derivation of E and P-values for SSSS plate

The values of E were all determined before the generation of P values. This was achieved by putting the E values in the various Equations of P as demonstrated below. The excel program gave account of the calculation process.

$$E_1 = L_{43}L_{35} \quad (3.550i)$$

$$E_2 = L_{33}L_{45} \quad (3.551)$$

$$E_3 = L_{43}L_{34} \quad , \quad (3.552)$$

$$E_4 = L_{33}L_{44} \tag{3.553}$$

$$P_5 = (E_1 - E_2)/(E_3 - E_4) \tag{3.554}$$

$$P_4 = ((L_{35} - L_{34}P_5)/L_{33}) \tag{3.555}$$

$$P_3 = ((L_{35} - L_{23}P_4 - L_{24}P_5)/L_{22}) \tag{3.556}$$

$$P_2 = ((L_{15} - L_{12}P_3 + L_{13}P_4 + L_{14}P_5)/L_{11}) \tag{3.557}$$

Derivation of K_T Values for SSSS plate

For the first case of K,

$$B_{11}(g_1 - g_2P_5 - (g_{c1}/t)P_2)k_1 = K_{T1} \tag{3.558}$$

also the second case of K can be expressed as

$$\frac{1}{\beta^2 * k_6}\left[2 * g_1(B_{12} + 2B_{33}) - g_2(B_{12} + 2B_{33})(P_5 + P_4) - \left(\frac{g_{c1}}{t}\right)(B_{12}(P_3 + P_2) + 2B_{33}(P_3 + 4P_2))\right]k_2 = K_{T2} \tag{3.559}$$

Further more

$$\frac{B_{22}}{\beta^4 k_6}\left(g_1 - g_2P_4 - \frac{g_{c1}}{t}P_3\right)k_3 = K_{T3} \tag{3.560}$$

and finally,

$$B_{13}/(\beta k_6)\left(-4g_1 + g_2 + \frac{g_{c1}}{t}(P_2 - P_3)\right)k_4 = K_{T4} \tag{3.561}$$

and

$$B_{23}/(\beta^3 k_6)\left(4g_1 - g_2P_4 - \frac{g_{c1}}{t}(P_2 + 3P_3)\right)k_5 = K_{T5} \tag{3.562}$$

From the various values of K_T the buckling Equation can be given as

$$\frac{N_x a^2}{D_0} = E_0(K_{T1} + K_{T2} + K_{T3} + K_{T4} + K_{T5})/k_6 D_0 \tag{3.563}$$

3.2.6.5 Three Laminated SSSS Thick Anisotropic Plate Condition

The total of three laminas were considered at this stage. For the cases of m = 1 , 2 and 3 were treated separately. The bending , coupling and membrane stiffness showed some differences as the number of the laminas increases. The values obtained, were queued into the appropriate equation for the derivations of the of the J_i. and g_i values needed for the next stage of the analysis.

Bending Stiffness For 3–Laminates, when m = 1

Three stiffness conditions, Bending, Coupling and Axial stiffness were considered. For the Bending stiffness, the values of J_{ij} when the total number of laminas, n is three, were derived as follows:

$$J_1 = \left[\frac{(s_m^3 - s_{m-1}^3)}{3}\right] * 3 \qquad (3.564)$$

Since this is the first lamina, that means m = 1. From Figure 3.5, all the possible values of s were clearly derived and substituting 1 for m in Equation (3.564) leaves it as

$$J_1 = \left[\frac{(s_1^3 - s_0^3)}{3}\right] * 3 = \left[\frac{((-0.333)^3 - (-0.5)^3)}{3}\right] * 3 \qquad (3.565)$$

Also

$$J_2 = \left[\frac{\left(s_m^3 - \frac{4}{5}s_m^5\right) - \left(s_{m-1}^3 - \frac{4}{5}s_{m-1}^5\right)}{3}\right] * 3 \qquad (3.566)$$

Substituting the values of the unknowns gives

$$J_2 = \left[\frac{\left((-0.3333)^3 - \frac{4}{5}(-0.3333)^5\right) - \left((-0.5)^3 - \frac{4}{5}(-0.5)^3\right)}{3}\right] * 3 \qquad (3.567)$$

while the

$$J_3 = \left[\frac{\left(s_m^3 - \frac{8}{5}s_m^5 + \frac{16}{21}s_m^7\right) - \left(s_{m-1}^3 - \frac{8}{5}s_{m-1}^5 + \frac{16}{21}s_{m-1}^7\right)}{3}\right] * 3 \qquad (3.568)$$

From Equation (3.568) comes

$$J_3 = \left[\frac{\left(s_1^3 - \frac{8}{5}s_1^5 + \frac{16}{21}s_1^7\right) - \left(s_0^3 - \frac{8}{5}s_0^5 + \frac{16}{21}s_0^7\right)}{3}\right] * 3 \qquad (3.569)$$

Putting the values of the non-dimensional coordinates into Equation (3.569) gives

$$J_3 = \left[\frac{\left((-0.3333)^3 - \frac{8}{5}(-0.3333)^5 + \frac{16}{21}(-0.3333)^7\right) - \left((-0.5)^3 - \frac{8}{5}(-0.5)^5 + \frac{16}{21}(-0.5)^7\right)}{3}\right] * 3 \qquad (3.570)$$

and finally for the bending stiffness

$$J_4 = \left[\frac{\left(3s_m^1 - 8s_m^3 + \frac{48}{5}s_m^5\right) - \left(3s_{m-1}^1 - 8s_{m-1}^3 + \frac{48}{5}s_{m-1}^5\right)}{3}\right] * 3 \qquad (3.571)$$

That is

$$J_4 = \left[\frac{\left(3s_1^1 - 8s_1^3 + \frac{48}{5}s_1^5\right) - \left(3s_0^1 - 8s_0^3 + \frac{48}{5}s_0^5\right)}{3}\right] * 3 \qquad (3.572)$$

Substituting the values of s into Equation (3.572) gives

$$J_4 = \left[\frac{\left(3(-0.3333)^1 - 8(-0.3333)^3 + \frac{48}{5}(-0.3333)^5\right) - \left(3(-0.5)^1 - 8(-0.5)^3 + \frac{48}{5}(-0.5)^5\right)}{3}\right] * 3 \qquad (3.573)$$

Coupling Stiffness for 3-Laminates, when m = 1

For the Coupling, the values of J_{ij} when the total number of laminas, n is three, were derived as follows:

For

$$J_{c1} = \left[\frac{\left(s_m^2 - s_{m-1}^2\right)}{2}\right] * 3 \ (3.876)$$

That is $J_{c1} = \left[\frac{\left(s_1^2 - s_0^2\right)}{2}\right] * 3 \qquad (3.574)$

Putting the values of unknown gives

$$J_{c1} = \left[\frac{\left((-0.3333)^2 - (-0.5)^2\right)}{2}\right] * 3 \qquad (3.575)$$

Also for the

$$J_{c2} = \left[\frac{\left(s_m^2 - \frac{2}{3}s_m^4\right) - \left(s_{m-1}^2 - \frac{2}{3}s_{m-1}^4\right)}{3}\right] * 3 \qquad (3.576)$$

That is

$$J_{c2} = \left[\frac{\left(s_1^2 - \frac{2}{3}s_1^4\right) - \left(s_0^2 - \frac{2}{3}s_0^4\right)}{3}\right] * 3 \qquad (3.577)$$

Substituting the values of s gives

$$J_{c2} = \left[\frac{\left((-0.3333)^2 - \frac{2}{3}(-0.3333)^4\right) - \left((-0.5)^2 - \frac{2}{3}(-0.5)^4\right)}{3}\right] * 3 \qquad (3.578)$$

and finally for

$$J_{c3} = \left[\left(\mathrm{s}_\mathrm{m}^1 - \frac{4}{3}\mathrm{s}_\mathrm{m}^3\right) - \left(\mathrm{s}_{\mathrm{m}-1}^1 - \frac{4}{3}\mathrm{s}_{\mathrm{m}-1}^3\right)\right] * 3 \qquad (3.579)$$

That is

$$J_{c3} = \left(\mathrm{s}_1^1 - \frac{4}{3}\mathrm{s}_1^3\right) - \left(\mathrm{s}_0^1 - \frac{4}{3}\mathrm{s}_0^3\right) * 3 \qquad (3.580)$$

Substituting the values of s gives

$$J_{c3} = \left[\left((-0.3333)^1 - \frac{4}{3}(-0.3333)^3\right) - \left((-0.5)^1 - \frac{4}{3}(-0.5)^3\right)\right] * 3 \qquad (3.581)$$

Membrane Stiffness for 3–Laminates when m = 1

$$J_\mathrm{M} = [(s_m^1) - (s_{m-1}^1)] * 3 \qquad (3.582)$$

That is

$$J_\mathrm{M} = [(s_1^1) - (s_0^1)] * 3 \qquad (3.583)$$

Putting the values of s gives

$$\mathrm{J_M} = [(-0.3333)^1 - (-0.5)^1] * 3 \qquad (3.584)$$

For the Bending Stiffness For 3–Laminates Plate when m = 2, three stiffness conditions were considered. The value of J_{ij} when the total number of laminas, n is three, were derived as follows

$$J_1 = \left[\frac{(s_2^3 - s_1^3)}{3}\right] * 3 \tag{3.585}$$

Since this is the second lamina, that means m = 2. From Figure 3.4, all the possible values of s were clearly derived and substituting 1 for m in Equation (3.889) leaves it as

$$J_1 = \left[\frac{(s_2^3 - s_1^3)}{3}\right] * 2 = \left[\frac{((-0.16667)^3 - (-0.333)^3)}{3}\right] * 3 \tag{3.586}$$

Also

$$J_2 = \left[\frac{\left(s_m^3 - \frac{4}{5}s_m^5\right) - \left(s_{m-1}^3 - \frac{4}{5}s_{m-1}^5\right)}{3}\right] * 3 \tag{3.587}$$

Substituting the values of the unknowns gives

$$J_2 = \left[\frac{\left((-0.16667)^3 - \frac{4}{5}(-0.16667)^5\right) - \left((-0.3333)^3 - \frac{4}{5}(-0.3333)^5\right)}{3}\right] * 3 \tag{3.588}$$

while the

$$J_3 = \left[\frac{\left(s_m^3 - \frac{8}{5}s_m^5 + \frac{16}{21}s_m^7\right) - \left(s_{m-1}^3 - \frac{8}{5}s_{m-1}^5 + \frac{16}{21}s_{m-1}^7\right)}{3}\right] * 3 \tag{3.589}$$

From Equation (3.893) comes

$$J_3 = \left[\frac{\left(s_2^3 - \frac{8}{5}s_2^5 + \frac{16}{21}s_2^7\right) - \left(s_1^3 - \frac{8}{5}s_1^5 + \frac{16}{21}s_1^7\right)}{3}\right] * 3 \tag{3.590}$$

Putting the values of the non dimensional coordinates into Equation (3.590) gives

$$J_3 = \left[\frac{\left((-0.16667)^3 - \frac{8}{5}(-0.16667)^5 + \frac{16}{21}(-0.16667)^7\right) - \left((-0.3333)^3 - \frac{8}{5}(-0.3333)^5 + \frac{16}{21}(-0.3333)^7\right)}{3}\right] * 3 \tag{3.591}$$

and finally for the bending stiffness

$$J_4 = \left[\frac{\left(3s_m^1 - 8s_m^3 + \frac{48}{5}s_m^5\right) - \left(3s_{m-1}^1 - 8s_{m-1}^3 + \frac{48}{5}s_{m-1}^5\right)}{3}\right] * 3 \qquad (3.592)$$

That is

$$J_4 = \left[\frac{\left(3s_2^1 - 8s_2^3 + \frac{48}{5}s_2^5\right) - \left(3s_1^1 - 8s_1^3 + \frac{48}{5}s_1^5\right)}{3}\right] * 3 \qquad (3.593)$$

Substituting the values of s into Equation (3.593) gives

$$J_4 = \left[\frac{J_{4r} - J_{4l}}{3}\right] * 3 \qquad (3.594)$$

where

$$J_{4r} = \left(3(-0.16667)^1 - 8(-0.16667)^3 + \frac{48}{5}(-0.16667)^5\right) \qquad (3.594a)$$

and

$$J_{4l} = \left(3(-0.333)^1 - 8(-0.333)^3 + \frac{48}{5}(-0.333)^5\right) \qquad (3.594b)$$

For the coupling Stiffness for 3-Laminates when m = 2, the value of J_{ij} when the total number of laminas, n is three, were derived as follows:

For

$$J_{c1} = \left[\frac{\left(s_m^2 - s_{m-1}^2\right)}{2}\right] * 3 \qquad (3.595)$$

That is $J_{c1} = \left[\frac{\left(s_2^2 - s_1^2\right)}{2}\right] * 3$ (3.596)

Putting the values of unknown gives

$$J_{c1} = \left[\frac{\left((-0.16667)^2 - (-0.333)^2\right)}{2}\right] * 3 \qquad (3.597)$$

Also for the

$$J_{c2} = \left[\frac{\left(s_m^2 - \frac{2}{3}s_m^4\right) - \left(s_{m-1}^2 - \frac{2}{3}s_{m-1}^4\right)}{3}\right] * 3 \qquad (3.597)$$

That is

$$J_{c2} = \left[\frac{\left(s_2^2 - \frac{2}{3}s_2^4\right) - \left(s_1^2 - \frac{2}{3}s_1^4\right)}{3}\right] * 3 \qquad (3.598)$$

Substituting the values of s gives

$$J_{c2} = \left[\frac{\left((-0.16667)^2 - \frac{2}{3}(-0.16667)^4\right) - \left((-0.333)^2 - \frac{2}{3}(-0.333)^4\right)}{3}\right] * 3 \qquad (3.599)$$

and finally for

$$J_{c3} = \left[\left(\mathrm{s}_{\mathrm{m}}^1 - \frac{4}{3}\mathrm{s}_{\mathrm{m}}^3\right) - \left(\mathrm{s}_{\mathrm{m}-1}^1 - \frac{4}{3}\mathrm{s}_{\mathrm{m}-1}^3\right)\right] * 3 \qquad (3.600)$$

That is

$$J_{c3} = \left(\mathrm{s}_2^1 - \frac{4}{3}\mathrm{s}_2^3\right) - \left(\mathrm{s}_1^1 - \frac{4}{3}\mathrm{s}_1^3\right) * 3 \qquad (3.601)$$

Substituting the values of s gives

$$J_{c3} = \left[\left((-0.16667)^1 - \frac{4}{3}(-0.16667)^3\right) - \left((-0.3333)^1 - \frac{4}{3}(-0.3333)^3\right)\right] * 3 \qquad (3.602)$$

Membrane Stiffness for 3–Laminates when m = 2

$$J_{\mathrm{M}} = [(s_m^1) - (s_{m-1}^1)] * 3 \qquad (3.603)$$

That is

$$J_{\mathrm{M}} = [(s_2^1) - (s_1^1)] * 3 \qquad (3.604)$$

Putting the values of s gives

$$\mathrm{J}_{\mathrm{M}} = [(-0.16667)^1 - (-0.3333)^1] * 3 \qquad (3.605)$$

For the Bending Stiffness For 3–Laminates when m = 3, three stiffness conditions were considered. The values of J_{ij} when the total number of laminas, n is three, were derived as follows

$$J_1 = \left[\frac{(s_m^3 - s_{m-1}^3)}{3}\right] * 3 \qquad (3.606)$$

Since this is the second lamina, that means m = 2. From Figure 3.5, all the possible values of s were clearly derived and substituting 1 for m in Equation (3.606) leaves it as

$$J_1 = \left[\frac{(s_3^3 - s_2^3)}{3}\right] * 3 = \left[\frac{((-0.16667)^3 - (-0.333)^3)}{3}\right] * 3 \qquad (3.607)$$

Also

$$J_2 = \left[\frac{\left(s_m^3 - \frac{4}{5}s_m^5\right) - \left(s_{m-1}^3 - \frac{4}{5}s_{m-1}^5\right)}{3}\right] * 3 \qquad (3.608)$$

Substituting the values of the unknowns gives

$$J_2 = \left[\frac{\left((0)^3 - \frac{4}{5}(0)^5\right) - \left((-0.16667)^3 - \frac{4}{5}(-0.16667)^5\right)}{3}\right] * 3 \qquad (3.609)$$

while the

$$J_3 = \left[\frac{\left(s_m^3 - \frac{8}{5}s_m^5 + \frac{16}{21}s_m^7\right) - \left(s_{m-1}^3 - \frac{8}{5}s_{m-1}^5 + \frac{16}{21}s_{m-1}^7\right)}{3}\right] * 3 \qquad (3.610)$$

then

$$J_3 = \left[\frac{\left(s_3^3 - \frac{8}{5}s_3^5 + \frac{16}{21}s_3^7\right) - \left(s_2^3 - \frac{8}{5}s_2^5 + \frac{16}{21}s_2^7\right)}{3}\right] * 3 \qquad (3.611)$$

Putting the values of the non dimensional coordinates into Equation (3.611) gives

$$J_3 = \left[\frac{\left((0)^3 - \frac{8}{5}(0)^5 + \frac{16}{21}(0)^7\right) - \left((-0.16667)^3 - \frac{8}{5}(-0.16667)^5 + \frac{16}{21}(-0.16667)^7\right)}{3}\right] * 3 \qquad (3.612)$$

and finally for the bending stiffness

$$J_4 = \left[\frac{\left(3s_m^1 - 8s_m^3 + \frac{48}{5}s_m^5\right) - \left(3s_{m-1}^1 - 8s_{m-1}^3 + \frac{48}{5}s_{m-1}^5\right)}{3}\right] * 3 \qquad (3.613)$$

That is

$$J_4 = \left[\frac{\left(3s_3^1 - 8s_3^3 + \frac{48}{5}s_3^5\right) - \left(3s_2^1 - 8s_2^3 + \frac{48}{5}s_2^5\right)}{3}\right] * 3 \quad (3.614)$$

Substituting the values of s into Equation (3.614) gives

$$J_4 = \left[\frac{\left(3(0)^1 - 8(0)^3 + \frac{48}{5}(0)^5\right) - \left(3(-0.16667)^1 - 8(-0.16667)^3 + \frac{48}{5}(-0.16667)^5\right)}{3}\right] * 3 \quad (3.615)$$

For the Coupling Stiffness for 3-Laminates when m = 3, the values of J_{ij} when the total number of laminas, n is three, were derived as follows:

For

$$J_{c1} = \left[\frac{\left(s_m^2 - s_{m-1}^2\right)}{2}\right] * 3 \quad (3.616)$$

That is $J_{c1} = \left[\frac{\left(s_3^2 - s_2^2\right)}{2}\right] * 3$ (3.617)

Putting the values of unknown gives

$$J_{c1} = \left[\frac{\left((0)^2 - (-0.16667)^2\right)}{2}\right] * 3 \quad (3.618)$$

Also for the

$$J_{c2} = \left[\frac{\left(s_m^2 - \frac{2}{3}s_m^4\right) - \left(s_{m-1}^2 - \frac{2}{3}s_{m-1}^4\right)}{3}\right] * 3 \quad (3.619)$$

That is

$$J_{c2} = \left[\frac{\left(s_3^2 - \frac{2}{3}s_3^4\right) - \left(s_2^2 - \frac{2}{3}s_2^4\right)}{3}\right] * 3 \quad (3.620)$$

Substituting the values of s gives

$$J_{c2} = \left[\frac{\left((0)^2 - \frac{2}{3}(0)^4\right) - \left((-0.16667)^2 - \frac{2}{3}(-0.16667)^4\right)}{3}\right] * 3 \quad (3.621)$$

and finally for

$$J_{c3} = \left[\left(s_m^1 - \frac{4}{3}s_m^3\right) - \left(s_{m-1}^1 - \frac{4}{3}s_{m-1}^3\right)\right] * 3 \quad (3.622)$$

That is

$$J_{c3} = \left(s_3^1 - \frac{4}{3}s_3^3\right) - \left(s_2^1 - \frac{4}{3}s_2^3\right) * 3 \quad (3.623)$$

Substituting the values of s gives

$$J_{c3} = \left[\left((0)^1 - \frac{4}{3}(0)^3\right) - \left((-0.16667)^1 - \frac{4}{3}(-0.16667)^3\right)\right] * 3 \quad (3.624)$$

Membrane Stiffness for 3–Laminates when m = 3

$$J_M = [(s_m^1) - (s_{m-1}^1)] * 3 \quad (3.625)$$

That is

$$J_M = [(s_3^1) - (s_2^1)] * 3 \quad (3.626)$$

Putting the values of s gives

$$J_M = [(0)^1 - (-0.16667)^1] * 3 \quad (3.627)$$

3.2.6.6 Three Laminated SSSS Thick Anisotropic Plate for 0^0 0^0 0^0 Arrangement

For the case of SSSS with three laminas, the plates were positioned at the angels 0^0 0^0 0^0 similarly as in the case of two laminated plates, the Z-values were introduced and that gave the buckling load as shown in Equation (3.628).

$$\frac{N_x a^2}{D_0} = E_0(K_{T1} + K_{T2} + K_{T3} + K_{T4} + K_{T5})/k_6 * D_0 \quad (3.628)$$

where

$$B_{11} * (g_1 - g_2P_5 - (g_{c1}/t) * P_2) * k_1 = K_{T1} \quad (3.629)$$

with

$$\frac{1}{\beta^2 * k_6}\left[2g_1(B_{12} + 2B_{33}) - g_2(B_{12} + 2B_{33})(P_5 + P_4) - \left(\frac{g_{c1}}{t}\right)(B_{12}(P_3 + P_2) + 2B_{33}(P_3 + 4P_2))\right] k_2 = K_{T2} \quad (3.630)$$

also

$$\frac{B_{22}}{\beta^4 k_6}\left(g_1 - g_2 P_4 - \frac{g_{c1}}{t} P_3\right) k_3 = K_{T3} \quad (3.631)$$

and then

$$B_{13}/(\beta k_6)\left(-4g_1 + g_2 + \frac{g_{c1}}{t}(P_2 - P_3)\right) k_4 = K_{T4} \quad (3.632)$$

The derived buckling equation was used to run the anaylsis of the palte considering different Properties and the results were as detailed in Chapter 4.

3.2.6.7 Three Laminted SSSS Thick Anisotropic Plate for 0^0 90^0 0^0 Arrangement

Similarly the case of SSSS with three laminas,when the plates were positioned at the angles 0^0 90^0 0^0 the buckling load was formulated on the introduction of the Z-values and the analysis was conducted at different conditions amd different material properties and the results were as shown in the Chapter 4.

3.2.7 Analysis of CCCC Thick Laminated Anisotropic Plate

In each case of the laminated CCCC analysis, the entire process was iterated severally depending the on the number of the laminas. In each iteration, the bending stiffness was considered as

$$J_{cccc_{Bi}} = \frac{(S_m^2 - S_{m-1}^2)}{3} \quad (3.633)$$

also for the case of Coupling Stifness as

$$J_{cccc_{Ci}} = \frac{(S_m^2 - S_{m-1}^2)}{3} \quad (3.634)$$

and finally for the Membrane stiffness as

$$J_{cccc_{Mi}} = \frac{(S_m^2 - S_{m-1}^2)}{3} \quad (3.635)$$

The introduction of the angle of rotations, out of plane , in plane displacements and other parameters as earlier listed, into the total potential energy gave rise to coupling, bending and membrane stiffnesses, upon further minimization. The various deived equations, were initially expressed as j_i and then as g_i values. For simplicity, the equations were reduced interms of Z and L-values, and the resultant equations were solved using Gauss Elimination method. The final values of P which are all functions of L were substituted into equation. The A-values were introduced into the equations and that further reduced the entire process into smaller expression. On introduction of the stiffness coefficients k_i, the various K_{Ti} values were generated. These were finally substituted into Equation (3.628) for the derivation of the Buckling loads.

$$. \frac{N_{ccccx}a^2}{D_0} = E_0 * (K_{ccccT1} + K_{ccccT2} + K_{ccccT3} + K_{ccccT4} + K_{ccccT5})/k_6 * D_0 \quad (3.636)$$

For the analysis of all the angles considered cccc plate, the same approach in the case ssss plate was adopted. Also the same procedures were adopted in the analysis of the entire plate considered in this work. The buckling load Equations derived as explained were used to conduct numerical analysis for all the laminated plate cases considered. In each case, the materials were assumed to passes the following properties:

0.5 to -0.5 thickness in compression and tension zone respectively,

Elastic Modulus, E_y = 1GPa E_x =25Pa

Poison Ratio $\mu_{xy} = 0.25$

Shear Modulus $G_{xy} = 0.5$Pa

In the analysis, two selected angles were considered, 0^0 and 90^0. The same process was adopted in the analysis of CCCC plate and other plates studied in this work. Firstly, the total of two laminas were considered. For the case of m = 1 and for the case of m = 2. The values obtained, were queued in to the appropriate equation for the derivations of the of the J_i. and g_i values needed for the next stage of the analysis. Three stiffness conditions also were considered. For the Bending stiffness, the values of J_{ij} when the total number of laminas, n is two, were derived. The results derived in all the treated plates were discussed in Chapter 4.

CHAPTER FOUR

RESULTS AND DICUSSIONS

4.1 Results

As detailed in Chapter 3, the analysis of thick laminated anisotropic plate was conducted , using 3rd order Energy functional. Upon substitution of the stiffness coefficients into the derived Equations, different observations and values were recorded. The processes involving the generation of the various formulas adopted in the analysis were all detailed in Chapter 3 while the results were all recorded in sub sections below.

4.1.1 The strain of Thick Laminated Anisotropic Plate

Equation (4.1) is the strain of thick laminated anisotropic plate.

$$[\varepsilon] = \begin{bmatrix} \left(\frac{du_0}{adR} - \frac{tS}{a^2}\frac{\partial^2 w}{\partial R^2} + \frac{tH}{a}.\frac{\partial \phi_x}{\partial R}\right) \\ \left(\frac{dv_0}{a\beta\partial Q} - \frac{tS}{\beta^2 a^2}\frac{\partial^2 w}{\partial Q^2} + \frac{tH}{a\beta}.\frac{\partial \phi_y}{\partial Q}\right) \\ \left(\frac{du_0}{a\beta\partial Q} + \frac{dv_0}{adR} - \frac{2tS}{\beta a^2}\frac{\partial^2 w}{\partial R\partial Q} + \frac{tH}{\beta a}.\left(\frac{\partial \phi_x}{\partial Q} + \beta.\frac{\partial \phi_y}{\partial R}\right)\right) \\ \left(\frac{du_0}{tdS} + \frac{\partial H}{\partial S}.\emptyset_x\right) \\ \left(\frac{dv_0}{tdS} + \frac{\partial H}{\partial S}.\emptyset_y\right) \end{bmatrix} \quad (4.1)$$

4.1.2 The stress of Thick Laminated Anisotropic Plate

The stress of thick laminted anisotrpic plate is also gives as:

$$[\sigma] = \frac{E_0}{1-\mu_{12}\mu_{21}}\begin{bmatrix} B_{11} & B_{12} & B_{13} & 0 & 0 \\ B_{21} & B_{22} & B_{23} & 0 & 0 \\ B_{31} & B_{32} & B_{33} & 0 & 0 \\ 0 & 0 & 0 & B_{44} & 0 \\ 0 & 0 & 0 & 0 & B_{55} \end{bmatrix}\begin{bmatrix} \frac{du_0}{adR} - \frac{tS}{a^2}\frac{\partial^2 w}{\partial R^2} + \frac{tH}{a}.\frac{\partial \phi_x}{\partial R} \\ \frac{dv_0}{a\beta\partial Q} - \frac{tS}{\beta^2 a^2}\frac{\partial^2 w}{\partial Q^2} + \frac{tH}{a\beta}.\frac{\partial \phi_y}{\partial Q} \\ \frac{du_0}{a\beta\partial Q} + \frac{dv_0}{adR} - \frac{2tS}{\beta a^2}\frac{\partial^2 w}{\partial R\partial Q} + \frac{tH}{\beta a}.\left(\frac{\partial \phi_x}{\partial Q} + \beta.\frac{\partial \phi_y}{\partial R}\right) \\ \frac{du_0}{tdS} + \frac{\partial H}{\partial S}.\emptyset_x \\ \frac{dv_0}{tdS} + \frac{\partial H}{\partial S}.\emptyset_y \end{bmatrix} \quad (4.2)$$

4.1.3 The Total Potential Energy

Based on the constitutive relationship developed, the total potential energy was formulated by bringing the strain and stress of the thick laminated anisotropic plate. Upon further minimization the result of the stress-strain equations, the Total Potential energy was formulated as

$$\pi = \frac{E_0}{2}\iiint \begin{bmatrix} \left(\frac{du_0}{adR} - \frac{tS}{a^2}\frac{\partial^2 w}{\partial R^2} + \frac{tH}{a}.\frac{\partial \phi_x}{\partial R}\right) \\ \left(\frac{dv_0}{a\beta\partial Q} - \frac{tS}{\beta^2 a^2}\frac{\partial^2 w}{\partial Q^2} + \frac{tH}{a\beta}.\frac{\partial \phi_y}{\partial Q}\right) \\ \left(\frac{du_0}{a\beta\partial Q} + \frac{dv_0}{adR} - \frac{2tS}{\beta a^2}\frac{\partial^2 w}{\partial R\partial Q} + \frac{tH}{\beta a}.\left(\frac{\partial \phi_x}{\partial Q} + \beta.\frac{\partial \phi_y}{\partial R}\right)\right) \\ \left(\frac{du_0}{tdS} + \frac{\partial H}{\partial S}.\emptyset_x\right) \\ \left(\frac{dv_0}{tdS} + \frac{\partial H}{\partial S}.\emptyset_y\right) \end{bmatrix}^T * \begin{bmatrix} B_{11} & B_{12} & B_{13} & 0 & 0 \\ B_{21} & B_{22} & B_{23} & 0 & 0 \\ B_{31} & B_{32} & B_{33} & 0 & 0 \\ 0 & 0 & 0 & B_{44} & 0 \\ 0 & 0 & 0 & 0 & B_{55} \end{bmatrix} *$$

$$\begin{bmatrix} \left(\frac{du_0}{adR} - \frac{tS}{a^2}\frac{\partial^2 w}{\partial R^2} + \frac{tH}{a}.\frac{\partial \phi_x}{\partial R}\right) \\ \left(\frac{dv_0}{a\beta\partial Q} - \frac{tS}{\beta^2 a^2}\frac{\partial^2 w}{\partial Q^2} + \frac{tH}{a\beta}.\frac{\partial \phi_y}{\partial Q}\right) \\ \left(\frac{du_0}{a\beta\partial Q} + \frac{dv_0}{adR} - \frac{2tS}{\beta a^2}\frac{\partial^2 w}{\partial R\partial Q} + \frac{tH}{\beta a}.\left(\frac{\partial \phi_x}{\partial Q} + \beta.\frac{\partial \phi_y}{\partial R}\right)\right) \\ \left(\frac{du_0}{tdS} + \frac{\partial H}{\partial S}.\emptyset_x\right) \\ \left(\frac{dv_0}{tdS} + \frac{\partial H}{\partial S}.\emptyset_y\right) \end{bmatrix} dxdydz - \iint (0 + \frac{N_x}{2}\left(\frac{dw}{dx}\right)^2 + 0)dxdy \qquad (4.3)$$

