

**SAMPLING STRATEGIES FOR HANDLING
NONRESPONSE AND MEASUREMENT ERRORS IN
POSTSTRATIFIED SAMPLING SCHEME**

By

IZUNOBI, CHINYEAKA HOSTENSIA

(B.Sc., M.Sc., STATISTICS)

(20154987798)

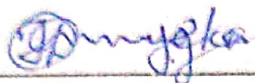
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Certification

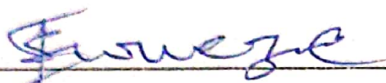
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Prof. A.C. Onyeka
Supervisor

27/12/2021

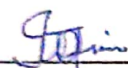
Date



Prof. I.S. Iwueze
PG Coordinator /Co-supervisor

27/12/2021

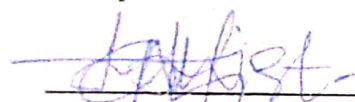
Date



Dr. C.J. Ogbonna
Co-supervisor

27/12/2021

Date



Dr. C.C. Nwaigwe
Head of Department

27/12/2021

Date



Prof. C.C.Z. Akaolisa
Dean, School of Physical Sciences (SOPS)

27/11/22

Date

Prof. C.C. Eze
Dean, Postgraduate School (PGS)



Prof. Ekaette Inyang Enang
External Examiner

07/12/2021

Date

Dedication

This work is dedicated to God and my lovely family!

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Contents

Certification Page	i
Dedication	ii
Acknowledgements	iii
Table of Contents	iv
Tables	vi
Abstract	x
1 CHAPTER ONE	
1.0 INTRODUCTION	1
1.1 Background of Study	1
1.2 Statement of Problem	5
1.3 Aim and Objectives	6
1.4 Significance of Study	7
1.5 Scope of Study	7
1.6 Notations	8
1.7 Outline of the Study	11
2 CHAPTER TWO	
2.0 REVIEW OF LITERATURE	12
2.1 Handling Nonresponse in Sample Surveys	12
2.2 Handling Measurement Errors in Sample Surveys	18
2.3 Joint Occurrence of Nonresponse and Measurement Errors in Surveys	20
2.4 Poststratified Sampling Scheme	22
3 CHAPTER THREE	
3.0 METHODOLOGY	27
3.1 Nonresponse in Poststratified Sampling Scheme	27
3.1.1 Nonresponse on both the study and auxiliary variables under the Post- stratified Sampling Scheme	28
3.1.2 Nonresponse on the study variable and complete response on the aux- iliary variable under the Poststratified Sampling Scheme	45

3.1.3	Nonresponse on the auxiliary variable and complete response on the study variable under the Poststratified Sampling Scheme	51
3.2	Measurement Errors in Poststratified Sampling Scheme	58
3.2.1	Measurement Errors on both the study and auxiliary variables under the Poststratified Sampling Scheme	59
3.2.2	Measurement Errors on the study variable and none on the Auxiliary variable under the Poststratified Sampling Scheme	67
3.2.3	Measurement Errors on the Auxiliary variable and none on the Study variable under the Poststratified Sampling Scheme	71
3.3	Joint Occurrence of Nonresponse and Measurement Errors in Poststratified Sampling Scheme	76
3.3.1	Joint Occurrence of Nonresponse and Measurement Errors on both the study and auxiliary variables under the Poststratified Sampling Scheme	77
3.3.2	Joint Occurrence of Nonresponse and Measurement Errors on the study variable and not on the auxiliary variable under the Poststratified Sampling Scheme	91
3.3.3	Joint Occurrence of Nonresponse and Measurement Errors on the auxiliary variable and not on the study variable under the Poststratified Sampling Scheme	99
3.4	Method of Data Collection and Analysis	108

4 CHAPTER FOUR

4.0	RESULTS AND DISCUSSIONS	111
4.1	Nonresponse on both the study and auxiliary variables under the Poststratified Sampling Scheme	112
4.2	Nonresponse on only the study variable and not on the auxiliary variable under the poststratified sampling scheme	119
4.3	Nonresponse on only the auxiliary variable and not on the study variable under the poststratified sampling scheme	124
4.4	Measurement Errors on both the study and auxiliary variables under the poststratified sampling scheme	130
4.5	Measurement Errors on only the study variable and not on the auxiliary variable under the Poststratified Sampling Scheme	135
4.6	Measurement Errors on only the auxiliary variable and not on the study variable under the Poststratified Sampling Scheme	139

4.7	Joint Occurrence of Nonresponse and Measurement Errors on both the study and auxiliary variables under the Poststratified Sampling Scheme	144
4.8	Joint Occurrence of Nonresponse and Measurement Errors on only the study variable and not on the auxiliary variable under the Poststratified Sampling Scheme	150
4.9	Joint Occurrence of Nonresponse and Measurement Errors on only the auxiliary variable and not on the study variable under the Poststratified Sampling Scheme	156
5	CHAPTER FIVE	
5.0	CONCLUSION AND RECOMMENDATIONS	163
5.1	Summary	163
5.2	Conclusions	168
5.3	Recommendations	170
5.4	Contribution to Knowledge	171
5.5	Areas of Further Research	171
	References	171
	Appendix I	187

Tables

Table	Title	Page
3.1	Special Cases of Proposed Class of Estimators	43
3.2	Summary Statistics for Australian Survey data (Dataset-1)	109
3.3	Summary Statistics for Fitness data (Dataset-2)	110
4.1	Percentage Relative Efficiency (PRE) of Some Proposed Estimators over the Usual Poststratified Mean Estimator when there is nonresponse on both the study and auxiliary variables for positively correlated study and auxiliary variables (Dataset-1)	114
4.2	Percentage Relative Efficiency (PRE) of Some Proposed Estimators over the Usual Poststratified Mean Estimator when there is nonresponse on both the study and auxiliary variables for negatively correlated study and auxiliary variables (Dataset-2)	115
4.3	Percentage Relative Efficiency (PRE) of Some Proposed Estimators over the Usual Poststratified Mean Estimator when there is nonresponse on only the study variable for positively correlated study and auxiliary variables (Dataset-1)	120
4.4	Percentage Relative Efficiency (PRE) of Some Proposed Estimators over the Usual Poststratified Mean Estimator when there is nonresponse on only the study variable for negatively correlated study and auxiliary variables (Dataset-2)	121
4.5	Percentage Relative Efficiency (PRE) of Some Proposed Estimators over the Usual Poststratified Mean Estimator when there is nonresponse on only the auxiliary variable for positively correlated study and auxiliary variables (Dataset-1)	126
4.6	Percentage Relative Efficiency (PRE) of Some Proposed Estimators over the Usual Poststratified Mean Estimator when there is nonresponse on only the auxiliary variable for negatively correlated study and auxiliary variables (Dataset-2)	127
4.7	Percentage Relative Efficiency (PRE) of Some Proposed Estimators over the Usual Poststratified Mean Estimator when there are measurement errors on both the study and auxiliary variables for positively correlated study and auxiliary variables (Dataset-1)	132

Tables Contd.

Table	Title	Page
4.8	Percentage Relative Efficiency (PRE) of Some Proposed Estimators over the Usual Poststratified Mean Estimator when there are measurement errors on both the study and auxiliary variables for negatively correlated study and auxiliary variables (Dataset-2)	133
4.9	Percentage Relative Efficiency (PRE) of Some Proposed Estimators over the Usual Poststratified Mean Estimator when there are measurement errors on only the study variable for positively correlated study and auxiliary variables (Dataset-1)	136
4.1	Percentage Relative Efficiency (PRE) of Some Proposed Estimators over the Usual Poststratified Mean Estimator when there are measurement errors on only the study variable for negatively correlated study and auxiliary variables (Dataset-2)	137
4.11	Percentage Relative Efficiency (PRE) of Some Proposed Estimators over the Usual Poststratified Mean Estimator when there are measurement errors on only the auxiliary variable for positively correlated study and auxiliary variables (Dataset-1)	141
4.12	Percentage Relative Efficiency (PRE) of Some Proposed Estimators over the Usual Poststratified Mean Estimator when there are measurement errors on only the auxiliary variable for negatively correlated study and auxiliary variables (Dataset-2)	142
4.13	Percentage Relative Efficiency (PRE) of Some Proposed Estimators over the Usual Poststratified Mean Estimator when there is joint occurrence of non-response and measurement errors on both the study and auxiliary variables for positively correlated study and auxiliary variables (Dataset-1)	146
4.14	Percentage Relative Efficiency (PRE) of Some Proposed Estimators over the Usual Poststratified Mean Estimator when there is joint occurrence of non-response and measurement errors on both the study and auxiliary variables for negatively correlated study and auxiliary variables (Dataset-2)	147
4.15	Percentage Relative Efficiency (PRE) of Some Proposed Estimators over the Usual Poststratified Mean Estimator when there is joint occurrence of non-response and measurement errors on only the study variable for positively correlated study and auxiliary variables (Dataset-1)	152
4.16	Percentage Relative Efficiency (PRE) of Some Proposed Estimators over the Usual Poststratified Mean Estimator when there is joint occurrence of non-response and measurement errors on only the study variable for negatively correlated study and auxiliary variables (Dataset-2)	152

Tables Contd.