The functional was further expanded and presented in terms of bending, coupling and membrane stiffness as

$$\int_{-0.5}^{0.5}\sigma.\varepsilon dS=\frac{E_0t^3}{[1-\mu_{XY}\mu_{YX}]}B_{11}\left(S^2\left(\frac{\partial^2w}{\partial R^2}\right)^2-aHS.\frac{\partial^2w}{\partial R^2}.\frac{\partial\phi_x}{\partial R}-aHS.\frac{\partial^2w}{\partial R^2}.\frac{\partial\phi_x}{\partial R}+a^2S^2.\left(\frac{\partial\phi_x}{\partial R}\right)^2\right)$$

$$+2B_{12}\left(\frac{S^2}{\beta^2}.\frac{\partial^2w}{\partial R^2}.\frac{\partial^2w}{\partial Q^2}-\frac{aHS}{\beta}.\frac{\partial^2w}{\partial R^2}.\frac{\partial\phi_y}{\partial Q}-\frac{aHS}{\beta^2}\frac{\partial^2w}{\partial Q^2}.\frac{\partial\phi_x}{\partial R}+\frac{a^2H^2}{\beta}.\frac{\partial\phi_y}{\partial Q}.\frac{\partial\phi_x}{\partial R}\right)$$

$$+2B_{13}\left(-\frac{2S^2}{\beta}\frac{\partial^2w}{\partial R\partial Q}.\frac{\partial^2w}{\partial R^2}-\frac{aHS}{\beta}\frac{\partial^2w}{\partial R^2}.\frac{\partial\phi_x}{\partial Q}-aHS\frac{\partial^2w}{\partial R^2}.\frac{\partial\phi_y}{\partial R}-\frac{2aHS}{\beta}\frac{\partial^2w}{\partial R\partial Q}.\frac{\partial\phi_x}{\partial R}+\frac{a^2H^2}{\beta}.\frac{\partial\phi_x}{\partial R}.\frac{\partial\phi_x}{\partial Q}+a^2H^2.\frac{\partial\phi_y}{\partial R}.\frac{\partial\phi_x}{\partial R}\right)$$

$$+B_{22}\left(\frac{S^2}{\beta^4}\left(\frac{\partial^2w}{\partial Q^2}\right)^2-\frac{aHS}{\beta^3}.\frac{\partial^2w}{\partial Q^2}.\frac{\partial\phi_y}{\partial Q}-\frac{aHS}{\beta^3}\frac{\partial^2w}{\partial Q^2}.\frac{\partial\phi_y}{\partial Q}+\frac{a^2H^2}{\beta^2}.\left(\frac{\partial\phi_y}{\partial Q}\right)^2\right)$$

$$+2B_{23}\left(\frac{2S^2}{\beta^3}\frac{\partial^2w}{\partial R\partial Q}.\frac{\partial^2w}{\partial Q^2}-\frac{aSH}{\beta^3}.\frac{\partial\phi_x}{\partial Q}.\frac{\partial^2w}{\partial Q^2}-\frac{aSH}{\beta^2}.\frac{\partial\phi_y}{\partial R}.\frac{\partial^2w}{\partial Q^2}-\frac{2aHS}{\beta^2}\frac{\partial^2w}{\partial R\partial Q}.\frac{\partial\phi_y}{\partial Q}+\frac{a^2H^2}{\beta^2}.\frac{\partial\phi_x}{\partial Q}.\frac{\partial\phi_y}{\partial Q}+\frac{a^2H^2}{\beta}.\frac{\partial\phi_y}{\partial R}.\frac{\partial\phi_y}{\partial Q}\right)$$

$$+B_{33}\left(\frac{4S^2}{\beta^2}\left(\frac{\partial^2w}{\partial R\partial Q}\right)^2-\frac{2aHS}{\beta^2}.\frac{\partial^2w}{\partial R\partial Q}.\frac{\partial\phi_x}{\partial Q}-\frac{2aHS}{\beta}.\frac{\partial^2w}{\partial R\partial Q}.\frac{\partial\phi_y}{\partial R}\right)$$

$$+B_{33}\left(-\frac{2aHS}{\beta^2}\frac{\partial^2w}{\partial R\partial Q}.\frac{\partial\phi_x}{\partial Q}+\frac{a^2H^2}{\beta^2}.\left(\frac{\partial\phi_x}{\partial Q}\right)^2+\frac{a^2H^2}{\beta}\frac{\partial\phi_y}{\partial R}.\frac{\partial\phi_x}{\partial Q}\right)$$

$$+B_{33}\left(-\frac{2aHS}{\beta}\frac{\partial^2w}{\partial R\partial Q}.\frac{\partial\phi_y}{\partial R}+\frac{a^2H^2}{\beta}.\frac{\partial\phi_x}{\partial Q}.\frac{\partial\phi_y}{\partial R}+a^2H^2\left(\frac{\partial\phi_y}{\partial R}\right)^2\right)$$

$$
\begin{aligned}
&+\frac{E_0 t^3}{[1-\mu_{XY}\mu_{YX}]a^4} B_{11}\left(-\frac{a}{t}S.\frac{\partial^2 w}{\partial R^2}.\frac{du_0}{dR}+\frac{a^2}{t}H.\frac{du_0}{dR}.\frac{\partial \phi_x}{\partial R}-\frac{aS}{t}.\frac{\partial^2 w}{\partial R^2}.\frac{du_0}{\partial R}\right.\\
&\qquad\left.+\frac{a^2}{t}H\frac{\partial \phi_x}{\partial R}.\frac{du_0}{dR}\right)\\
&\qquad+2B_{12}\left(-\frac{a}{t\beta}S.\frac{\partial^2 w}{\partial R^2}.\frac{dv_0}{\partial Q}+\frac{a^2 H}{t\beta}\frac{dv_0}{\partial Q}.\frac{\partial \phi_x}{\partial R}-\frac{aS}{t\beta^2}\frac{\partial^2 w}{\partial Q^2}.\frac{du_0}{dR}\right.\\
&\qquad\left.+\frac{a^2 H}{t\beta}.\frac{\partial \phi_y}{\partial Q}.\frac{du_0}{dR}\right)+2B_{13}\left(-\frac{a}{t\beta}S\frac{\partial^2 w}{\partial R^2}.\frac{du_0}{\partial Q}-\frac{a}{t}S\frac{\partial^2 w}{\partial R^2}.\frac{dv_0}{dR}\right.\\
&\qquad+\frac{a^2 H}{t}\frac{du_0}{\beta\partial Q}.\frac{\partial \phi_x}{\partial R}+\frac{a^2 H}{t}\frac{dv_0}{dR}.\frac{\partial \phi_x}{\partial R}+\frac{2aS}{t\beta}\frac{\partial^2 w}{\partial R\partial Q}.\frac{du_0}{dR}+\frac{a^2 H}{t\beta}\frac{\partial \phi_x}{\partial Q}.\frac{du_0}{dR}\\
&\qquad\left.+\frac{a^2 H}{t\beta}.\frac{\partial \phi_y}{\partial R}.\frac{du_0}{dR}\right)
\end{aligned}
$$

$$
+B_{22}\left(-\frac{aS}{t\beta^3}.\frac{\partial^2 w}{\partial Q^2}.\frac{dv_0}{\partial Q}+\frac{a^2 H}{t\beta^2}\frac{dv_0}{\partial Q}.\frac{\partial \phi_y}{\partial Q}-\frac{aS}{t\beta^3}\frac{\partial^2 w}{\partial Q^2}.\frac{dv_0}{\partial Q}+\frac{a^2 H}{t\beta^2}.\frac{\partial \phi_y}{\partial Q}.\frac{dv_0}{\partial Q}\right)
$$

$$
\begin{aligned}
&+2B_{23}\left(-\frac{aS}{t\beta^3}\frac{du_0}{\partial Q}.\frac{\partial^2 w}{\partial Q^2}-\frac{aS}{t\beta^2}\frac{dv_0}{dR}.\frac{\partial^2 w}{\partial Q^2}+\frac{a^2 H}{t\beta^2}\frac{du_0}{\partial Q}.\frac{\partial \phi_y}{\partial Q}+\frac{a^2 H}{t\beta}\frac{dv_0}{dR}.\frac{\partial \phi_y}{\partial Q}-\frac{2aS}{t\beta^2}\frac{\partial^2 w}{\partial R\partial Q}.\frac{dv_0}{\partial Q}\right.\\
&\qquad\left.+\frac{a^2 H}{t\beta^2}.\frac{\partial \phi_x}{\partial Q}.\frac{dv_0}{\partial Q}+\frac{a^2 H}{t\beta}.\frac{\partial \phi_y}{\partial R}.\frac{dv_0}{\partial Q}\right)
\end{aligned}
$$

$$
+B_{33}\left(-\frac{2aS}{t\beta^2}.\frac{\partial^2 w}{\partial R\partial Q}.\frac{du_0}{\partial Q}-\frac{2aS}{t\beta}.\frac{\partial^2 w}{\partial R\partial Q}.\frac{dv_0}{dR}\right)+B_{33}\left(\frac{a^2 H}{t\beta^2}\frac{du_0}{\partial Q}.\frac{\partial \phi_x}{\partial Q}+\frac{a^2 H}{t\beta}\frac{dv_0}{dR}.\frac{\partial \phi_x}{\partial Q}\right)
$$

$$
+B_{33}\left(+\frac{a^2 H}{t\beta}\frac{du_0}{\partial Q}.\frac{\partial \phi_y}{\partial R}+\frac{a^2 H}{t}\frac{dv_0}{dR}.\frac{\partial \phi_y}{\partial R}\right)+B_{33}\left(-\frac{2aS}{t\beta^2}\frac{\partial^2 w}{\partial R\partial Q}.\frac{du_0}{\partial Q}+\frac{a^2 H}{t\beta^2}.\frac{\partial \phi_x}{\partial Q}.\frac{du_0}{\partial Q}+\frac{a^2 H}{t\beta}\frac{\partial \phi_y}{\partial R}.\frac{du_0}{\partial Q}\right)
$$

$$
+B_{33}\left(-\frac{2aS}{t\beta}\frac{\partial^2 w}{\partial R\partial Q}.\frac{dv_0}{dR}+\frac{a^2 H}{t\beta}.\frac{\partial \phi_x}{\partial Q}.\frac{dv_0}{dR}+\frac{a^2 H}{t}\frac{\partial \phi_y}{\partial R}.\frac{dv_0}{dR}\right)
$$

$$
+\frac{E_0 t^3}{[1-\mu_{XY}\mu_{YX}]a^4} B_{11}\left(+\frac{a^2}{t^2}.\left(\frac{du_0}{dR}\right)^2\right)+2B_{12}\left(+\frac{a^2}{t^2\beta}\frac{du_0}{dR}.\frac{dv_0}{\partial Q}\right)
$$

$$
+2B_{13}\left(\frac{a^2}{t^2}\frac{du_0}{\beta\partial Q}.\frac{du_0}{dR}+\frac{a^2}{t^2}.\frac{dv_0}{dR}.\frac{du_0}{dR}\right)+2B_{22}\left(\frac{a^2}{t^2\beta^2}.\left(\frac{dv_0}{\partial Q}\right)^2\right)
$$

$$
+2B_{23}\left(\frac{a^2}{t^2\beta^2}\frac{du_0}{\partial Q}\frac{dv_0}{\partial Q}+\frac{a^2}{t^2\beta}\frac{dv_0}{dR}.\frac{dv_0}{\partial Q}\right)+B_{33}\left(\frac{a^2}{t^2\beta^2}.\left(\frac{du_0}{\partial Q}\right)^2+\frac{a^2}{t^2\beta}\frac{dv_0}{dR}.\frac{du_0}{\partial Q}\right)
$$

$$+B_{33}\left(\frac{a^2}{t^2\beta}\frac{du_0}{\partial Q}.\frac{dv_0}{dR}+\frac{a^2}{t^2}\left(\frac{dv_0}{dR}\right)^2\right)$$

$$+B_{44}\frac{a^4}{t^2}\left(\left(\emptyset_x.\frac{\partial H}{\partial S}\right)^2+\emptyset_x.\frac{du_0}{dS}.\frac{\partial H}{\partial S}+\emptyset_x.\frac{\partial H}{\partial S}.\frac{du_0}{dS}+\left(\frac{du_0}{dS}\right)^2\right)$$

$$+B_{55}\frac{a^4}{t^2}\left(\left(\emptyset_y.\frac{\partial H}{\partial S}\right)^2+.\emptyset_y.\frac{dv_0}{dS}.\frac{\partial H}{\partial S}+\emptyset_y.\frac{\partial H}{\partial S}.\frac{dv_0}{dS}+\left(\frac{dv_0}{dS}\right)^2\right) \qquad (4.4)$$

Categorically grouped as

$$U=\frac{abt}{2}(\sigma.\varepsilon)_B+(\sigma.\varepsilon)_C+(\sigma.\varepsilon)_M \quad dR\ dQ\ dS \qquad (4.5)$$

where Strain Energy for case of Bending Stiffness, U_b is

$$Ub=$$

$$\frac{E_0t^3}{[1-\mu_{XY}\mu_{YX}]}B_{11}\left(S^2\left(\frac{\partial^2w}{\partial R^2}\right)^2-aHS.\frac{\partial^2w}{\partial R^2}.\frac{\partial\phi_x}{\partial R}-aHS.\frac{\partial^2w}{\partial R^2}.\frac{\partial\phi_x}{\partial R}+a^2S^2.\left(\frac{\partial\phi_x}{\partial R}\right)^2\right)$$

$$+2B_{12}\left(\frac{S^2}{\beta^2}.\frac{\partial^2w}{\partial R^2}.\frac{\partial^2w}{\partial Q^2}-\frac{aHS}{\beta}.\frac{\partial^2w}{\partial R^2}.\frac{\partial\phi_y}{\partial Q}-\frac{aHS}{\beta^2}\frac{\partial^2w}{\partial Q^2}.\frac{\partial\phi_x}{\partial R}+\frac{a^2H^2}{\beta}.\frac{\partial\phi_y}{\partial Q}.\frac{\partial\phi_x}{\partial R}\right)$$

$$+2B_{13}\left(-\frac{2S^2}{\beta}\frac{\partial^2w}{\partial R\partial Q}.\frac{\partial^2w}{\partial R^2}-\frac{aHS}{\beta}\frac{\partial^2w}{\partial R^2}.\frac{\partial\phi_x}{\partial Q}-aHS\frac{\partial^2w}{\partial R^2}.\frac{\partial\phi_y}{\partial R}-\frac{2aHS}{\beta}\frac{\partial^2w}{\partial R\partial Q}.\frac{\partial\phi_x}{\partial R}\right.$$
$$\left.+\frac{a^2H^2}{\beta}.\frac{\partial\phi_x}{\partial R}.\frac{\partial\phi_x}{\partial Q}+a^2H^2.\frac{\partial\phi_y}{\partial R}.\frac{\partial\phi_x}{\partial R}\right)$$

$$+B_{22}\left(\frac{S^2}{\beta^4}\left(\frac{\partial^2w}{\partial Q^2}\right)^2-\frac{aHS}{\beta^3}.\frac{\partial^2w}{\partial Q^2}.\frac{\partial\phi_y}{\partial Q}-\frac{aHS}{\beta^3}\frac{\partial^2w}{\partial Q^2}.\frac{\partial\phi_y}{\partial Q}+\frac{a^2H^2}{\beta^2}.\left(\frac{\partial\phi_y}{\partial Q}\right)^2\right)$$

$$+2B_{23}\left(\frac{2S^2}{\beta^3}\frac{\partial^2w}{\partial R\partial Q}.\frac{\partial^2w}{\partial Q^2}-\frac{aSH}{\beta^3}.\frac{\partial\phi_x}{\partial Q}.\frac{\partial^2w}{\partial Q^2}-\frac{aSH}{\beta^2}.\frac{\partial\phi_y}{\partial R}.\frac{\partial^2w}{\partial Q^2}-\frac{2aHS}{\beta^2}\frac{\partial^2w}{\partial R\partial Q}.\frac{\partial\phi_y}{\partial Q}\right.$$
$$\left.+\frac{a^2H^2}{\beta^2}.\frac{\partial\phi_x}{\partial Q}.\frac{\partial\phi_y}{\partial Q}+\frac{a^2H^2}{\beta}.\frac{\partial\phi_y}{\partial R}.\frac{\partial\phi_y}{\partial Q}\right)$$

$$+B_{33}\left(\frac{4S^2}{\beta^2}\left(\frac{\partial^2w}{\partial R\partial Q}\right)^2-\frac{2aHS}{\beta^2}.\frac{\partial^2w}{\partial R\partial Q}.\frac{\partial\phi_x}{\partial Q}-\frac{2aHS}{\beta}.\frac{\partial^2w}{\partial R\partial Q}.\frac{\partial\phi_y}{\partial R}\right)$$

$$+B_{33}\left(-\frac{2aHS}{\beta^2}\frac{\partial^2w}{\partial R\partial Q}.\frac{\partial\phi_x}{\partial Q}+\frac{a^2H^2}{\beta^2}.\left(\frac{\partial\phi_x}{\partial Q}\right)^2+\frac{a^2H^2}{\beta}\frac{\partial\phi_y}{\partial R}.\frac{\partial\phi_x}{\partial Q}\right)$$

$$+B_{33}\left(-\frac{2aHS}{\beta}\frac{\partial^2 w}{\partial R\partial Q}.\frac{\partial\phi_y}{\partial R}+\frac{a^2H^2}{\beta}.\frac{\partial\phi_x}{\partial Q}.\frac{\partial\phi_y}{\partial R}+a^2H^2\left(\frac{\partial\phi_y}{\partial R}\right)^2\right) \quad (4.6)$$

Also for the case of Coupling Stiffness, U_C is given as

$$U_C = \frac{E_0t^3}{[1-\mu_{XY}\mu_{YX}]a^4}B_{11}$$

$$\left(-\frac{a}{t}S.\frac{\partial^2 w}{\partial R^2}.\frac{du_0}{dR}+\frac{a^2}{t}H.\frac{du_0}{dR}.\frac{\partial\phi_x}{\partial R}-\frac{aS}{t}.\frac{\partial^2 w}{\partial R^2}.\frac{du_0}{\partial R}+\frac{a^2}{t}H\frac{\partial\phi_x}{\partial R}.\frac{du_0}{dR}\right)+2B_{12}$$

$$\left(-\frac{a}{t\beta}S.\frac{\partial^2 w}{\partial R^2}.\frac{dv_0}{\partial Q}+\frac{a^2H}{t\beta}\frac{dv_0}{\partial Q}.\frac{\partial\phi_x}{\partial R}-\frac{aS}{t\beta^2}\frac{\partial^2 w}{\partial Q^2}.\frac{du_0}{dR}+\frac{a^2H}{t\beta}.\frac{\partial\phi_y}{\partial Q}.\frac{du_0}{dR}\right)$$

$$+2B_{12}\left(-\frac{a}{t\beta}S\frac{\partial^2 w}{\partial R^2}.\frac{du_0}{\partial Q}-\frac{a}{t}S\frac{\partial^2 w}{\partial R^2}.\frac{dv_0}{dR}+\frac{a^2H}{t}\frac{du_0}{\beta\partial Q}.\frac{\partial\phi_x}{\partial R}+\frac{a^2H}{t}\frac{dv_0}{dR}.\frac{\partial\phi_x}{\partial R}-\frac{2aS}{t\beta}\frac{\partial^2 w}{\partial R\partial Q}.\frac{du_0}{dR}+\frac{a^2H}{t\beta}\frac{\partial\phi_x}{\partial Q}.\frac{du_0}{dR}+\frac{a^2H}{t\beta}.\frac{\partial\phi_y}{\partial R}.\frac{du_0}{dR}\right)$$

$$+B_{22}\left(-\frac{aS}{t\beta^3}.\frac{\partial^2 w}{\partial Q^2}.\frac{dv_0}{\partial Q}+\frac{a^2H}{t\beta^2}\frac{dv_0}{\partial Q}.\frac{\partial\phi_y}{\partial Q}-\frac{aS}{t\beta^3}\frac{\partial^2 w}{\partial Q^2}.\frac{dv_0}{\partial Q}+\frac{a^2H}{t\beta^2}.\frac{\partial\phi_y}{\partial Q}.\frac{dv_0}{\partial Q}\right)$$

$$+2B_{23}\left(-\frac{aS}{t\beta^3}\frac{du_0}{\partial Q}.\frac{\partial^2 w}{\partial Q^2}-\frac{\boldsymbol{a}S}{t\beta^2}\frac{dv_0}{dR}.\frac{\partial^2 w}{\partial Q^2}+\frac{a^2H}{t\beta^2}\frac{du_0}{\partial Q}.\frac{\partial\phi_y}{\partial Q}+\frac{a^2H}{t\beta}\frac{dv_0}{dR}.\frac{\partial\phi_y}{\partial Q}-\frac{2aS}{t\beta^2}\frac{\partial^2 w}{\partial R\partial Q}.\frac{dv_0}{\partial Q}+\frac{a^2H}{t\beta^2}.\frac{\partial\phi_x}{\partial Q}.\frac{dv_0}{\partial Q}+\frac{a^2H}{t\beta}.\frac{\partial\phi_y}{\partial R}.\frac{dv_0}{\partial Q}\right)$$

$$+B_{33}\left(-\frac{2aS}{t\beta^2}.\frac{\partial^2 w}{\partial R\partial Q}.\frac{du_0}{\partial Q}-\frac{2aS}{t\beta}.\frac{\partial^2 w}{\partial R\partial Q}.\frac{dv_0}{dR}\right)+B_{33}\left(\frac{a^2H}{t\beta^2}\frac{du_0}{\partial Q}.\frac{\partial\phi_x}{\partial Q}+\frac{a^2H}{t\beta}\frac{dv_0}{dR}.\frac{\partial\phi_x}{\partial Q}\right)$$

$$+B_{33}\left(+\frac{a^2H}{t\beta}\frac{du_0}{\partial Q}.\frac{\partial\phi_y}{\partial R}+\frac{a^2H}{t}\frac{dv_0}{dR}.\frac{\partial\phi_y}{\partial R}\right)$$

$$+B_{33}\left(-\frac{2aS}{t\beta^2}\frac{\partial^2 w}{\partial R\partial Q}.\frac{du_0}{\partial Q}+\frac{a^2H}{t\beta^2}.\frac{\partial\phi_x}{\partial Q}.\frac{du_0}{\partial Q}+\frac{a^2H}{t\beta}\frac{\partial\phi_y}{\partial R}.\frac{du_0}{\partial Q}\right)$$

$$+B_{33}\left(-\frac{2aS}{t\beta}\frac{\partial^2 w}{\partial R\partial Q}.\frac{dv_0}{dR}+\frac{a^2H}{t\beta}.\frac{\partial\phi_x}{\partial Q}.\frac{dv_0}{dR}+\frac{a^2H}{t}\frac{\partial\phi_y}{\partial R}.\frac{dv_0}{dR}\right) \quad (4.7)$$

and finally for Axial/Membrane Stiffness, U_M is expressed as

$$U_M = \frac{E_0t^3}{[1-\mu_{XY}\mu_{YX}]a^4}B_{11}\left(\frac{a^2}{t^2}.\left(\frac{du_0}{dR}\right)^2\right)+2B_{12}\left(\frac{a^2}{t^2\beta}\frac{du_0}{dR}.\frac{dv_0}{\partial Q}\right)$$

$$+2B_{13}\left(\frac{a^2}{t^2}\frac{du_0}{\beta\partial Q}.\frac{du_0}{dR}+\frac{a^2}{t^2}.\frac{dv_0}{dR}.\frac{du_0}{dR}\right)+B_{22}\left(\frac{a^2}{t^2\beta^2}.\left(\frac{dv_0}{\partial Q}\right)^2\right)$$

$$+2B_{23}\left(\frac{a^2}{t^2\beta^2}\frac{du_0}{\partial Q}\frac{dv_0}{\partial Q}+\frac{a^2}{t^2\beta}\frac{dv_0}{dR}.\frac{dv_0}{\partial Q}\right)+B_{33}\left(\frac{a^2}{t^2\beta^2}\left(\frac{du_0}{\partial Q}\right)^2+\frac{a^2}{t^2\beta}\frac{dv_0}{dR}.\frac{du_0}{\partial Q}\right)$$

$$+B_{33}\left(\frac{a^2}{t^2\beta}\frac{du_0}{\partial Q}.\frac{dv_0}{dR}+\frac{a^2}{t^2}\left(\frac{dv_0}{dR}\right)^2\right)$$

$$+\ B_{44}\frac{a^4}{t^2}\left(\left(\emptyset_x.\frac{\partial H}{\partial S}\right)^2+\emptyset_x.\frac{du_0}{dS}.\frac{\partial\ H}{\partial S}+\emptyset_x.\frac{\partial H}{\partial S}.\frac{du_0}{dS}+\left(\frac{du_0}{dS}\right)^2\right)$$

$$+B_{55}\ \frac{a^4}{t^2}\left(\left(\emptyset_y.\frac{\partial H}{\partial S}\right)^2+.\emptyset_y.\frac{dv_0}{dS}.\frac{\partial H}{\partial S}+\emptyset_y.\frac{\partial H}{\partial S}.\frac{dv_0}{dS}+\left(\frac{dv_0}{dS}\right)^2\right) \qquad (4.8)$$

4.1.4 **Resultant Governing Equation of Laminated Thick Anisotropic Plate**

The derived Total potential energy as shown in subsection 4.1.2 was further differentiated with respect to the out of plane displacement (deflection) and the Governing Equation as shown in Equation (4.9)

$$\frac{\partial\pi}{\partial w}=\int_0^1\int_0^1\{2\,B_{11}S^2\frac{\partial^4 w}{\partial R^4}+\frac{2S^2}{\beta^2}\left[B_{xy}\right]\frac{\partial^4 w}{\partial R^2\,\partial Q^2}+\frac{4B_{13}S^2}{\beta}\frac{\partial^4 w}{\partial R^3\,\partial Q}+\frac{8B_{23}S^2}{\beta^3}\frac{\partial^4 w}{\partial R\,\partial Q^3}$$
$$+\frac{B_{22}2S^2}{\beta^4}\frac{\partial^4 w}{\partial Q^4}-2B_{11}aHS.\frac{\partial^3\phi_x}{\partial R^3}-\frac{2aHS}{\beta^2}\left[B_{xy}\right]\frac{\partial^3\phi_x}{\partial R\,\partial Q^2}$$

$$+\frac{2B_{13}aHS}{\beta}\frac{\partial^3\phi_x}{\partial R^2\,\partial Q}-\frac{2B_{23}aSH}{\beta^3}.\frac{\partial^3\phi_x}{\partial Q^3}-2B_{13}aHS\frac{\partial^3\phi_y}{\partial R^3}-\frac{2aHS}{\beta}\left[B_{xy}\right]\frac{\partial^3\phi_y}{\partial R^2\,\partial Q}$$

$$-\frac{2aSH}{\beta^2}\left[B_{23}+2B_{23}\right]\frac{\partial^2\phi_y}{\partial R\,\partial Q^2}-\frac{2B_{22}aHS}{\beta^3}.\frac{\partial^3\phi_y}{\partial Q^3}-2B_{11}\frac{aS}{t}.\frac{\partial^3 u_0}{\partial R^3}+\frac{2B_{13}aS}{t\beta}\frac{\partial^3 u_0}{\partial R^2\,\partial Q}$$

$$-\ \frac{aS}{t\beta^2}\left[B_{xy}\right]\frac{\partial^3 u_0}{\partial R\,\partial Q^2}-\frac{2B_{23}aS}{t\beta^3}.\frac{\partial^3 u_0}{\partial Q^3}-\frac{2B_{13}a}{t}S\frac{\partial^3 v_0}{\partial R^3}-\frac{2aS}{t\beta}\left[B_{xy}\right]\frac{\partial^3 v_0}{\partial R^2\,\partial Q}$$

$$- \frac{6B_{23}aS}{t\beta^2}\frac{\partial^3 v_0}{\partial R\,\partial Q^2} - \frac{2B_{22}aS}{t\beta^3}.\frac{\partial^3 v_0}{\partial Q^3} - \frac{N_x a^2}{D_0}\Bigg\} \, dRdQ$$

$$= 0 \qquad (4.9)$$

4.1.5 Compatibility Equation

The general Total potential Energy derived, was further differentiated with respect to middle layer inplane displacement and shear rotaions on both x and y components respectively and these formed the four compatibility Equation as shown in Equation (4.10), (4.11), (4.12) and (4.13). The middle layer in-plane displacemet in x direction is givens as:

$$u_o = B_{11}\left[-\frac{2aS}{t}.\frac{\partial^3 w}{\partial R^3} + \frac{2a^2}{t^2}.\frac{\partial^2 u_0}{\partial R^2} + \frac{2a^2 H}{t}.\frac{\partial^2 \phi_x}{\partial R^2}\right]$$

$$\frac{B_{12}}{\beta^2}\left[-\frac{2aS}{t}\frac{\partial^3 w}{dR\,\partial Q^2} + \frac{2a^2\beta}{t^2}.\frac{\partial^2 v_0}{dR\,\partial Q} + \frac{2a^2 H\beta}{t}.\frac{\partial^2 \phi_y}{\partial R\,\partial Q}\right]$$

$$+\frac{B_{13}}{\beta}\left[-\frac{3aS}{t}.\frac{\partial^3 w}{\partial R^2\,\partial Q} + \frac{2a^2}{t^2}\left[4.\frac{\partial^2 u_0}{\partial R\,\partial Q} + \beta.\frac{\partial^2 v_0}{\partial R^2}\right] + \frac{a^2 H}{t}\left[2.\frac{\partial^2 \phi_x}{\partial R\,\partial Q} + \frac{\partial^2 \phi_y}{\partial R^2}\right]\right]$$

$$+\frac{B_{23}}{\beta^3}\left[-\frac{2B_{23}aS}{t}.\frac{\partial^3 w}{\partial Q^3} + \frac{2B_{23}a^2\beta}{t^2}.\frac{\partial^2 v_0}{\partial Q^2} + \frac{2B_{23}a^2 H\beta}{t}.\frac{\partial^2 \phi_y}{\partial Q^2}\right]$$

$$+\frac{B_{33}}{\beta^2}\left[-\frac{4aS}{t}.\frac{\partial^3 w}{\partial R\,\partial\,\partial Q^2} + \frac{2a^2}{t^2}.\left[\frac{\partial^2 u_0}{\partial Q^2} + \beta\frac{\partial^2 v_0}{\partial Q\,\partial R}\right] + \frac{2a^2 H}{t}.\left[\frac{\partial^2 \phi_x}{\partial Q^2} + \beta.\frac{\partial^2 \phi_y}{\partial Q\,\partial R}\right]\right] \qquad (4.10)$$

While the middle layer in-plane displacement in y direction, is expressed as:

$$\frac{B_{12}}{\beta}\left[-\frac{2aS}{t}.\frac{\partial^3 w}{\partial Q\,\partial R^2} + \frac{2a^2}{t}\left[\frac{H\,\partial^2 \phi_x}{\partial R\,\partial Q} + \frac{\partial^2 v_0}{tdR\,\partial Q}\right]\right]$$

$$+\frac{B_{13}}{\beta}\left[-\frac{2aS\beta}{t}.\frac{\partial^3 w}{\partial R^3} + \frac{2a^2\beta}{t}.\left[\frac{\partial^2 u_0}{t\,\partial R^2} + H\frac{\partial^2 \phi_x}{\partial R^2}\right]\right]$$

$$+\frac{B_{22}}{\beta^3}\left[-\frac{2aS}{t}.\frac{\partial^3 w}{\partial Q^3} + \frac{4a^2\beta}{t^2}.\frac{\partial^2 v_0}{\partial Q^2} + \frac{2a^2\beta H}{t}.\frac{\partial^2 \phi_y}{\partial Q^2}\right]$$

$$+\frac{B_{23}}{\beta^2}\left[-\frac{6aS}{t}.\frac{\partial^3 w}{\partial R\,\partial Q^2} + \frac{4a^2}{t^2}\left[\frac{\partial^2 u_0}{\partial Q^2} + \beta\frac{\partial^2 v_0}{\partial Q\,\partial R}\right] + \frac{2a^2 H}{t}.\left[\frac{2\beta\,\partial^2 \phi_y}{\partial R\,\partial Q} + \frac{\partial^2 \phi_x}{\partial Q^2}\right]\right]$$

$$\frac{B_{33}}{\beta}\left[-\frac{4aS}{t}\frac{\partial^3 w}{\partial R^2\,\partial Q}+\frac{2a^2}{t^2}\frac{\partial^2 u_0}{\partial Q\,\partial R}+\frac{2a^2\beta}{t^2}.\frac{\partial^2 v_0}{\partial R^2}+\frac{2a^2H\beta}{t}.\frac{\partial^2\phi_y}{\partial R^2}+\frac{2a^2H}{t}.\frac{\partial^2\phi_x}{\partial R\,\partial Q}\right]$$

$$+B_{55}.\frac{a^2}{t^2}.2\frac{\partial^2 v_0}{\partial S^2}.\}dR\,dQ \qquad (4.11)$$

The shear rotation on the y-z components were given as:

$$\emptyset_y=\frac{B_{12}}{\beta}\left[-2aHS.\frac{\partial^3 w}{\partial Q\,\partial R^2}+\frac{2a^2H}{t}.\frac{\partial^2 u_0}{\partial R\,\partial Q}+2a^2H^2.\frac{\partial^2\phi_x}{\partial R\,\partial Q}\right]$$

$$+\frac{B_{13}}{\beta}\left[-2aHS\beta\frac{\partial^3 w}{\partial R^3}+\frac{2a^2H}{t}.\frac{\partial^2 u_0}{\partial R^2}+2a^2H^2\beta.\frac{\partial^2\phi_x}{\partial R^2}\right]$$

$$+\frac{B_{22}}{\beta^3}\left[-2aHS.\frac{\partial^3 w}{\partial Q^3}+2a^2H^2\beta.\frac{\partial^2\phi_y}{\partial Q^2}+\frac{2a^2H\beta}{t}\frac{\partial^2 v_0}{\partial Q^2}\right]$$

$$+\frac{B_{23}}{\beta^2}\left[-6aHS.\frac{\partial^3 w}{\partial R\,\partial Q^2}+\frac{2a^2H}{t}.\frac{\partial^2 u_0}{\partial Q^2}+\frac{4a^2H\beta}{t}\frac{\partial^2 v_0}{\partial Q\,\partial R}+2a^2H^2.\frac{\partial^2\phi_x}{\partial Q^2}+2H^2\beta.\frac{\partial^2\phi_y}{\partial Q\,\partial R}\right]$$

$$\frac{B_{33}}{\beta}\left[-4aHS.\frac{\partial^3 w}{\partial R^2\,\partial Q}+\frac{2a^2H}{t}.\frac{\partial^2 u_0}{\partial R\,\partial Q}+\frac{2a^2H\beta}{t}.\frac{\partial^2 v_0}{\partial R^2}+2a^2H^2.\frac{\partial\phi_x}{\partial Q\,\partial R}+2a^2H^2\beta\frac{\partial^2\phi_y}{\partial R^2}\right]$$

$$+2B_{55}.\frac{a^4}{t^2}.g_4.\emptyset_y+B_{55}.\frac{a^4}{t^3}.\left[2g_{C3}.\frac{dv_0}{dS}\right].\}dR\,dQ \qquad (4.12)$$

The shear rotation on X-Z components is also given as

$$\emptyset_x=B_{11}\left[-2aHS.\frac{\partial^3 w}{\partial R^3}+\frac{2a^2H}{t}.\frac{\partial^2 u_0}{\partial R^2}+2a^2S^2.\frac{\partial^2\phi_x}{\partial R^2}\right]$$

$$+\frac{B_{12}}{\beta^2}\left[-2aHS\frac{\partial^3 w}{\partial R\,\partial Q^2}+\frac{2a^2H\beta}{t}\frac{\partial^2 v_0}{\partial R\,\partial Q}+2a^2H^2\beta.\frac{\partial^2\phi_y}{\partial R\,\partial Q}\right]$$

$$+\frac{B_{13}}{\beta}\left[-5aHS.\frac{\partial^3 w}{\partial Q\,\partial R^2}+\frac{4a^2H}{t}.\frac{\partial^2 u_0}{\partial R\,\partial Q}+\frac{2a^2H\beta}{t}\frac{\partial^2 v_0}{\partial R^2}+2a^2H^2\beta.\frac{\partial^2\phi_y}{\partial R^2}\right]$$

$$+\frac{B_{23}}{\beta^3}\left[-2aSH.\frac{\partial^3 w}{\partial Q^3}+\frac{2a^2H\beta}{t}.\frac{\partial^2 v_0}{\partial Q^2}+2a^2H^2\beta.\frac{\partial^2\phi_y}{\partial Q^2}\right]$$

$$+\frac{B_{33}}{\beta^2}\left[-4aHS.\frac{\partial^3 w}{\partial R\,\partial Q^2}+\frac{2a^2H}{t}.\left[\frac{\partial^2 u_0}{\partial Q^2}+\beta.\frac{\partial^2 v_0}{\partial R\,\partial Q}\right]+2a^2H^2\left[.\frac{\partial^2 \phi_x}{\partial Q^2}+\beta.\frac{\partial^2 \phi_y}{\partial R\,\partial Q}\right]\right]$$

$$+B_{44}.\frac{a^4}{t^3}\left[2g_{C3}.\frac{du_0}{dS}\right]+2B_{44}.\frac{a^4}{t^2}.g_4.\emptyset_x\}dR\,dQ \tag{4.13}$$

4.1.5.1 Stiffness Coefficients

The bending stiffnesses for n-laminates J_1 $(i = 1 \text{ to } 3)$ are given respectively as:

$$J_1 = \left[\frac{(s_m^3 - s_{m-1}^3)}{3}\right] * n \tag{4.14}$$

$$J_2 = \left[\frac{\left(s_m^3 - \frac{4}{5}s_m^5\right) - \left(s_{m-1}^3 - \frac{4}{5}s_{m-1}^5\right)}{3}\right] * n \tag{4.15}$$

$$J_3 = \left[\frac{\left(s_m^3 - \frac{8}{5}s_m^5 + \frac{16}{21}s_m^7\right) - \left(s_{m-1}^3 - \frac{8}{5}s_{m-1}^5 + \frac{16}{21}s_{m-1}^7\right)}{3}\right] * n \tag{4.16}$$

$$J_4 = \left[\frac{\left(3s_m^1 - 8s_m^3 + \frac{48}{5}s_m^5\right) - \left(3s_{m-1}^1 - 8s_{m-1}^3 + \frac{48}{5}s_{m-1}^5\right)}{3}\right] * n \tag{4.17}$$

The coupling stiffnesses for n-laminates J_1 $(i = 1 \text{ to } 3)$ are given respectively as:

$$J_{c1} = \left[\frac{(s_m^2 - s_{m-1}^2)}{2}\right] * n \tag{4.18}$$

$$J_{c2} = \left[\frac{\left(s_m^2 - \frac{2}{3}s_m^4\right) - \left(s_{m-1}^2 - \frac{2}{3}s_{m-1}^4\right)}{3}\right] * n \tag{4.19}$$

$$J_{c3} = \left[\left(s_m^1 - \frac{4}{3}s_m^3\right) - \left(s_{m-1}^1 - \frac{4}{3}s_{m-1}^3\right)\right] * n \tag{4.20}$$

Finally for Membrane Stiffness with n–Laminates , it is gives as

$$J_M = [(s_m^1) - (s_{m-1}^1)] * n \tag{4.21}$$

4.1.5.2 Buckling Load Equation and the Stiffness Coefficients

The resolution of the four compatibility Equation, finally produced Equation needed for the numerical analysis. The buckling Equation for the laminated thick anisotropic plate is as shown in Equation (4.22)

$$\frac{N_x a^2}{D_0} = E_0 * \left(\frac{K_{T1}+K_{T2}+K_{T3}+K_{T4}+K_{T5}}{K_{T6}}\right)/D_0 \quad (4.22)$$

where

$$K_{T1} = B_{11}\left(g_1 - g_2 P_5 - \frac{g_{c1}}{t} P_2\right) k_1 \quad (4.23)$$

$$K_{T2} = \frac{1}{\beta^2 k_6}\left[2g_1(B_{12}+2B_{33}) - g_2(B_{12}+2B_{33})[P_5+P_4] - \frac{g_{c1}}{t}(B_{12}[P_3+P_2] + 2B_{33}[P_3+4P_2])\right] k_2 \quad (4.24)$$

$$K_{T3} = \frac{B_{22}}{\beta^4 k_6}\left(g_1 - g_2 P_4 - \frac{g_{c1}}{t} P_3\right) k_3 \quad (4.25)$$

$$K_{T4} = \frac{B_{13}}{\beta k_6}\left(-4g_1 + g_2[P_5 - P_4] + \frac{g_{c1}}{t}[P_2 - P_3]\right) k_4 \quad (4.26)$$

$$K_{T5} = \frac{B_{23}}{\beta^3 k_6}\left(4g_1 - g_2 P_4 - \frac{g_{c1}}{t}[P_2 + 3P_3]\right) k_5 \quad (4.27)$$

4.1.6 Results of the Numerical Analysis For Different Laminated Anisotropic Thick Plate Cases.

In the process of deriving the buckling load equation which was needed for the proper numerical analysis, other engineering material properties were introduced at different levels. These values were introduced into the equation to reduce the lengthy mathematical expressions into simpler form. The process helped in converting the derived values into solvable equation using Gauss elimination method.

The results derived show some differences when different angles were considered and also on introduction of other engineering properties. These properties were adopted in the analysis of various plate conditions as ealier discussed and findings were as detailed preceding sectons.

4.1.6.1 Detailed Output For SSSS Thick Laminated Anisotropic Plate.

For different lamina combination for SSSS plate, the results of the analysis are as detailed in this section.

4.1.6.2 The case of Two Combined SSSS Plate with 0^0 0^0 arrangement.

The results from the analysis when the laminates were positioned at the angles 0^0 0^0 are as presented in Table 4.1

Table 4.1 The Bending, Coupling and Membrane stiffness for the SSSS at 0^0 0^0 angle

Lamina, m = 1				
Bending	$J_1 = 0.08333$ $J_2 = 0.06667$ $J_3 = 0.05397$ $J_4 = 0.53333$	Coupling	$Jc_1 = -0.125$ $Jc_2 = -0.20833$ $Jc_3 = 0.6666$	Membrane $J_M = 1$
Lamina, m = 2				
Bending	$J_1 = 0.08333$ $J_2 = 0.06667$ $J_3 = 0.05397$ $J_4 = 0.53333$	Coupling	$Jc_1 = 0.125$ $Jc_2 = 0.20833$ $Jc_3 = 0.66667$	Membrane, $J_M = 1$

4.1.6.3 The case of Two combined SSSS Plate with 0^0 90^0 arrangement.

The formulated values from the analysis when the laminates were positioned at the angles 0^0 90^0 are as presented in Table 4.2

Table 4.2 The Bending, Coupling and Membrane stiffness for the SSSS plate with 0^0 90^0

Lamina, m = 1				
Bending	$J_1 = 0.08333$ $J_2 = 0.06667$ $J_3 = 0.05397$ $J_4 = 0.53333$	Coupling	$Jc_1 = -0.125$ $Jc_2 = -0.20833$ $Jc_3 = 0.6666$	Membrane $J_M = 1$
Lamina, m = 2				
Bending	$J_1 = 0.08333$ $J_2 = 0.06667$ $J_3 = 0.05397$ $J_4 = 0.53333$	Coupling	$Jc_1 = 0.125$ $Jc_2 = 0.20833$ $Jc_3 = 0.66667$	Membrane, $J_M = 1$

4.1.6.4 Results of Three laminated Simple Simple Simple Simple Thick Plate

Results of Three laminated Simple Simple Simple Simple Thick Plate were detailed at different plate arrangements

4.1.6.5 The case of Three Combined SSSS plate with 0^0 0^0 0^0 angle arrangement.

The formulated values from the analysis when the layer orientation is 0^0 0^0 0^0

are as presented in Table 4.3

Table 4.3 The Bending, Coupling and Membrane stiffness for the SSSS plate with 0^0 0^0 0^0

Lamina, m = 1				
Bending	J_1 =-0.12037 J_2 = -0.09547 J_3 =-0.07653 J_4 =-0.3358	Coupling	Jc_1 = -0.33333 Jc_2 = -0.2716 Jc_3 =-0.51852	Membrane J_M =-1
Lamina, m = 2				
Bending	J_1 =-0.00926 J_2 = -0.00905 J_3 =-0.00885 J_4 = -0.92841	Coupling	Jc_1 =0.0000 Jc_2 = 0.0000 Jc_3 =-0.96298	Membrane, J_M =-1.0002
Lamina, m = 3				
Bending	J_1 = -0.12037 J_2 = -0.09547 J_3 = -0.07653 J_4 = -0.33579	Coupling	Jc_1 = 0.3333 Jc_2 = 0.2716 Jc_3 =-0.51851	Membrane J_M =-0.9999

4.1.6.6 The case of Three Combined SSSS Plate with angle 0^0 90^0 0^0.

The values derived for the case of the angles 0^0 90^0 0^0 , are as presented in Table 4.4

Table 4.4 The Bending, Coupling and Membrane stiffness for the SSSS plate with 0^0 90^0 0^0

Lamina, m = 1				
Bending	J_1 = -0.12037 J_2 = -0.09547 J_3 = -0.07653 J_4 = -0.3358	Coupling	Jc_1 = -0.33333 Jc_2 = -0.2716 Jc_3 = -0.51852	Membrane J_M =-1
Lamina, m = 2				
Bending	J_1 = -0.00926 J_2 = -0.00905 J_3 = -0.00885 J_4 = -0.92841	Coupling	Jc_1 = 0.0000 Jc_2 = 0.0000 Jc_3 =-0.96298	Membrane, J_M =-1.0002

Lamina, m = 3				
Bending	J_1 = -0.12037 J_2 = -0.09547 J_3 = -0.07653 J_4 = -0.33579	Coupling	Jc_1 = 0.3333 Jc_2 = 0.2716 Jc_3 =-0.51851	Membrane J_M =-0.9999

As ealier explained, the lengthy mathematical expressions formulated were further collapsed into smaller fractions yielding the coefficients, which were considered as the unknown parameters. These were derived using the guass elimination method as expalained in chapter three. Tables 4.5, 4.6, 4.7 4.8 and 4.9 show the values of the unknown parameters and the buckling Load equations, for the case of SSSS plate,

Table 4.5 The Coefficicents of The Formulated Parameters for the SSSS plate

P_2	P_3	P_4	P_5	P_6
-1.71315	-0.84346	-0.02373	-1.70727	
A_2	A_3	A_4	A_5	A_6
-1.71315 A_1	-0.84346 A_1	-0.02373A_1	-1.70727A_1	

The Buckling Load Equation for SSSS at Orientattion of 0^0 0^0

Table 4.6 The Buckling Load Equation for SSSS Lamina 1

when Ø = 0^0 SSSS K_T – Values				
$Kssss_{T1}$	$Kssss_{T2}$	$Kssss_{T3}$	$Kssss_{T4}$	$Kssss_{T5}$
-58.861689	535.451846	-40.4594018	0	0
$\frac{N_X}{D_O}$	18248.1488			

Table 4.7 The Buckling Load Equation for SSSS Lamina 2

when Ø = 0^0 SSSS K_T – Values				
$Kssss_{T1}$	$Kssss_{T2}$	$Kssss_{T3}$	$Kssss_{T4}$	$Kssss_{T5}$
-58.861689	535.451846	-40.4594018	0	0
$\frac{N_X}{D_O}$	18248.1488			

The Buckling Load Equation for SSSS at Orientattion of 0^0 90^0

Table 4.8 The Buckling Load Equation for SSSS Lamina 1

when Ø = 0^0 SSSS K_T – Values

K_{ssssT1}	K_{ssssT2}	K_{ssssT3}	K_{ssssT4}	K_{ssssT5}
-58.861689	535.451846	-40.4594018	0	0
$\frac{N_X}{D_O}$	18248.1488			

Table 4.9 The Buckling Load Equation for SSSS Lamina 2

when Ø = 90^0 SSSS K_T – Values				
K_{ssssT1}	K_{ssssT2}	K_{ssssT3}	K_{ssssT4}	K_{ssssT5}
0.541604	-13.7691	725.3688	0	0
$\frac{N_X}{D_O}$	29796.71			

The Buckling Load Equation for SSSS at Orientattion of 0^0 0^0 0^0

Table 4.10 The Buckling Load Equation for SSSS Lamina 1

when Ø = 0^0 SSSS K_T – Values				
K_{ssssT1}	K_{ssssT2}	K_{ssssT3}	K_{ssssT4}	K_{ssssT5}
-354.763	-850.473	-236.822	0	0
$\frac{N_X}{D_O}$	-60337.2			

Table 4.11 The Buckling Load Equation for SSSS Lamina 2

when Ø = 0^0 SSSS K_T – Values				
K_{ssssT1}	K_{ssssT2}	K_{ssssT3}	K_{ssssT4}	K_{ssssT5}
-2.77491	-2.78887	-1.11691	0	0
$\frac{N_X}{D_O}$	-279.527			

Table 4.12 The Buckling Load Equation for SSSS Lamina 3

when Ø = 0^0 SSSS K_T – Values				
K_{ssssT1}	K_{ssssT2}	K_{ssssT3}	K_{ssssT4}	K_{ssssT5}
115.399	113.441	1967.58	0	0
$\frac{N_X}{D_O}$	82407.4			

4.1.7 Results of different aspect ratios for Two laminates of SSSS plate with 0^0 0^0 Arrangement.

On introduction of different aspect ratios, which ranges from 1 to 2 with arithmetic increase of 0.1, gave the results as detailed in Table 4.13

considering

$$K_{ssssT} = \left(\frac{K_{SSSST1} + K_{SSSST2} + K_{SSSST3} + K_{SSSST4} + K_{SSSST5}}{K_{SSSST6}}\right) \quad (4.28)$$

Table 4.13 Buckling results of different aspect ratios for SSSS plate with $0^0\,0^0$ arrangement

m = 1						m = 2				
$\beta = \frac{a}{b}$	$\frac{E_0}{D_0}.K_{ssss}$		1.5	-98312.4		$\beta = \frac{a}{b}$	$\frac{E_0}{D_0}.K_{ssss}$		1.5	4784.868
1	-60337.2		1.6	-109181		1	3374.795		1.6	5166.446
1.1	-65137.2		1.7	-120924		1.1	3531.464		1.7	5564.728
1.2	-71619		1.8	-133509		1.2	3780.832		1.8	5978.708
1.3	-79431.2		1.9	-146911		1.3	4084.393		1.9	6408.17
1.4	-88370		2.0	161115		1.4	4422.462		2.0	6853.303

4.1.7.1 Results of different aspect ratios for Two laminates of SSSS plate with $0^0\,90^0$ Arrangement.

Similarly, on introducing different aspect ratios, which ranges from 1 to 2 with arithmetic increase of 0.1, gave the results as detailed in Table 4.14

where

$$K_{ssssT} = \left(\frac{K_{ssssT1} + K_{ssssT2} + K_{ssssT3} + K_{ssssT4} + K_{ssssT5}}{K_{ssssT6}}\right) \quad (4.29)$$

Table 4.14 Buckling results of different aspect ratios for SSSS plate with $0^0\ 90^0$ arrangement

m = 1						m = 2				
$\beta = \frac{a}{b}$	$\frac{E_0}{D_0}.K_{ssss}$		1.5	-98312.4		$\beta = \frac{a}{b}$	$\frac{E_0}{D_0}.K_{SSSS}$		1.5	8067.506
1	-60337.2		1.6	-109181		1	29796.71		1.6	7671.104

1.1	-65137.2		1.7	-120924		1.1	19760.6		1.7	7665.017
1.2	-71619		1.8	-133509		1.2	14120.64		1.8	7936.754
1.3	-88370		1.9	-146911		1.3	10875.66		1.9	8413.808
1.4	-98312.4		2.0	161115		1.4	9034.924		2.0	9047.97

4.1.7.2 Results of different aspect ratios for Three laminates of SSSS plate with $0^0\ 0^0\ 0^0$ Arrangement.

On introduction of different aspect ratios, which ranges from 1 to 2 with arithmetic increase of 0.1, gave the results as detailed in Table 4.15

where

$$K_{ssss} = \left(\frac{K_{ssssT1} + K_{ssssT2} + K_{ssssT3} + K_{ssssT4} + K_{ssssT5}}{K_{ssssT6}}\right) \quad (4.30)$$

Table 4.15a Buckling results of different aspect ratios for SSSS plate with $0^0\ 0^0\ 0^0$ arrangement

m = 1						m = 2				
$\beta = \frac{a}{b}$	$\frac{E_0}{D_0}.K_{ssss}$		1.5	-98312.4		$\beta = \frac{a}{b}$	$\frac{E_0}{D_0}.K_{ssss}$		1.5	-388.789
1	-60337.2		1.6	-109181		1	-279.527		1.6	-423.222
1.1	-65137.2		1.7	-120924		1.1	-289.29		1.7	-460.627
1.2	-71619		1.8	-133509		1.2	-306.873		1.8	-500.839
1.3	-79431.2		1.9	-146911		1.3	-330.048		1.9	-543.75
1.4	-88370		2.0	161115		1.4	-357.59		2.0	-589.292

Table 4.15b

m = 1					
$\beta = \frac{a}{b}$	$\frac{E_0}{D_0}.K_{SSSS}$		1.5	-15787.3	
1	82407.4		1.6	-27545.9	
1.1	50768.8		1.7	-38851.9	
1.2	28433		1.8	50000.6	
1.3	11207.8		1.9	-61186.9	
1.4	-3116.64		2.0	-72542.8	

4.1.7.3 Detailed Output For CCCC Thick Laminated Anisotropic Plate.

For different lamina combination for CCCC plate, the results of the analysis are as detailed in this section.

4.1.7.4 The case of Two Combined CCCC with angles of 0^0 0^0

The results from the analysis when the laminates were positioned at the angles 0^0 0^0 are as presented in Table 4.16

Table 4.16 The Bending, Coupling and Membrane stiffness for the CCCC at 0^0 0^0 angle

Lamina, m = 1				
Bending	$J_1 = 0.08333$ $J_2 = 0.06667$ $J_3 = 0.05397$ $J_4 = 0.53333$	Coupling	$Jc_1 = -0.1250$ $Jc_2 = -0.20833$ $Jc_3 = 0.66667$	Membrane $J_M = 1$
Lamina, m = 2				
Bending	$J_1 = 0.08333$ $J_2 = 0.06667$ $J_3 = 0.05397$ $J_4 = 0.53333$	Coupling	$Jc_1 = 0.125$ $Jc_2 = 0.20833$ $Jc_3 = 0.66667$	Membrane, $J_M = 1$

4.1.7.5 The case of Two combined CCCC plate layer at angle 0^0 90^0

The formulated values from the analysis when the laminates were positioned at the angles 0^0 90^0 are as presented in Table 4.17

Table 4.17 The Bending, Coupling and Membrane stiffness for the CCCC plate with 0^0 90^0

Lamina, m = 1				
Bending	J_1 = 0.08333 J_2 = 0.06667 J_3 = 0.05397 J_4 = 0.53333	Coupling	Jc_1 = -0.125 Jc_2 = -0.20833 Jc_3 = 0.6666	Membrane J_M =1
Lamina, m = 2				
Bending	J_1 = 0.08333 J_2 = 0.06667 J_3 = 0.05397 J_4 = 0.53333	Coupling	Jc_1 = 0.125 Jc_2 = 0.20833 Jc_3 = 0.66667	Membrane, J_M = 1

4.1.8 Results of Three laminated Clamped Clamped Clamped Clamped Thick Plate

Results of Three laminated Clamped Clamped Clamped Clamped Thick Plate were detailed at different plate orientation

4.1.8.1 The case of Three Combined CCCC with angles of 0^0 0^0 0^0

The formulated values from the analysis when the layer orientation is 0^0 0^0 0^0

are as presented in Table 4.18

Table 4.18 The Bending, Coupling and Membrane stiffness for the CCCC plate with 0^0 0^0 0^0

Lamina, m = 1				
Bending	J_1 =-0.12037 J_2 = -0.09547 J_3 =-0.07653 J_4 =-0.3358	Coupling	Jc_1 = -0.33333 Jc_2 = -0.2716 Jc_3 =-0.51852	Membrane J_M =-1
Lamina, m = 2				
Bending	J_1 =-0.00926 J_2 = -0.00905 J_3 =-0.00885 J_4 = -0.92841	Coupling	Jc_1 =0.0000 Jc_2 = 0.0000 Jc_3 =-0.96298	Membrane, J_M =-1.0002
Lamina, m = 3				
Bending	J_1 = -0.12037 J_2 = -0.09547 J_3 = -0.07653 J_4 = -0.33579	Coupling	Jc_1 = 0.3333 Jc_2 = 0.2716 Jc_3 =-0.51851	Membrane J_M =-0.9999

4.1.8.2 The Buckling Values For The Case of CCCC Plate

As ealier explained, the lengthy mathematical expressions formulated were further collapsed into smaller fractions yielding the coefficients, which were considered as the unknown parameters. These were derived using the guass elimination method as explained in chapter three. Tables 4.19. 4.20, 4.21, 4.22 and 4.23 show the values of the unknown parameters and the buckling Load equations, for the case of CCCC plate,

Table 4.19 The Coefficicents of The Formulated Parameters for the CCCC plate

P_2	P_3	P_4	P_5	P_6
-1.62510668	-0.18390167	-0.02291	0.078824	
A_2	A_3	A_4	A_5	A_6
-1.62510668 A_1	-0.18390167A_1	-0.02291A_1	0.078824A_1	

The Buckling Load Equation for CCCC at Orientattion of 0^0 0^0

Table 4.20 The Buckling Load Equation for CCCC Plate for Lamina 1

Ø = 0^0 CCCC K_T – Values				
K_{T1}	K_{T2}	K_{T3}	K_{T4}	K_{T5}
28.1026	60.1968	-3.474301	0	0
$\frac{N_X}{D_O}$	3549.17272			

Table 4.21 The Buckling Load Equation for CCCC Plate for Lamina 2

Ø = 0^0 CCCC K_T – Values				
K_{T1}	K_{T2}	K_{T3}	K_{T4}	K_{T5}
24.14129	29.32363	0.242578	0	0
$\frac{N_X}{D_O}$	2247.176			

The Buckling Load Equation for CCCC at Orientattion of 0^0 90^0

Table 4.22 The Buckling Load Equation for CCCC Plate for Lamina 1

$Ø = 0^0$ CCCC K_T – Values				
K_{T1}	K_{T2}	K_{T3}	K_{T4}	K_{T5}
28.1026	60.1968	-3.474301	0	0
$\frac{N_X}{D_O}$	3549.17272			

Table 4.23 The Buckling Load Equation for CCCC Plate for Lamina 2

$Ø = 90^0$ CCCC K_T – Values				
K_{T1}	K_{T2}	K_{T3}	K_{T4}	K_{T5}
0.909098	-287.342	8.448973	0	0
$\frac{N_X}{D_O}$	-11631.1			

The Buckling Load Equation for CCCC at Orientattion of 0^0 0^0 0^0 as shown in Fig 4.24, 4.25 and 4.26 .

Table 4.24 The Buckling Load Equation for CCCC plate fof Lamina 1

$Ø = 0^0$ CCCC K_T – Values				
$Kcccc_{T1}$	$Kcccc_{T2}$	$Kcccc_{T3}$	$Kcccc_{T4}$	$Kccc_{T5}$
-10.3757	-118.72	-56.2501	0	0
$\frac{N_X}{D_O}$	-7755.05			

Table 4.25 The Buckling Load Equation for CCCC plate fof Lamina 2

$Ø = 0^0$ CCCC K_T – Values				
$Kcccc_{T1}$	$Kcccc_{T2}$	$Kcccc_{T3}$	$Kcccc_{T4}$	$Kccc_{T5}$
-2.77539	7.3529	0.35498	0	0
$\frac{N_X}{D_O}$	206.38			

Table 4.26 The Buckling Load Equation for CCCC plate Lamina 3

$Ø = 0^0$ CCCC K_T – Values				
$Kcccc_{T1}$	$Kcccc_{T2}$	$Kcccc_{T3}$	$Kcccc_{T4}$	$Kccc_{T5}$
-142.78	-357.792	-6.98085	0	0
$\frac{N_X}{D_O}$	-21236.5			

4.1.8.3 Results of different aspect ratios for Two laminates of CCCC plate with 0^0 0^0 Arrangement.

On introduction of different aspect ratios, which ranges from 1 to 2 with arithmetic increase of 0.1, gave the results as detailed in Table 4.27

where

$$K_{cccc} = \left(\frac{K_{ccccT1} + K_{ccccT2} + K_{ccccT3} + K_{ccccT4} + K_{ccccT5}}{K_{ccccT6}}\right) \qquad (4.31)$$

Table 4.27 Buckling results of different aspect ratios for CCCC plate with $0^0\,0^0$ arrangement

m = 1						m = 2				
$\beta = \frac{a}{b}$	$\frac{E_0}{D_0}.K_{cscs}$		1.5	6424.126		$\beta = \frac{a}{b}$	$\frac{E_0}{D_0}.K_{cscs}$		1.5	1510.281
1	3549.173		1.6	7110.54		1	971.792		1.6	1648.629
1.1	4060.632		1.7	7838.47		1.1	1044.116		1.7	1792.207
1.2	4599.265		1.8	8608.794		1.2	1141.366		1.8	1940.211
1.3	5170.54		1.9	9422.159		1.3	1254.391		1.9	2092.03
1.4	5778.022		2.0	10279.04		1.4	1378.296		2.0	2247.176

4.1.8.4 Results of different aspect ratios for Two laminates of CCCC plate with 90^0 0^0 Arrangement.

For case of CCCC, on introducting different aspect ratios, which ranges from 1 to 2 with arithmetic increase of 0.1, gave the results as detailed in Table 4.28

where

$$K_{cccc} = \left(\frac{K_{ccccT1} + K_{ccccT2} + K_{ccccT3} + K_{ccccT4} + K_{ccccT5}}{K_{ccccT6}}\right) \qquad (4.32)$$

Table 4.28 Buckling results of different aspect ratios for CCCC plate with 0^0 90^0arrangement

m = 1						m = 2				
$\beta = \frac{a}{b}$	$\frac{E_0}{D_0}.K_{cccc}$		1.5	6424.126		$\beta = \frac{a}{b}$	$\frac{E_0}{D_0}.K_{cccc}$		1.5	-11908.9
1	3549.173		1.6	7110.54		1	-8361.07		1.6	-11965.7

1.1	4060.632		1.7	7838.47		1.1	-9945.92		1.7	-11949.7
1.2	4599.265		1.8	8608.794		1.2	-10881.6		1.8	-11880.7
1.3	5170.54		1.9	9422.159		1.3	-11433.8		1.9	-11771.6
1.4	5778.022		2.0	10279.04		1.4	-11748.3		2.0	-11631.1

4.1.8.5 Results of different aspect ratios for Three laminates of CCCC plate with $0^0\ 0^0\ 0^0$ Arrangement.

On introduction of different aspect ratios, which ranges from 1 to 2 with arithmetic increase of 0.1, gave the results as detailed in Table 4.29

where

$$K_{cccc} = \left(\frac{K_{ccccT1} + K_{ccccT2} + K_{ccccT3} + K_{ccccT4} + K_{ccccT5}}{K_{ccccT6}}\right) \quad (4.33)$$

Table 4.29a Buckling results of different aspect ratios for CCCC plate with $0^0\ 0^0\ 0^0$ arrangement

m = 1						m = 2				
$\beta = \frac{a}{b}$	$\frac{E_0}{D_0}.K_{cccc}$		1.5	-11848.6		$\beta = \frac{a}{b}$	$\frac{E_0}{D_0}.K_{cccc}$		1.5	35.8437
1	-7755.05		1.6	12908.9		1	40.4146		1.6	57.3488
1.1	-8346.65		1.7	-14034.7		1.1	18.4929		1.7	84.9027
1.2	-9088.11		1.8	15226.5		1.2	10.2222		1.8	118.683

1.3	-9931.49		1.9	16485.3		1.3	11.5626		1.9	159.026
1.4	-10854.9		2.0	-17812.3		1.4	20.4424		2.0	206.38

Table 4.29b

m = 3				
$\beta = \frac{a}{b}$	$\frac{E_0}{D_0}.K$cccc		1.5	18687.1
1	-234422.8		1.6	-18772.4
1.1	-21416.2		1.7	-19074.9
1.2	-20109		1.8	-19583.4
1.3	-19293.7		1.9	-20299.4
1.4	18845.3		2.0	21236.5

4.1.8.6 Detailed Output For CSCS Thick Laminated Anisotropic Plate.

For different lamina combination for CSCS plate, the results of the analysis are as detailed in this section.

4.1.8.7 The case of Two Combined CSCS with angles of 0^0 0^0

The results from the analysis when the laminates were positioned at the angles 0^0 0^0 are as presented in Table 4.30

Table 4.30 The Bending, Coupling and Membrane stiffness for the CSCS at 0^0 0^0 angle

Lamina, m = 1				
Bending	$J_1 = 0.08333$ $J_2 = 0.06667$ $J_3 = 0.05397$ $J_4 = 0.53333$	Coupling	$Jc_1 = -0.125$ $Jc_2 = -0.20833$ $Jc_3 = 0.6666$	Membrane $J_M = 1$
Lamina, m = 2				
Bending	$J_1 = 0.08333$ $J_2 = 0.06667$ $J_3 = 0.05397$ $J_4 = 0.53333$	Coupling	$Jc_1 = 0.125$ $Jc_2 = 0.20833$ $Jc_3 = 0.66667$	Membrane, $J_M = 1$

4.1.9 The case of Two combined CSCS plate layer at angle 0^0 90^0

The formulated values from the analysis when the laminates were positioned at the angles 0^0 90^0 are as presented in Table 4.31

Table 4.31 The Bending, Coupling and Membrane stiffness for the CSCS plate with 0^0 90^0

Lamina, m = 1				
Bending	$J_1 = 0.08333$ $J_2 = 0.06667$ $J_3 = 0.05397$ $J_4 = 0.53333$	Coupling	$Jc_1 = -0.125$ $Jc_2 = -0.20833$ $Jc_3 = 0.6666$	Membrane $J_M = 1$
Lamina, m = 2				
Bending	$J_1 = 0.08333$ $J_2 = 0.06667$ $J_3 = 0.05397$ $J_4 = 0.53333$	Coupling	$Jc_1 = 0.125$ $Jc_2 = 0.20833$ $Jc_3 = 0.66667$	Membrane, $J_M = 1$

4.1.9.1 Results of Three laminated Clamped Simple Clamped Simple Thick Plate

After analyzing a thick plate considering different angles of orientation, the results for a three laminated Clamped Simple Clamped Simple Thick Plate condititon as as detailed.