Table	Title	Page
4.17	Percentage Relative Efficiency (PRE) of Some Proposed Estimators over the Usual Poststratified Mean Estimator when there is joint occurrence of nonresponse and measurement errors on only the auxiliary variable for positively correlated study and auxiliary variables (Dataset-1)	158
4.18	Percentage Relative Efficiency (PRE) of Some Proposed Estimators over the Usual Poststratified Mean Estimator when there is joint occurrence of nonresponse and measurement errors on only the auxiliary variable for negatively correlated study and auxiliary variables (Dataset-2)	159
5.1	Summary of Proposed Classes of Estimators	165

Abstract

Nonresponse and measurement errors, often referred to as nonsampling errors, pose serious challenges in surveys. The focus of the present study is on the development of appropriate sampling strategies for handling of these problems under the poststratified sampling scheme. Three (3) broad cases considered in the study were the occurrence of nonresponse alone, the occurrence of measurement errors alone and the joint occurrence of nonresponse and measurement errors. These three (3) broad cases were considered when each occurred on both the study and auxiliary variables, on only the study variable, and on only the auxiliary variable, giving rise to nine (9) distinct cases. Three (3) separate-type estimators of the population mean of the study variable were then proposed for each of the nine (9) distinct cases, giving a total of twenty-seven (27) sampling strategies. These three (3) types of estimators are the customary-type sample mean estimators, the difference-type estimators, and a class of ratio/product-type estimators. Properties of the proposed estimators were obtained both for an achieved sample configuration (conditional argument) and for repeated samples of fixed size n (unconditional argument). The proposed customary-type and difference-type estimators were unbiased, while the proposed classes of ratio/product-type estimators were generally biased for the study variable population mean. The results showed that there exists an optimum class of estimators among the proposed classes of estimators. The theoretical results were subjected to numerical validation and verification, using two datasets, namely, Dataset-1, for positively correlated study and auxiliary variables, and Dataset-2, for negatively correlated study and auxiliary variables. Both the theoretical and empirical results showed that the optimum estimators in the proposed classes of estimators performed better than the usual poststratified sample mean estimator (when there are no nonresponse and measurement errors). The optimum estimators also generally performed better than the proposed customary-type estimators and other proposed ratio/product-type estimators. The study recommends, among others, the use of auxiliary information when developing improved sampling strategies for the handling of the problem of nonresponse and measurement errors in poststratified sampling scheme.

KEYWORDS: Sampling strategies, Nonresponse, Measurement errors, Poststratified Sampling, Auxiliary Information, difference, ratio, product estimators

CHAPTER ONE

1.0 INTRODUCTION

1.1 Background of Study

This work is a research on a topic entitled “Nonresponse and Measurement Errors in Poststratified Sampling Scheme.” The study reviewed some works on nonsampling errors when estimating population parameters in simple random sampling and other sampling schemes, such as stratified random sampling, systematic sampling, successive sampling and two-phase or double sampling schemes. Nonsampling errors are generally divided into two categories, namely, nonresponse and measurement errors. Consequently, the concepts of nonresponse, measurement errors and poststratified sampling scheme, are discussed and some existing works on nonsampling errors and poststratified sampling scheme are reviewed with a view to developing sampling strategies for the handling of the problem of nonresponse, as well as that of measurement errors, in poststratified sampling scheme.

It is obvious that informed decisions are often preferred to decisions made on the basis of trial and error. Usually, data are required to make informed decisions. However errors may occur in the course of data collection. Generally, there are two types of errors in surveys, namely, sampling errors and nonsampling errors. Sampling errors occur as a result of the use of information obtained on only a part of the population instead of the entire population. Nonsampling errors involve all other errors that cannot be attributed to errors due to sampling. In this work,

we will focus on nonsampling errors, which involve nonresponse and measurement errors, as already mentioned.

The Concept of Nonresponse in Surveys: Nonresponse is the process of not getting information from some units in the selected sample, due to various reasons like refusals to participate in the survey, lack of information, non-availability of the respondents, etc. This implies that no information is obtained from certain units already included in the sample or those that are supposed to be included in the sample. Basically, three types of nonresponse are:

1. Non-coverage: This is when some units that are supposed to be in the population are not included in the sampling frame.
2. Unit nonresponse: It arises when no information is obtained from certain sample units. This may be because respondents refuse to participate in the survey when contacted or they cannot be contacted.
3. Item nonresponse: It is a situation where some units have incomplete data on one or more items. Item nonresponse is therefore noticed by gaps in the data records for responding sample units. Reasons may be due to refusals or omissions by enumerators.

Many authors have investigated the problem of nonresponse in surveys. Hansen and Hurwitz (1946) and Rachokarn and Lawson (2017) estimated the population mean in the presence of nonresponse under the simple random sampling scheme. Still under the simple random sampling scheme, authors like Koyuncu, Shahzad, Hanif and Luengo (2017) and Sinha (2016) respectively considered estimation of population variance and ratio/product in the presence of nonresponse. Sinha and Kumar (2015) and Bandyopadhyay, Singh and Das (2016) considered the problem of nonresponse under two phase sampling scheme, Maji, Singh, and Bandyopadhyay (2018) worked on nonresponse under successive sampling scheme, while Onyeka, Ogbumuo and Izunobi (2019), Oyoo, Manene, Ouma and Muhua (2019), Iseh and Bassey (2021) and Unal and Kadilar (2021) investigated the problem of nonresponse under the stratified random sampling scheme. No literature was found on the handling of the problem of nonresponse under the poststratified sampling scheme, and this is a vacuum the present study intends to fill.

The Concept of Measurement Errors in Surveys: Most statistical theories are built on the assumption that the observed values in a survey are the same as the true values of the characteristics of interest. In other words, the observed values are assumed to be the same as the true values without making provision for any errors in measurement. However, this is not always the case because there are situations where the observed values might slightly differ from the true values. Consequently, this slight difference between the observed and true values is often referred to as measurement error. Many authors believe that measurement errors should be incorporated in the development of statistical theories. Hence, in the present work, attempts will be made to incorporate measurement errors in developing sampling strategies for parameter estimation in poststratified sampling scheme.

In practice, measurement errors can occur due to a number of reasons. Possible reasons include the following:

1. It is usually difficult to obtain the exact measurement of some qualitative or categorical data, like intelligence, education, taste or poverty level.
2. In certain quantitative variables that are clearly defined, like age, height or weight of students, it might still be hard to record the exact and correct values.
3. Measurement errors also occur when some variables are measured only in terms of some related yardsticks, like measuring level of education by number of years in school.

Generally, measurement errors occur due to differences between the observed and true values of some characteristics under study. These differences often occur as a result of imperfect measurements of the true values of the variables. Some authors have considered the problem of measurement errors in surveys under various sampling schemes. Recently, authors like Pourjafar and Zardasht (2020) and Zahid and Shabbir (2019) considered the estimation of the population mean in simple random sampling scheme when there are measurement errors. Similarly, Singh and Karpe (2009a) estimated the population ratio/product, while Singh and Surya (2016) and Singh and Karpe (2009c) estimated the population variance in simple

random sampling scheme when there are measurement errors. Also, Khalil, Gupta and Hanif (2017), Khalil, Gupta and Hanif (2018) and Vishwakarma, Singh and Singh (2020) developed sampling strategies for the handling of measurement errors under the stratified random sampling scheme. However, literature did not reveal any works in the handling of the problem of measurement errors under the poststratified sampling scheme, and this study intends to fill this gap.

The Poststratified Sampling Scheme: In sample surveys, stratification is implored when there are heterogeneous sub-populations that are internally homogeneous. The essence is to ensure that all the various heterogeneous sub-populations are adequately represented in the sample. When the sampling frames of the sub-populations, otherwise called strata, are available, then stratification can be carried out prior to sample selection. The process is usually referred to as stratified random sampling. However, when the strata sampling frames are not readily available, stratification could still be implemented but only after sample selection. The process is generally referred to as poststratified sampling scheme. The sampling design for the poststratified sampling scheme can be described as follows.

Sampling Design for Poststratified Sampling Scheme

Let a population of N units consists of L sub-populations or strata of sizes, $N_1, N_2, \dots, N_L$ such that $\sum_{h=1}^L N_h = N$. Because the strata frames are not readily available, we first select a sample of size, n units from the N units in the population, and information obtained on both the study and stratification variables. Then the n sampled units are allocated into their various strata with n_h units falling into stratum h , such that $\sum_{h=1}^L n_h = n$. It is generally assumed that n is large enough such that: $n_h > 0$ or $P(n_h = 0) = 0 \quad \forall h$.

It is worthy of note that the stratum size n_h is fixed for an achieved sample configuration, say $\underline{n} = (n_1, n_2, \dots, n_L)$, while n_h is a random variable for repeated samples of fixed size, n units. These two scenarios give rise to what is usually referred to as conditional and unconditional arguments, respectively. In other words, conditional argument refers to when properties are ob-

tained for any achieved sample configuration, say $\underline{n} = (n_1, n_2, \dots, n_L)$, while the unconditional argument refers to when properties are obtained for repeated samples of fixed size, n units. Most survey statisticians hold the view that properties of estimators under the unconditional argument should be employed at the planning stage, while properties under the conditional argument should be utilized at the estimation stage, after sample selection.

1.2 Statement of Problem

Nonresponse usually leads to a reduction in the originally intended sample size. One of the implications of the reduced sample size is the distortion of the randomness of the sampled units. Invariably, this distortion in randomness introduces into the estimation process, a bias, often referred to as nonresponse bias. The higher the size of the nonresponse units in the sample, the higher the nonresponse bias. Another problem associated with the reduction in the original sample size when there is nonresponse is the case of reduced efficiency. Because nonresponse reduces the sample size, estimators under nonresponse utilize sample sizes that are smaller than the originally intended sample size. And with the reduced sample size, there is likely to be reduced efficiencies of estimators under nonresponse. Consequently, to deal with both bias and reduced efficiencies arising from nonresponse, attempts should be made to increase the size of the response units in the sample that would be used for estimation purposes.

Measurement errors, on the other hand, give rise to observed values that are incorrect, inaccurate, and not exactly the same as the true values. This, in turn, makes the results obtained on the basis of the observed values alone to be incorrect and different from what should have been obtained if the true values were known and utilized. In other words, incorrect results are likely to be generated when measurement errors are not integrated or included in the estimation process. Hence the need to deal with measurement errors in surveys.

Generally, the two types of nonsampling errors, namely, nonresponse and measurement errors, usually constitute challenging problems that need to be addressed and handled in surveys.

The two challenges can be handled separately or jointly, depending on the assumptions we make about their occurrences. From the literature, the problem of both separate and simultaneous occurrence of nonresponse and measurement errors has been addressed under some sampling schemes like the simple random sampling, stratified random sampling and systematic sampling schemes. However, previous works did not reveal any work in the area of the handling of the problem of nonresponse and measurement errors under the poststratified sampling scheme. This is the vacuum the present study intends to fill. In this research work, sampling strategies are developed for addressing the problem both separate and simultaneous occurrence of nonresponse and measurement errors in poststratified sampling scheme.

1.3 Aim and Objectives

The aim of this work is to develop sampling strategies for the handling of the problem of both separate and joint occurrences of nonresponse and measurement errors under the poststratified sampling scheme. The specific objectives of the study are as follows:

1. To develop sampling strategies for the handling of the problem of:
 - (i) Separate occurrence of nonresponse alone in poststratified sampling scheme.
 - (ii) Separate occurrence of measurement errors alone in poststratified sampling scheme.
 - (iii) Joint occurrence of nonresponse and measurement errors in poststratified sampling scheme.
2. To obtain the conditional and unconditional properties of the proposed sampling strategies/estimators.
3. To carry out both analytical and empirical efficiency comparisons of the proposed estimators for the handling of the problem of both separate and joint occurrence of nonresponse and measurement errors in poststratified sampling scheme.

1.4 Significance of Study

This work will give readers, researchers and policy makers some basic insight on ways of handling both separate and joint occurrences of nonresponse and measurement errors in poststratified sampling scheme. Specifically, information would be available on various possible sampling strategies for addressing the problem of nonresponse and measurement errors occurring either separately or simultaneously in poststratified sampling scheme. Furthermore, the analytical and empirical comparisons of the efficiencies of the proposed estimators would give clue to researchers, survey statisticians and other interest groups as to which estimators that should be preferred to the other proposed and existing estimators.

1.5 Scope of Study

In this work, we formulated sampling strategies for the handling of both separate and joint occurrences of nonresponse and measurement errors in poststratified sampling scheme. Both conditional and unconditional properties of the proposed estimators are discussed and supported with empirical results. Estimation of the population mean as the parameter of interest is considered and the separate-type estimators are utilized in all cases. In all, twenty-seven estimators/sampling strategies are discussed for the handling of different cases of occurrence of nonresponse alone, occurrence of measurement errors alone and joint occurrence of nonresponse and measurement errors when using poststratified sampling scheme. The efficiencies of the proposed estimators were compared with that of the existing usual poststratified sample mean estimators, as well as among the proposed estimators.

1.6 Notations

Let (Y_{hi}, X_{hi}) , $i = 1, \dots, N_h$ denote observations on the study and auxiliary variables for the i^{th} unit in the population in stratum h , while (y_{hi}, x_{hi}) , $i = 1, \dots, n_h$ denote observations on the study and auxiliary variables for the i^{th} unit in the sample in the h^{th} stratum. Then, we define the following notations.

$$\bar{Y}_h = \frac{1}{N_h} \sum_{i=1}^{N_h} Y_{hi} \quad \text{is the population mean of } y \text{ in stratum } h$$

$$\bar{X}_h = \frac{1}{N_h} \sum_{i=1}^{N_h} X_{hi} \quad \text{is the population mean of } x \text{ in stratum } h$$

$$\bar{y}_h = \frac{1}{n_h} \sum_{i=1}^{n_h} y_{hi} \quad \text{is the sample mean of } y \text{ in stratum } h, \text{ based on the sample size, } n_h$$

$$\bar{x}_h = \frac{1}{n_h} \sum_{i=1}^{n_h} x_{hi} \quad \text{is the sample mean of } x \text{ in stratum } h, \text{ based on the sample size, } n_h$$

$$\bar{y}_{1h} = \frac{1}{n_{1h}} \sum_{i=1}^{n_{1h}} y_{hi} \quad \text{is the sample mean of } y \text{ in stratum } h, \text{ based on the}$$

response group sample size, n_{1h}

$$\bar{x}_{1h} = \frac{1}{n_{1h}} \sum_{i=1}^{n_{1h}} x_{hi} \quad \text{is the sample mean of } x \text{ in stratum } h, \text{ based on the}$$

response group sample size, n_{1h}

$$\bar{y}_{2mh} = \frac{1}{m_h} \sum_{i=1}^{m_h} y_{hi} \quad \text{is the sample mean of } y \text{ in stratum } h, \text{ based on the}$$

nonresponse sub-sample group sample size, m_h

$$\bar{x}_{2mh} = \frac{1}{m_h} \sum_{i=1}^{m_h} x_{hi} \quad \text{is the sample mean of } x \text{ in stratum } h, \text{ based on the}$$

nonresponse sub-sample group sample size, m_h

$$g_h = n_{2h}/m_h > 1 \quad \text{is the sub-sampling factor in stratum } h$$

$$w_{2h} = n_{2h}/n_h \quad \text{is the } h^{th} \text{ stratum nonresponse group sample weight}$$

$$W_{2h} = N_{2h}/N_h \quad \text{is the } h^{th} \text{ stratum nonresponse group population weight}$$

$W_h = N_h/N$ is the h^{th} stratum population weight

$$\bar{Y} = \frac{1}{N} \sum_{h=1}^L \sum_{i=1}^{N_h} Y_{hi} = \frac{1}{N} \sum_{h=1}^L N_h \bar{Y}_h = \sum_{h=1}^L W_h \bar{Y}_h \quad \text{is the population mean of y}$$

$$\bar{X} = \frac{1}{N} \sum_{h=1}^L \sum_{i=1}^{N_h} X_{hi} = \frac{1}{N} \sum_{h=1}^L N_h \bar{X}_h = \sum_{h=1}^L W_h \bar{X}_h \quad \text{is the population mean of x}$$

$R_h = \bar{Y}_h/\bar{X}_h$ is the population ratio of study and auxiliary variables in stratum h

$R = \bar{Y}/\bar{X}$ is the population ratio of study and auxiliary variables

$f = n/N$ is the sampling fraction

$$S_{yh}^2 = \frac{1}{N_h - 1} \sum_{i=1}^{N_h} (Y_{hi} - \bar{Y}_h)^2 \quad \text{is the population variance of y in stratum h}$$

$$S_{xh}^2 = \frac{1}{N_h - 1} \sum_{i=1}^{N_h} (X_{hi} - \bar{X}_h)^2 \quad \text{is the population variance of x in stratum h}$$

$$S_{yjh} = \frac{1}{N_h - 1} \sum_{i=1}^{N_h} (Y_{hi} - \bar{Y}_h)(X_{hi} - \bar{X}_h) \quad \text{is the population covariance of y and x}$$

in the h^{th} stratum

$$S_{2yh}^2 = \frac{1}{N_{2h} - 1} \sum_{i=1}^{N_{2h}} (Y_{hi} - \bar{Y}_{2h})^2 \quad \text{is the nonresponse group population}$$

variance of y in stratum h

$$S_{2xh}^2 = \frac{1}{N_{2h} - 1} \sum_{i=1}^{N_{2h}} (X_{hi} - \bar{X}_{2h})^2 \quad \text{is the nonresponse group population}$$

variance of x in stratum h

$$s_{2yh}^2 = \frac{1}{n_{2h} - 1} \sum_{i=1}^{n_{2h}} (y_{hi} - \bar{y}_{2h})^2 \quad \text{is the nonresponse group sample}$$

variance of y in stratum h

$$s_{2xh}^2 = \frac{1}{n_{2h} - 1} \sum_{i=1}^{n_{2h}} (x_{hi} - \bar{x}_{2h})^2 \quad \text{is the nonresponse group sample}$$

variance of x in stratum h

$$S_{2yjh} = \frac{1}{N_{2h} - 1} \sum_{i=1}^{N_{2h}} (Y_{hi} - \bar{Y}_{2h})(X_{hi} - \bar{X}_{2h}) \quad \text{is the nonresponse group population}$$

covariance of y and x in stratum h

$$s_{2yxh} = \frac{1}{n_{2h} - 1} \sum_{i=1}^{n_{2h}} (y_{hi} - \bar{y}_{2h})(x_{hi} - \bar{x}_{2h}) \quad \text{is the nonresponse group sample}$$

covariance of y and x in stratum h

$$C_{yh} = S_{yh}/\bar{Y}_h \quad \text{is the coefficient of variation of y in stratum h}$$

$$C_{xh} = S_{xh}/\bar{X}_h \quad \text{is the coefficient of variation of x in stratum h}$$

$$\rho_{xyh} = S_{xyh}/S_{xh}S_{yh} \quad \text{is the population correlation coefficient of x and y in stratum h}$$

$$W_2 = N_2/N \quad \text{is the nonresponse group weight}$$

$$S_y^2 = \frac{1}{N-1} \sum_{i=1}^N (Y_i - \bar{Y})^2 \quad \text{is the population variance of y}$$

$$S_x^2 = \frac{1}{N-1} \sum_{i=1}^N (X_i - \bar{X})^2 \quad \text{is the population variance of x}$$

$$S_{xy} = \frac{1}{N-1} \sum_{i=1}^N (X_i - \bar{X})(Y_i - \bar{Y}) \quad \text{is the population covariance of x and y}$$

$$S_{2y}^2 = \frac{1}{N_2-1} \sum_{i=1}^{N_2} (Y_i - \bar{Y}_2)^2 \quad \text{is the nonresponse group population variance of y}$$

$$S_{2x}^2 = \frac{1}{N_2-1} \sum_{i=1}^{N_2} (X_i - \bar{X}_2)^2 \quad \text{is the nonresponse group population variance of x}$$

$$S_{2xy} = \frac{1}{N_2-1} \sum_{i=1}^{N_2} (X_i - \bar{X}_2)(Y_i - \bar{Y}_2) \quad \text{is the nonresponse group population}$$

covariance of x and y

$$C_y = S_y/\bar{Y} \quad \text{is the coefficient of variation of y}$$

$$C_x = S_x/\bar{X} \quad \text{is the coefficient of variation of x}$$

$$\rho_{yx} = S_{yx}/S_yS_x \quad \text{is the population correlation coefficient of y and x}$$

$$n_1 \quad \text{is the size of the sample response group}$$

$$n_2 \quad \text{is the size of the sample nonresponse group}$$

$$m \quad \text{is the size of the subsample nonresponse group}$$

$$g = n_2/m \quad \text{is the size of the sample nonresponse group}$$

1.7 Outline of the Study

The present work consists of only the first two chapters of the proposed doctorate degree thesis entitled “Nonresponse and Measurement Errors in Poststratified Sampling Scheme.” Chapter One introduces the study by dealing with basic issues like the concepts of nonresponse, measurement errors and poststratified sampling scheme, statement of problems, aim and objectives and significance of the study. Chapter two deals with the literature review. The review is on works carried out in the areas dealing with nonresponse alone in surveys, dealing with measurement errors alone in surveys, dealing with joint or simultaneous occurrence of nonresponse and measurement errors in surveys, and estimation of parameters in poststratified sampling scheme. Chapter three discusses the methodology for the handling of nonresponse and measurement errors in poststratified sampling scheme. Chapter Four deals with results and discussion, while the summary, conclusions and recommendations of the study are presented in Chapter Five.