4.1.9.2 The case of Three Combined CSCS with angles of 0^0 0^0 0^0

The formulated values from the analysis when the layer orientation is 0^0 0^0 0^0

are as presented in Table 4.32

Table 4.32 The Bending, Coupling and Membrane stiffness for the CSCS plate with $0^0\ 0^0\ 0^0$

Lamina, m = 1				
Bending	J_1 =-0.12037 J_2 = -0.09547 J_3 =-0.07653 J_4 =-0.3358	Coupling	Jc_1 = -0.33333 Jc_2 = -0.2716 Jc_3 =-0.51852	Membrane J_M =-1
Lamina, m = 2				
Bending	J_1 =-0.00926 J_2 = -0.00905 J_3 =-0.00885 J_4 = -0.92841	Coupling	Jc_1 =0.0000 Jc_2 = 0.0000 Jc_3 =-0.96298	Membrane, J_M =-1.0002
Lamina, m = 3				
Bending	J_1 = -0.12037 J_2 = -0.09547 J_3 = -0.07653 J_4 = -0.33579	Coupling	Jc_1 = 0.3333 Jc_2 = 0.2716 Jc_3 =-0.51851	Membrane J_M =-0.9999

4.1.9.3 The Buckling Values For The Case of CSCS Plate

As ealier explained, the lengthy mathematical expressions formulated were further collapsed into smaller fractions yielding the coefficients, which were considered as the unknown parameters. These were derived using the guass elimination method as expalained in chapter three. Tables 4.33, 4.34, 4.35, 4.36 and 4.37 show values of the unknown parameters and the buckling Load equations, for the case of CSCS plate,

Table 4.33 The Coefficicents of The Formulated Parameters for the CSCS plate

P_2	P_3	P_4	P_5	P_6
-1.71790018	-1.33466069	-0.72489	-0.87140285	
A_2	A_3	A_4	A_5	A_6
-1.71790018 A_1	-1.33466069 A_1	-0.72489 A_1	-0.87140285 A_1	

The Buckling Load Equation for CSCS at Orientattion of $0^0\ 0^0$

Table 4.34 The Buckling Load Equation for CSCS plate lamina 1

Ø = 0^0 CSCS K_T – Values				
$Kcscs_{T1}$	K_{T2}	K_{T3}	K_{T4}	K_{T5}

25.5196245	2517.2375	-5.97988635	0	0
$\frac{N_X}{D_O}$	106141.307			

Table 4.35 The Buckling Load Equation for CSCS plate lamina 2

Ø = 0^0 CSCS K_T – Values				
$Kcscs_{T1}$	K_{T2}	K_{T3}	K_{T4}	K_{T5}
				0
$\frac{N_X}{D_O}$				

The Buckling Load Equation for CSCS at Orientattion of 0^0 90^0

Table 4.36 The Buckling Load Equation for CSCS plate lamina 1

Ø = 0^0 CSCS K_T – Values				
$Kcscs_{T1}$	K_{T2}	K_{T3}	K_{T4}	K_{T5}
25.5196245	2517.2375	-5.97988635	0	0
$\frac{N_X}{D_O}$	106141.3037			

Table 4.37 The Buckling Load Equation for CSCS plate lamina 2

Ø = 90^0 CSCS K_T – Values				
$Kcscs_{T1}$	K_{T2}	K_{T3}	K_{T4}	K_{T5}
0.949074	113.2725	24.03654	0	0
$\frac{N_X}{D_O}$	5784.859			

The Buckling Load Equation for CSCS at Orientattion of 0^0 0^0 0^0

Table 4.38 The Buckling Load Equation for CSCS plate lamina 1

Ø = 0^0 CSCS K_T – Values				
$Kcscs_{T1}$	K_{T2}	K_{T3}	K_{T4}	K_{T5}
-153.038	-960.098	-263.35	0	0
$\frac{N_X}{D_O}$	-57593.6			

Table 4.39 The Buckling Load Equation for CSCS plate lamina 2

Ø = 0^0 CSCS K_T – Values				
$Kcscs_{T1}$	K_{T2}	K_{T3}	K_{T4}	K_{T5}
-2.76531	-2.80237	-0.92845	0	0

$\frac{N_X}{D_O}$	-271.805

Table 4.40 The Buckling Load Equation for CSCS plate lamina 3

Ø = 0^0 CSCS K_T – Values				
$Kcscs_{T1}$	K_{T2}	K_{T3}	K_{T4}	K_{T5}
-71.0673	-178.127	-203.917	0	0
$\frac{N_X}{D_O}$	18958.6			

4.1.9.4 Results of different aspect ratios for Two laminates of CSCS plate with 0^0 0^0 Arrangement.

On introduction of different aspect ratios, which ranges from 1 to 2 with arithmetic increase of 0.1, gave the results as detailed in Table 4.41

where

$$K_{cscs} = \left(\frac{K_{cscsT1} + K_{cscsT2} + K_{cscsT3} + K_{cscsT4} + K_{cscsT5}}{K_{cscsT6}}\right) \quad (4.34)$$

Table 4.41 Buckling results of different aspect ratios for CSCS plate with 0^0 0^0 arrangement

m = 1						m = 2				
$\beta = \frac{a}{b}$	$\frac{E_0}{D_0}.Kcscs$		1.5	57890.26		$\beta = \frac{a}{b}$	$\frac{E_0}{D_0}.Kcscs$		1.5	2646.98
1	23437.04		1.6	66406.28		1	1832.743		1.6	2925.146
1.1	29313.24		1.7	75487.72		1.1	1856.741		1.7	3224.44
1.2	35665.48		1.8	85136.94		1.2	1981.82		1.8	3543.681
1.3	42531.46		1.9	95354.81		1.3	2167.318		1.9	3882.445
1.4	49934.87		2.0	106141.3		1.4	2392.575		2.0	4240.683

4.1.9.5 Results of different aspect ratios for Two laminates of CSCS plate with 0^0 90^0 Arrangement.

On introduction of different aspect ratios, which ranges from 1 to 2 with arithmetic increase of 0.1, gave the results as detailed in Table 4.42

where

$$K_{cscs} = \left(\frac{K_{cscsT1} + K_{cscsT2} + K_{cscsT3} + K_{cscsT4} + K_{cscsT5}}{K_{cscsT6}}\right) \quad (4.35)$$

Table 4.42 Buckling results of different aspect ratios for CSCS plate with 0^0 90^0 arrangement

m = 1						m = 2				
$\beta = \frac{a}{b}$	$\frac{E_0}{D_0}.K_{cscs}$		1.5	57890.26		$\beta = \frac{a}{b}$	$\frac{E_0}{D_0}.K_{cscs}$		1.5	3211.923
1	23437.04		1.6	66406.28		1	14687.88		1.6	3297.836
1.1	29313.24		1.7	75487.72		1.1	9222.368		1.7	3658.217
1.2	35665.48		1.8	85136.94		1.2	6105.374		1.8	4221.755
1.3	42531.46		1.9	95354.81		1.3	4376.233		1.9	4941.316
1.4	49934.87		2.0	106141.3		1.4	3511.66		2.0	5784.857

4.1.9.6 Results of different aspect ratios for Three laminates of CSCS plate with 0^0 0^0 0^0 Arrangement.

On introduction of different aspect ratios, which ranges from 1 to 2 with arithmetic increase of 0.1, gave the results as detailed in Table 4.43

where

$$K_{cscs} = \left(\frac{K_{cscsT1} + K_{cscsT2} + K_{cscsT3} + K_{cscsT4} + K_{cscsT5}}{K_{T6}}\right) \quad (4.36)$$

Table 4.43a Buckling results of different aspect ratios for CSCS plate with 0^0 0^0 0^0 arrangement

m = 1						m = 2				
$\beta = \frac{a}{b}$	$\frac{E_0}{D_0}.K_{csc s}$		1.5	-103275		$\beta = \frac{a}{b}$	$\frac{E_0}{D_0}.K_{csc s}$		1.5	-388.58
1	-57593.6		1.6	-116193		1	-271.805		1.6	-423.75
1.1	-63610.2		1.7	-130147		1.1	-284.236		1.7	-461.864
1.2	-71456		1.8	-145100		1.2	-303.596		1.8	-502.794
1.3	-80802.6		1.9	-161022		1.3	-328.041		1.9	-546.462
1.4	-91451.1		2.0	-177894		1.4	-356.563		2.0	-592.82

Table 4.43b

m = 3				
$\beta = \frac{a}{b}$	$\frac{E_0}{D_0}.K_{csc s}$		1.5	14310.6
1	-18958.6		1.6	12894.8
1.1	-18095.8		1.7	-11180.8
1.2	-17289		1.8	-9132.46
1.3	16443.3		1.9	-6721.96
1.4	15474.2		2.0	-3928.38

4.1.10 The case of Two Combined CSSS with angles of 0^0 0^0

The results from the analysis when the laminates were positioned at the angles 0^0 0^0 are as presented in Table 4.44

Table 4.44 The Bending, Coupling and Membrane stiffness for the CSSS at 0^0 0^0 angle

Lamina, m = 1

Bending	J_1 = 0.08333 J_2 = 0.06667 J_3 = 0.05397 J_4 = 0.53333	Coupling	Jc_1 = -0.125 Jc_2 = -0.20833 Jc_3 = 0.6666	Membrane J_M =1
Lamina, m = 2				
Bending	J_1 = 0.08333 J_2 = 0.06667 J_3 = 0.05397 J_4 = 0.53333	Coupling	Jc_1 = 0.125 Jc_2 = 0.20833 Jc_3 = 0.66667	Membrane, J_M = 1

4.1.11 The case of Two combined CSSS plate layer at angle 0^0 90^0

The formulated values from the analysis when the laminates were positioned at the angles 0^0 90^0 are as presented in Table 4.45

Table 4.45 The Bending, Coupling and Membrane stiffness for the CSSS plate with 0^0 90^0

Lamina, m = 1				
Bending	J_1 = 0.08333 J_2 = 0.06667 J_3 = 0.05397 J_4 = 0.53333	Coupling	Jc_1 = -0.125 Jc_2 = -0.20833 Jc_3 = 0.6666	Membrane J_M =1
Lamina, m = 2				
Bending	J_1 = 0.08333 J_2 = 0.06667 J_3 = 0.05397 J_4 = 0.53333	Coupling	Jc_1 = 0.125 Jc_2 = 0.20833 Jc_3 = 0.66667	Membrane, J_M = 1

4.1.11.1 Results of Three laminated Clamped Simple Simple Simple Thick Plate

The case of Three Combined CSSS with angles of 0^0 0^0 0^0 . The formulated values from the analysis when the layer orientation is 0^0 0^0 $0^0$are as presented in Table 4.46

Table 4.46 The Bending, Coupling and Membrane stiffness for the CSSS plate with 0^0 0^0 0^0

Lamina, m = 1				
Bending	J_1 =-0.12037 J_2 = -0.09547 J_3 =-0.07653 J_4 =-0.3358	Coupling	Jc_1 = -0.33333 Jc_2 = -0.2716 Jc_3 =-0.51852	Membrane J_M =-1
Lamina, m = 2				
Bending	J_1 =-0.00926 J_2 = -0.00905 J_3 =-0.00885 J_4 = -0.92841	Coupling	Jc_1 =0.0000 Jc_2 = 0.0000 Jc_3 =-0.96298	Membrane, J_M =-1.0002
Lamina, m = 3				
Bending	J_1 = -0.12037 J_2 = -0.09547 J_3 = -0.07653 J_4 = -0.33579	Coupling	Jc_1 = 0.3333 Jc_2 = 0.2716 Jc_3 =-0.51851	Membrane J_M =-0.9999

4.1.11.2 The Buckling Values For The Case of CSSS Plate

As ealier explained, the lengthy mathematical expressions formulated were further collapsed into smaller fractions yielding the coefficients, which were considered as the unknown parameters. These were derived using the guass elimination method as expalained in chapter three. Tables 4.47, 4.48, 4.49 4.50 and 4.51 show the values of the unknown parameters and the buckling Load equations respectively, for the case of CSSS plate,

Table 4.47 The Coefficicents of The Formulated Parameters for the CSSS plate

P_2	P_3	P_4	P_5	P_6
-1.70516953	-0.65796316	-0.39344	-0.74941697	
A_2	A_3	A_4	A_5	A_6
-1.70516953 A_1	-0.65796316 A_1	-0.39344 A_1	-0.74941697A_1	

The Buckling Load Equation for CSSS at Orientattion of 0^0 0^0

Table 4.48 The Buckling Load Equation for CSSS plate Lamina 1

Ø = 0^0 CSSS K_T – Values				
K_{csssT1}	K_{csssT2}	K_{csssT3}	K_{csssT4}	K_{csssT5}

2.23119684	566.673229	-45.2604353	0	0
$\frac{N_X}{D_O}$	21909.7904			

Table 4.49 The Buckling Load Equation for CSSS plate Lamina 2

Ø = 0^0 CSSS K_T – Values				
K_{csssT1}	K_{csssT2}	K_{csssT3}	K_{csssT4}	K_{csssT5}
17.13819	39.96587	10.46921	0	0
$\frac{N_X}{D_O}$	2827.333			

The Buckling Load Equation for CSSS at Orientattion of 0^0 90^0

Table 4.50 The Buckling Load Equation for CSSS plate Lamina 1

Ø = 0^0 CSSS K_T – Values				
K_{csssT1}	K_{csssT2}	K_{csssT3}	K_{csssT4}	K_{csssT5}
2.23119684	566.673229	-45.2604353	0	0
$\frac{N_X}{D_O}$	21909.7904			

Table 4.51 The Buckling Load Equation for CSSS plate Lamina 2

Ø = 90^0 CSSS K_T – Values				
K_{csssT1}	K_{csssT2}	K_{csssT3}	K_{csssT4}	K_{csssT5}
0.737247	-33.3125	530.5905	0	0
$\frac{N_X}{D_O}$	20837.45			

The Buckling Load Equation for CSSS at Orientattion of 0^0 0^0 0^0

Table 4.52 The Buckling Load Equation for CSSS plate Lamina 1

Ø = 0^0 CSSS K_T – Values				
K_{csssT1}	K_{csssT2}	K_{csssT3}	K_{csssT4}	K_{csssT5}
-203.367	-920.297	-249.569	0	0
$\frac{N_X}{D_O}$	-57457.4			

Table 4.53 The Buckling Load Equation for CSSS plate lamina 2

Ø = 0^0 CSSS K_T – Values

K_{csssT1}	K_{csssT2}	K_{csssT3}	K_{csssT4}	K_{csssT5}
-2.77161	-2.81097	-0.98294	0	0
$\frac{N_X}{D_O}$	-274.708			

Table 4.54 The Buckling Load Equation for CSSS plate lamina 3

$\varnothing = 0^0$ CSSS K_T – Values				
K_{csssT1}	K_{csssT2}	K_{csssT3}	K_{csssT4}	K_{csssT5}
-89.8141	-164.077	-443.437	0	0
$\frac{N_X}{D_O}$	-29176.9			

4.1.11.3 Results of different aspect ratios for Two laminates of CSSS plate with $0^0\,0^0$ Arrangement.

On introduction of different aspect ratios, which ranges from 1 to 2 with arithmetic increase of 0.1, gave the results as detailed in Table 4.55

where

$$K_{csss} = \left(\frac{K_{csssT1} + K_{csssT2} + K_{csssT3} + K_{csssT4} + K_{csssT5}}{K_{csssT6}}\right) \quad (4.38)$$

Table 4.55 Buckling results of different aspect ratios for CSSS plate with $0^0\,0^0$ arrangement

m = 1						m = 2				
$\beta = \frac{a}{b}$	$\frac{E_0}{D_0}.K_{csssc}$		1.5	54489.03		$\beta = \frac{a}{b}$	$\frac{E_0}{D_0}.K_{csssc}$		1.5	4134.957
1	21909.79		1.6	62484.22		1	2827.333		1.6	4495.923
1.1	27498.82		1.7	70992.39		1.1	2970.348		1.7	4875
1.2	33523.84		1.8	80015.84		1.2	3199.07		1.8	5271.582
1.3	40018.4		1.9	89555.81		1.3	3794.623		1.9	5684.823
1.4	47002.78		2.0	99612.99		1.4	3794.623		2.0	6114.86

4.1.11.4 Results of different aspect ratios for Two laminates of CSSS plate with 90^0

0^0 Arrangement.

On introduction of different aspect ratios, which ranges from 1 to 2 with arithmetic increase of 0.1, gave the results as detailed in Table 4.56

where

$$K_{cscs} = \left(\frac{K_{csssT1} + K_{csssT2} + K_{csssT3} + K_{csssT4} + K_{csssT5}}{K_{csssT6}}\right) \quad (4.39)$$

Table 4.56 Buckling results of different aspect ratios for CSSS Splate with 0^0 90^0 arrangement.

m = 1						m = 2				
$\beta = \frac{a}{b}$	$\frac{E_0}{D_0}.K_{csssc}$		1.5	54489.03		$\beta = \frac{a}{b}$	$\frac{E_0}{D_0}.K_{csssc}$		1.5	54489.03
1	21909.79		1.6	62484.22		1	20837.45		1.6	6181.931
1.1	27498.82		1.7	70992.39		1.1	14097.3		1.7	6390.926
1.2	33523.84		1.8	80015.84		1.2	10225.16		1.8	6824.337
1.3	40018.4		1.9	89555.81		1.3	8014.104		1.9	7427.155
1.4	47002.78		2.0	99612.99		1.4	6820.436		2.0	8162.242

4.1.11.5 Results of different aspect ratios for Three laminates of CSSS plate with 0^0 0^0 0^0 Arrangement.

A structure with different laminas was treated considering the layers in compression and

On introduction of different aspect ratios, which ranges from 1 to 2 with arithmetic increase of 0.1, gave the results as detailed in Table 4.57

where

$$K_{csss} = \left(\frac{K_{csssT1} + K_{csssT2} + K_{csssT3} + K_{csssT4} + K_{csssT5}}{K_{csssT6}}\right) \quad (4.40)$$

Table 4.57a Buckling results of different aspect ratios for CSSS plate with 0^0 0^0 0^0

Arrangement.

m = 1						m = 2				
$\beta = \frac{a}{b}$	$\frac{E_0}{D_0}.K_{csssc}$		1.5	-100325		$\beta = \frac{a}{b}$	$\frac{E_0}{D_0}.K_{csssc}$		1.5	-389.978
1	-57457.4		1.6	-112491		1	-274.708		1.6	-425.121
1.1	-63033.2		1.7	-125636		1.1	-286.468		1.7	-463.236
1.2	-70383.3		1.8	-139724		1.2	-305.43		1.8	-504.188
1.3	-79169.3		1.9	154729		1.3	-329.642		1.9	-547.89
1.4	-79169.3		2.0	-170632		1.4	-358.029		2.0	-594.291

Table 4.57b.

m = 3				
$\beta = \frac{a}{b}$	$\frac{E_0}{D_0}.K_{csssc}$		1.5	-11181.3
1	-29176.9		1.6	-6659.44
1.1	-25620.2		1.7	-1548.33
1.2	-22254.8		1.8	4221.61
1.3	-18846		1.9	10703.2
1.4	-15204.6		2.0	17935.9

4.1.12 Detailed Output For CCSS Thick Laminated Anisotropic Plate.

For different lamina combination for CCSS plate, the results of the analysis are as detailed in this section.

4.1.12.1 The case of Two Combined CCSS with angles of 0^0 0^0

The results from the analysis when the laminates were positioned at the angles 0^0 0^0 are as presented in Table 4.58

Table 4.58 The Bending, Coupling and Membrane stiffness for the CCSS at 0^0 0^0 angle

Lamina, m = 1				
Bending	J_1 = 0.08333 J_2 = 0.06667 J_3 = 0.05397 J_4 = 0.53333	Coupling	Jc_1 = -0.125 Jc_2 = -0.20833 Jc_3 = 0.6666	Membrane J_M =1
Lamina, m = 2				
Bending	J_1 = 0.08333 J_2 = 0.06667 J_3 = 0.05397 J_4 = 0.53333	Coupling	Jc_1 = 0.125 Jc_2 = 0.20833 Jc_3 = 0.66667	Membrane, J_M = 1

4.1.12.2 The case of Two combined CCSS plate layer at angle 0^0 90^0

The formulated values from the analysis when the laminates were positioned at the angles 0^0 90^0 are as presented in Table 4.59

Table 4.59 The Bending, Coupling and Membrane stiffness for the CCSS plate with 0^0 90^0

Lamina, m = 1				
Bending	J_1 = 0.08333 J_2 = 0.06667 J_3 = 0.05397 J_4 = 0.53333	Coupling	Jc_1 = -0.125 Jc_2 = -0.20833 Jc_3 = 0.6666	Membrane J_M =1
Lamina, m = 2				
Bending	J_1 = 0.08333 J_2 = 0.06667 J_3 = 0.05397 J_4 = 0.53333	Coupling	Jc_1 = 0.125 Jc_2 = 0.20833 Jc_3 = 0.66667	Membrane, J_M = 1

4.1.12.3 Results of Three laminated Clamped Clamped Simple Simple Thick Plate

The case of Three Combined CCSS with angles of 0^0 0^0 0^0 .The formulated values from the analysis when the layer orientation is 0^0 0^0 0^0 are as presented in Table 4.60

Table 4.60 The Bending, Coupling and Membrane stiffness for the CCSS plate with 0^0 0^0 0^0

Lamina, m = 1				
Bending	J_1 =-0.12037 J_2 = -0.09547 J_3 =-0.07653 J_4 =-0.3358	Coupling	Jc_1 = -0.33333 Jc_2 = -0.2716 Jc_3 =-0.51852	Membrane J_M =-1
Lamina, m = 2				
Bending	J_1 =-0.00926 J_2 = -0.00905 J_3 =-0.00885 J_4 = -0.92841	Coupling	Jc_1 =0.0000 Jc_2 = 0.0000 Jc_3 =-0.96298	Membrane, J_M =-1.0002
Lamina, m = 3				
Bending	J_1 = -0.12037 J_2 = -0.09547 J_3 = -0.07653 J_4 = -0.33579	Coupling	Jc_1 = 0.3333 Jc_2 = 0.2716 Jc_3 =-0.51851	Membrane J_M =-0.9999

4.1.12.4 The Buckling Values For The Case of CCSS Plate

As ealier explained, the lengthy mathematical expressions formulated were further collapsed into smaller fractions yielding the coefficients, which were considered as the unknown parameters. These were derived using the guass elimination method as expalained in chapter three. Tables 4.61, 4.62, 4.63, 4.64, 4.65 and 4.66 show the values of the unknown parameters and the buckling Load equations, for the case of CCSS plate,

Table 4.61 The Coefficicents of The Formulated Parameters for the CSSS plate

P_2	P_3	P_4	P_5	P_6
-1.1557213	17.1059846	0.02912	0.09553618	
A_2	A_3	A_4	A_5	A_6
-1.1557213 A_1	17.1059846 A_1	0.02912 A_1	0.09553618 A_1	

The Buckling Load Equation for CCSS at Orientation of 0^0 0^0

Table 4.62 The Buckling Load Equation for CCSS plate Lamina 1

Ø = 0^0 CCSS K_T – Values				
K_{ccssT1}	K_{ccssT2}	K_{ccssT3}	K_{ccssT4}	K_{ccssT5}
19.1684919	-4887.501	226.21473	0	0
$\frac{N_X}{D_O}$	-194230.87			

Table 4.63 The Buckling Load Equation for CCSS plate Lamina 2

Ø = 0^0 CCSS K_T – Values				
K_{ccssT1}	K_{ccssT2}	K_{ccssT3}	K_{ccssT4}	K_{ccssT5}
28.5734	-1143.16	-52.2499	0	0
$\frac{N_X}{D_O}$	-48821.8			

The Buckling Load Equation for CCSS at Orientation of 0^0 90^0

Table 4.64 The Buckling Load Equation for CCSS plate Lamina 1

Ø = 0^0 CCSS K_T – Values				
K_{ccssT1}	K_{ccssT2}	K_{ccssT3}	K_{ccssT4}	K_{ccssT5}
19.1684919	-4887.501	226.21473	0	0
$\frac{N_X}{D_O}$	-194230.87			

Table 4.65 The Buckling Load Equation for CCSS plate Lamina 2

Ø = 90^0 CCSS K_T – Values				
K_{ccssT1}	K_{ccssT2}	K_{ccssT3}	K_{ccssT4}	K_{ccssT5}
1.01412	384.364	111.683	0	0
$\frac{N_X}{D_O}$	20797.5			

The Buckling Load Equation for CCSS at Orientattion of 0^0 0^0 0^0

Table 4.66 The Buckling Load Equation for CCSS plate Lamina 1

Ø = 0^0 CSSS K_T – Values				
K_{csssT1}	K_{csssT2}	K_{csssT3}	K_{csssT4}	K_{csssT5}
-25.1863	-486.795	1667.89	0	0
$\frac{N_X}{D_O}$	48364.2			

Table 4.67 The Buckling Load Equation for CCSS plate Lamina 2

Ø = 0^0 CSSS K_T – Values				
K_{csssT1}	K_{csssT2}	K_{csssT3}	K_{csssT4}	K_{csssT5}
-2.77896	-3.23106	-2.3378	0	0
$\frac{N_X}{D_O}$	-349.282			

Table 4.68 The Buckling Load Equation for CCSS plate Lamina 3

Ø = 0^0 CSSS K_T – Values				
K_{csssT1}	K_{csssT2}	K_{csssT3}	K_{csssT4}	K_{csssT5}
-46.626	-484.331	867.593	0	0
$\frac{N_X}{D_O}$	14085.2			

4.1.12.5 Results of different aspect ratios for Two laminates of CCSS plate with 0^0 0^0 Arrangement.

On introduction of different aspect ratios, which ranges from 1 to 2 with arithmetic increase of 0.1, gave the results as detailed in Table 4.69.

Where

$$K_{ccss} = \left(\frac{K_{ccssT1} + K_{ccssT2} + K_{ccssT3} + K_{ccssT4} + K_{ccssT5}}{K_{csssT6}}\right) \quad (4.41)$$

Table 4.69 Buckling results of different aspect ratios for CCSS plate with 0^0 0^0 arrangement.

m = 1						m = 2				
$\beta = \frac{a}{b}$	$\frac{E_0}{D_0}.K_{ccss}$		1.5	-34775.3		$\beta = \frac{a}{b}$	$\frac{E_0}{D_0}.K_{ccss}$		1.5	1248.71
1	37522.5		1.6	-58262.1		1	17134.2		1.6	-4689.39
1.1	27405.7		1.7	-85698.6		1.1	14111.1		1.7	-12436.1
1.2	15687.5		1.8	-117363		1.2	11622.3		1.8	-22244.2
1.3	1764.66		1.9	-153482		1.3	8995.02		1.9	-34318.6
1.4	-14900.1		2.0	-194231		1.4	5686.29		2.0	-48821.8

4.1.12.6 Results of different aspect ratios for Two laminates of CCSS plate with 90^0 0^0 Arrangement.

On introduction of different aspect ratios, which ranges from 1 to 2 with arithmetic increase of 0.1, gave the results as detailed in Table 4.70

where

$$K_{ccss} = \left(\frac{K_{ccssT1} + K_{ccssT2} + K_{ccssT3} + K_{ccssT4} + K_{ccssT5}}{K_{ccssT6}}\right) \quad (4.42)$$

Table 4.70 Buckling results of different aspect ratios for CCSS Splate with 0^0 90^0 arrangement

m = 1						m = 2				
$\beta = \frac{a}{b}$	$\frac{E_0}{D_0}.K_{CSSSc}$		1.5	-34775.3		$\beta = \frac{a}{b}$	$\frac{E_0}{D_0}.K_{CSSSc}$		1.5	3782.97
1	37522.5		1.6	-58262.1		1	1681235		1.6	9174.53
1.1	27405.7		1.7	-85698.6		1.1	-309217		1.7	12801.7
1.2	15687.5		1.8	-117363		1.2	-78054.6		1.8	15666.8
1.3	1764.66		1.9	-153482		1.3	-25555		1.9	18249.6
1.4	1764.66		2.0	-194231		1.4	-5696.46		2.0	20797.5

4.1.12.7 Results of different aspect ratios for Three laminates of CCSS plate with 0^0 0^0 0^0 Arrangement.

On introduction of different aspect ratios, which ranges from 1 to 2 with arithmetic increase of 0.1, gave the results as detailed in Table 4.71

where

$$K_{ccss} = \left(\frac{K_{ccssT1} + K_{ccssT2} + K_{ccssT3} + K_{ccssT4} + K_{ccssT5}}{K_{ccssT6}}\right) \tag{4.43}$$

Table 4.71a Buckling results of different aspect ratios for CCSS plate with 0^0 0^0 0^0

arrangement

m = 1						m = 2				
$\beta = \frac{a}{b}$	$\frac{E_0}{D_0}.K_{ccss}$		1.5	19784.2		$\beta = \frac{a}{b}$	$\frac{E_0}{D_0}.K_{ccss}$		1.5	-440.674
1	48364.2		1.6	22067.5		1	-349.282		1.6	-478.492
1.1	36922.5		1.7	26617.6		1.1	-346.769		1.7	-520.241
1.2	28340		1.8	33286.2		1.2	-358.364		1.8	-565.56
1.3	22725.8		1.9	41946.1		1.3	-379.415		1.9	-614.21
1.4	19939.5		2.0	52487.6		1.4	-407.349		2.0	-666.03

Table 4.71b

m = 3				
$\beta = \frac{a}{b}$	$\frac{E_0}{D_0}.K_{ccss}$		1.5	-19481.5
1	14085.2		1.6	-10329.1
1.1	-4067.3		1.7	4185.66
1.2	-15937.4		1.8	24261.1
1.3	-22280.8		1.9	50096.9
1.4	-23437.6		2.0	81874.8

4.1.13 Detailed Output For SCCS Thick Laminated Anisotropic Plate.

For different lamina combination for SCCS plate, the results of the analysis are as detailed in this section.

4.1.13.1 The case of Two Combined SCCS with angles of 0^0 0^0

The results from the analysis when the laminates were positioned at the angles 0^0 0^0 are as presented in Table 4.72

Table 4.72 The Bending, Coupling and Membrane stiffness for the SCCS at 0^0 0^0 angle

Lamina, m = 1				
Bending	$J_1 = 0.08333$ $J_2 = 0.06667$ $J_3 = 0.05397$ $J_4 = 0.53333$	Coupling	$Jc_1 = -0.125$ $Jc_2 = -0.20833$ $Jc_3 = 0.6666$	Membrane $J_M = 1$
Lamina, m = 2				
Bending	$J_1 = 0.08333$ $J_2 = 0.06667$ $J_3 = 0.05397$ $J_4 = 0.53333$	Coupling	$Jc_1 = 0.125$ $Jc_2 = 0.20833$ $Jc_3 = 0.66667$	Membrane, $J_M = 1$

4.1.13. 2 The case of Two combined SCCS plate layer at angle 0^0 90^0

The formulated values from the analysis when the laminates were positioned at the angles 0^0 90^0 are as presented in Table 4.73

Table 4.73 The Bending, Coupling and Membrane stiffness for the SCCS plate with 0^0 90^0

Lamina, m = 1				
Bending	$J_1 = 0.08333$ $J_2 = 0.06667$ $J_3 = 0.05397$ $J_4 = 0.53333$	Coupling	$Jc_1 = -0.125$ $Jc_2 = -0.20833$ $Jc_3 = 0.6666$	Membrane $J_M = 1$
Lamina, m = 2				
Bending	$J_1 = 0.08333$ $J_2 = 0.06667$ $J_3 = 0.05397$ $J_4 = 0.53333$	Coupling	$Jc_1 = 0.125$ $Jc_2 = 0.20833$ $Jc_3 = 0.66667$	Membrane, $J_M = 1$

4.1.13.3 Results of Three laminated Simple Clamped Clamped Simple Thick Plate

The case of Three Combined SCCS with angles of 0^0 0^0 0^0**.** The formulated values from the analysis when the layer orientation is 0^0 0^0 0^0 are as presented in Table 4.74

Table 4.74 The Bending, Coupling and Membrane stiffness for the SCCS plate with 0^0 0^0 0^0

Lamina, m = 1				
Bending	J_1 =-0.12037 J_2 = -0.09547 J_3 =-0.07653 J_4 =-0.3358	Coupling	Jc_1 = -0.33333 Jc_2 = -0.2716 Jc_3 =-0.51852	Membrane J_M =-1
Lamina, m = 2				
Bending	J_1 =-0.00926 J_2 = -0.00905 J_3 =-0.00885 J_4 = -0.92841	Coupling	Jc_1 =0.0000 Jc_2 = 0.0000 Jc_3 =-0.96298	Membrane, J_M =-1.0002
Lamina, m = 3				
Bending	J_1 = -0.12037 J_2 = -0.09547 J_3 = -0.07653 J_4 = -0.33579	Coupling	Jc_1 = 0.3333 Jc_2 = 0.2716 Jc_3 =-0.51851	Membrane J_M =-0.9999

4.1.13.4 The Buckling Values For The Case of SCCS Plate

As ealier explained, the lengthy mathematical expressions formulated were further collapsed into smaller fractions yielding the coefficients, which were considered as the unknown parameters. These were derived using the guass elimination method as expalained in chapter three. Tables 4.75, 4.76, 4.77, 4.78 and 4.79 shows values of the unknown parameters and the buckling Load equations respectively, for the case of SCCS plate,

Table 4.75 The Coefficicents of The Formulated Parameters for the SCCS plate

P_2	P_3	P_4	P_5	P_6
-2.087033	-0.8435068	-0.46911	0.49518372	
A_2	A_3	A_4	A_5	A_6
-2.087033 A_1	-0.8435068 A_1	-0.46911 A_1	0.49518372 A_1	

Buckling Load Equation for SCCS at Orientattion of 0^0 0^0

Table 4.76 The Buckling Load Equation for SCCS plate Lamina 1

Ø = 0^0 SCCS K_T – Values				
K_{sccsT1}	K_{sccsT2}	K_{sccsT3}	K_{sccsT4}	K_{sccsT5}
10.152872	729.891629	-55.541193	0	0
$\frac{N_X}{D_O}$	28640.3058			

Table 4.77 The Buckling Load Equation for SCCS plate Lamina 2

$Ø = 0^0$ SCCS K_T – Values				
K_{sccsT1}	K_{sccsT2}	K_{sccsT3}	K_{sccsT4}	K_{sccsT5}
36.3161	125.938	25.3229	0	0
$\frac{N_X}{D_O}$	7848.41			

The Buckling Load Equation for SCCS at Orientattion of 0^0 90^0

Table 4.78 The Buckling Load Equation for SCCS plate Lamina 1

$Ø = 0^0$ CCSS K_T – Values				
K_{sccsT1}	K_{sccsT2}	K_{sccsT3}	K_{sccsT4}	K_{sccsT5}
10.152872	729.891629	-55.541193	0	0
$\frac{N_X}{D_O}$	28640.3058			

Table 4.79 The Buckling Load Equation for SCCS plate Lamina 2

$Ø = 90^0$ CCSS K_T – Values				
K_{sccsT1}	K_{sccsT2}	K_{sccsT3}	K_{sccsT4}	K_{sccsT5}
1.23766	106.487	1298.29	0	0
$\frac{N_X}{D_O}$	58828.9			

The Buckling Load Equation for SCCS at Orientattion of 0^0 0^0 0^0

Table 4.80 The Buckling Load Equation for SCCS plate Lamina 1

$Ø = 0^0$ SCCS K_T – Values				
K_{sccsT1}	K_{sccsT2}	K_{sccsT3}	K_{sccsT4}	K_{sccsT5}
-4.06398	-1019.87	-310.184	0	0
$\frac{N_X}{D_O}$	-55820.7			

Table 4.81 The Buckling Load Equation for SCCS plate Lamina 2

$Ø = 0^0$ SCCS K_T – Values				
K_{sccsT1}	K_{sccsT2}	K_{sccsT3}	K_{sccsT4}	K_{sccsT5}
-2.82067	-3.94909	-2.44069	0	0
$\frac{N_X}{D_O}$	-385.375			

Table 4.82 The Buckling Load Equation for SCCS plate Lamina 3

Ø = 0^0 SCCS K_T – Values				
K_{sccsT1}	K_{sccsT2}	K_{sccsT3}	K_{sccsT4}	K_{sccsT5}
-65.6531	-309.749	15.8151	0	0
$\frac{N_X}{D_O}$	-15045.5			

4.1.13.5 Results of different aspect ratios for Two laminates of SCCS plate with 0^0 0^0 Arrangement.

On introduction of different aspect ratios, which ranges from 1 to 2 with arithmetic increase of 0.1, gave the results as detailed in Table 4.83

Where

$$K_{sccs} = \left(\frac{K_{sccsT1} + K_{sccsT2} + K_{sccsT3} + K_{sccsT4} + K_{sccsT5}}{K_{sccsT6}}\right) \quad (4.44)$$

Table 4.83 Buckling results of different aspect ratios for SCCS plate with 0^0 0^0 arrangement

m = 1						m = 2				
$\beta = \frac{a}{b}$	$\frac{E_0}{D_0}.K_{csssc}$		1.5	70548.5		$\beta = \frac{a}{b}$	$\frac{E_0}{D_0}.K_{csssc}$		1.5	12096.6
1	28640.3		1.6	81003.5		1	7848.41		1.6	13340.4
1.1	35732.3		1.7	92194.6		1.1	8301.63		1.7	14671.8
1.2	43433.8		1.8	104134		1.2	9017.38		1.8	14671.8
1.3	51784.7		1.9	116830		1.3	9914.79		1.9	17576.1
1.4	60815.1		2.0	130291		1.4	10949.6		2.0	19142.7

4.1.13.6 Results of different aspect ratios for Two laminates of SCCS plate with 0^0 90^0 Arrangement.

On introduction of different aspect ratios, which ranges from 1 to 2 with arithmetic increase of 0.1, gave the results as detailed in Table 4.84

where

$$K_{sccs} = \left(\frac{K_{sccsT1} + K_{sccsT2} + K_{sccsT3} + K_{sccsT4} + K_{sccsT5}}{K_{sccsT6}}\right) \quad (4.45)$$

Table 4.84 Buckling results of different aspect ratios for SCCS Splate with 0^0 90^0 arrangement

m = 1						m = 2				
$\beta = \frac{a}{b}$	$\frac{E_0}{D_0}.K_{csssc}$		1.5	70548.5		$\beta = \frac{a}{b}$	$\frac{E_0}{D_0}.K_{csssc}$		1.5	19573.5
1	28640.3		1.6	81003.5		1	41717.7		1.6	18519.5
1.1	35732.3		1.7	92194.6		1.1	41717.7		1.7	18203.5
1.2	43433.8		1.8	104134		1.2	31538.7		1.8	18421.1
1.3	51784.7		1.9	116830		1.3	25381.1		1.9	19038.4
1.4	60815.1		2.0	130291		1.4	21686.6		2.0	19965

4.1.13.7 Results of different aspect ratios for Three laminates of SCCS plate with 0^0 0^0 0^0 Arrangement.

On introduction of different aspect ratios, which ranges from 1 to 2 with arithmetic increase of 0.1, gave the results as detailed in Table 4.85

where

$$K_{sccs} = \left(\frac{K_{sccsT1} + K_{sccsT2} + K_{sccsT3} + K_{sccsT4} + K_{sccsT5}}{K_{sccsT6}}\right) \quad (4.46)$$

Table 4.85a Buckling results of different aspect ratios for SCCS plate with 0^0 0^0 0^0 arrangement

m = 1						m = 2				
$\beta = \frac{a}{b}$	$\frac{E_0}{D_0}.K_{sccs}$		1.5	-99456.7		$\beta = \frac{a}{b}$	$\frac{E_0}{D_0}.K_{sccs}$		1.5	-509.734
1	-55820.7		1.6	-112507		1	-385.375		1.6	-556.941
1.1	-60619.7		1.7	-126660		1.1	-387.049		1.7	-608.869
1.2	-67887.1		1.8	-141871		1.2	-404.325		1.8	-665.143
1.3	-76989.1		1.9	-158115		1.3	-432.167		1.9	-725.519
1.4	-87578.4		2.0	-175374		1.4	-467.828		2.0	-789.83

Table 4.85b

m = 3				
$\beta = \frac{a}{b}$	$\frac{E_0}{D_0}.K_{sccs}$		1.5	-28356.3
1	-15045.5		1.6	-30613.2
1.1	18439.9		1.7	-32848.3
1.2	-21221.7		1.8	-35062
1.3	-23712.7		1.9	-37250
1.4	-26066.7		2.0	-39405.7

4.1.13.8 Detailed Output For SCSC Thick Laminated Anisotropic Plate.

For different lamina combination for SCSC plate, the results of the analysis are as detailed in this section.

4.1.13.9 The case of Two Combined SCSC with angles of 0^0 0^0

The results from the analysis when the laminates were positioned at the angles 0^0 0^0 are as presented in Table 4.86

Table 4.86 The Bending, Coupling and Membrane stiffness for the SCSC at 0^0 0^0 angle

Lamina, m = 1				
Bending	$J_1 = 0.08333$ $J_2 = 0.06667$ $J_3 = 0.05397$ $J_4 = 0.53333$	Coupling	$Jc_1 = -0.125$ $Jc_2 = -0.20833$ $Jc_3 = 0.6666$	Membrane $J_M = 1$
Lamina, m = 2				
Bending	$J_1 = 0.08333$ $J_2 = 0.06667$ $J_3 = 0.05397$ $J_4 = 0.53333$	Coupling	$Jc_1 = 0.125$ $Jc_2 = 0.20833$ $Jc_3 = 0.66667$	Membrane, $J_M = 1$

4.1.14 The case of Two combined SCSC plate layer at angle 0^0 90^0

The formulated values from the analysis when the laminates were positioned at the angles 0^0 90^0 are as presented in Table 4.87

Table 4.87 The Bending, Coupling and Membrane stiffness for the SCSC plate with **0^0 90^0**

Lamina, m = 1				
Bending	$J_1 = 0.08333$ $J_2 = 0.06667$ $J_3 = 0.05397$ $J_4 = 0.53333$	Coupling	$Jc_1 = -0.125$ $Jc_2 = -0.20833$ $Jc_3 = 0.6666$	Membrane $J_M = 1$
Lamina, m = 2				
Bending	$J_1 = 0.08333$ $J_2 = 0.06667$ $J_3 = 0.05397$ $J_4 = 0.53333$	Coupling	$Jc_1 = 0.125$ $Jc_2 = 0.20833$ $Jc_3 = 0.66667$	Membrane, $J_M = 1$

4.1.14.1 Results of Three laminated Simple Clamped Simple Clamped Thick Plate

The case of Three Combined SCSC with angles of 0^0 0^0 0^0 · The formulated values from the analysis when the layer orientation is 0^0 0^0 0^0

are as presented in Table 4.88.

Table 4.88 The Bending, Coupling and Membrane stiffness for the SCSC plate with $0^0\ 0^0\ 0^0$

Lamina, m = 1				
Bending	J_1 =-0.12037 J_2 = -0.09547 J_3 =-0.07653 J_4 =-0.3358	Coupling	Jc_1 = -0.33333 Jc_2 = -0.2716 Jc_3 =-0.51852	Membrane J_M =-1
Lamina, m = 2				
Bending	J_1 =-0.00926 J_2 = -0.00905 J_3 =-0.00885 J_4 = -0.92841	Coupling	Jc_1 =0.0000 Jc_2 = 0.0000 Jc_3 =-0.96298	Membrane, J_M =-1.0002
Lamina, m = 3				
Bending	J_1 = -0.12037 J_2 = -0.09547 J_3 = -0.07653 J_4 = -0.33579	Coupling	Jc_1 = 0.3333 Jc_2 = 0.2716 Jc_3 =-0.51851	Membrane J_M =-0.9999

Table 4.90 The Buckling Load Equation for SCSC plate Lamina 1

Ø = 0^0 SCSC K_T – Values				
K_{scscT1}	K_{scscT2}	K_{scscT3}	K_{scscT4}	K_{scscT5}
-2.77527	-3.53329	-6.31396	0	0
$\frac{N_X}{D_O}$	-528.138			

Table 4.91 The Buckling Load Equation for SCSC plate Lamina 2

Ø = 90^0 SCSC K_T – Values				
K_{scscT1}	K_{scscT2}	K_{scscT3}	K_{scscT4}	K_{scscT5}
0.39314	-1355.02	3530.65	0	0
$\frac{N_X}{D_O}$	91047.3			

The Buckling Load Equation for SCSC at Orientattion of $0^0\ 0^0\ 0^0$

Table 4.92 The Buckling Load Equation for SCSC Plate Lamina 1

Ø = 0^0 SCSC K_T – Values				
K_{scscT1}	K_{scscT2}	K_{scscT3}	K_{scscT4}	K_{scscT5}
-63.2779	-1020.75	-820.003	0	0
$\frac{N_X}{D_O}$	-79666.7			

Table 4.93 The Buckling Load Equation for SCSC Plate Lamina 2

Ø = 0^0 SCSC K_T – Values				
K_{scscT1}	K_{scscT2}	K_{scscT3}	K_{scscT4}	K_{scscT5}
-2.77527	-3.53329	-6.31396	0	0
$\frac{N_x}{D_O}$	-528.138			

Table 4.94 The Buckling Load Equation for SCSC Plate Lamina 3

Ø = 0^0 SCSC K_T – Values				
K_{scscT1}	K_{scscT2}	K_{scscT3}	K_{scscT4}	K_{scscT5}
-2.77527	-3.53326	-3.53326	0	0
$\frac{N_x}{D_O}$	-528.138			

4.1.14.2 Results of different aspect ratios for Two laminates of SCSC plate with 0^0 0^0 Arrangement.

On introduction of different aspect ratios, which ranges from 1 to 2 with arithmetic increase of 0.1, gave the results as detailed in Table 4.95

where

$$K_{scsc} = \left(\frac{K_{scscT1} + K_{scscT2} + K_{scscT3} + K_{scscT4} + K_{scscT5}}{K_{scscT6}}\right) \quad (4.47)$$

Table 4.95 Buckling results of different aspect ratios for SCSC plate with 0^0 0^0 arrangement

m = 1						m = 2				
$\beta = \frac{a}{b}$	$\frac{E_0}{D_0}.K_{csssc}$		1.5	66021.5		$\beta = \frac{a}{b}$	$\frac{E_0}{D_0}.K_{csssc}$		1.5	6663.76
1	26072.3		1.6	74406		1	6220.5		1.6	7367.29
1.1	31584.2		1.7	37524		1.1	5520.1		1.7	8149.16
1.2	37524		1.8	83314.2		1.2	5395.63		1.8	8991.12
1.3	50789.5		1.9	92752.8		1.3	5619.53		1.9	9880..79
1.4	58152.8		2.0	102727		1.4	6066.47		2.0	10809.6

4.1.14.3 Results of different aspect ratios for Two laminates of SCSC plate with 0^0 90^0 Arrangement.

On introduction of different aspect ratios, which ranges from 1 to 2 with arithmetic increase of 0.1, gave the results as detailed in Table 4.96

where $K_{sccs} = \left(\frac{K_{sccsT1} + K_{sccsT2} + K_{sccsT3} + K_{sccsT4} + K_{sccsT5}}{K_{sccsT6}}\right)$ (4.48)

Table 4.96 Buckling results of different aspect ratios for SCSC Splate with 90^0 0^0 arrangement

m = 1					m = 2			
$\beta = \frac{a}{b}$	$\frac{E_0}{D_0}.K_{csssc}$	1.5	66021.5		$\beta = \frac{a}{b}$	$\frac{E_0}{D_0}.K_{csssc}$	1.5	-14296.3
1	26072.3	1.6	74406		1	91047.3	1.6	-18172.9
1.1	31584.2	1.7	37524		1.1	44572.2	1.7	-20486.7
1.2	37524	1.8	83314.2		1.2	17928.8	1.8	-21729.9
1.3	50789.5	1.9	83314.2		1.3	1886.6	1.9	-22215.9
1.4	58152.8	2.0	102727		1.4	-8059	2.0	-22150.1

4.1.14.4 Results of different aspect ratios for Three laminates of SCSC plate with 0^0 0^0 0^0 Arrangement.

On introduction of different aspect ratios, which ranges from 1 to 2 with arithmetic increase of 0.1, gave the results as detailed in Table 4.97

where

$$K_{scsc} = \left(\frac{K_{scscT1} + K_{scscT2} + K_{scscT3} + K_{scscT4} + K_{scscT5}}{K_{scscT6}}\right) \quad (4.49)$$

Table 4.97a Buckling results of different aspect ratios for SCSC plate with 0^0 0^0 0^0 arrangement

m = 1						m = 2				
$\beta = \frac{a}{b}$	$\frac{E_0}{D_0}.K_{scsc}$		1.5	-106194		$\beta = \frac{a}{b}$	$\frac{E_0}{D_0}.K_{scsc}$		1.5	-493.038
1	-79666.7		1.6	-117603		1	-528.138		1.6	-526.018
1.1	-78686.5		1.7	-130301		1.1	-472.095		1.7	-565.084
1.2	-81900.8		1.8	-144180		1.2	-451.415		1.8	-609.264
1.3	-87996		1.9	-159168		1.3	-452.399		1.9	-657.929
1.4	-96237.2		2.0	-175217		1.4	-467.67		2.0	-710.658

Table 4.97b.

m = 3				
$\beta = \frac{a}{b}$	$\frac{E_0}{D_0}.K_{scsc}$		1.5	-94074.6
1	-110208		1.6	
1.1	-100932		1.7	-94074.6
1.2	-96782.4		1.8	-102457
1.3	-93428.8		1.9	-106287
1.4	-93040.5		2.0	-110351

4.1.14.5 Detailed Output For SSCS Thick Laminated Anisotropic Plate.

For different lamina combination for SSCS plate, the results of the analysis are as detailed in this section.

4.1.14.6 The case of Two Combined SSCS with angles of 0^0 0^0

The results from the analysis when the laminates were positioned at the angles 0^0 0^0 are as presented in Table 4.98

Table 4.98 The Bending, Coupling and Membrane stiffness for the SSCS at 0^0 0^0 angle

Lamina, m = 1				
Bending	$J_1 = 0.08333$ $J_2 = 0.06667$ $J_3 = 0.05397$ $J_4 = 0.53333$	Coupling	$Jc_1 = -0.125$ $Jc_2 = -0.20833$ $Jc_3 = 0.6666$	Membrane $J_M = 1$
Lamina, m = 2				
Bending	$J_1 = 0.08333$ $J_2 = 0.06667$ $J_3 = 0.05397$ $J_4 = 0.53333$	Coupling	$Jc_1 = 0.125$ $Jc_2 = 0.20833$ $Jc_3 = 0.66667$	Membrane, $J_M = 1$

4.1.14. 7 The case of Two combined SSCS plate layer at angle 0^0 90^0

The formulated values from the analysis when the laminates were positioned at the angles 0^0 90^0 are as presented in Table 4.99

Table 4.99 The Bending, Coupling and Membrane stiffness for the SSCS plate with 0^0 90^0

Lamina, m = 1				
Bending	$J_1 = 0.08333$ $J_2 = 0.06667$ $J_3 = 0.05397$ $J_4 = 0.53333$	Coupling	$Jc_1 = -0.125$ $Jc_2 = -0.20833$ $Jc_3 = 0.6666$	Membrane $J_M = 1$
Lamina, m = 2				
Bending	$J_1 = 0.08333$ $J_2 = 0.06667$ $J_3 = 0.05397$ $J_4 = 0.53333$	Coupling	$Jc_1 = 0.125$ $Jc_2 = 0.20833$ $Jc_3 = 0.66667$	Membrane, $J_M = 1$

4.1.14.8 Results of Three laminated Simple Simple Clamped Simple Thick Plate

The case of Three Combined SSCS with angles of 0^0 0^0 0^0 **.** The formulated values from the analysis when the layer orientation is 0^0 0^0 0^0 are as presented in Table 4.100

Table 4.100 The Bending, Coupling and Membrane stiffness for the SSCS plate with $0^0\ 0^0\ \ 0^0$

Lamina, m = 1				
Bending	J_1 =-0.12037 J_2 = -0.09547 J_3 =-0.07653 J_4 =-0.3358	Coupling	Jc_1 = -0.33333 Jc_2 = -0.2716 Jc_3 =-0.51852	Membrane J_M =-1
Lamina, m = 2				
Bending	J_1 =-0.00926 J_2 = -0.00905 J_3 =-0.00885 J_4 = -0.92841	Coupling	Jc_1 =0.0000 Jc_2 = 0.0000 Jc_3 =-0.96298	Membrane, J_M =-1.0002
Lamina, m = 3				
Bending	J_1 = -0.12037 J_2 = -0.09547 J_3 = -0.07653 J_4 = -0.33579	Coupling	Jc_1 = 0.3333 Jc_2 = 0.2716 Jc_3 =-0.51851	Membrane J_M =-0.9999

4.1.14.9 The Buckling Values For The Case of SSCS Plate

As ealier explained, the lengthy mathematical expressions formulated were further collapsed into smaller fractions yielding the coefficients, which were considered as the unknown parameters. These were derived using the guass elimination method as expalained in chapter three. Tables 4.101 and 4.102 shows values of the unknown parameters and the buckling Load equations respectively, for the case of SSCS plate,

Table 4.101 The Coefficicents of The Formulated Parameters for the SSCS plate

P_2	P_3	P_4	P_5	P_6
-2.8170182	-0.4638003	-0.24131	-0.0048141	
A_2	A_3	A_4	A_5	A_6
-2.8170182 A_1	-0.4638003 A_1	-0.24131 A_1	-0.0048141 A_1	

The Buckling Load Equation for SSCS at Orientattion of 0^0 0^0

Table 4.102 The Buckling Load Equation for SSCS plate Lamina 1

Ø = 0^0 SSCS K_T – Values				
K_{sscsT1}	K_{sscsT2}	K_{sscsT3}	K_{sscsT4}	K_{sscsT5}
18.8586291	852.65048	-29.615188	0	0
$\frac{N_X}{D_O}$	35225.6871			

Table 4.103 The Buckling Load Equation for SSCS plate Lamina 2

Ø = 0^0 SSCS K_T – Values				
K_{sscsT1}	K_{sscsT2}	K_{sscsT3}	K_{sscsT4}	K_{sscsT5}
25.4795	68.3211	6.14581	0	0
$\frac{N_X}{D_O}$	4181.86			

The Buckling Load Equation for SSCS at Orientattion of 0^0 90^0

Table 4.104 The Buckling Load Equation for SSCS plate Lamina 1

Ø = 0^0 SSCS K_T – Values				
K_{sscsT1}	K_{sscsT2}	K_{sscsT3}	K_{sscsT4}	K_{sscsT5}
18.8586291	852.65048	-29.615188	0	0
$\frac{N_X}{D_O}$	35225.6871			

Table 4.105 The Buckling Load Equation for SSCS plate Lamina 2

Ø = 90^0 SSCS K_T – Values				
K_{sscsT1}	K_{sscsT2}	K_{sscsT3}	K_{sscsT4}	K_{sscsT5}
1.00104	79.849	28.0257	0	0
$\frac{N_X}{D_O}$	4555.47			

The Buckling Load Equation for SSCS at Orientattion of 0^0 0^0 0^0

Table 4.106 The Buckling Load Equation for SSCS plate Lamina 1

Ø = 0^0 SSCS K_T – Values				
K_{sscsT1}	K_{sscsT2}	K_{sscsT3}	K_{sscsT4}	K_{sscsT5}
-46.1784	-912.289	-142.169	0	0
$\frac{N_X}{D_O}$	-46051.7			

Table 4.107 The Buckling Load Equation for SSCS plate Lamina 2

Ø = 0^0	SSCS K_T – Values			
K_{sscsT1}	K_{sscsT2}	K_{sscsT3}	K_{sscsT4}	K_{sscsT5}
-2.77801	-2.77621	-0.95859	0	0
$\frac{N_X}{D_O}$	-272.502			

Table 4.108 The Buckling Load Equation for SSCS plate Lamina 3

Ø = 0^0	SSCS K_T – Values			
K_{sscsT1}	K_{sscsT2}	K_{sscsT3}	K_{sscsT4}	K_{sscsT5}
-37.8302	-220.863	24.863	0	0
$\frac{N_X}{D_O}$	-9797.04			

4.1.15 Results of different aspect ratios for Two laminates of SSCS plate with 0^0 0^0 Arrangement.

On introduction of different aspect ratios, which ranges from 1 to 2 with arithmetic increase of 0.1, gave the results as detailed in Table 4.1.109

where

$$K_{sscs} = \left(\frac{K_{sscsT1} + K_{sscsT2} + K_{sscsT3} + K_{sscsT4} + K_{sscsT5}}{K_{sscsT6}}\right) \quad (4.50)$$

Table 4.109 Buckling results of different aspect ratios for SSCS plate with 0^0 0^0 arrangement

m = 1						m = 2				
$\beta = \frac{a}{b}$	$\frac{E_0}{D_0}.K_{sscs}$		1.5	73278.7		$\beta = \frac{a}{b}$	$\frac{E_0}{D_0}.K_{sscs}$		1.5	7007.92
1	35225.7		1.6	87790.3		1	4181.86		1.6	7747.91
1.1	40403.2		1.7	109831		1.1	4597.6		1.7	8544.55
1.2	46582.1		1.8	149110		1.2	5109.24		1.8	9400.76
1.3	53840.7		1.9	230865		1.3	5687.86		1.9	10318.8
1.4	62498.2		2.0	394270		1.4	6321.7		2.0	11300

4.1.15.1 Results of different aspect ratios for Two laminates of SSCS plate with 0^0 90^0 Arrangement.

On introduction of different aspect ratios, which ranges from 1 to 2 with arithmetic increase of 0.1, gave the results as detailed in Table 4.111

where

$$K_{sscs} = \left(\frac{K_{sscsT1} + K_{sscsT2} + K_{sscsT3} + K_{sscsT4} + K_{sscsT5}}{K_{sscsT6}}\right) \quad (4.51)$$

Table 4.111 Buckling results of different aspect ratios for SSCS Splate with 90^0 0^0 arrangement

m = 1						m = 2				
$\beta = \frac{a}{b}$	$\frac{E_0}{D_0}.Kcsss_c$		1.5	73278.7		$\beta = \frac{a}{b}$	$\frac{E_0}{D_0}.Kcsss_c$		1.5	3077.24
1	35225.7		1.6	87790.3		1	57533.7		1.6	2593.64
1.1	40403.2		1.7	109831		1.1	23295.5		1.7	2623.22
1.2	46582.1		1.8	149110		1.2	12062.9		1.8	3014.97
1.3	53840.7		1.9	230865		1.3	6931.81		1.9	3678.25
1.4	62498.2		2.0	394270		1.4	4344.78		2.0	4555.47

4.1.15.2 Results of different aspect ratios for Three laminates of SSCS plate with 0^0 0^0 0^0 Arrangement.

On introduction of different aspect ratios, which ranges from 1 to 2 with arithmetic increase of 0.1, gave the results as detailed in Table 4.112

considering

$$K_{sscs} = \left(\frac{K_{sscsT1} + K_{sscsT2} + K_{sscsT3} + K_{sscsT4} + K_{sscsT5}}{K_{sscsT6}}\right) \quad (4.52)$$

Table 4.112a Buckling results of different aspect ratios for SSCS plate with 0^0 0^0 0^0 Arrangement

m = 1						m = 2				
$\beta = \frac{a}{b}$	$\frac{E_0}{D_0}.K_{csssc}$		1.5	-89577.8		$\beta = \frac{a}{b}$	$\frac{E_0}{D_0}.K_{csssc}$		1.5	-89577.8
1	-46051.7		1.6	-101211		1	-46051.7		1.6	-101211
1.1	-52489.8		1.7	-113707		1.1	-52489.8		1.7	-113707
1.2	-60205		1.8	-127045		1.2	-60205		1.8	-127045
1.3	-69021.4		1.9	-127045		1.3	-69021.4		1.9	141214
1.4	-78834		2.0	-156204		1.4	-78834		2.0	-156204

Table 4.112b.

m = 3				
$\beta = \frac{a}{b}$	$\frac{E_0}{D_0}.K_{csssc}$		1.5	-421.435
1	-284.352		1.6	-421.435
1.1	-303.233		1.7	-458.9
1.2	-327.235		1.8	-499.074
1.3	-355.303		1.9	-541.86
1.4	-386.826		2.0	-587.194

4.1.15.3 The case of Two Combined SCCC with angles of 0^0 0^0

The results from the analysis when the laminates were positioned at the angles 0^0 0^0 are as presented in Table 4.113

Table 4.113 The Bending, Coupling and Membrane stiffness for the SCCC at 0^0 0^0 angle

Lamina, m = 1				
Bending	J_1 = 0.08333 J_2 = 0.06667 J_3 = 0.05397 J_4 = 0.53333	Coupling	Jc_1 = -0.125 Jc_2 = -0.20833 Jc_3 = 0.6666	Membrane J_M =1
Lamina, m = 2				
Bending	J_1 = 0.08333 J_2 = 0.06667 J_3 = 0.05397 J_4 = 0.53333	Coupling	Jc_1 = 0.125 Jc_2 = 0.20833 Jc_3 = 0.66667	Membrane, J_M = 1

4.1.15.4 The case of Two combined SCCC plate layer at angle 0^0 90^0

The formulated values from the analysis when the laminates were positioned at the angles 0^0 90^0 are as presented in Table 4.114

Table 4.114 The Bending, Coupling and Membrane stiffness for the SCCC plate with 0^0 90^0

Lamina, m = 1				
Bending	J_1 = 0.08333 J_2 = 0.06667 J_3 = 0.05397 J_4 = 0.53333	Coupling	Jc_1 = -0.125 Jc_2 = -0.20833 Jc_3 = 0.6666	Membrane J_M =1
Lamina, m = 2				
Bending	J_1 = 0.08333 J_2 = 0.06667 J_3 = 0.05397 J_4 = 0.53333	Coupling	Jc_1 = 0.125 Jc_2 = 0.20833 Jc_3 = 0.66667	Membrane, J_M = 1

4.1.15.5 Results of Three laminated Simple Clamped Clamped Clamped Thick Plate

The case of Three Combined SCCC with angles of 0^0 0^0 0^0 .The formulated values from the analysis when the layer orientation is 0^0 0^0 0^0 are as presented in Table 4.115

Table 4.115 The Bending, Coupling and Membrane stiffness for the SCCC plate with 0^0 0^0 0^0

Lamina, m = 1				
Bending	J_1 =-0.12037 J_2 = -0.09547 J_3 =-0.07653 J_4 =-0.3358	Coupling	Jc_1 = -0.33333 Jc_2 = -0.2716 Jc_3 =-0.51852	Membrane J_M =-1
Lamina, m = 2				
Bending	J_1 =-0.00926 J_2 = -0.00905 J_3 =-0.00885 J_4 = -0.92841	Coupling	Jc_1 =0.0000 Jc_2 = 0.0000 Jc_3 =-0.96298	Membrane, J_M =-1.0002
Lamina, m = 3				
Bending	J_1 = -0.12037 J_2 = -0.09547 J_3 = -0.07653 J_4 = -0.33579	Coupling	Jc_1 = 0.3333 Jc_2 = 0.2716 Jc_3 =-0.51851	Membrane J_M =-0.9999

4.1.15.6 The Buckling Values For The Case of SCCC Plate

As ealier explained, the lengthy mathematical expressions formulated were further collapsed into smaller fractions yielding the coefficients, which were considered as the unknown parameters. These were derived using the guass elimination method as expalained in chapter three. Tables 4.115, 4.116, 4.117, 4.118 and 4.119 show the values of the unknown parameters and the buckling Load equations , for the case of SCCC plate,

Table 4.115b The Coefficients of The Formulated Parameters for the SCCC plate

P_2	P_3	P_4	P_5	P_6
-1.8517343	-0.211339	-0.05305	0.02400065	
A_2	A_3	A_4	A_5	A_6
-1.8517343 A_1	-0.211339 A_1	-0.05305 A_1	0.02400065 A_1	

The Buckling Load Equation for SCCC at Orientattion of 0^0 0^0

Table 4.116 The Buckling Load Equation for SCCC plate Lamina 1

$Ø = 0^0$ SCCC K_T – Values				
K_{scccT1}	K_{scccT2}	K_{scccT3}	K_{scccT4}	K_{scccT5}
23.1973858	678.238721	-47.49981	0	0
$\frac{N_X}{D_O}$	27361.3513			

Table 4.117 The Buckling Load Equation for SCCC plate Lamina 2

$Ø = 0^0$ SCCC K_T – Values				
K_{scccT1}	K_{scccT2}	K_{scccT3}	K_{scccT4}	K_{scccT5}
27.8916	106.634	77.35	0	0
$\frac{N_x}{D_O}$	8865.1			