CHAPTER TWO

2.0 REVIEW OF LITERATURE

This chapter deals with review of some works involving poststratification as well as the handling of nonresponse and measurement errors in sample surveys. As mentioned earlier in Chapter one, nonresponse and measurement errors are non-sampling errors that affect results obtained in sample surveys. In the literature, there are works on ways of handling nonresponse and measurement errors in different sampling schemes, such as simple random sampling, stratified random sampling, systematic sampling, successive sampling and others. Previous works included surveys in which nonresponse and measurement errors occurred separately and others involving joint occurrence of nonresponse and measurement errors. Again, population parameters considered for estimation in the presence of nonresponse and measurement errors included population mean, total, variance, median, ratio and product. We start by reviewing works involving estimation of population parameters in the presence of nonresponse.

2.1 Handling Nonresponse in Sample Surveys

The earliest work on handling the problem of nonresponse in sample surveys is attributed to Hansen and Hurwitz (1946). They considered the method of subsampling the nonresponse group in order to adjust for the nonresponse in a mail survey. They considered a subsample of size m from the nonresponse group n_2 and assumed that responses are available from the subsampled group. They proposed an estimator of population mean in simple random sampling scheme under nonresponse and also obtained conditions under which their proposed estima-

tor would perform better than some existing estimators in the literature. A number of authors followed Hansen and Hurwitz (1946) in discussing ways of handling nonresponse when estimating the population mean of the study variable under simple random sampling scheme. These include Kish and Lansing (1954), Little (1982), Rao (1986), Rao (1990), Khare and Srivastava (1993), Tripathi, Das and Khare (1994) and Khare and Srivastava (1997). The work carried out by Rao (1986) involved utilization of auxiliary information by considering ratio-type estimator for handling nonresponse under simple random sampling scheme, while Rao (1990), suggested a regression-type estimator. Khare and Srivastava (1993) and Tripathi et al. (1994) also utilized auxiliary information for estimating population mean in the presence of nonresponse under simple random sampling scheme, and Khare and Srivastava (1997) utilized transformed ratio-type estimators for the population mean in the presence of nonresponse under simple random sampling scheme. Sub-sampling the non-respondents in order to solve the problem of nonresponse when estimating finite population mean vector was considered by Tripathi and Khare (1997).

Several other authors have also contributed to the problem of estimation of population mean in simple random sampling scheme in the presence of nonresponse using auxiliary information., while some assumed nonresponse on only the study variable, others assumed nonresponse on both study and auxiliary variables. Authors who utilized auxiliary information for the estimation of population mean in simple random sampling scheme in the presence of nonresponse include Singh and Kumar (2008a), Ismail, Shahbaz and Hanif (2011), Singh and Kumar (2011), Kumar and Bhogal (2011), Kumar and Viswanathaiah (2013), Grover and Kaur (2014), Rachokarn and Lawson (2017) and Rachokarn and Lawson (2018). Regression-type estimator was utilized by Singh and Kumar(2008b), while Singh and Kumar (2011) considered a combination of regression and ratio-type estimators. Kumar and Viswanathaiah (2013) considered a generalized family of transformed estimators, while Grover and Kaur (2014) suggested an exponential ratio-type class of estimators. General class of estimators of population mean in the presence of nonresponse under simple random sampling scheme was also considered by Ismail et al. (2011), Rachokarn and Lawson (2017) and Singh and Kumar (2009a).

The works reviewed so far involved dealing with nonresponse in simple random sampling scheme when the population mean is the parameter of interest. However, many authors have considered estimation of population parameters other than the population mean. For instance, some authors have considered the variance as their population parameter of interest. Studies on nonresponse under simple random sampling scheme with the population variance as the parameter of interest includes those carried out by Singh and Joarder (1998), Singh, Chandra and Singh (2003), Ahmeda, Al-Titi, Al-Rawi, Abu-Dayyeh (2005), Singh, Tailor, Kim and Singh (2012) and Koyuncu et al. (2017). The estimation of finite population variance using random non-response in simple random sampling was considered by Singh and Joarder (1998) and Ahmeda et al. (2005), while Singh et al. (2003) utilized multi auxiliary information on variance estimation in the presence of non-response in simple random sampling. Singh et al. (2012) considered some classes of estimators of finite population variance, while Koyuncu et al. (2017) discussed the estimation of population variance using quartiles in absence and presence of non-response in survey sampling.

In addition to dealing with nonresponse when the parameters of interest are population mean or variance, as already reviewed, some other authors have addressed estimation of population ratio when handling nonresponse in simple random sampling. Estimation of population ratio of two population means in the presence of nonresponse in simple random sampling scheme has been considered by authors like Toutenberg and Srivastava (1998), Khare and Pandey (2000), Khare and Sinha (2002), Khare and Sinha (2007), Singh, Singh, Tailor, Allen and Kozak (2009), Khare, Srivastava, and Kumar (1993), Singh et al. (2012), Khare, Srivastava and Kumar (2013), Sinha (2014), Kumar (2014), Kumar and Patel (2015) and Sinha (2016). The use of auxiliary information in estimating population ratio in simple random sampling scheme under nonresponse was discussed by Khare and Pandey (2000), Khare and Sinha (2002) and Sinha (2014), while Khare and Sinha (2007) utilized multi-auxiliary variables. On their own part, Khare and Srivastava (1993) and Khare et al. (2013) proposed chain-type estimators, while Kumar and Patel (2015) proposed a regression-type estimator for estimating population ratio in the pres-

ence of nonresponse. Furthermore, estimation of product of two population means, in addition to estimation of population ratio under nonresponse in simple random sampling scheme, was considered by Singh et al. (2012), Kumar and Patel (2015) and Sinha (2016).

The discussions so far have focused on nonresponse under simple random sampling scheme. However, treatment of nonresponse in surveys has been extended to other sampling schemes or techniques. For instance, Shahzad, Hanif, Koyuncu and Luengo (2017) considered estimation of population mean under systematic random sampling in the absence and presence non-response. Furthermore, nonresponse under stratified random sampling scheme has been investigated by Saghir and Shabbir (2012) and Singh and Malik (2014) who estimated population mean in stratified random sampling scheme, using auxiliary attribute(s) under nonresponse. Estimation of ratio and product of two population means in stratified random sampling scheme under nonresponse was considered by Khare and Sinha (2012), while Clement, Udofia and Enang (2014) estimated domain means in stratified sampling design in the presence of nonresponse. More recently, Onyeka, Ogbumuo and Izunobi (2019) considered a class of estimators of population mean in stratified random sampling scheme when using auxiliary information in the presence of non-response, while Oyoo, Manene, Ouma and Muhua (2019) discussed estimation of finite population total in stratified random sampling scheme under nonresponse.

Investigations on the incidence of nonresponse in surveys have not been restricted to one-time surveys only. Several authors have considered nonresponse in successive sampling or sampling on more than one occasion. These include the works carried out by authors like Choudhary, Bathal and Sud (2004), Singh and Kumar (2009), Singh and Kumar (2010), Singh, Kumar and Bhogal (2011a), Singh, Kumar and Bhogal (2011b), Mehmood, Ahmad and Shahid (2012), Kuma (2012), Bandyopadhyay and Singh (2005), Singh and Karna (2014) and Karna and Nath (2015). Use of multivariate ratio estimator for population mean in successive sampling under nonresponse was considered by Singh et al. (2011), while Singh and Kumar (2010) considered the estimation of product of two population means in the presence of nonresponse in sampling over two occasions. Similarly, Kuma (2012) investigated the estimation of ratio of two popula-

tion means in successive sampling under nonresponse, while Mehmood et al. (2012) discussed the estimation of population variance in successive sampling under nonresponse.

Furthermore, Sharma and Singh (2016) discussed the estimation of current occasion population mean under nonresponse in h-occasion successive sampling scheme. The use of a ratio-cum-product estimator of current occasion population mean in presence of nonresponse under successive sampling was employed by Garcia-Luengo (2016), while Pal and Singh (2017) utilized power transformation for estimation of the population mean in presence of nonresponse under successive sampling. Chatterjee, Singh and Bandyopadhyay (2018) considered the estimation of current occasion population mean in successive sampling under super-population model in the presence of random nonresponse, while Maji et al. (2018) considered estimation of ratio as well as product of two population means in the presence of random nonresponse under successive sampling scheme.

The use of double sampling or two-phase sampling scheme has also been given some considerations by many authors when dealing with nonresponse in sample surveys. Population parameters of interest considered included population mean, variance, median, ratio and product of two population means. Authors who considered estimation of population mean using double sampling scheme in the presence of nonresponse include Khare and Srivastava (1995), Tabasum and Khan (2004), Tabasum and Khan (2006), Singh and Kumar (2008b), Singh and Kumar (2009), Khare and Srivastava (2010), Khare, Pandey and Srivastava (2011), Ismail, Shahbaz and Hanif (2013) and Hazra (2015). Different forms of ratio, product and regression-type estimators were considered by Khare and Srivastava (1995), Tabasum and Khan (2004) and Tabasum and Khan (2006), while classes of estimators of population mean using double sampling scheme in the presence of nonresponse were discussed by Singh and Kumar (2009), Khare and Srivastava (2010) and Khare, Pandey and Srivastava (2011). The use of exponential ratio-type estimator for population mean was considered by Hazra (2015).