The Buckling Load Equation for SCCC at Orientattion of 0^0 90^0

Table 4.118 The Buckling Load Equation for SCCC plate Lamina 1

$Ø = 0^0$ SCCC K_T – Values				
K_{scccT1}	K_{scccT2}	K_{scccT3}	K_{scccT4}	K_{scccT5}
23.1973858	678.238721	-47.49981	0	0
$\frac{N_x}{D_O}$	27361.3513			

Table 4.119 The Buckling Load Equation for SCCC plate Lamina 2

$Ø = 90^0$ SCCC K_T – Values				
K_{scccT1}	K_{scccT2}	K_{scccT3}	K_{scccT4}	K_{scccT5}
1.13953	67.9388	2545	0	0
$\frac{N_x}{D_O}$	109376			

The Buckling Load Equation for SCCC at Orientattion of 0^0 0^0 0^0

Table 4.120 The Buckling Load Equation for SCCC plate Lamina 1

$Ø = 0^0$ SCCC K_T – Values				
K_{scccT1}	K_{scccT2}	K_{scccT3}	K_{scccT4}	K_{scccT5}
-40	-1116.87	-677.413	0	0
$\frac{N_x}{D_O}$	-76748.1			

Table 4.121 The Buckling Load Equation for SCCC plate Lamina 2

$Ø = 0^0$ SCCC K_T – Values				
K_{scccT1}	K_{scccT2}	K_{scccT3}	K_{scccT4}	K_{scccT5}
-2.77813	-3.71966	-5.97843	0	0
$\frac{N_x}{D_O}$	-522.018			

Table 4.122 The Buckling Load Equation for SCCC plate Lamina 3

Ø = 0^0 SCCC K_T – Values				
K_{scccT1}	K_{scccT2}	K_{scccT3}	K_{scccT4}	K_{scccT5}
-59.7418	-339.412	422.808	0	0
$\frac{N_X}{D_O}$	989.701			

4.1.16.7 Results of different aspect ratios for Two laminates of SCCC plate with 0^0 0^0 Arrangement.

On introduction of different aspect ratios, which ranges from 1 to 2 with arithmetic increase of 0.1, gave the results as detailed in Table 4.1.123

$$\text{where}\quad K_{sccc} = \left(\frac{K_{scccT1}+K_{scccT2}+K_{scccT3}+K_{scccT4}+K_{scccT5}}{K_{scccT6}}\right) \qquad (4.53)$$

Table 4.123 Buckling results of different aspect ratios for SCCC plate with 0^0 0^0 arrangement

m = 1						m = 2				
$\beta = \frac{a}{b}$	$\frac{E_0}{D_0}.K_{sccc}$		1.5	64221.4		$\beta = \frac{a}{b}$	$\frac{E_0}{D_0}.K_{sccc}$		1.5	11208.4
1	27361.4		1.6	73349		1	8865.1		1.6	12302.1
1.1	33616.8		1.7	83091		1.1	8616.2		1.7	13503.6
1.2	40411.6		1.8	83091		1.2	8869.9		1.8	14798.6
1.3	47766.1		1.9	104451		1.3	9445.93		1.9	16177.5
1.4	55697.8		2.0	116081		1.4	10244.7		2.0	17633.4

4.1.16.8 Results of different aspect ratios for Two laminates of SCCC plate with 0^0 90^0 Arrangement.

On introduction of different aspect ratios, which ranges from 1 to 2 with arithmetic increase of 0.1, gave the results as detailed in Table 4.1.124

$$\text{where}\quad K_{sccc} = \left(\frac{K_{scccT1}+K_{scccT2}+K_{scccT3}+K_{scccT4}+K_{scccT5}}{K_{scccT6}}\right) \qquad (4.54)$$

Table 4.124 Buckling results of different aspect ratios for SCCC Splate with 0^0 90^0 arrangement

m = 1						m = 2				
$\beta = \frac{a}{b}$	$\frac{E_0}{D_0}.K_{csssc}$		1.5	64221.4		$\beta = \frac{a}{b}$	$\frac{E_0}{D_0}.K_{csssc}$		1.5	28438.5
1	27361.4		1.6	73349		1	109376		1.6	25146.2
1.1	33616.8		1.7	83091		1.1	75575.6		1.7	23218.7
1.2	40411.6		1.8	93455.8		1.2	54951.4		1.8	22259.7
1.3	47766.1		1.9	104451		1.3	41990.7		1.9	2209.3
1.4	55697.8		2.0	116081		1.4	33717.6		2.0	22292.6

4.1.15.9 Results of different aspect ratios for Three laminates of SCCC plate with 0^0 0^0 0^0 Arrangement.

On introduction of different aspect ratios, which ranges from 1 to 2 with arithmetic increase of 0.1, gave the results as detailed in Table 4.1.125

$$\text{where } K_{sccc} = \left(\frac{K_{scccT1} + K_{scccT2} + K_{scccT3} + K_{scccT4} + K_{scccT5}}{K_{scccT6}}\right) \quad (4.55)$$

Table 4.125a Buckling results of different aspect ratios for SCCC plate with 0^0 0^0 0^0 arrangement

m = 1						m = 2				
$\beta = \frac{a}{b}$	$\frac{E_0}{D_0}.K_{sccc}$		1.5	-99166.8		$\beta = \frac{a}{b}$	$\frac{E_0}{D_0}.K_{sccc}$		1.5	-502.017
1	-76748.1		1.6	-122993		1	-522.018		1.6	-537.167
1.1	-77487.4		1.7	-136908		1.1	-469.931		1.7	-578.353
1.2	-82192.7		1.8	-152014		1.2	-452.656		1.8	-624.66
1.3	-89645.3		1.9	-168245		1.3	-456.542		1.9	-675.522
1.4	-99166.8		2.0	-185560		1.4	-474.349		2.0	-730.538

Table 4.125b.

m = 3				
$\beta = \frac{a}{b}$	$\frac{E_0}{D_0}.K_{sccc}$		1.5	-29353.9
1	989.701		1.6	-33640.4
1.1	-7968.63		1.7	-37873
1.2	-14652.6		1.8	-42117.1
1.3	-20110		1.9	-42117.1
1.4	-24904.9		2.0	-50783.5

4.1.16 Detailed Output For CCCS Thick Laminated Anisotropic Plate

The results from the analysis when the laminates were positioned at the angles 0^0 0^0 are as presented in Table 4.1.126

4.1.16.1 The case of Two Combined CCCS with angles of 0^0 0^0

Table 4.126 The Bending, Coupling and Membrane stiffness for the CCCS at 0^0 0^0 angle

Lamina, m = 1				
Bending	$J_1 = 0.08333$ $J_2 = 0.06667$ $J_3 = 0.05397$ $J_4 = 0.53333$	Coupling	$Jc_1 = -0.125$ $Jc_2 = -0.20833$ $Jc_3 = 0.6666$	Membrane $J_M = 1$
Lamina, m = 2				
Bending	$J_1 = 0.08333$ $J_2 = 0.06667$ $J_3 = 0.05397$ $J_4 = 0.53333$	Coupling	$Jc_1 = 0.125$ $Jc_2 = 0.20833$ $Jc_3 = 0.66667$	Membrane, $J_M = 1$

4.1.16. 2 The case of Two combined CCCS plate layer at angle 0^0 90^0

The formulated values from the analysis when the laminates were positioned at the angles 0^0 90^0 are as presented in Table 4.127

Table 4.127 The Bending, Coupling and Membrane stiffness for the CCCS plate with 0^0 90^0

Lamina, m = 1				
Bending	J_1 = 0.08333 J_2 = 0.06667 J_3 = 0.05397 J_4 = 0.53333	Coupling	Jc_1 = -0.125 Jc_2 = -0.20833 Jc_3 = 0.6666	Membrane J_M =1
Lamina, m = 2				
Bending	J_1 = 0.08333 J_2 = 0.06667 J_3 = 0.05397 J_4 = 0.53333	Coupling	Jc_1 = 0.125 Jc_2 = 0.20833 Jc_3 = 0.66667	Membrane, J_M = 1

4.1.16.3 Results of Three laminated Clamped Clamped Clamped Simple Thick Plate

The Laminated cases of the CCCS were considered, at different angle orientations. It was observed that the noticeable changes occurred as the plates were positioned at different inclination.

4.1.16.4 The case of Three Combined CCCS with angles of 0^0 0^0 0^0

The formulated values from the analysis when the layer orientation is 0^0 0^0 0^0

are as presented in Table 4.128

Table 4.128 The Bending, Coupling and Membrane stiffness for the CCCS plate with 0^0 0^0 0^0

Lamina, m = 1				
Bending	J_1 =-0.12037 J_2 = -0.09547 J_3 =-0.07653 J_4 =-0.3358	Coupling	Jc_1 = -0.33333 Jc_2 = -0.2716 Jc_3 =-0.51852	Membrane J_M =-1
Lamina, m = 2				
Bending	J_1 =-0.00926 J_2 = -0.00905 J_3 =-0.00885 J_4 = -0.92841	Coupling	Jc_1 =0.0000 Jc_2 = 0.0000 Jc_3 =-0.96298	Membrane, J_M =-1.0002
Lamina, m = 3				
Bending	J_1 = -0.12037 J_2 = -0.09547 J_3 = -0.07653 J_4 = -0.33579	Coupling	Jc_1 = 0.3333 Jc_2 = 0.2716 Jc_3 =-0.51851	Membrane J_M =-0.9999

4.1.16.5 The Buckling Values For The Case of CCCS Plate

As ealier explained, the lengthy mathematical expressions formulated were further collapsed into smaller fractions yielding the coefficients, which were considered as the unknown parameters. These were derived using the guass elimination method as expalained in chapter three. Tables 4.129, 4.130, 4.131, 4.132 and 4.132 show the values of the unknown parameters and the buckling Load equations, for the case of CCCS plate,

Table 4.129 The Coefficicents of The Formulated Parameters for the CCCS plate

P_2	P_3	P_4	P_5	P_6
-1.651565	7.49428833	0.00786	-1.0705183	
A_2	A_3	A_4	A_5	A_6
-1.651565 A_1	7.49428833 A_1	0.00786 A_1	-1.0705183 A_1	

The Buckling Load Equation for CCCS at Orientattion of 0^0 0^0

Table 4.130 The Buckling Load Equation for CCCS plate Lamina 1

Ø = 0^0 CCCS K_T – Values				
K_{cccsT1}	K_{cccsT2}	K_{cccsT3}	K_{cccsT4}	K_{cccsT5}
43.9198068	-13.778319	1517.13882	0	0
$\frac{N_X}{D_O}$	64739.7619			

Table 4.131 The Buckling Load Equation for CCCS plate Lamina 2

Ø = 0^0 CCCS K_T – Values				
K_{cccsT1}	K_{cccsT2}	K_{cccsT3}	K_{cccsT4}	K_{cccsT5}
-6.11488	60.5749	1292.91	0	0
$\frac{N_X}{D_O}$	56375.2			

The Buckling Load Equation for CCCS at Orientattion of 0^0 90^0

Table 4.132 The Buckling Load Equation for CCCS Plate Lamina 1

Ø = 0^0 CCCS K_T – Values				
K_{cccsT1}	K_{cccsT2}	K_{cccsT3}	K_{cccsT4}	K_{cccsT5}
43.9198068	-13.778319	1517.13882	0	0
$\frac{N_X}{D_O}$	64739.7619			

Table 4.133 The Buckling Load Equation for CCCS Plate Lamina 2

$\varnothing = 90^0$ CCCS K_T – Values				
K_{cccsT1}	K_{cccsT2}	K_{cccsT3}	K_{cccsT4}	K_{cccsT5}
2.37232	144.412	-155.004	0	0
$\frac{N_X}{D_O}$	-343.929			

The Buckling Load Equation for CCCS at Orientattion of 0^0 0^0 0^0

Table 4.134 The Buckling Load Equation for CCCS plate Lamina 1

$\varnothing = 0^0$ CCCS K_T – Values				
K_{cccsT1}	K_{cccsT2}	K_{cccsT3}	K_{cccsT4}	K_{cccsT5}
-114.646	57.2149	5830.55	0	0
$\frac{N_X}{D_O}$	241553			

Table 4.135 The Buckling Load Equation for CCCS plate Lamina 2

$\varnothing = 0^0$ CCCS K_T – Values				
K_{cccsT1}	K_{cccsT2}	K_{cccsT3}	K_{cccsT4}	K_{cccsT5}
-0.1111	-1.27291	-3.4689	0	0
$\frac{N_X}{D_O}$	-203.05			

Table 4.136 The Buckling Load Equation for CCCS plate Lamina 3

$\varnothing = 0^0$ CCCS K_T – Values				
K_{cccsT1}	K_{cccsT2}	K_{cccsT3}	K_{cccsT4}	K_{cccsT5}
-128.153	-1693.53	944.543	0	0
$\frac{N_X}{D_O}$	-36700.3			

4.1.16.6 Results of different aspect ratios for Two laminates of CCCS plate with 0^0 0^0 Arrangement.

On introduction of different aspect ratios, which ranges from 1 to 2 with arithmetic increase of 0.1, gave the results as detailed in Table 4.137

Where

$$K_{cccs} = \left(\frac{K_{cccsT1} + K_{cccsT2} + K_{cccsT3} + K_{cccsT4} + K_{cccsT5}}{K_{cccsT6}}\right) \quad (4.56)$$

Table 4.137 Buckling results of different aspect ratios for CCCS plate with 0^0 0^0 arrangement

m = 1						m = 2				
$\beta = \frac{a}{b}$	$\frac{E_0}{D_0}.K_{cccs}$		1.5	20973.2		$\beta = \frac{a}{b}$	$\frac{E_0}{D_0}.K_{cccs}$		1.5	30637.3
1	64739.8		1.6	13431.9		1	56375.2		1.6	29985.5
1.1	53424		1.7	5424.28		1.1	46774		1.7	30214.4
1.2	44071.7		1.8	-3313.92		1.2	40014		1.8	31177.7
1.3	35901.1		1.9	-13025.4		1.3	35366.7		1.9	32750.1
1.4	28346.1		2.0	-23939.2		1.4	32351.5		2.0	34817.9

4.1.16.7 Results of different aspect ratios for Two laminates of CCCS plate with 0^0 90^0 Arrangement.

On introduction of different aspect ratios, which ranges from 1 to 2 with arithmetic increase of 0.1, gave the results as detailed in Table 4.138

where

$$K_{cccs} = \left(\frac{K_{cccsT1} + K_{cccsT2} + K_{cccsT3} + K_{cccsT4} + K_{cccsT5}}{K_{scccT6}}\right) \quad (4.57)$$

Table 4.138 Buckling results of different aspect ratios for CCCS Splate with $90^0\, 0^0$ arrangement

m = 1						m = 2				
$\beta = \frac{a}{b}$	$\frac{E_0}{D_0}.K_{cccs}$		1.5	20973.2		$\beta = \frac{a}{b}$	$\frac{E_0}{D_0}.K_{cccs}$		1.5	24721.6
1	64739.8		1.6	13431.9		1	-343.929		1.6	22977.8
1.1	53424		1.7	5424.28		1.1	20303.2		1.7	21331.8
1.2	44071.7		1.8	-3313.92		1.2	26397.7		1.8	19871.3
1.3	35901.1		1.9	-13025.4		1.3	27325.4		1.9	18626
1.4	28346.1		2.0	-23939.2		1.4	26344.8		2.0	17600.1

4.1.16.8 Results of different aspect ratios for Three laminates of CCCS plate with $0^0\, 0^0\, 0^0$ Arrangement.

On introduction of different aspect ratios, which ranges from 1 to 2 with arithmetic increase of 0.1, gave the results as detailed in Table 4.139

where

$$K_{sssc} = \frac{K_{ssscT1} + K_{ssscT2} + K_{ssscT3} + K_{ssscT4} + K_{ssscT5}}{K_{ssscT6}} \quad (4.58)$$

Table 4.139 Buckling results of different aspect ratios for CCCS plate with $0^0\ 0^0\ 0^0$ arrangement

m = 1						m = 2				
$\beta = \frac{a}{b}$	$\frac{E_0}{D_0}.K_{sssc}$		1.5	124383		$\beta = \frac{a}{b}$	$\frac{E_0}{D_0}.K_{sssc}$		1.5	-163.989
1	241553		1.6	117973		1	-221.892		1.6	-163.942
1.1	202262		1.7	115388		1.1	-195,242		1.7	-165.284
1.2	172776		1.8	116283		1.2	-179.655		1.8	-167.671
1.3	150899		1.9	120424		1.3	-170.709		1.9	-170.878
1.4	135117		2.0	127655		1.4	-165.96		2.0	-174.751

Table 4.139

m = 3				
$\beta = \frac{a}{b}$	$\frac{E_0}{D_0}.K_{sssc}$		1.5	55048.5
1	225051		1.6	34035.9
1.1	1760.95		1.7	14769.2
1.2	137263		1.8	-3241.82
1.3	105436		1.9	-20328.3
1.4	78512.7		2.0	-36700.3

4.1.17 Detailed Output For SSSC Thick Laminated Anisotropic Plate.

For different lamina combination for SSSC plate, the results of the analysis are as detailed in this section.

4.1.17.1 The case of Two Combined SSSC with angles of $0^0\ 0^0$

The results from the analysis when the laminates were positioned at the angles $0^0\ 0^0$ are as presented in Table 4.140

Table 4.140 The Bending, Coupling and Membrane stiffness for the SSSC at 0^0 0^0 angle

Lamina, m = 1				
Bending	$J_1 = 0.08333$ $J_2 = 0.06667$ $J_3 = 0.05397$ $J_4 = 0.53333$	Coupling	$Jc_1 = -0.125$ $Jc_2 = -0.20833$ $Jc_3 = 0.6666$	Membrane $J_M = 1$
Lamina, m = 2				
Bending	$J_1 = 0.08333$ $J_2 = 0.06667$ $J_3 = 0.05397$ $J_4 = 0.53333$	Coupling	$Jc_1 = 0.125$ $Jc_2 = 0.20833$ $Jc_3 = 0.66667$	Membrane, $J_M = 1$

4.1.17. 2 The case of Two combined SSSC plate layer at angle 0^0 90^0

The formulated values from the analysis when the laminates were positioned at the angles 0^0 90^0 are as presented in Table 4.141

Table 4.141 The Bending, Coupling and Membrane stiffness for the SSSC plate with 0^0 90^0

Lamina, m = 1				
Bending	$J_1 = 0.08333$ $J_2 = 0.06667$ $J_3 = 0.05397$ $J_4 = 0.53333$	Coupling	$Jc_1 = -0.125$ $Jc_2 = -0.20833$ $Jc_3 = 0.6666$	Membrane $J_M = 1$
Lamina, m = 2				
Bending	$J_1 = 0.08333$ $J_2 = 0.06667$ $J_3 = 0.05397$ $J_4 = 0.53333$	Coupling	$Jc_1 = 0.125$ $Jc_2 = 0.20833$ $Jc_3 = 0.66667$	Membrane, $J_M = 1$

4.1.17.3 Results of Three laminated Simple Simple Simple Clamped Thick Plate

Results of Three laminated Simple Simple Simple Clamped Thick Plate were recorded at different material properties

4.1.17.4 The case of Three Combined SSSC with angles of 0^0 0^0 0^0

The formulated values from the analysis when the layer orientation is 0^0 0^0 0^0

are as presented in Table 4.142

Table 4.142 The Bending, Coupling and Membrane stiffness for the SSSC plate with 0^0 0^0 0^0

Lamina, m = 1				
Bending	J_1 =-0.12037 J_2 = -0.09547 J_3 =-0.07653 J_4 =-0.3358	Coupling	Jc_1 = -0.33333 Jc_2 = -0.2716 Jc_3 =-0.51852	Membrane J_M =-1
Lamina, m = 2				
Bending	J_1 =-0.00926 J_2 = -0.00905 J_3 =-0.00885 J_4 = -0.92841	Coupling	Jc_1 =0.0000 Jc_2 = 0.0000 Jc_3 =-0.96298	Membrane, J_M =-1.0002
Lamina, m = 3				
Bending	J_1 = -0.12037 J_2 = -0.09547 J_3 = -0.07653 J_4 = -0.33579	Coupling	Jc_1 = 0.3333 Jc_2 = 0.2716 Jc_3 =-0.51851	Membrane J_M =-0.9999

4.1.17.5 The Buckling Values For The Case of SSSC Plate

As ealier explained, the lengthy mathematical expressions formulated were further collapsed into smaller fractions yielding the coefficients, which were considered as the unknown parameters. These were derived using the guass elimination method as expalained in chapter three. Tables 4.143, 4.144 , 4.145 , 4.146 and 4.147 show the values of the unknown parameters and the buckling Load equations, for the case of SSSC plate,

Table 4.143 The Coefficicents of The Formulated Parameters for the SSSC plate

P_2	P_3	P_4	P_5	P_6
-1.2709609	0.12048523	0.33585	-0.7381277	
A_2	A_3	A_4	A_5	A_6
-1.2709609 A_1	0.12048523 A_1	0.33585 A_1	-0.7381277 A_1	

The Buckling Load Equation for SSSC at Orientattion of 0^0 0^0

Table 4.144 The Buckling Load Equation for SSSC plate Lamina 1

Ø = 0^0 SSSC K_T - Values				
K_{ssscT1}	K_{ssscT2}	K_{ssscT3}	K_{ssscT4}	K_{ssscT5}
28.2628994	429.943337	46.8279743	0	0
$\frac{N_X}{D_O}$	21131.1385			

Table 4.145 The Buckling Load Equation for SSSC plate Lamina 2

Ø = 0^0 SSSC K_T – Values				
K_{ssscT1}	K_{ssscT2}	K_{ssscT3}	K_{ssscT4}	K_{ssscT5}
1.11341	410.717	-27.6844	0	0
$\frac{N_x}{D_o}$	16073.1			

The Buckling Load Equation for SSSC at Orientattion of 0^0 90^0

Table 4.146 The Buckling Load Equation for SSSC plate Lamina 1

Ø = 0^0 SSSC K_T – Values				
K_{ssscT1}	K_{ssscT2}	K_{ssscT3}	K_{sscsT4}	K_{ssscT5}
28.2628994	429.943337	46.8279743	0	0
$\frac{N_x}{D_o}$	21131.1385			

Table 4.147 The Buckling Load Equation for SSSC plate Lamina 2

Ø = 90^0 SSSC K_T – Values				
K_{ssscT1}	K_{ssscT2}	K_{ssscT3}	K_{sscsT4}	K_{ssscT5}
-0.36719	-903.839	6186.35	0	0
$\frac{N_x}{D_o}$	221010			

The Buckling Load Equation for SSSC at Orientattion of 0^0 0^0 0^0

Table 4.148 The Buckling Load Equation for SSSC plate Lamina 1

Ø = 0^0 SSSC K_T – Values				
K_{ssscT1}	K_{ssscT2}	K_{ssscT3}	K_{ssscT4}	K_{ssscT5}
-128.562	-784.181	9.03391	0	0
$\frac{N_x}{D_o}$	-37812.1			

Table 4.149 The Buckling Load Equation for SSSC plate Lamina 2

Ø = 0^0 SSSC K_T – Values				
K_{ssscT1}	K_{ssscT2}	K_{ssscT3}	K_{ssscT4}	K_{ssscT5}
-2.77483	-3.17585	-2.68758	0	0
$\frac{N_x}{D_o}$	-361.433			

Table 4.150 The Buckling Load Equation for SSSC plate Lamina 3

$\emptyset = 0^0$	**SSSC** K_T – Values			
K_{ssscT1}	K_{ssscT2}	K_{ssscT3}	K_{ssscT4}	K_{ssscT5}
77.5737	40.8818	191.911	0	0
$\frac{N_x}{D_O}$	12986			

4.1.17.8 Results of different aspect ratios for Two laminates of SSSC plate with 0^0 0^0 Arrangement.

On introduction of different aspect ratios, which ranges from 1 to 2 with arithmetic increase of 0.1, gave the results as detailed in Table 4.151

where

$$K_{sssc} = \left(\frac{K_{ssscT1} + K_{ssscT2} + K_{ssscT3} + K_{ssscT4} + K_{ssscT5}}{K_{ssscT6}}\right) \quad (4.59)$$

Table 4.151 Buckling results of different aspect ratios for SSSC plate with 0^0 0^0 arrangement

m = 1						m = 2				
$\beta = \frac{a}{b}$	$\frac{E_0}{D_0}.K_{sssc}$		1.5	18894.6		$\beta = \frac{a}{b}$	$\frac{E_0}{D_0}.K_{sssc}$		1.5	4092.8
1	21131.1		1.6	13675.9		1	9549.88		1.6	2139.65
1.1	26404.3		1.7	7007.23		1.1	8297.14		1.7	-510.691
1.2	26783.1		1.8	1224.75		1.2	7381.77		1.8	-3985
1.3	25415.6		1.9	-11122.1		1.3	6526.77		1.9	-8390.09
1.4	22778.3		2.0	-22772.1		1.4	5497.47		2.0	-13814.6

4.1.17.9 Results of different aspect ratios for Two laminates of SSSC plate with 0^0 90^0 Arrangement.

On introduction of different aspect ratios, which ranges from 1 to 2 with arithmetic increase of 0.1, gave the results as detailed in Table 4.152

where

$$K_{sssc} = \left(\frac{K_{ssscT1} + K_{ssscT2} + K_{ssscT3} + K_{ssscT4} + K_{ssscT5}}{K_{ssscT6}}\right) \quad (4.60)$$

Table 4.152 Buckling results of different aspect ratios for SSSC Splate with $90^0\,0^0$ arrangement

m = 1						m = 2			
$\beta = \frac{a}{b}$	$\frac{E_0}{D_0}.K_{csssc}$		1.5	18894.6		$\beta = \frac{a}{b}$	$\frac{E_0}{D_0}.K_{csssc}$		1.5	-6367.46
1	21131.1		1.6	13675.9		1	221010		1.6	7088.78
1.1	26404.3		1.7	7007.23		1.1	187750		1.7	11922.6
1.2	26783.1		1.8	1224.75		1.2	208261		1.8	14052.3
1.3	25415.6		1.9	-11122.1		1.3	1611859		1.9	15220.7
1.4	22778.3		2.0	-22772.1		1.4	-61999.9		2.0	16073.1

4.1.18 Results of different aspect ratios for Three laminates of SSSC plate with $0^0\,0^0\,0^0$ Arrangement.

On introduction of different aspect ratios, which ranges from 1 to 2 with arithmetic increase of 0.1, gave the results as detailed in Table 4.153

Where

$$K_{sssc} = \left(\frac{K_{ssscT1} + K_{ssscT2} + K_{ssscT3} + K_{ssscT4} + K_{ssscT5}}{K_{ssscT6}}\right) \quad (4.61)$$

Table 4.153 Buckling results of different aspect ratios for SSSC plate with 0^0 0^0 0^0 arrangement

m = 1						m = 2				
$\beta = \frac{a}{b}$	$\frac{E_0}{D_0}.K_{sssc}$		1.5	-46310.7		$\beta = \frac{a}{b}$	$\frac{E_0}{D_0}.K_{sssc}$		1.5	-438.088
1	-37812.1		1.6	-50849.8		1	-361.433		1.6	-474.481
1.1	-31153.4		1.7	-55038.6		1.1	-353.792		1.7	-514.925
1.2	-32748.5		1.8	-58833		1.2	-361.906		1.8	-559.006
1.3	-36795.3		1.9	-62222.7		1.3	-380.426		1.9	-606.448
1.4	-41535.4		2.0	-65213.3		1.4	-406.391		2.0	-657.066

Table 4.2.153

m = 3				
$\beta = \frac{a}{b}$	$\frac{E_0}{D_0}.K_{sssc}$		1.5	19470
1	12986		1.6	26475
1.1	11370.6		1.7	35841.5
1.2	10839.3		1.8	47730.8
1.3	11845.2		1.9	62290
1.4	14654.3		2.0	79651.2

4.1.18.1 Detailed Output For CCSC Thick Laminated Anisotropic Plate.

The results from the analysis when the laminates were positioned at the angles 0^0 0^0 are as resented in Table 4.154

4.1.18 .2 The case of Two Combined CCSC with angles of 0^0 0^0

Table 4. 154 The Bending, Coupling and Membrane stiffness for the SSSC at 0^0 0^0 angle

Lamina, m = 1				
Bending	$J_1 = 0.08333$ $J_2 = 0.06667$ $J_3 = 0.05397$ $J_4 = 0.53333$	Coupling	$Jc_1 = -0.125$ $Jc_2 = -0.20833$ $Jc_3 = 0.6666$	Membrane $J_M = 1$
Lamina, m = 2				
Bending	$J_1 = 0.08333$ $J_2 = 0.06667$ $J_3 = 0.05397$ $J_4 = 0.53333$	Coupling	$Jc_1 = 0.125$ $Jc_2 = 0.20833$ $Jc_3 = 0.66667$	Membrane, $J_M = 1$

4.1.18. 3 The case of Two combined CCSC plate layer at angle 0^0 90^0

The formulated values from the analysis when the laminates were positioned at the angles 0^0 90^0 are as presented in Table 4.155

Table 4.155 The Bending, Coupling and Membrane stiffness for the CCSC plate with 0^0 90^0

Lamina, m = 1				
Bending	$J_1 = 0.08333$ $J_2 = 0.06667$ $J_3 = 0.05397$ $J_4 = 0.53333$	Coupling	$Jc_1 = -0.125$ $Jc_2 = -0.20833$ $Jc_3 = 0.6666$	Membrane $J_M = 1$
Lamina, m = 2				
Bending	$J_1 = 0.08333$ $J_2 = 0.06667$ $J_3 = 0.05397$ $J_4 = 0.53333$	Coupling	$Jc_1 = 0.125$ $Jc_2 = 0.20833$ $Jc_3 = 0.66667$	Membrane, $J_M = 1$

4.1.19 Results of Three laminated Clamped Clamped Simple Clamped Thick Plate

The case of Three Combined CCSC with angles of 0^0 0^0 0^0 ·The formulated values from the analysis when the layer orientation is 0^0 0^0 0^0 are as presented in Table 4.156

Table 4.156 The Bending, Coupling and Membrane stiffness for the CCSC plate with 0^0 0^0 0^0

Lamina, m = 1				
Bending	J_1 =-0.12037 J_2 = -0.09547 J_3 =-0.07653 J_4 =-0.3358	Coupling	Jc_1 = -0.33333 Jc_2 = -0.2716 Jc_3 =-0.51852	Membrane J_M =-1
Lamina, m = 2				
Bending	J_1 =-0.00926 J_2 = -0.00905 J_3 =-0.00885 J_4 = -0.92841	Coupling	Jc_1 =0.0000 Jc_2 = 0.0000 Jc_3 =-0.96298	Membrane, J_M =-1.0002
Lamina, m = 3				
Bending	J_1 = -0.12037 J_2 = -0.09547 J_3 = -0.07653 J_4 = -0.33579	Coupling	Jc_1 = 0.3333 Jc_2 = 0.2716 Jc_3 =-0.51851	Membrane J_M =-0.9999

4.1.19.1 The Buckling Values For The Case of CCSC Plate

As ealier explained, the lengthy mathematical expressions formulated were further collapsed into smaller fractions yielding the coefficients, which were considered as the unknown parameters. These were derived using the guass elimination method as expalained in chapter three. Tables 4.157, 4.158, 4.159, 4.160 and 4.161 show the values of the unknown parameters and the buckling Load equations , for the case of CCSC plate,

Table 4.157 The Coefficicents of The Formulated Parameters for the CCSC plate

P_2	P_3	P_4	P_5	P_6
-1.8522734	-0.2127422	-0.05621	0.02140941	
A_2	A_3	A_4	A_5	A_6
-1.8522734 A_1	-0.2127422 A_1	-0.05621 A_1	0.02140941 A_1	

The Buckling Load Equation for CCSC at Orientattion of 0^0 0^0

Table 4.158 The Buckling Load Equation for CCSC plate Lamina 1

Ø = 0^0 CCSC K_T – Values				
K_{ccscT1}	K_{ccscT2}	K_{ccscT3}	K_{ccscT4}	K_{ccscT5}
23.2488256	678.629808	-48.010003	0	0
$\frac{N_X}{D_O}$	27358.5201			

Table 4.159 The Buckling Load Equation for CCSC plate Lamina 2

Ø = 0^0 CCSC K_T – Values				
K_{ccscT1}	K_{ccscT2}	K_{ccscT3}	K_{ccscT4}	K_{ccscT5}
27.9501	106.825	77.7656	0	0
$\frac{N_x}{D_O}$	8892.93			

The Buckling Load Equation for CCSC at Orientattion of 0^0 90^0

Table 4.160 The Buckling Load Equation for CCSC plate Lamina 1

Ø = 0^0 SSSC K_T – Values				
K_{ccscT1}	K_{ccscT2}	K_{ccscT3}	K_{cccsT4}	K_{ccscT5}
23.2488256	678.629808	-48.010003	0	0
$\frac{N_x}{D_O}$	27358.5201			

Table 4.161 The Buckling Load Equation for CCSC plate Lamina 2

Ø = 90^0 SSSC K_T – Values				
K_{ccscT1}	K_{ccscT2}	K_{ccscT3}	K_{cccsT4}	K_{ccscT5}
1.13981	67.9664	2546.44	0	0
$\frac{N_x}{D_O}$	109437			

The Buckling Load Equation for CCSC at Orientattion of 0^0 0^0 0^0

Table 4.162 The Buckling Load Equation for CCSC plate Lamina 1

Ø = 0^0 CCSC K_T – Values				
K_{ccscT1}	K_{ccscT2}	K_{ccscT3}	K_{ccscT4}	K_{ccscT5}
-40.0912	-1117.84	-679.204	0	0
$\frac{N_x}{D_O}$	-76867.8			

Table 4.163 The Buckling Load Equation for CCSC plate Lamina 2

Ø = 0^0 CCSC K_T – Values				
K_{ccscT1}	K_{ccscT2}	K_{ccscT3}	K_{ccscT4}	K_{ccscT5}
-0.11112	-14.259	-8.7434	0	0
$\frac{N_x}{D_O}$	-967.092			

Table 4.164 The Buckling Load Equation for CCSC plate Lamina 3

$\O = 0^0$ CCSC K_T – Values				
K_{ccscT1}	K_{ccscT2}	K_{ccscT3}	K_{ccscT4}	K_{ccscT5}
-59.8892	-339.693	424.459	0	0
$\frac{N_x}{D_O}$	1040.89			

4.1.19.2 Results of different aspect ratios for Two laminates of CCSC plate with 0^0 0^0 Arrangement.

On introduction of different aspect ratios, which ranges from 1 to 2 with arithmetic increase of 0.1, gave the results as detailed in Table 4.165

Where

$$K_{ccsc} = \left(\frac{K_{ccscT1} + K_{ccscT2} + K_{ccscT3} + K_{ccscT4} + K_{ccscT5}}{K_{ccscT6}}\right) \quad (4.62)$$

Table 4.165 Buckling results of different aspect ratios for CCSC plate with 0^0 0^0 arrangement

m = 1						m = 2				
$\beta = \frac{a}{b}$	$\frac{E_0}{D_0}.Kccsc$		1.5	64292.7		$\beta = \frac{a}{b}$	$\frac{E_0}{D_0}.Kccsc$		1.5	11211.5
1	27358.5		1.6	73446.5		1	8892.93		1.6	12298.9
1.1	33623.5		1.7	93622.7		1.1	8638.2		1.7	13492.9
1.2	40429.9		1.8	93622.7		1.2	8887.14		1.8	14778.8
1.3	47798.6		1.9	1044662		1.3	9458.76		1.9	16146.7
1.4	55747.7		2.0	116346		1.4	10253		2.0	17589.8

4.1.19.3 Results of different aspect ratios for Two laminates of CCSC plate with 0^0 90^0 Arrangement.

On introduction of different aspect ratios, which ranges from 1 to 2 with arithmetic increase of 0.1, gave the results as detailed in Table 4.166

where $K_{ccsc} = \left(\frac{K_{ccscT1} + K_{ccscT2} + K_{ccscT3} + K_{ccscT4} + K_{ccscT5}}{K_{ccscT6}}\right)$ (4.63)

Table 4.166 Buckling results of different aspect ratios for CCSC Splate with $90^0\ 0^0$ arrangement

m = 1						m = 2				
$\beta = \frac{a}{b}$	$\frac{E_0}{D_0}.K_{ccsc}$		1.5	64292.7		$\beta = \frac{a}{b}$	$\frac{E_0}{D_0}.K_{ccsc}$		1.5	28463
1	27358.5		1.6	73446.5		1	109437		1.6	25168.2
1.1	33623.5		1.7	83220		1.1	75623.8		1.7	23238.9
1.2	40429.9		1.8	93622.7		1.2	54990.4		1.8	22278.7
1.3	47798.6		1.9	1044662		1.3	42023.2		1.9	22027.5
1.4	55747.7		2.0	116346		1.4	33745.5		2.0	22310.5

4.1.19.4 Results of different aspect ratios for Three laminates of CCSC plate with $0^0\ 0^0\ 0^0$ Arrangement.

A structure with different laminas was treated considering the layers in compression and

On introduction of different aspect ratios, which ranges from 1 to 2 with arithmetic increase of 0.1, gave the results as detailed in Table 4.167

where

$$K_{ccsc} = \left(\frac{K_{ccscT1} + K_{ccscT2} + K_{ccscT3} + K_{ccscT4} + K_{ccscT5}}{K_{ccscT6}}\right) \quad (4.64)$$

Table 4.167 Buckling results of different aspect ratios for CCSC plate with 0^0 0^0 0^0 arrangement.

$\beta = \frac{a}{b}$	$\frac{E_0}{D_0}.K_{ccsc}$		1.5	-110520		$\beta = \frac{a}{b}$	$\frac{E_0}{D_0}.K_{ccsc}$		1.5	-478.195
1	-76867.8		1.6	-123170		1	-470.888		1.6	-512.886
1.1	-77605.5		1.7	-137109		1.1	-433.121		1.7	-553.01
1.2	-82314.5		1.8	-152241		1.2	-423.162		1.8	-597.789
1.3	-89775.1		1.9	-168503		1.3	-430.724		1.9	-646.709
1.4	-99308.7		2.0	-185853		1.4	-450.155		2.0	-699.425

Table 4.167.

m = 3				
$\beta = \frac{a}{b}$	$\frac{E_0}{D_0}.K_{ccsc}$		1.5	-29360.8
1	1040.89		1.6	-33649.9
1.1	-7937		1.7	-37882.8
1.2	-14635.9		1.8	-42125
1.3	-20104.1		1.9	-46416.3
1.4	-24906.7		2.0	-50780.6

4.1.19.5 The case of Two Combined CSSC with angles of 0^0 0^0

The results from the analysis when the laminates were positioned at the angles 0^0 0^0 are as presented in Table 4.2.42

Table 4.168 The Bending, Coupling and Membrane stiffness for the CSSC at 0^0 0^0 angle

Lamina, m = 1				
Bending	$J_1 = 0.08333$ $J_2 = 0.06667$ $J_3 = 0.05397$ $J_4 = 0.53333$	Coupling	$Jc_1 = -0.125$ $Jc_2 = -0.20833$ $Jc_3 = 0.6666$	Membrane $J_M = 1$
Lamina, m = 2				
Bending	$J_1 = 0.08333$ $J_2 = 0.06667$ $J_3 = 0.05397$ $J_4 = 0.53333$	Coupling	$Jc_1 = 0.125$ $Jc_2 = 0.20833$ $Jc_3 = 0.66667$	Membrane, $J_M = 1$

4.1.19.6 The case of Two combined CSSC plate layer at angle 0^0 90^0

The formulated values from the analysis when the laminates were positioned at the angles 0^0 90^0 are as presented in Table 4.169

Table 4.169 The Bending, Coupling and Membrane stiffness for the CSSC plate with 0^0 90^0

Lamina, m = 1				
Bending	$J_1 = 0.08333$ $J_2 = 0.06667$ $J_3 = 0.05397$ $J_4 = 0.53333$	Coupling	$Jc_1 = -0.125$ $Jc_2 = -0.20833$ $Jc_3 = 0.6666$	Membrane $J_M = 1$
Lamina, m = 2				
Bending	$J_1 = 0.08333$ $J_2 = 0.06667$ $J_3 = 0.05397$ $J_4 = 0.53333$	Coupling	$Jc_1 = 0.125$ $Jc_2 = 0.20833$ $Jc_3 = 0.66667$	Membrane, $J_M = 1$

4.1.19.7 Results of Three laminated Clamped Simple Simple Clamped Thick Plate

The case of Three Combined CSSC with angles of 0^0 0^0 0^0 · The formulated values from the analysis when the layer orientation is 0^0 0^0 0^0 are as presented in Table 4.170

Table 4.170 The Bending, Coupling and Membrane stiffness for the CSSC plate with $0^0\ 0^0\ 0^0$

Lamina, m = 1				
Bending	J_1 =-0.12037 J_2 = -0.09547 J_3 =-0.07653 J_4 =-0.3358	Coupling	Jc_1 = -0.33333 Jc_2 = -0.2716 Jc_3 =-0.51852	Membrane J_M =-1
Lamina, m = 2				
Bending	J_1 =-0.00926 J_2 = -0.00905 J_3 =-0.00885 J_4 = -0.92841	Coupling	Jc_1 =0.0000 Jc_2 = 0.0000 Jc_3 =-0.96298	Membrane, J_M =-1.0002
Lamina, m = 3				
Bending	J_1 = -0.12037 J_2 = -0.09547 J_3 = -0.07653 J_4 = -0.33579	Coupling	Jc_1 = 0.3333 Jc_2 = 0.2716 Jc_3 =-0.51851	Membrane J_M =-0.9999

4.1.19.8 The Buckling Values For The Case of CSSC Plate

As ealier explained, the lengthy mathematical expressions formulated were further collapsed into smaller fractions yielding the coefficients, which were considered as the unknown parameters. These were derived using the guass elimination method as expalained in chapter three. Tables 4.171, 4.172, 4.173, 4.174 and 4.175 show the values of the unknown parameters and the buckling Load equations , for the case of CSSC plate,

Table 4.171 The Coefficicents of The Formulated Parameters for the CSSC plate

P_2	P_3	P_4	P_5	P_6
-2.0786844	-0.3780362	-0.17041	0.38968325	
A_2	A_3	A_4	A_5	A_6
-2.0786844 A_1	-0.3780362A_1	-0.17041A_1	0.38968325A_1	

The Buckling Load Equation for CSSC at Orientattion of $0^0\ 0^0$

Table 4.172 The Buckling Load Equation for CSSC plate Lamina 1

Ø = 0^0 CSSC K_T – Values				
K_{csscT1}	K_{csscT2}	K_{csscT3}	K_{csscT4}	K_{csscT5}
10.1543982	729.896495	-55.541141	0	0
$\frac{N_X}{D_O}$	28640.5754			

Table 4.173 The Buckling Load Equation for CSSC plate Lamina 2

Ø = 0^0 CSSC K_T – Values				
K_{csscT1}	K_{csscT2}	K_{csscT3}	K_{csscT4}	K_{csscT5}
34.5877	119.871	26.3029	0	0
$\frac{N_X}{D_O}$	7563.24			

The Buckling Load Equation for CSSC at Orientattion of 0^0 90^0

Table 4.174 The Buckling Load Equation for CSSC plate Lamina 1

Ø = 0^0 CSSC K_T – Values				
K_{csscT1}	K_{csscT2}	K_{csscT3}	K_{csscT4}	K_{csscT5}
10.1543982	729.896495	-55.541141	0	0
$\frac{N_X}{D_O}$	28640.5754			

Table 4.175 The Buckling Load Equation for CSSC plate Lamina 2

Ø = 90^0 CSSC K_T – Values				
K_{csscT1}	K_{csscT2}	K_{csscT3}	K_{csscT4}	K_{csscT5}
1.21089	97.9503	1297.48	0	0
$\frac{N_X}{D_O}$	58436.9			

The Buckling Load Equation for CSSC at Orientattion of 0^0 0^0 0^0

Table 4.175 The Buckling Load Equation for CSSC plate Lamina 1

Ø = 0^0 CSSC K_T – Values				
K_{csscT1}	K_{csscT2}	K_{csscT3}	K_{csscT4}	K_{csscT5}
-4.07233	-1019.87	-310.172	0	0
$\frac{N_X}{D_O}$	-55820.7			

Table 4.175 The Buckling Load Equation for CSSC plate Lamina 2

Ø = 0^0 CSSC K_T – Values				
K_{csscT1}	K_{csscT2}	K_{csscT3}	K_{csscT4}	K_{csscT5}
-2.82064	-3.9487	-2.44054	0	0
$\frac{N_X}{D_O}$	-385.351			

Table 4.176 The Buckling Load Equation for CSSC plate Lamina 3

$\O = 0^0$	CSSC K_T – Values			
K_{csscT1}	K_{csscT2}	K_{csscT3}	K_{csscT4}	K_{csscT5}
-65.6511	-309.746	15.8055	0	0
$\frac{N_x}{D_O}$	-15045.7			

4.1.19.9 Results of different aspect ratios for Two laminates of CSSC plate with 0^0 0^0 Arrangement.

On introduction of different aspect ratios, which ranges from 1 to 2 with arithmetic increase of 0.1, gave the results as detailed in Table 4.177

where

$$K_{ccsc} = \left(\frac{K_{csscT1} + K_{csscT2} + K_{csscT3} + K_{csscT4} + K_{csscT5}}{K_{csscT6}}\right) \quad (4.65)$$

Table 4.177 Buckling results of different aspect ratios for CSSC plate with 0^0 0^0 arrangement.

m = 1						m = 2				
$\beta = \frac{a}{b}$	$\frac{E_0}{D_0}.K_{csssc}$		1.5	70549.1		$\beta = \frac{a}{b}$	$\frac{E_0}{D_0}.K_{csssc}$		1.5	11629.6
1	28640.6		1.6	81004.2		1	7563.24		1.6	1286.6
1.1	35732.6		1.7	92195.4		1.1	7985.2		1.7	14107.1
1.2	43434.2		1.8	104135		1.2	8668.06		1.8	15465.1
1.3	51785.1		1.9	116831		1.3	9529.81		1.9	16896.9
1.4	60815.6		2.0	130292		1.4	10525.6		2.0	18400

4.1.20 Results of different aspect ratios for Two laminates of CSSC plate with 0^0 90^0 Arrangement.

On introduction of different aspect ratios, which ranges from 1 to 2 with arithmetic increase of 0.1, gave the results as detailed in Table 4.178

where K_{cssc}

$$= \left(\frac{K_{csscT1} + K_{csscT2} + K_{csscT3} + K_{csscT4} + K_{csscT5}}{K_{csscT6}}\right) \quad (4.66)$$

Table 4.178 Buckling results of different aspect ratios for CSSC Splate with $90^0\ 0^0$ arrangement

m = 1						m = 2				
$\beta = \frac{a}{b}$	$\frac{E_0}{D_0}.Kcssc$		1.5	70549.1		$\beta = \frac{a}{b}$	$\frac{E_0}{D_0}.Kcssc$		1.5	19247.1
1	28640.6		1.6	81004.2		1	58436.9		1.6	18196.5
1.1	35732.6		1.7	92195.4		1.1	41351		1.7	17882.4
1.2	43434.2		1.8	104135		1.2	31188.5		1.8	18100.8
1.3	51785.1		1.9	116831		1.3	25042.1		1.9	18718
1.4	60815.6		2.0	130292		1.4	21355.2		2.0	19644.4

4.1.21.1 Results of different aspect ratios for Three laminates of CSSC plate with $0^0\ 0^0\ 0^0$ Arrangement.

On introduction of different aspect ratios, which ranges from 1 to 2 with arithmetic increase of 0.1, gave the results as detailed in Table 4.179

where

$$K_{sssc} = \left(\frac{K_{csscT1} + K_{csscT2} + K_{csscT3} + K_{csscT4} + K_{csscT5}}{K_{csscT6}}\right) \quad (4.67)$$

Table 4.179a Buckling results of different aspect ratios for CSSC plate with $0^0\ 0^0\ 0^0$ arrangement

m = 1						m = 2				
$\beta = \frac{a}{b}$	$\frac{E_0}{D_0}.K_{cssc}$		1.5	-99458.1		$\beta = \frac{a}{b}$	$\frac{E_0}{D_0}.K_{cssc}$		1.5	509.696
1	-55820.7		1.6	-112509		1	-385.351		1.6	-556.9
1.1	-60620.1		1.7	-126662		1.1	-387.024		1.7	-608.823
1.2	-67887.7		1.8	-141873		1.2	-494.298		1.8	-665.092
1.3	-76990		1.9	-158117		1.3	-432.138		1.9	-725.461
1.4	-87579.5		2.0	-175377		1.4	-467.795		2.0	-789.766

Table 4.179b

m = 3				
$\beta = \frac{a}{b}$	$\frac{E_0}{D_0}.K_{cssc}$		1.5	-28356
1	-15045.7		1.6	-30612.9
1.1	-18439.9		1.7	-32848
1.2	-21221.**6**		1.8	-35061.8
1.3	-23712.4		1.9	-37249.8
1.4	-26066.4		2.0	-39405.5

4.1.20.2 Detailed Output For SSCC Thick Laminated Anisotropic Plate.

For different lamina combination for SSCC plate, the results of the analysis are as detailed in this section.

4.1.20.3 The case of Two Combined SSCC with angles of $0^0\ 0^0$

The results from the analysis when the laminates were positioned at the angles $0^0\ 0^0$ are as presented in Table 4.180

Table 4.180 The Bending, Coupling and Membrane stiffness for the SSCC at 0^0 0^0 angle

Lamina, m = 1				
Bending	$J_1 = 0.08333$ $J_2 = 0.06667$ $J_3 = 0.05397$ $J_4 = 0.53333$	Coupling	$Jc_1 = -0.125$ $Jc_2 = -0.20833$ $Jc_3 = 0.6666$	Membrane $J_M = 1$
Lamina, m = 2				
Bending	$J_1 = 0.08333$ $J_2 = 0.06667$ $J_3 = 0.05397$ $J_4 = 0.53333$	Coupling	$Jc_1 = 0.125$ $Jc_2 = 0.20833$ $Jc_3 = 0.66667$	Membrane, $J_M = 1$

4.1.20.4 The case of Two combined SSCC plate layer at angle 0^0 90^0

The formulated values from the analysis when the laminates were positioned at the angles 0^0 90^0 are as presented in Table 4.181

Table 4.181 The Bending, Coupling and Membrane stiffness for the SSCC plate with **0^0 90^0**

Lamina, m = 1				
Bending	$J_1 = 0.08333$ $J_2 = 0.06667$ $J_3 = 0.05397$ $J_4 = 0.53333$	Coupling	$Jc_1 = -0.125$ $Jc_2 = -0.20833$ $Jc_3 = 0.6666$	Membrane $J_M = 1$
Lamina, m = 2				
Bending	$J_1 = 0.08333$ $J_2 = 0.06667$ $J_3 = 0.05397$ $J_4 = 0.53333$	Coupling	$Jc_1 = 0.125$ $Jc_2 = 0.20833$ $Jc_3 = 0.66667$	Membrane, $J_M = 1$

4.1.20.5 Results of Three laminated Simple Simple Clamped Clamped Thick Plate

The case of Three Combined SSCC with angles of 0^0 0^0 0^0 The formulated values from the analysis when the layer orientation is 0^0 0^0 0^0 are as presented in Table 4.182

Table 4.182 The Bending, Coupling and Membrane stiffness for the SSCC plate with $0^0\ 0^0\ 0^0$

Lamina, m = 1				
Bending	J_1 =-0.12037 J_2 = -0.09547 J_3 =-0.07653 J_4 =-0.3358	Coupling	Jc_1 = -0.33333 Jc_2 = -0.2716 Jc_3 =-0.51852	Membrane J_M =-1
Lamina, m = 2				
Bending	J_1 =-0.00926 J_2 = -0.00905 J_3 =-0.00885 J_4 = -0.92841	Coupling	Jc_1 =0.0000 Jc_2 = 0.0000 Jc_3 =-0.96298	Membrane, J_M =-1.0002
Lamina, m = 3				
Bending	J_1 = -0.12037 J_2 = -0.09547 J_3 = -0.07653 J_4 = -0.33579	Coupling	Jc_1 = 0.3333 Jc_2 = 0.2716 Jc_3 =-0.51851	Membrane J_M =-0.9999

4.1.20.6 The Buckling Values For The Case of SSCC Plate

As ealier explained, the lengthy mathematical expressions formulated were further collapsed into smaller fractions yielding the coefficients, which were considered as the unknown parameters. These were derived using the guass elimination method as expalained in chapter three. Tables 4.183, 4.184, 4.185, 4.186 and 4.187 show the values of the unknown parameters and the buckling Load equations , for the case of SSCC plate,

Table 4.183 The Coefficicents of The Formulated Parameters for the SSCC plate

P_2	P_3	P_4	P_5	P_6
-1.1551685	17.1064216	0.0291	0.09542725	
A_2	A_3	A_4	A_5	A_6
-1.1551685A_1	17.1064216A_1	0.0291A_1	0.09542725 A_1	

The Buckling Load Equation for SSCC at Orientattion of $0^0\ 0^0$

Table 4.184 The Buckling Load Equation for SSCC plate Lamina 1

Ø = 0^0 SSCC K_T – Values				
K_{ssccT1}	K_{ssccT2}	K_{ssccT3}	K_{ssccT4}	K_{ssccT5}
19.1395653	11.4850383	864.30001	0	0
$\frac{N_X}{D_O}$	37444.5445			

Table 4.185 The Buckling Load Equation for SSCC plate Lamina 2

$Ø = 0^0$ SSCC K_T – Values				
K_{ssccT1}	K_{ssccT2}	K_{ssccT3}	K_{ssccT4}	K_{ssccT5}
28.362	176.288	204.285	0	0
$\frac{N_X}{D_O}$	17110.2			

The Buckling Load Equation for SSCC at Orientattion of 0^0 90^0

Table 4.186 The Buckling Load Equation for SSCC plate Lamina 1

$Ø = 0^0$ SSCC K_T – Values				
K_{ssccT1}	K_{ssccT2}	K_{ssccT3}	K_{ssccT4}	K_{ssccT5}
19.1395653	11.4850383	864.30001	0	0
$\frac{N_X}{D_O}$	37444.5445			

Table 4.187 The Buckling Load Equation for SSCC plate Lamina 2

$Ø = 90^0$ SSCC K_T – Values				
K_{ssccT1}	K_{ssccT2}	K_{ssccT3}	K_{ssccT4}	K_{ssccT5}
3.31589	934.152	39078	0	0
$\frac{N_X}{D_O}$	1674286			

The Buckling Load Equation for SSCC at Orientattion of 0^0 0^0 0^0

Table 4.188 The Buckling Load Equation for SSCC plate Lamina 1

$Ø = 0^0$ SSCC K_T – Values				
K_{ssccT1}	K_{ssccT2}	K_{ssccT3}	K_{ssccT4}	K_{ssccT5}
-28.6657	983.767	296.705	0	0
$\frac{N_X}{D_O}$	52376.8			

Table 4.189 The Buckling Load Equation for SSCC plate Lamina 2

$Ø = 0^0$ SSCC K_T – Values				
K_{ssccT1}	K_{ssccT2}	K_{ssccT3}	K_{ssccT4}	K_{ssccT5}
-0.11112	-11.4138	-4.47733	0	0
$\frac{N_X}{D_O}$	-669.551			

Table 4.190 The Buckling Load Equation for SSCC plate Lamina 3

Ø = 0^0	SSCC K_T – Values			
K_{ssccT1}	K_{ssccT2}	K_{ssccT3}	K_{ssccT4}	K_{ssccT5}
-46.6296	-483.342	866.602	0	0
$\frac{N_x}{D_O}$	14085			

4.1.20.7 Results of different aspect ratios for Two laminates of SSCC plate with 0^0 0^0 Arrangement.

On introduction of different aspect ratios, which ranges from 1 to 2 with arithmetic increase of 0.1, gave the results as detailed in Table 4.191

where

$$K_{sscc} = \left(\frac{K_{ssccT1} + K_{ssccT2} + K_{ssccT3} + K_{ssccT4} + K_{ssccT5}}{K_{ssccT6}}\right) \quad (4.68)$$

Table 4.191 Buckling results of different aspect ratios for SSCC plate with 0^0 0^0 arrangement

m = 1						m = 2				
$\beta = \frac{a}{b}$	$\frac{E_0}{D_0}.K_{sscc}$		1.5	-34690.8		$\beta = \frac{a}{b}$	$\frac{E_0}{D_0}.K_{sscc}$		1.5	1268.94
1	37444.5		1.6	-58124.3		1	17110.2		1.6	-4651.51
1.1	27349.5		1.7	-85498.8		1.1	14093.6		1.7	-12375.8
1.2	15657.2		1.8	-117093		1.2	11611.2		1.8	-22156.6
1.3	1765.89		1.9	-153131		1.3	8991.57		1.9	-34198.1
1.4	-14860.9		2.0	-193790		1.4	5692.99		2.0	-48662.7

4.1.20.8 Results of different aspect ratios for Two laminates of SSCC plate with 0^0 90^0 Arrangement.

On introduction of different aspect ratios, which ranges from 1 to 2 with arithmetic increase of 0.1, gave the results as detailed in Table 4.192

where

$$K_{sscc} = \left(\frac{K_{ssccT1} + K_{ssccT2} + K_{ssccT3} + K_{ssccT4} + K_{ssccT5}}{K_{ssccT6}}\right) \quad (4.69)$$

Table 4.192 Buckling results of different aspect ratios for SSCC Splate with $90^0\ 0^0$ arrangement

m = 1						m = 2				
$\beta = \frac{a}{b}$	$\frac{E_0}{D_0}.K_{csssc}$		1.5	-34690.8		$\beta = \frac{a}{b}$	$\frac{E_0}{D_0}.K_{csssc}$		1.5	3769.69
1	37444.5		1.6	-58124.3		1	1674286		1.6	9151.57
1.1	27349.5		1.7	-85498.8		1.1	-308891		1.7	12771.7
1.2	15657.2		1.8	-117093		1.2	-77946.5		1.8	15630.8
1.3	1765.89		1.9	-153131		1.3	-25521.3		1.9	18207.8
1.4	-14860.9		2.0	-193790		1.4	-5693.87		2.0	20749.9

4.1.20.9 Results of different aspect ratios for Three laminates of SSCC plate with $0^0\ 0^0\ 0^0$ Arrangement.

On introduction of different aspect ratios, which ranges from 1 to 2 with arithmetic increase of 0.1, gave the results as detailed in Table 4.193

where

$$K_{sscc} = \left(\frac{K_{ssccT1} + K_{ssccT2} + K_{ssccT3} + K_{ssccT4} + K_{ssccT5}}{K_{ssccT6}}\right) \quad (4.70)$$

Table 4.193a Buckling results of different aspect ratios for SSCC plate with $0^0\ 0^0\ 0^0$ arrangement

m = 1						m = 2				
$\beta = \frac{a}{b}$	$\frac{E_0}{D_0} . K_{csssc}$		1.5	19745.8		$\beta = \frac{a}{b}$	$\frac{E_0}{D_0} . K_{csssc}$		1.5	-439.902
1	48276.5		1.6	22022.7		1	-348.774		1.6	-477.626
1.1	36856.6		1.7	26562.2		1.1	-346.249		1.7	-564.481
1.2	28289.7		1.8	33216.1		1.2	-357.805		1.8	-564.481
1.3	22685.1		1.9	41857.4		1.3	-378.798		1.9	-613.013
1.4	19902.5		2.0	52376.8		1.4	-406.66		2.0	-664.707

Table 4.193b.

m = 3				
$\beta = \frac{a}{b}$	$\frac{E_0}{D_0} . K_{csssc}$		1.5	-19469.3
1	14085		1.6	-10351.8
1.1	-4038.47		1.7	4115.39
1.2	-15893.8		1.8	24130.5
1.3	-2223.5		1.9	49892.8
1.4	-23402.7		2.0	81583.9

4.2 Discussions of Results

The work done in this project is large enough and elaborate to accomadate other plate conditions and different angle orientatations which were not treated in the work. The developed computer programes also, is very flexible and can handle plate structure of any number of layers or laminas. In each case of plate treated, the same buckling load equation were applied in deriving their various coefficients. It was observed that values derived by substitutitng the 3^{rd} order stiffness coefficients are very similar to those gotten using 2^{nd} and

4^{th} stiffness coefficients as in the case of Ritz and Garlerkin. From the direct and general variation, the formulated four compatible equation were resolved using Guass elimination method. The gauss soluuons provided the P-values, which were used to calculate the various Buckling Coefficients. The iteration processes continued repeatedly for ech lamina or layer up to the laminate.

4.2.1 Stiffness Coefficients

Three major stiffness were formulated in the course of the work. They include bending stiffness, coupling stiffness and axil stiffness. Theaxial stiffness is sometimes knowns as membrane stiffness. The derived strain energy equation can be grouped categorically under these three stifnesses or as a function of the engineering properties which includes $(w, u_o, v_o, \emptyset_x, \emptyset_y)$. These properties are either out of plane displacement, middle layer inplane displacement or shear rotation.

4.2.2 Derivation of The Various Stiffnesses

From the mathematical expression, it was observed that some of the coefficients have values like S^2 or H^2 other have S or H and the rest have neither S nor H. A close look shows that the first category with S^2 or H^2 are multiplying either w.w or $\emptyset.\emptyset$ while the second category with S or H is either multiplying w.u or w.v and final categories which have neither S nor H. They just possess either u_o, v_o or $u_o.v_o$ values. The first category is considered the bending stiffness, while the the second and the third are called the coupling and axial stiffness respectively.

4.2.3 Derivation of The Compatibility Equation

The total Potential Energy, which is the sum of the Strain Energy andwork done by external Load applied was differentiated at five different stages. The first stage was with respect to the amplitude otherwise known as the Governing Equation. Secondly it was differentiated with the second amplitude A_2 while the third and forth compatibility equations were derived using A_3 and A_4 respectively. The stages involved in the formulation of the four equations were solved using Gauss elimination method as detailed in chapter 3.

4.2.4 Iteration Process For the SSSS Laminated Thick Plate

The First Laminated thick SSSS plate considered, has 0^o, 0^o arrangement other shape orientation 90^o 0^o , 0^o 90^o 0^o and 0^o 0^o 0^o were considered. On the introduction of the shear Modulus, Elastic Modulus and Elastic Modulus tangible changes were observed when the number of the laminas were increased, thereby increasing the number of the angles of inclination. Also the values of the buckling coefficients were significantly ultered when the values of the angle were changed. The angles considered are 0^o and 90^o . The entire process was programmed to iterate up to the number of the Laminates. For the case of 2 Lamintes, the programme authomatically iterated twice and more, when the angle of inclination is ultered. For the case of 3 Laminates, the programmes iterates thrice and more, when the values of the angles are changed. Also on introduction of different aspect ratio, the buckling Equation shows remarkable arithemetic increase.

4.2.5 Iteration Process for The CCCC Laminated Thick Plate

The First Laminated thick CCCC plate considered, has 0^o, 0^o arrangement other shape orientation 90^o 0^o , 0^o 90^o 0^o and 0^o 0^o 0^o were considered. On the introduction of the shear Modulus, Elastic Modulus and Elastic Modulus tangible changes were observed when the number of the laminas were increased, thereby increasing the number of the angles of inclination. Also the values of the buckling coefficients were significantly ultered when the values of the angle were changed. The angles considered are 0^o and 90^o . The entire process was programmed to iterate up to the number of the Laminates. For the case of 2 Lamintes, the programme authomatically iterated twice and more, when the angle of inclination is ultered. For the case of 3 Laminates, the programmes iterates thrice and more, when the values of the angles are changed. Also on introduction of different aspect ratio, the buckling Equation shows remarkable arithemetic increase.

4.2.6 Iteration Process for The CSCS Laminated Thick Plate

The First Laminated thick CSCS plate considered, has 0^o, 0^o arrangement other shape orientation 90^o 0^o , 0^o 90^o 0^o and 0^o 0^o 0^o were considered. On the introduction of the shear Modulus, Elastic Modulus and Elastic Modulus tangible changes were observed when the number of the laminas were increased, thereby increasing the number of the angles of inclination. Also the values of the buckling coefficients were significantly ultered when the values of the angle were changed. The angles considered are 0^{o} and 90^{o} . The entire process was programmed to iterate up to the number of the Laminates. For the case of 2 Lamintes, the programme authomatically iterated twice and more, when the angle of inclination is ultered. For the case of 3 Laminates, the programmes iterates thrice and more, when the values of the angles are changed. Also on introduction of different aspect ratio, the buckling Equation shows remarkable arithemetic increase as already demonstrated.

4.2.7 Iteration Process for The CSSS Laminated Thick Plate

The First Laminated thick CSSS plate considered, has 0^{o}, 0^{o} arrangement other shape orientation 90^{o} 0^{o} , 0^{o} 90^{o} 0^{o} and 0^{o} 0^{o} 0^{o} were considered. On the introduction of the shear Modulus, Elastic Modulus and Elastic Modulus tangible changes were observed when the number of the laminas were increased, thereby increasing the number of the angles of inclination. Also the values of the buckling coefficients were significantly ultered when the values of the angle were changed. The angles considered are 0^{o} and 90^{o} . The entire process was programmed to iterate up to the number of the Laminates. For the case of 2 Lamintes, the programme authomatically iterated twice and more, when the angle of inclination is ultered. For the case of 3 Laminates, the programmes iterates thrice and more, when the values of the angles are changed. Also on introduction of different aspect ratio, the buckling Equation shows remarkable arithemetic increase as shown in the tables.