As already mentioned, parameters other than the population mean has also been considered

by authors using double sampling or two-phase sampling in the presence of nonresponse. Estimation of population variance was considered by Sinha and Kumar (2015), while estimation of population median under nonresponse using double sampling scheme was considered by Bandyopadhyay et al. (2016). Estimation of ratio of two population means was discussed by Khare and Sinha (2004), Khare and Sinha (2012) and Patel and Narendra (2015), while estimation of both ratio and product of two population means was considered by Singh and Kumar (2008b). The use of double sampling under stratified random sampling scheme has been considered by Varshney and Ahsan (2012) and Iftekhar, Ali and Ahsan (2014), who utilized some multivariate estimators of the population mean in stratified double sampling design in the presence of nonresponse. Similarly, the use of double sampling procedure on successive occasions in the presence of nonresponse has been considered by Singh, Khetan and Maurya (2016) for the estimation of population mean, while recently, Singh and Khalid (2019) investigated estimation of population variance in two-phase successive sampling in the presence of random nonresponse.

Some recent works in handling the problem of nonresponse in surveys include the study carried out by Kumar and Kumar (2019) who estimated population mean in simple random sampling using auxiliary attribute in the presence of nonresponse. Singh and Rai (2019) proposed log-type estimators for population mean in systematic sampling in the presence of nonresponse. Iseh and Bassey (2021) employed calibration approach in estimating population mean in small sample size with nonresponse under stratified random sampling. Unal and Kadilar (2021) considered a new family of exponential type estimators of population mean in simple random sampling in the presence of nonresponse. However, the literature did not reveal any methodology for the handling of the problem of nonresponse under the poststratified sampling scheme, and this is a vacuum the present study intends to fill.

2.2 Handling Measurement Errors in Sample Surveys

Nonresponse and measurement errors are two types of nonsampling errors in surveys. Some authors have handled them separately, while others focused on joint occurrence of both nonresponse and measurement errors in surveys. We have reviewed some works on the handling of nonresponse when estimating different population parameters of interest under different sampling schemes. These were cases in which only nonresponse was assumed to be present in the surveys. We shall now proceed to review surveys in which only measurement errors are assumed to be present. Authors who have contributed to estimation of population mean in the presence of measurement errors under simple random sampling scheme include Hansen, Hurwitz, Marks and Mauldin (1946), Hansen, Hurwitz and Bershad (1946), Cochran (1968), Casady and Valliant, Fuller (1966), Shalabb (1996), Manisha and Singh (2001), Srivastava and Shalabh (2001), Manisha and Singh (2002), Singh and Karpe (2008), Singh and Karpe (2009b) and Timothy and Christian (2009). The study carried out by Hansen et al. (1946) on response errors in national or large-scale census surveys is considered as one of the earliest works on measurement errors. The work dealt with some basic and fundamental concepts of measurement errors in surveys. Subsequent works, such as the studies carried out by Shalabb (1996) and Timothy and Christian (2009) involved the use of auxiliary information in handling measurement errors. Both authors considered ratio-type estimators for the estimation of population mean under simple random sampling scheme in the presence of measurement errors. Regression-type estimators were considered by Srivastava and Shalabh (2001) and Manisha and Singh (2002), while Singh and Karpe (2008) discussed both ratio-type and product-type estimators for population mean in the presence of measurement errors under the simple random sampling scheme.

Estimation of population mean in the presence of measurement errors under the simple random sampling scheme has also been given further consideration by authors like Allen, Singh and Smarandache (2003), Kumar, Singh, Singh and Smarandache (2011), Kumar, Singh, Sawan and Chauhan (2011), Shukla, Pathak and Thakur (2012), Shukla, Pathak and Thakur (2012),

Malik, Singh and Singh (2013) Singh, Singh and Smarandache (2014), Pourjafar and Zardasht (2020) and Zahid and Shabbir (2019). The use of difference-type estimator was considered by Singh et al. (2014), while Kumar et al. (2011a) and Kumar et al. (2011b) utilized exponential-type estimators for the population mean in the presence of measurement errors under simple random sampling scheme. Furthermore, Allen et al. (2003) and Shukla et al. (2012) considered general classes of estimators, while Shukla et al. (2012) and Zahid and Shabbir (2019) utilized variable transformation for the purpose of estimating the population mean under measurement errors in simple random sampling scheme.

In addition to estimation of population mean in the presence of measurement errors under simple random sampling scheme, some authors have also considered estimation of other population parameters of interest like population variance, ratio and product of two population means. These include the works carried out by Singh and Karpe (2008), Singh and Karpe (2009c), Singh and Karpe (2009a), Singh and Karpe (2009), Diana and Giordan (2012), Sharma and Singh (2013) and Singh and Surya (2016). The estimation of ratio and product of two population was considered by Singh and Karpe (2009a), while Singh and Karpe (2009) discussed a general procedure for estimating general parameter in presence of measurement errors under simple random sampling scheme. Estimation of population variance in the presence of measurement errors under simple random sampling scheme was considered by Singh and Karpe (2008) and Diana and Giordan (2012). Furthermore, general classes of estimators were considered by Singh and Karpe (2009c), Sharma and Singh (2013), Singh (2013) and Singh and Surya (2016) for estimating population variance in simple random sampling scheme under measurement errors.

Some authors have also extended the handling of measurement errors in simple random sampling scheme to stratified random sampling scheme. These include Singh and Karpe (2010), Khalil et al. (2017) and Khalil et al. (2018). The effect of measurement errors in stratified random sampling scheme was considered by Singh and Karpe (2010) using separate and combined ratio, product and regression-type estimators. Properties of the proposed estimators were

derived under large sample approximation and it was observed that the separate-type estimators performed better than the combined estimators under certain conditions. Khalil et al. (2017) considered estimation of population mean of a sensitive variable in stratified random sampling scheme, based on randomized response technique (RRT) in the presence of measurement errors. A generalized estimator of population mean was proposed by using additively scrambled responses for the sensitive variable. Further consideration of population mean estimation in stratified random sampling using scrambled responses in the presence of measurement errors was carried out by Khalil et al. (2018) and Vishwakarma, Singh and Singh (2020). Like in the case of nonresponse, there is no methodology in the literature for the handling of the problem of measurement errors under the poststratified sampling scheme; a vacuum this study intends to fill.

2.3 Joint Occurrence of Nonresponse and Measurement Errors in Surveys

As we mentioned earlier, some authors have addressed the handling of nonresponse and measurement errors separately, while some discussed the joint effects of nonresponse and measurement errors in surveys. So far, we have reviewed works where nonresponse and measurement errors occurred separately. We shall now proceed with the review of studies that investigated the joint occurrence of nonresponse and measurement errors when estimating some population parameters of interest under different sampling schemes. The assumption is that the two types of nonsampling errors, that is, nonresponse and measurement errors, occurred jointly or simultaneously in a given study. In other words, the presence of both nonresponse and measurement errors in a given survey is assumed. For instance, estimation of population mean in the presence of both nonresponse and measurement errors under the simple random sampling scheme has been discussed by authors like Singh and Sharma (2015), Kumar, Bhogal, Nataraja and Viswanathaiah (2015), Azeem and Hanif (2016), Bhushan and Kumar (2017), Irfan, Javed and Lin (2018), Singh and Singh and Bouza (2018), Zahid and Shabbir (2018) and Singh and Vishwakarma (2019). In handling the problem of estimating the population mean in simple random

sampling scheme when both nonresponse and measurement errors occurred jointly, Singh and Sharma (2015) suggested a class of estimators after discussing the customary sample mean estimator, the ratio-type estimator and the regression-type estimator. Their proposed class of estimators is a linear combination of the sample mean and the ratio-type estimators on the assumption that the nonresponse and measurement errors occurred jointly in both the study and auxiliary variables. Kumar et al. (2015), in their own work, also assumed the presence of nonresponse and measurement errors in both the study and auxiliary variable but suggested an exponential ratio-type estimator for estimating the population mean in the presence of both nonresponse and measurement errors under the simple random sampling scheme.

The use of transformed auxiliary variable was considered by Azeem and Hanif (2016), who proposed three estimators that utilized variable transformation in estimating the population mean under the joint influence of nonresponse and measurement errors in simple random sampling scheme. Kumar (2016), like Azeem and Hanif (2016), discussed estimation of population mean in the presence of nonresponse and measurement errors, using variable transformation in simple random sampling scheme. He proposed a general class of estimators for estimating the population mean, using variable transformation, under the joint occurrence of nonresponse and measurement errors in simple random sampling scheme. Irfan et al. (2018) suggested a class of estimators, which is a linear function of ratio-type and exponential-type estimators, while Singh et al. (2018) considered a class of estimators, which is a combination of regression-type, exponential ratio-type and exponential product-type estimators. Singh and Vishwakarma (2019) also discussed a general class of estimators of the population mean using auxiliary information with the combined effect of nonresponse and measurement errors in simple random sampling scheme.

In dealing with the problem of joint or simultaneous occurrence of nonresponse and measurement errors, some authors have employed the use of the double sampling technique in situations where some population parameters of the auxiliary variable were unknown. These include the works carried out by Bhushan et al. (2017), Yadav, Devi and Yadav (2018) and

Kumar and Bhogal (2018), who utilized the double sampling procedure to address estimation of population mean under the joint influence of nonresponse and measurement errors when the population mean of an auxiliary variable is unknown. Bhushan et al. (2017) proposed some modified ratio, product and difference-type estimators, while Kumar and Bhogal (2018) utilized variable transformation under two-phase or double sampling scheme, and proposed a class of estimators, which is a linear combination of ratio and product-type estimators for estimating the population mean in the presence of both nonresponse and measurement errors when the population mean of an auxiliary variable is unknown.