4.2.8 Iteration Process for The SCCS Laminated Thick Plate

The First Laminated thick SCCS plate considered, has 0^{o}, 0^{o} arrangement other shape orientation 90^{o} 0^{o} , 0^{o} 90^{o} 0^{o} and 0^{o} 0^{o} 0^{o} were considered. On the introduction of the shear Modulus, Elastic Modulus and Elastic Modulus tangible changes were observed when the number of the laminas were increased, thereby increasing the number of the angles of inclination. Also the values of the buckling coefficients were significantly ultered when the values of the angle were changed. The angles considered are 0^{o} and 90^{o} . The entire process was programmed to iterate up to the number of the Laminates. For the case of 2 Lamintes, the programme authomatically iterated twice and more, when the angle of inclination is ultered. For the case of 3 Laminates, the programmes iterates thrice and more, when the values of the angles are changed. Also on introduction of different aspect ratio, the buckling Equation shows remarkable arithemetic increase.

4.2.9 Iteration Process for The SCSC Laminated Thick Plate

The First Laminated thick SCSC plate considered, has 0^o, 0^o arrangement other shape orientation 90^o 0^o , 0^o 90^o 0^o and 0^o 0^o 0^o were considered. On the introduction of the shear Modulus, Elastic Modulus and Elastic Modulus tangible changes were observed when the number of the laminas were increased, thereby increasing the number of the angles of inclination. Also the values of the buckling coefficients were significantly ultered when the values of the angle were changed. The angles considered are 0^o and 90^o . The entire process was programmed to iterate up to the number of the Laminates. For the case of 2 Lamintes, the programme authomatically iterated twice and more, when the angle of inclination is ultered. For the case of 3 Laminates, the programmes iterates thrice and more, when the values of the angles are changed. Also on introduction of different aspect ratio, the buckling Equation shows remarkable arithemetic increase.

4.2.10 Iteration Process for The SSCS Laminated Thick Plate

The First Laminated thick SSCS plate considered, has 0^o, 0^o arrangement other shape orientation 90^o 0^o , 0^o 90^o 0^o and 0^o 0^o 0^o were considered. On the introduction of the shear Modulus, Elastic Modulus and Elastic Modulus tangible changes were observed when the number of the laminas were increased, thereby increasing the number of the angles of inclination. Also the values of the buckling coefficients were significantly ultered when the values of the angle were changed. The angles considered are 0^o and 90^o . The entire process was programmed to iterate up to the number of the Laminates. For the case of 2 Lamintes, the programme authomatically iterated twice and more, when the angle of inclination is ultered. For the case of 3 Laminates, the programmes iterates thrice and more, when the values of the angles are changed. Also on introduction of different aspect ratio, the buckling Equation shows remarkable arithemetic increase.

4.2.11 Iteration Process for The SCCC Laminated Thick Plate

The First Laminated thick SCCC plate considered, has 0^o, 0^o arrangement other shape orientation 90^o 0^o , 0^o 90^o 0^o and 0^o 0^o 0^o were considered. On the introduction of the shear Modulus, Elastic Modulus and Elastic Modulus tangible changes were observed when the number of the laminas were increased, thereby increasing the number of the angles of inclination. Also the values of the buckling coefficients were significantly ultered when the values of the angle were changed. The angles considered are 0^o and 90^o . The entire process was programmed to iterate up to the number of the Laminates. For the case of 2 Lamintes, the programme authomatically iterated twice and more, when the angle of inclination is ultered. For the case of 3 Laminates, the programmes iterates thrice and more, when the values of the angles are changed. Also on introduction of different aspect ratio, the buckling Equation shows remarkable arithemetic increase.

4.2.12 Iteration Process for The CCCS Laminated Thick Plate

The First Laminated thick CCCS plate considered, has 0^o, 0^o arrangement other shape orientation 90^o 0^o , 0^o 90^o 0^o and 0^o 0^o 0^o were considered. On the introduction of the shear Modulus, Elastic Modulus and Elastic Modulus tangible changes were observed when the number of the laminas were increased, thereby increasing the number of the angles of inclination. Also the values of the buckling coefficients were significantly ultered when the values of the angle were changed. The angles considered are 0^o and 90^o . The entire process was programmed to iterate up to the number of the Laminates. For the case of 2 Lamintes, the programme authomatically iterated twice and more, when the angle of inclination is ultered. For the case of 3 Laminates, the programmes iterates thrice and more, when the values of the angles are changed. Also on introduction of different aspect ratio, the buckling Equation shows remarkable arithemetic increase.

4.2.13 Iteration Process for The SSSC Laminated Thick Plate

The First Laminated thick SSSC plate considered, has 0^o, 0^o arrangement other shape orientation 90^o 0^o , 0^o 90^o 0^o and 0^o 0^o 0^o were considered. On the introduction of the shear Modulus, Elastic Modulus and Elastic Modulus tangible changes were observed when the number of the laminas were increased, thereby increasing the number of the angles of inclination. Also the values of the buckling coefficients were significantly ultered when the values of the angle were changed. The angles considered are 0^o and 90^o . The entire process was programmed to iterate up to the number of the Laminates. For the case of 2 Lamintes, the programme authomatically iterated twice and more, when the angle of inclination is ultered. For the case of 3 Laminates, the programmes iterates thrice and more, when the values of the angles are changed. Also on introduction of different aspect ratio, the buckling Equation shows remarkable arithemetic increase.

4.2.14 Iteration Process for The CCSC Laminated Thick Plate

The First Laminated thick CCSC plate considered, has 0^o, 0^o arrangement other shape orientation 90^o 0^o , 0^o 90^o 0^o and 0^o 0^o 0^o were considered. On the introduction of the shear Modulus, Elastic Modulus and Elastic Modulus tangible changes were observed when the number of the laminas were increased, thereby increasing the number of the angles of inclination. Also the values of the buckling coefficients were significantly ultered when the values of the angle were changed. The angles considered are 0^o and 90^o . The entire process was programmed to iterate up to the number of the Laminates. For the case of 2 Lamintes, the programme authomatically iterated twice and more, when the angle of inclination is ultered. For the case of 3 Laminates, the programmes iterates thrice and more, when the values of the angles are changed. Also on introduction of different aspect ratio, the buckling Equation shows remarkable arithemetic increase.

4.2.15 Iteration Process for The CSSC Laminated Thick Plate

The First Laminated thick CSSC plate considered, has 0^o, 0^o arrangement other shape orientation 90^o 0^o , 0^o 90^o 0^o and 0^o 0^o 0^o were considered. On the introduction of the shear Modulus, Elastic Modulus and Elastic Modulus tangible changes were observed when the number of the laminas were increased, thereby increasing the number of the angles of inclination. Also the values of the buckling coefficients were significantly ultered when the values of the angle were changed. The angles considered are 0^o and 90^o . The entire process was programmed to iterate up to the number of the Laminates. For the case of 2 Lamintes, the programme authomatically iterated twice and more, when the angle of inclination is ultered. For the case of 3 Laminates, the programmes iterates thrice and more, when the values of the angles are changed. Also on introduction of different aspect ratio, the buckling Equation shows remarkable arithemetic increase.

4.2.16 Iteration Process for The SSCC Laminated Thick Plate

The First Laminated thick SSCC plate considered, has 0^o, 0^o arrangement other shape orientation 90^o 0^o , 0^o 90^o 0^o and 0^o 0^o 0^o were considered. On the introduction of the shear Modulus, Elastic Modulus and Elastic Modulus tangible changes were observed when the number of the laminas were increased, thereby increasing the number of the angles of inclination. Also the values of the buckling coefficients were significantly ultered when the values of the angle were changed. The angles considered are 0^o and 90^o . The entire process was programmed to iterate up to the number of the Laminates. For the case of 2 Lamintes, the programme authomatically iterated twice and more, when the angle of inclination is ultered. For the case of 3 Laminates, the programmes iterates thrice and more, when the values of the angles are changed. Also on introduction of different aspect ratio, the buckling Equation shows remarkable arithemetic increase.

4.3 Comparison of the Present Work with Previuos Establishment

Using Equation (4.71) results from the present work were compared with those from existing literature, it was discovered that the derived values gotten from this present research were in good agreement with the previous. The percentage differences were very infitestimal and in some cases almost the same.

$$\text{Percentage Difference} = \left(\frac{V_{pr} - V_{pv}}{V_{pr}} \times 100\%\right) \quad (4.71)$$

where

V_{pr} and V_{pv} are Present and Previous study respectively.

Table 4.2.70 shows the comparison the stiffness coefficients generated from the present work (Third order principle) with those from previous work. The previous work were done based on Ritz and Garlekin approach.

Table 4.194: Comparison of Third order stiffeness with those from previous work done using Ritz and Garlekin approach for the cases of CSCS and CCCS thick laminated anisotropic plates .

Period	Plate	Stiffeness		Period	Plate	Stiffness
Present	CSCS	k_1 = 0.00762		Previous	CSCS	k_1 = 0.007612
		k_2 = 0.009252				k_2 = 0.00925
		k_3= 0.03937				k_3 = 0.03937
Present	CCCS	k_1 = 0.002857		Previous	CCCS	k_1 = 0.002856
		k_2 = 0.0016327				k_2 = 0. 001633
		k_3 = 0.0060317				k_3 = 0.00603
Percentage Diff. for CSCS	k_1	0.053%		Percentage Diff. for CSSS	k_1	0.033%
	k_2	0.001%			k_2	0.0012%
	k_3	0.0363%			k_3	0.0298

Table 4.195: Comparison of Third order stiffeness with those from previous work done using Ritz and Garlekin approach for the cases of CCCC and CCSS thick laminated anisotropic plates .

Period	Plate	Stiffeness		Period	Plate	Stiffness
Present	CCCC	k_1 = 0.00127		Previous	CCCC	k_1 = 0.00195
		k_2 =0.00036				k_2 = 0.003600
		k_3 = 0.00127				k_3 = 0.00127
Present	CCSS	k_1 = 0.013572		Previous	CCSS	k_1 = 0.013566
		k_2 = 0.007347				k_2 = 0.007346
		k_3 = 0.013572				k_3 = 0.013571
Percentage Diff. for CCCC	k_1	0.02%		Percentage Diff. for CCSS	k_1	0.0441%
	k_2	0.001%			k_2	0.0091%
	k_3	0.063%			k_3	0.0081%

The various buckling loads were detailed in Table 4.2.70 and compared with previous researchers. The values derived from the works of Megson (2010), Chajes (1794) were achieved using 2nd and 4th order functional. The work of Ventsel & Krauthermmer (2001) also showed great similarity both in their stiffnesses and buckling load. For the cases of SCSC and

CSSS thick laminated anisotropic plates. The consideration was done for different intervals of the aspect as shown in Table 4.2.70 and Table 4.2.71. The thick laminated anisotropic plates were considered to be in angle orientation of 0^0 0^0.

Table 4.196: Comparison of Buckling Loads results from previous works done using Ritz and Garlekin approach, with the results from this present work for the case of CSSS thick laminated anisotropic plates.

Aspect Ratios (p = b/a)	Present Study	(Previous Study) Ventse l& Krauthermmer (2001),	% Diff. Present and Previous work
1	21909.79	21909.85	-0.00028
1.1	27498.82	27498.73	0.000329
1.6	62484.22	62483.25	0.001547
1.9	89555.81	89554.27	0.001723
2	99612.99	99612.6	0.000392

Table 4.197: Comparison of Buckling Loads results from previous works done using Ritz and Garlekin approach, with the results from this present work for the case of SCSC thick laminated anisotropic plates.

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Aspect Ratios (p = b/a)	Present Study	(Previous Study) Ventse l& Krauthermmer (2001),	% Diff. Present and Previous work
1	26072.3	26072.23	-0.00028
1.1	31584.2	31584.1	0.000329
1.6	74406	74405.76	0.001547
1.9	92752.8	92751.2	0.001723
2	102727	102726.6	0.000392

CHAPTER 5

CONCLUSIONS AND RECOMMEDATIONS

5.1 Conclusions

The shape functions of the various plates were formulated. This made it possible to know the various stiffness coefficients of all the plate elements. The kinematic relation were derived, together with stress – strain relation. The formulation of the above mention parameters contributed immensely in the determination of strain Energy of Thick a laminated plate. The strain was transformed from global to local coordinate system and also the relationship between strain vector and the strain tensor was established. The strain energy was categorized in to 3 stiffness namely Bending stiffness U_B, Coupling stiffness, U_C and Membrane / Axial stiffness, U_M. These values of strain energy were differentiated to obtain the equation of equilibrium of forces. The total potential functional of thick rectangular anisotropic plate was derived by adding the differential values of strain Energy and the external work done on the plate .From the various process adopted in the course of the work the following conclusion can be drawn from the entire research work.

(i) The strain equations of the thick laminated anisotropic plates was formulated .
(ii) The stress equation of the thick laminated anisotropic plates was derived.
(ii) The Stress-Strain Relation for a lamina of the thick laminated plates were determined and translated from Local coordinate to Global coordinate system.
(iv) The Strain Energy, Total Potential Energy functional and Governing Equation of Equilibrium were obtained by minimizing the Total Potential Energy for a thick laminated rectangular anisotropic plate
(v) The numerical analysis of thick laminated Anisotropic plate was conducted considering different edge/boundary orientations.

5.2 Recommendations

The following recommendations were made:

i. More research work on laminated thick plate should be carried out to determine free vibration analysis.

ii. Other researchers should carry out further investigation on pure bending analysis of thick plate

iii. Third other energy functional should be adopted in carry out research on laminated case of thin plate element

iv. .Gauss Elimination Program developed can be used in resolving high engineering problems.

5.3 Contributions To Knowledge

This research work "Stability Analysis of Thick Laminated anisotropic plate using third other energy Functional" has contribution in the following knowledge :

i. The research work has created platform for the resolution of thin plate and other thick plates using third order energy functional.

The research work clearly shows the different results of the Strain Energy when the dealing with thick anisotropic plate, in terms of Bending, Coupling and Axial Stiffness mathematically as:

$$U = \frac{abt}{2}(\sigma.\varepsilon)_B + (\sigma.\varepsilon)_C + (\sigma.\varepsilon)_M \quad dR\, dQ\, dS \tag{5.1}$$

where Strain Energy for case of Bending Stiffness, U_b is

$$\frac{E_0 t^3}{[1-\mu_{XY}\mu_{YX}]} B_{11}\left(S^2\left(\frac{\partial^2 w}{\partial R^2}\right)^2 - aHS.\frac{\partial^2 w}{\partial R^2}.\frac{\partial \phi_x}{\partial R} - aHS.\frac{\partial^2 w}{\partial R^2}.\frac{\partial \phi_x}{\partial R} + a^2S^2.\left(\frac{\partial \phi_x}{\partial R}\right)^2\right)$$

$$+2B_{12}\left(\frac{S^2}{\beta^2}.\frac{\partial^2 w}{\partial R^2}.\frac{\partial^2 w}{\partial Q^2} - \frac{aHS}{\beta}.\frac{\partial^2 w}{\partial R^2}.\frac{\partial \phi_y}{\partial Q} - \frac{aHS}{\beta^2}\frac{\partial^2 w}{\partial Q^2}.\frac{\partial \phi_x}{\partial R} + \frac{a^2H^2}{\beta}.\frac{\partial \phi_y}{\partial Q}.\frac{\partial \phi_x}{\partial R}\right)$$

$$+2B_{13}\left(-\frac{2S^2}{\beta}\frac{\partial^2 w}{\partial R\,\partial Q}.\frac{\partial^2 w}{\partial R^2} - \frac{aHS}{\beta}\frac{\partial^2 w}{\partial R^2}.\frac{\partial \phi_x}{\partial Q} - aHS\frac{\partial^2 w}{\partial R^2}.\frac{\partial \phi_y}{\partial R} - \frac{2aHS}{\beta}\frac{\partial^2 w}{\partial R\,\partial Q}.\frac{\partial \phi_x}{\partial R} + \frac{a^2H^2}{\beta}.\frac{\partial \phi_x}{\partial R}.\frac{\partial \phi_x}{\partial Q} + a^2H^2.\frac{\partial \phi_y}{\partial R}.\frac{\partial \phi_x}{\partial R}\right)$$

$$+B_{22}\left(\frac{S^2}{\beta^4}\left(\frac{\partial^2 w}{\partial Q^2}\right)^2 - \frac{aHS}{\beta^3}.\frac{\partial^2 w}{\partial Q^2}.\frac{\partial \phi_y}{\partial Q} - \frac{aHS}{\beta^3}\frac{\partial^2 w}{\partial Q^2}.\frac{\partial \phi_y}{\partial Q} + \frac{a^2H^2}{\beta^2}.\left(\frac{\partial \phi_y}{\partial Q}\right)^2\right)$$

$$+2B_{23}\left(\frac{2S^2}{\beta^3}\frac{\partial^2 w}{\partial R\,\partial Q}.\frac{\partial^2 w}{\partial Q^2}-\frac{aSH}{\beta^3}.\frac{\partial\phi_x}{\partial Q}.\frac{\partial^2 w}{\partial Q^2}-\frac{aSH}{\beta^2}.\frac{\partial\phi_y}{\partial R}.\frac{\partial^2 w}{\partial Q^2}-\frac{2aHS}{\beta^2}\frac{\partial^2 w}{\partial R\,\partial Q}.\frac{\partial\phi_y}{\partial Q}\right.$$
$$\left.+\frac{a^2H^2}{\beta^2}.\frac{\partial\phi_x}{\partial Q}.\frac{\partial\phi_y}{\partial Q}+\frac{a^2H^2}{\beta}.\frac{\partial\phi_y}{\partial R}.\frac{\partial\phi_y}{\partial Q}\right)$$

$$+B_{33}\left(\frac{4S^2}{\beta^2}\left(\frac{\partial^2 w}{\partial R\,\partial Q}\right)^2-\frac{2aHS}{\beta^2}.\frac{\partial^2 w}{\partial R\,\partial Q}.\frac{\partial\phi_x}{\partial Q}-\frac{2aHS}{\beta}.\frac{\partial^2 w}{\partial R\,\partial Q}.\frac{\partial\phi_y}{\partial R}\right)$$

$$+B_{33}\left(-\frac{2aHS}{\beta^2}\frac{\partial^2 w}{\partial R\,\partial Q}.\frac{\partial\phi_x}{\partial Q}+\frac{a^2H^2}{\beta^2}.\left(\frac{\partial\phi_x}{\partial Q}\right)^2+\frac{a^2H^2}{\beta}\frac{\partial\phi_y}{\partial R}.\frac{\partial\phi_x}{\partial Q}\right)$$

$$+B_{33}\left(-\frac{2aHS}{\beta}\frac{\partial^2 w}{\partial R\,\partial Q}.\frac{\partial\phi_y}{\partial R}+\frac{a^2H^2}{\beta}.\frac{\partial\phi_x}{\partial Q}.\frac{\partial\phi_y}{\partial R}+a^2H^2\left(\frac{\partial\phi_y}{\partial R}\right)^2\right) \qquad (5.2)$$

also for the case of Coupling Stiffness, U_C is given as

$$\frac{E_0t^3}{[1-\mu_{XY}\mu_{YX}]a^4}B_{11}\left(-\frac{a}{t}S.\frac{\partial^2 w}{\partial R^2}.\frac{du_0}{dR}+\frac{a^2}{t}H.\frac{du_0}{dR}.\frac{\partial\phi_x}{\partial R}-\frac{aS}{t}.\frac{\partial^2 w}{\partial R^2}.\frac{du_0}{\partial R}+\frac{a^2}{t}H\frac{\partial\phi_x}{\partial R}.\frac{du_0}{dR}\right)$$
$$+2B_{12}\left(-\frac{a}{t\beta}S.\frac{\partial^2 w}{\partial R^2}.\frac{dv_0}{\partial Q}+\frac{a^2H}{t\beta}\frac{dv_0}{\partial Q}.\frac{\partial\phi_x}{\partial R}-\frac{aS}{t\beta^2}\frac{\partial^2 w}{\partial Q^2}.\frac{du_0}{dR}+\frac{a^2H}{t\beta}.\frac{\partial\phi_y}{\partial Q}.\frac{du_0}{dR}\right)$$
$$+2B_{13}\left(-\frac{a}{t\beta}S\frac{\partial^2 w}{\partial R^2}.\frac{du_0}{\partial Q}-\frac{a}{t}S\frac{\partial^2 w}{\partial R^2}.\frac{dv_0}{dR}+\frac{a^2H}{t}\frac{du_0}{\beta\,\partial Q}.\frac{\partial\phi_x}{\partial R}+\frac{a^2H}{t}\frac{dv_0}{dR}.\frac{\partial\phi_x}{\partial R}\right.$$
$$\left.-\frac{2aS}{t\beta}\frac{\partial^2 w}{\partial R\,\partial Q}.\frac{du_0}{dR}+\frac{a^2H}{t\beta}\frac{\partial\phi_x}{\partial Q}.\frac{du_0}{dR}+\frac{a^2H}{t\beta}.\frac{\partial\phi_y}{\partial R}.\frac{du_0}{dR}\right)$$

$$+B_{22}\left(-\frac{aS}{t\beta^3}.\frac{\partial^2 w}{\partial Q^2}.\frac{dv_0}{\partial Q}+\frac{a^2H}{t\beta^2}\frac{dv_0}{\partial Q}.\frac{\partial\phi_y}{\partial Q}-\frac{aS}{t\beta^3}\frac{\partial^2 w}{\partial Q^2}.\frac{dv_0}{\partial Q}+\frac{a^2H}{t\beta^2}.\frac{\partial\phi_y}{\partial Q}.\frac{dv_0}{\partial Q}\right)$$

$$+2B_{23}\left(-\frac{aS}{t\beta^3}\frac{du_0}{\partial Q}.\frac{\partial^2 w}{\partial Q^2}-\frac{aS}{t\beta^2}\frac{dv_0}{dR}.\frac{\partial^2 w}{\partial Q^2}+\frac{a^2H}{t\beta^2}\frac{du_0}{\partial Q}.\frac{\partial\phi_y}{\partial Q}+\frac{a^2H}{t\beta}\frac{dv_0}{dR}.\frac{\partial\phi_y}{\partial Q}-\frac{2aS}{t\beta^2}\frac{\partial^2 w}{\partial R\,\partial Q}.\frac{dv_0}{\partial Q}\right.$$
$$\left.+\frac{a^2H}{t\beta^2}.\frac{\partial\phi_x}{\partial Q}.\frac{dv_0}{\partial Q}+\frac{a^2H}{t\beta}.\frac{\partial\phi_y}{\partial R}.\frac{dv_0}{\partial Q}\right)$$

$$+B_{33}\left(-\frac{2aS}{t\beta^2}.\frac{\partial^2 w}{\partial R\,\partial Q}.\frac{du_0}{\partial Q}-\frac{2aS}{t\beta}.\frac{\partial^2 w}{\partial R\,\partial Q}.\frac{dv_0}{dR}\right)+B_{33}\left(\frac{a^2H}{t\beta^2}\frac{du_0}{\partial Q}.\frac{\partial\phi_x}{\partial Q}+\frac{a^2H}{t\beta}\frac{dv_0}{dR}.\frac{\partial\phi_x}{\partial Q}\right)$$

$$+B_{33}\left(+\frac{a^2H}{t\beta}\frac{du_0}{\partial Q}.\frac{\partial\phi_y}{\partial R}+\frac{a^2H}{t}\frac{dv_0}{dR}.\frac{\partial\phi_y}{\partial R}\right)$$

$$+\ B_{33}\left(-\frac{2aS}{t\beta^2}\frac{\partial^2 w}{\partial R\,\partial Q}.\frac{du_0}{\partial Q}+\frac{a^2H}{t\beta^2}.\frac{\partial\phi_x}{\partial Q}.\frac{du_0}{\partial Q}+\frac{a^2H}{t\beta}\frac{\partial\phi_y}{\partial R}.\frac{du_0}{\partial Q}\right)$$

$$+B_{33}\left(-\frac{2aS}{t\beta}\frac{\partial^2 w}{\partial R\,\partial Q}.\frac{dv_0}{dR}+\frac{a^2H}{t\beta}.\frac{\partial\phi_x}{\partial Q}.\frac{dv_0}{dR}+\frac{a^2H}{t}\frac{\partial\phi_y}{\partial R}.\frac{dv_0}{dR}\right) \qquad (5.3)$$

finally for Axial/Membrane Stiffness, U_M is expressed as

$$\frac{E_0t^3}{[1-\mu_{XY}\mu_{YX}]a^4}B_{11}\left(\frac{a^2}{t^2}.\left(\frac{du_0}{dR}\right)^2\right)+2B_{12}\left(\frac{a^2}{t^2\beta}\frac{du_0}{dR}.\frac{dv_0}{\partial Q}\right)$$

$$+2B_{13}\left(\frac{a^2}{t^2}\frac{du_0}{\beta\,\partial Q}.\frac{du_0}{dR}+\frac{a^2}{t^2}.\frac{dv_0}{dR}.\frac{du_0}{dR}\right)+B_{22}\left(\frac{a^2}{t^2\beta^2}.\left(\frac{dv_0}{\partial Q}\right)^2\right)$$

$$+2B_{23}\left(\frac{a^2}{t^2\beta^2}\frac{du_0}{\partial Q}\frac{dv_0}{\partial Q}+\frac{a^2}{t^2\beta}\frac{dv_0}{dR}.\frac{dv_0}{\partial Q}\right)+B_{33}\left(\frac{a^2}{t^2\beta^2}\left(\frac{du_0}{\partial Q}\right)^2+\frac{a^2}{t^2\beta}\frac{dv_0}{dR}.\frac{du_0}{\partial Q}\right)$$

$$+B_{33}\left(\frac{a^2}{t^2\beta}\frac{du_0}{\partial Q}.\frac{dv_0}{dR}+\frac{a^2}{t^2}\left(\frac{dv_0}{dR}\right)^2\right)$$

$$+B_{44}\frac{a^4}{t^2}\left(\left(\emptyset_x.\frac{\partial H}{\partial S}\right)^2+\emptyset_x.\frac{du_0}{dS}.\frac{\partial H}{\partial S}+\emptyset_x.\frac{\partial H}{\partial S}.\frac{du_0}{dS}+\left(\frac{du_0}{dS}\right)^2\right)$$

$$+B_{55}\frac{a^4}{t^2}\left(\left(\emptyset_y.\frac{\partial H}{\partial S}\right)^2+.\,\emptyset_y.\frac{dv_0}{dS}.\frac{\partial H}{\partial S}+\emptyset_y.\frac{\partial H}{\partial S}.\frac{dv_0}{dS}+\left(\frac{dv_0}{dS}\right)^2\right) \qquad (5.4)$$

ii. This research has produced mathematical expression of thick plate as

$$[\varepsilon]=\begin{bmatrix}\left(\frac{du_0}{adR}-\frac{tS}{a^2}\frac{\partial^2 w}{\partial R^2}+\frac{tH}{a}.\frac{\partial\phi_x}{\partial R}\right)\\ \left(\frac{dv_0}{a\beta\,\partial Q}-\frac{tS}{\beta^2a^2}\frac{\partial^2 w}{\partial Q^2}+\frac{tH}{a\beta}.\frac{\partial\phi_y}{\partial Q}\right)\\ \left(\frac{du_0}{a\beta\,\partial Q}+\frac{dv_0}{adR}-\frac{2tS}{\beta a^2}\frac{\partial^2 w}{\partial R\,\partial Q}+\frac{tH}{\beta a}.\left(\frac{\partial\phi_x}{\partial Q}+\beta.\frac{\partial\phi_y}{\partial R}\right)\right)\\ \left(\frac{du_0}{tdS}+\frac{\partial H}{\partial S}.\emptyset_x\right)\\ \left(\frac{dv_0}{tdS}+\frac{\partial H}{\partial S}.\emptyset_y\right)\end{bmatrix} \qquad (5.5)$$

and for the strain of Thick Laminated Anisotropic Plate while for the stress as

$$[\sigma] = \frac{E_0}{1-\mu_{12}\mu_{21}}\begin{bmatrix} B_{11} & B_{12} & B_{13} & 0 & 0 \\ B_{21} & B_{22} & B_{23} & 0 & 0 \\ B_{31} & B_{32} & B_{33} & 0 & 0 \\ 0 & 0 & 0 & B_{44} & 0 \\ 0 & 0 & 0 & 0 & B_{55} \end{bmatrix}\begin{bmatrix} \left(\frac{du_0}{adR} - \frac{tS}{a^2}\frac{\partial^2 w}{\partial R^2} + \frac{tH}{a}.\frac{\partial \phi_x}{\partial R}\right) \\ \left(\frac{dv_0}{a\beta\,\partial Q} - \frac{tS}{\beta^2 a^2}\frac{\partial^2 w}{\partial Q^2} + \frac{tH}{a\beta}.\frac{\partial \phi_y}{\partial Q}\right) \\ \left(\frac{du_0}{a\beta\,\partial Q} + \frac{dv_0}{adR} - \frac{2tS}{\beta a^2}\frac{\partial^2 w}{\partial R\,\partial Q} + \frac{tH}{\beta a}.\left(\frac{\partial \phi_x}{\partial Q} + \beta.\frac{\partial \phi_y}{\partial R}\right)\right) \\ \left(\frac{du_0}{tdS} + \frac{\partial H}{\partial S}.\emptyset_x\right) \\ \left(\frac{dv_0}{tdS} + \frac{\partial H}{\partial S}.\emptyset_y\right) \end{bmatrix} \qquad (5.6)$$

Based on the constitutive relationship developed, the total potential energy was formulated in the course of the work mathematically as

$$= \frac{E_0}{2}\iiint \begin{bmatrix} \left(\frac{du_0}{adR} - \frac{tS}{a^2}\frac{\partial^2 w}{\partial R^2} + \frac{tH}{a}.\frac{\partial \phi_x}{\partial R}\right) \\ \left(\frac{dv_0}{a\beta\,\partial Q} - \frac{tS}{\beta^2 a^2}\frac{\partial^2 w}{\partial Q^2} + \frac{tH}{a\beta}.\frac{\partial \phi_y}{\partial Q}\right) \\ \left(\frac{du_0}{a\beta\,\partial Q} + \frac{dv_0}{adR} - \frac{2tS}{\beta a^2}\frac{\partial^2 w}{\partial R\,\partial Q} + \frac{tH}{\beta a}.\left(\frac{\partial \phi_x}{\partial Q} + \beta.\frac{\partial \phi_y}{\partial R}\right)\right) \\ \left(\frac{du_0}{tdS} + \frac{\partial H}{\partial S}.\emptyset_x\right) \\ \frac{dv_0}{tdS} + \frac{\partial H}{\partial S}.\emptyset_y \end{bmatrix}^T * \begin{bmatrix} B_{11} & B_{12} & B_{13} & 0 & 0 \\ B_{21} & B_{22} & B_{23} & 0 & 0 \\ B_{31} & B_{32} & B_{33} & 0 & 0 \\ 0 & 0 & 0 & B_{44} & 0 \\ 0 & 0 & 0 & 0 & B_{55} \end{bmatrix} *$$

$$\begin{bmatrix} \left(\frac{du_0}{adR} - \frac{tS}{a^2}\frac{\partial^2 w}{\partial R^2} + \frac{tH}{a}.\frac{\partial \phi_x}{\partial R}\right) \\ \left(\frac{dv_0}{a\beta\,\partial Q} - \frac{tS}{\beta^2 a^2}\frac{\partial^2 w}{\partial Q^2} + \frac{tH}{a\beta}.\frac{\partial \phi_y}{\partial Q}\right) \\ \left(\frac{du_0}{a\beta\,\partial Q} + \frac{dv_0}{adR} - \frac{2tS}{\beta a^2}\frac{\partial^2 w}{\partial R\,\partial Q} + \frac{tH}{\beta a}.\left(\frac{\partial \phi_x}{\partial Q} + \beta.\frac{\partial \phi_y}{\partial R}\right)\right) \\ \left(\frac{du_0}{tdS} + \frac{\partial H}{\partial S}.\emptyset_x\right) \\ \left(\frac{dv_0}{tdS} + \frac{\partial H}{\partial S}.\emptyset_y\right) \end{bmatrix} dxdydz - \iint (0 + \frac{N_x}{2}\left(\frac{dw}{dx}\right)^2 + 0)dxdy \qquad (5.7)$$

iii. The computer program developed in the course of the work can be used to further derived other thick plate properties.

iv. This research has developed excel program for resolving higher equations involving Gauss Elimination method .

v. The research has added to the literature, in the area of thick plate analysis

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