Furthermore, the problem of mean estimation in the presence of both nonresponse and measurement under the stratified random sampling scheme has been given some consideration by Zahid and Shabbir (2018). They proposed an improved class of estimators of the population mean under the joint influence of nonresponse and measurement errors in stratified random sampling scheme. West, Conard, Kreuter and Mittereder (2018), on the other hand, investigated the joint influence of nonresponse and measurement errors when the population variance of the study variable is the parameter of interest. They discussed sampling strategies for the population variance of the study variable among interviewers in standardized and conversational interviewing in the presence of joint or simultaneous occurrence of nonresponse and measurement errors under the simple random sampling scheme. Olayiwola, Ajayi, Akintunde, Wale-Orojo, Ajibade and Ojelabi (2020) considered estimation of population mean in simple random sampling in the presence of both measurement and response errors. The literature does not reveal any methodology for handling the problem of joint occurrence of nonresponse and measurement errors in poststratified sampling scheme, hence, the present study addresses this challenge.

2.4 Poststratified Sampling Scheme

Under the poststratified sampling scheme described in subsection (1), properties of the usual poststratified estimator have been discussed by authors like Hansen, Hurwitz and Madow

(1953), Kish (1965) and Cochran (1977) and Onyeka (2012). An alternative procedure to the usual collapsing of strata when there are strata with no or very few observations, was considered by Fuller (1966). He proposed a procedure within the unconditional framework that is superior to that of collapsing two strata. The proposed procedure is a generalization of collapsing two post strata to collapsing any number of strata in a poststratified sampling scheme. The study obtained unbiased estimators when the populations are divided into a large number of small post strata.

Other developments in the theory of poststratified sampling scheme included an important study carried out by Holt and Smith (1979). They examined the concept of post stratification and it's framework for statistical inference and suggested that inferences should be made under the conditional argument, that is, for achieved sample configurations. Part of their findings was the assertion that neither the sample mean nor the poststratification estimator could be said to be uniformly better than the other in all situations. However, their empirical investigation showed that poststratification has more advantage when there are unfavourable sample configurations.

In the literature, discussions on the use of either conditional or unconditional arguments in post-stratified sampling scheme have continued to dominate the scene over the years. Casady and Valliant (1992) considered conditional properties of poststratified estimators for normally distributed study variables. They proposed a new poststratified regression-type estimator, which is similar to the usual regression-type estimator under the simple random sampling scheme. The efficiency of their proposed estimator was compared with those of some existing estimators. Relative efficiency of poststratified estimators over the simple sample mean was also considered by Intrama, Suwattee, and Vonghirandecha (2011). The study employed the use of simulation method to generate relevant datasets and underscored the usefulness of poststratification, especially when dealing with skewed populations. The work carried out by Casady et al. (1992) involved study variables with normal distribution, while Intrama et al. (2011) highlighted the importance of poststratification for skewed populations. However, Dahlke, Breidt, Opsomer and Van-Keilegom (2013) considered nonparametric endogenous poststratification

estimation. Properties of the proposed Endogenous Poststratification Estimator (EPSE) were obtained. Discussions were mainly based on cases of sample-fitted nonparametric models, giving particular emphasis and attention on monotone regression model. Asymptotic properties of the nonparametric EPSE were investigated under a superpopulation model framework.

Similar to what is obtainable in simple random sampling and other sampling schemes, another issue that has been considered in poststratification by many authors is the use of auxiliary information in poststratified sampling scheme. Singh, Housila and Vishwakarma (2010) considered estimation of the population mean of the study variable, using auxiliary information under poststratification. They examined both combined and separate-type estimators under large sample sizes, and obtained the asymptotic properties of the proposed estimators. Similarly, Bacanlı and Evin Aksu (2012), while utilizing auxiliary information in poststratified sampling scheme, considered some ratio-type estimators. They proposed five ratio-type estimators and compared their efficiencies with some already existing estimators. Furthermore, Onyeka (2013) considered a general family or class of combined estimators of the population mean in poststratified sampling scheme, using known values of some population parameters of an auxiliary variable. Properties of the proposed class of estimators, including efficiency conditions, were obtained up to first order approximations.

On using auxiliary information in poststratified sampling scheme, some authors, like Onyeka, Izunobi and Iwueze (2013) and Lone and Tailor (2015), have also incorporated variable transformation, that is, the use of transformed auxiliary variables. Onyeka et al. (2013) considered a class of dual to ratio-type estimators of population mean in poststratified sampling, using known value of some population parameters of the auxiliary variable. Lone and Tailor (2015), on the other hand, considered a class of dual to separate product-type exponential estimators, under poststratified sampling scheme. In the studies carried out by Onyeka et al. (2013) and Lone and Tailor (2015), properties of the proposed classes of estimators were obtained, as well as efficiency conditions over some existing estimators. The theoretical results were also verified using appropriate numerical illustrations.

Further use of auxiliary information in poststratified sampling scheme was considered by Lone and Tailor (2016). They proposed a class of ratio-cum-product type estimators for the purpose of estimating the population mean in poststratified sampling scheme. Similarly, Vishwakarma (2017) considered an alternative to ratio-type estimator in poststratification. The study obtained properties of the proposed estimator both for the achieved sample configuration (conditional argument) and over repeated samples of fixed sized, n (unconditional argument). Simulated dataset was used to confirm the theoretical results. Also, Tailor, Tailor and Chouhan (2017) considered an improved ratio and product-type exponential estimators for population mean in poststratified sampling scheme. Properties of the proposed estimators were obtained up to first degree of approximation. Efficiency comparison was also carried out and conditions under which the proposed estimators will perform better than the usual sample mean estimator, the ratio-type estimator and the product-type estimator were obtained.

Extension of population mean estimation to estimation of population ratio of two variables in poststratified sampling scheme was considered by Onyeka et al. (2013) and Onyeka, Izunobi and Iwueze (2015), using variable transformation. They proposed six combined-type estimators and six separate-type estimators of population ratio of two population means in poststratified sampling scheme, using variable transformation. Properties of the proposed estimators were obtained both for achieved sample configurations and over repeated samples of fixed size n , and efficiency comparisons were carried out. Olayiwola, Ajayi and Okunlade (2016) contributed to the use of poststratification on successive occasions, by considering modified separate and combined poststratified regression-type estimators on two successive occasions. Properties of proposed estimators were obtained and efficiency comparisons were carried out accordingly.

As we earlier mentioned, there are works in the literature that considered the occurrence of nonresponse alone in a given survey, while others considered the occurrence of measurement errors alone. There are also works that considered the joint or simultaneous occurrence of both nonresponse and measurement errors. Furthermore, occurrence of nonresponse and mea-

surement errors have been largely considered under the simple random sampling scheme, the stratified sampling scheme, systematic sampling scheme and two-phase or double sampling scheme. However, the handling of the problem of nonresponse and measurement errors under the poststratified sampling scheme has not received adequate consideration over the years. Consequently, the present work is intended to fill the gap. The focus shall be on the formulation and development of sampling strategies for handling the problem of nonresponse and measurement errors when estimating population parameters under the poststratified sampling scheme.

CHAPTER THREE

3.0 METHODOLOGY

In this chapter, we develop methodologies for three broad situations involving nonresponse and measurement errors in surveys that utilize poststratified sampling scheme. First we develop sampling strategies for handling situations where there is occurrence of nonresponse alone. Secondly, we develop sampling strategies for handling situations where there is occurrence of measurement errors alone. Thirdly, we develop sampling strategies for handling situations where there is joint occurrence of nonresponse and measurement errors when using poststratified sampling scheme. Furthermore, this chapter also discusses methodology for both data collection and data analysis of datasets used for empirical illustration of the sampling strategies developed in this study.

3.1 Nonresponse in Poststratified Sampling Scheme

In this section, we consider poststratified sampling schemes in which there is the occurrence of nonresponse alone. The focus is to develop sampling strategies for estimating the population mean of the study variable or other population parameters of the study variable in the presence of nonresponse when using poststratified sampling scheme. Based on the assumption that auxiliary information would be employed in developing the sampling strategies, there are three possible ways the nonresponse could occur with respect to the study and auxiliary variables. First the nonresponse could occur on both the study and auxiliary variables. Secondly the nonresponse could be only on the study variable while there is complete response on the

auxiliary variable. Thirdly, the nonresponse could occur only on the auxiliary variable while there is complete response on the study variable. These three scenarios shall be handled in turn, starting with situations when there is nonresponse on both the study and auxiliary variables in poststratified sampling scheme. Specifically, we shall develop both sampling designs and estimators when there is nonresponse or missing data on both the study and auxiliary variables under the poststratified sampling scheme.

3.1.1 Nonresponse on both the study and auxiliary variables under the Poststratified Sampling Scheme

Following Hansen and Hurwitz (1946), we shall propose a sub-sampling design for the handling of nonresponse in poststratified sampling scheme, when there is nonresponse on both the study and auxiliary variables. The sampling design we propose, is as follows:

Sampling Design 3.1.1:

Consider a population, $\Omega = \{U_1, U_2, \dots, U_N\}$, of N units, divided into subpopulations or strata of sizes, N_h , $h = 1, 2, \dots, L$, such that $\sum_{h=1}^L N_h = N$. For the purpose of estimating the population mean of a study variable under the poststratified sampling scheme, we use the simple random sampling without replacement (SRSWOR) method to select a sample of n units from the N units in the population and obtain information on the study, auxiliary and stratification variables. We assume that n'_1 units responded out of the selected n units in the sample, while n'_2 units did not respond. The n'_1 units that responded were allocated to their various strata where n_{1h} units fall into stratum h such that $\sum_{h=1}^L n_{1h} = n'_1$ units. To reduce the effect of the nonresponse, a subsample of m units is selected from the nonresponse sample group of n'_2 units, and for these m subsampled units, extra efforts are made to obtain information on the missing study and auxiliary variables as well as the stratification variable. The m subsampled units were then allocated to their various strata with m_h units falling into stratum h such that $\sum_{h=1}^L m_h = m$ units. We assume that both n'_1 and m are large enough such that $P(n_{1h} = 0) = P(m_h = 0) = 0 \forall h$.

Clearly, the Sampling Design 3.1.1 seeks to reduce the effect of nonresponse when there are missing values on both the study and auxiliary variables when using the poststratified sampling scheme. Under the proposed sampling design, the available sample data on the study (y) and auxiliary (x) variables consist of the observed values of y and x from the response sample group of n_{1h} units in stratum h, supplemented with the observed values of y and x from the nonresponse subsample group of m_h units in stratum h. Consequently, the associated sample means, $\bar{y}_{1h}(\bar{x}_{1h})$ and $\bar{y}_{2mh}(\bar{x}_{2mh})$, form the basis for constructing estimators of the population mean ($\bar{Y}$) of the study variable, y, when there is nonresponse on both the study and auxiliary variables in poststratified sampling scheme. For the h^{th} stratum, we give a sample mean of the study variable (y) as a linear function of the response and nonresponse subsample means:

$$\bar{y}_h^* = \left(\frac{n_{1h}}{n_h}\right)\bar{y}_{1h} + \left(\frac{n_{2h}}{n_h}\right)\bar{y}_{2mh} \quad (3.1)$$

and the corresponding sample mean of the auxiliary variable, x, given by:

$$\bar{x}_h^* = \left(\frac{n_{1h}}{n_h}\right)\bar{x}_{1h} + \left(\frac{n_{2h}}{n_h}\right)\bar{x}_{2mh} \quad (3.2)$$

where $\bar{y}_{1h} = \frac{1}{n_{1h}} \sum_{i=1}^{n_{1h}} y_{hi}$ is the sample mean of y in stratum h, based on the response group sample size, n_{1h}

$\bar{y}_{2mh} = \frac{1}{m_h} \sum_{i=1}^{m_h} y_{hi}$ is the sample mean of y in stratum h, based on the nonresponse sub-sample group sample size, m_h

$\bar{x}_{1h} = \frac{1}{n_{1h}} \sum_{i=1}^{n_{1h}} x_{hi}$ is the sample mean of x in stratum h, based on the response group sample size, n_{1h}

$\bar{x}_{2mh} = \frac{1}{m_h} \sum_{i=1}^{m_h} x_{hi}$ is the sample mean of x in stratum h, based on the nonresponse sub-sample group sample size, m_h

Based on the Sampling Design 3.1.1 and using the nonresponse sample means (3.1) and (3.2), we propose three estimators of the population mean ($\bar{Y}$) of the study variable, y, when there

is nonresponse on both study and auxiliary variables under the poststratified sampling scheme. The three estimators we suggest are as follows:

1. The customary nonresponse sample mean estimator given by

$$\bar{y}_{ns1} = \sum_{h=1}^L W_h \bar{y}_h^* \quad (3.3)$$

2. A nonresponse difference-type estimator given by

$$\bar{y}_{nd1} = \sum_{h=1}^L W_h [\bar{y}_h^* - k_h(\bar{x}_h^* - \bar{X}_h)] \quad (3.4)$$

3. A general class of estimators that makes use of known values of some population parameters of the auxiliary variable in the h^{th} stratum, given by

$$\bar{y}_{nc1} = \sum_{h=1}^L W_h \bar{y}_h^* \left(\frac{a_h \bar{X}_h + b_h}{a_h \bar{x}_h^* + b_h} \right)^{\alpha_h} \quad (3.5)$$

where $k_h(\alpha_h)$ are any real numbers and $a_h(b_h)$ are also any real numbers or known values of population parameters of an auxiliary variable, x , in the h^{th} stratum, like coefficient of variation, skewness, kurtosis, etc. It must be noted that the estimator (3.3) does not make use of any auxiliary information, while the estimators (3.4) and (3.5) utilize information on one auxiliary variable. Particularly, the proposed general class of estimators (3.5) utilizes information on known values of population parameters of an auxiliary variable, and it is a simplified version of the class of estimators suggested by Khoshnevisan, Singh, Chauhan, Sawan, and Smarandache (2007) under the simple random sampling scheme. Properties of the above proposed estimators shall be obtained both under the conditional argument or given an achieved sample configuration, say $\underline{n} = (n_1, n_2, \dots, n_L)$ and under the unconditional argument or for repeated samples of fixed size, n . We start with the proposed customary nonresponse sample mean estimator, $\bar{y}_{ns1}$, given in (3.3).

Estimator 1: Customary nonresponse sample mean estimator, $\bar{y}_{ns1}$, when there is nonresponse on both the study and auxiliary variables under the Poststratified Sampling Scheme.

The proposed customary nonresponse sample mean estimator, $\bar{y}_{ns1}$, does not utilize any auxiliary information. The conditional and unconditional properties of the estimator, $\bar{y}_{ns1}$, are given in theorem 3.1.

Theorem 3.1: Under the conditional argument or for an achieved sample configuration, say $\underline{n} = (n_1, n_2, \dots, n_L)$, the proposed customary nonresponse sample mean estimator, $\bar{y}_{ns1}$, is unbiased for the population mean ($\bar{Y}$), of the study variable, y, with its conditional variance obtained as:

$$V_c(\bar{y}_{ns1}) = \sum_{h=1}^L W_h^2 \left(\frac{1}{n_h} - \frac{1}{N_h} \right) S_{yh}^2 + \sum_{h=1}^L W_h^2 \left(\frac{g_h - 1}{n_h} \right) W_{2h} S_{2yh}^2 \quad (3.6)$$

where $S_{yh}^2 = \frac{1}{N_h - 1} \sum_{i=1}^{N_h} (Y_{ih} - \bar{Y}_h)^2$, $g_h = \frac{n_{2h}}{m_{2h}}$

Furthermore, under the unconditional argument or for repeated samples of fixed size, n, the nonresponse sample mean estimator, $\bar{y}_{ns1}$, is unbiased for the population mean ($\bar{Y}$) of the study variable y, with its unconditional variance obtained up to first order of approximations as:

$$V(\bar{y}_{ns1}) = \left(\frac{1-f}{n} \right) \sum_{h=1}^L W_h S_{yh}^2 + \left(\frac{1}{n} \right) \sum_{h=1}^L W_h (g_h - 1) W_{2h} S_{2yh}^2 \quad (3.7)$$

Proof: Let E_1, V_1 and C_1 respectively denote expectation, variance and covariance over the initial or first phase sampling of n units out of N units in the population, while E_2, V_2 and C_2 respectively denote expectation, variance and covariance over the subsample or second phase sampling of m units out of the n'_2 nonresponse units. Then from (3.1) we have:

$$E_2(\bar{y}_h^*) = \left(\frac{n_{1h}}{n_h} \right) E_2(\bar{y}_{1h}) + \left(\frac{n_{2h}}{n_h} \right) E_2(\bar{y}_{2mh}) = \left(\frac{n_{1h}}{n_h} \right) (\bar{y}_{1h}) + \left(\frac{n_{2h}}{n_h} \right) (\bar{y}_{2h}) = \bar{y}_h \quad (3.8)$$

and

$$V_2(\bar{y}_h^*) = \left(\frac{n_{1h}}{n_h} \right)^2 V_2(\bar{y}_{1h}) + \left(\frac{n_{2h}}{n_h} \right)^2 V_2(\bar{y}_{2mh}) + 2 \left(\frac{n_{1h}}{n_h} \right) \left(\frac{n_{2h}}{n_h} \right) C_2(\bar{y}_{1h}, \bar{y}_{2mh}) \quad (3.9)$$

Considering the fact that $V_2(\bar{y}_{1h}) = C_2(\bar{y}_{1h}, \bar{y}_{2mh}) = 0$ and applying the theory of simple random

sampling without replacement (SRSWOR) method, we write (3.9) as

$$V_2(\bar{y}_h^*) = \left(\frac{n_{2h}}{n_h}\right)^2 V_2(\bar{y}_{2mh}) = \left(\frac{n_{2h}}{n_h}\right)^2 \left(\frac{1}{m_h} - \frac{1}{n_{2h}}\right) s_{2yh}^2 = \left(\frac{g_h - 1}{n_h}\right) \left(\frac{n_{2h}}{n_h}\right) s_{2yh}^2 \quad (3.10)$$

Applying the theories of double phase sampling and SRSWOR method and using (3.8), we obtain the conditional expectation of $\bar{y}_h^*$ as

$$E_c(\bar{y}_h^*) = E_1 E_2(\bar{y}_h^*) = E_1(\bar{y}_h) = \bar{Y}_h \quad (3.11)$$

Using (3.3) and (3.11), the conditional expectation of the proposed customary nonresponse sample mean estimator, $\bar{y}_{ns1}$, is obtained as

$$E_c(\bar{y}_{ns1}) = \sum_{h=1}^L W_h E_c(\bar{y}_h^*) = \sum_{h=1}^L W_h \bar{Y}_h = \bar{Y} \quad (3.12)$$

Hence, $\bar{y}_{ns1}$ is unbiased for $\bar{Y}$ under the conditional argument. To obtain the conditional variance of $\bar{y}_{ns1}$, we first obtain the conditional variance of $\bar{y}_h^*$ in the h^{th} stratum. Applying the theory of double sampling, the conditional variance of $\bar{y}_h^*$ can be written as:

$$V_c(\bar{y}_h^*) = V_1 E_2(\bar{y}_h^*) + E_1 V_2(\bar{y}_h^*) \quad (3.13)$$

Substituting (3.8) and (3.10) into (3.13) and applying results from SRSWOR gives:

$$V_c(\bar{y}_h^*) = V_1(\bar{y}_h) + E_1 \left[\left(\frac{g_h - 1}{n_h}\right) \left(\frac{n_{2h}}{n_h}\right) s_{2yh}^2 \right] = V_1(\bar{y}_h) + \left(\frac{g_h - 1}{n_h}\right) E_1 \left(\frac{n_{2h}}{n_h}\right) E_1(s_{2yh}^2) \quad (3.14)$$

or

$$V_c(\bar{y}_h^*) = \left(\frac{1}{n_h} - \frac{1}{N_h}\right) S_{yh}^2 + \left(\frac{g_h - 1}{n_h}\right) W_{2h} S_{2yh}^2 \quad (3.15)$$

From (3.3), we write the conditional variance of $\bar{y}_{ns1}$ as

$$V_c(\bar{y}_{ns1}) = V_c\left(\sum_{h=1}^L W_h \bar{y}_h^*\right) = \sum_{h=1}^L W_h^2 V_c(\bar{y}_h^*) + 2 \sum_{h=1}^L \sum_{j>h}^L W_h W_j C_c(\bar{y}_h^*, \bar{y}_j^*) = \sum_{h=1}^L W_h^2 V_c(\bar{y}_h^*) \quad (3.16)$$

since the covariance term between two different strata means vanishes. Substituting (3.15)

in (3.16) gives the conditional variance of the proposed customary nonresponse sample mean estimator, $\bar{y}_{ns1}$, as

$$V_c(\bar{y}_{ns1}) = \sum_{h=1}^L W_h^2 \left(\frac{1}{n_h} - \frac{1}{N_h} \right) S_{yh}^2 + \sum_{h=1}^L W_h^2 \left(\frac{g_h - 1}{n_h} \right) W_{2h} S_{2yh}^2 \quad (3.17)$$

as stated in the theorem. This proves the first part of the theorem that bothers on the conditional properties of the proposed customary nonresponse sample mean estimator, $\bar{y}_{ns1}$, of the population mean ($\bar{Y}$) of the study variable, y . To prove the second part of the theorem, which bothers on the unconditional properties of the proposed nonresponse estimator, $\bar{y}_{ns1}$, we consider repeated samples of fixed size of n units selected under the poststratified sampling scheme in which the h^{th} stratum sample size, n_h , is a random variable. Following Stephan (1945), the stratum sample size, n_h , has a hypergeometric distribution with the probability mass function given as

$$P(n_h) = \frac{\binom{N_h}{n_h} \binom{N - N_h}{n - n_h}}{\binom{N}{n}}, \quad n_h = 1, 2, \dots, n \quad (3.18)$$

with its mean and variance obtained respectively as

$$E(n_h) = nW_h \quad (3.19)$$

and

$$V(n_h) = \left(\frac{N - n}{N - 1} \right) (nW_h)(1 - W_h) \quad (3.20)$$

Consider an adjusted mean deviation of the random variable, n_h , given by

$$d_0 = \frac{n_h - E(n_h)}{E(n_h)} = \frac{n_h - nW_h}{nW_h} \quad \text{or} \quad n_h = (nW_h)(1 + d_0) \quad (3.21)$$

so that $E(d_0)=0$, using (3.19) and (3.21). Then, using (3.21) and (3.20), we obtain

$$E(d_0^2) = E\left(\frac{n_h - nW_h}{nW_h}\right)^2 = \frac{E(n_h - nW_h)^2}{(nW_h)^2} = \frac{V(n_h)}{(nW_h)^2} = \left(\frac{N - n}{N - 1}\right) \left(\frac{1 - W_h}{nW_h}\right) \quad (3.22)$$

Using (3.21) and expanding by Taylor's series up to second order of approximations or terms

of order $1/n^2$ in expected value, we obtain the reciprocal of n_h as

$$\frac{1}{n_h} = (nW_h)^{-1}(1 + d_0)^{-1} = (nW_h)^{-1}(1 - d_0 + d_0^2) \quad (3.23)$$

Using (3.23) and (3.22) and the fact that $E(d_0) = 0$, we obtain the expected value of the reciprocal of n_h up to second order of approximations as

$$E\left(\frac{1}{n_h}\right) = (nW_h)^{-1}\left[1 - E(d_0) + E(d_0^2)\right] = \frac{1}{nW_h} + \left(\frac{N-n}{N-1}\right)\left(\frac{1-W_h}{n^2W_h^2}\right) \quad (3.24)$$

Approximating (3.24) to terms of order n^{-1} or first order of approximations, simply gives

$$E\left(\frac{1}{n_h}\right) = \frac{1}{nW_h} \quad (3.25)$$

Therefore, to obtain the unconditional expectation of the proposed customary nonresponse sample mean estimator, $\bar{y}_{ns1}$, we take the (unconditional) expectation of (3.12) over repeated samples of fixed size, n , to obtain:

$$E(\bar{y}_{ns1}) = EE_c(\bar{y}_{ns1}) = E(\bar{Y}) = \bar{Y} \quad (3.26)$$

Hence, the proposed customary nonresponse sample mean estimator, $\bar{y}_{ns1}$, is unbiased for the population mean ($\bar{Y}$) under the unconditional argument. Furthermore, taking the unconditional expectation of (3.17) gives the unconditional variance of $\bar{y}_{ns1}$ as:

$$V(\bar{y}_{ns1}) = EV_c(\bar{y}_{ns1}) = \sum_{h=1}^L W_h^2 \left[E\left(\frac{1}{n_h}\right) - \frac{1}{N_h} \right] S_{yh}^2 + \sum_{h=1}^L W_h^2 E\left(\frac{1}{n_h}\right) (g_h - 1) W_{2h} S_{2yh}^2 \quad (3.27)$$

and using (3.25) to make the necessary substitutions in (3.27) gives the unconditional variance of the proposed sample mean estimator, $\bar{y}_{ns1}$, up to first order of approximations, as:

$$V(\bar{y}_{ns1}) = \left(\frac{1-f}{n}\right) \sum_{h=1}^L W_h S_{yh}^2 + \left(\frac{1}{n}\right) \sum_{h=1}^L W_h (g_h - 1) W_{2h} S_{2yh}^2 \quad (3.28)$$

as stated in the theorem. And this completes the proof.

Estimator 2: Difference-type estimator, $\bar{y}_{nd1}$, when there is nonresponse on both the study and auxiliary variables under the Poststratified Sampling Scheme.

The proposed difference-type estimator, $\bar{y}_{nd1}$, given in (3.4), utilizes auxiliary information with known population mean ($\bar{X}_h$) of the auxiliary variable, x , in the h^{th} stratum. We give the conditional and unconditional properties of the estimator, $\bar{y}_{nd1}$, in theorem 3.2.

Theorem 3.2: Under the conditional argument or for an achieved sample configuration, say $\underline{n} = (n_1, n_2, \dots, n_L)$, the proposed difference-type estimator, $\bar{y}_{nd1}$, is an unbiased estimator of the population mean ($\bar{Y}$), of the study variable, y , with its conditional variance obtained as:

$$V_c(\bar{y}_{nd1}) = \sum_{h=1}^L W_h^2 \left(\frac{1}{n_h} - \frac{1}{N_h} \right) (S_{yh}^2 + k_h^2 S_{xh}^2 - 2k_h S_{yxh}) + \sum_{h=1}^L W_h^2 \left(\frac{g_h - 1}{n_h} \right) W_{2h} (S_{2yh}^2 + k_h^2 S_{2xh}^2 - 2k_h S_{2yxh}) \quad (3.29)$$

Furthermore, under the unconditional argument or for repeated samples of fixed size, n , the difference-type estimator, $\bar{y}_{nd1}$, is an unbiased estimator of the population mean ($\bar{Y}$) of the study variable y , with its unconditional variance obtained up to first order of approximations as:

$$V(\bar{y}_{nd1}) = \left(\frac{1-f}{n} \right) \sum_{h=1}^L W_h (S_{yh}^2 + k_h^2 S_{xh}^2 - 2k_h S_{yxh}) + \left(\frac{1}{n} \right) \sum_{h=1}^L W_h (g_h - 1) W_{2h} (S_{2yh}^2 + k_h^2 S_{2xh}^2 - 2k_h S_{2yxh}) \quad (3.30)$$

Proof: The proposed difference-type estimator, $\bar{y}_{nd1}$, given in (3.4), can be written as:

$$\bar{y}_{nd1} = \sum_{h=1}^L W_h [\bar{y}_h^* - k_h (\bar{x}_h^* - \bar{X}_h)] = \sum_{h=1}^L W_h \bar{y}_{nd1h} \quad (3.31)$$

where

$$\bar{y}_{nd1h} = \bar{y}_h^* - k_h (\bar{x}_h^* - \bar{X}_h) \quad (3.32)$$

with its conditional expectation given by

$$E_c(\bar{y}_{nd1h}) = E_c(\bar{y}_h^*) - k_h [E_c(\bar{x}_h^*) - \bar{X}_h] \quad (3.33)$$

It follows from (3.11) that

$$E_c(\bar{x}_h^*) = \bar{X}_h \quad (3.34)$$

Substituting (3.11) and (3.34) in (3.33) gives the conditional expectation of $\bar{y}_{nd1h}$ as

$$E_c(\bar{y}_{nd1h}) = \bar{Y}_h - k_h(\bar{X}_h - \bar{X}_h) = \bar{Y}_h \quad (3.35)$$

Consequently, it follows from (3.31) and (3.35) that the conditional expectation of the proposed difference-type estimator, $\bar{y}_{nd1}$, is obtained as

$$E_c(\bar{y}_{nd1}) = \sum_{h=1}^L W_h E_c(\bar{y}_{nd1h}) = \sum_{h=1}^L W_h \bar{Y}_h = \bar{Y} \quad (3.36)$$

Hence, the proposed difference-type estimator, $\bar{y}_{nd1}$, is unbiased for $\bar{Y}$ under the conditional argument. To obtain the conditional variance of $\bar{y}_{nd1}$, we first obtain the conditional variance of the stratum mean estimator, $\bar{y}_{nd1h}$, given in (3.32), from where we can write:

$$V_c(\bar{y}_{nd1h}) = V_c[\bar{y}_h^* - k_h(\bar{x}_h^* - \bar{X}_h)] = V_c(\bar{y}_h^*) + k_h^2 V_c(\bar{x}_h^*) - 2k_h C_c(\bar{y}_h^*, \bar{x}_h^*) \quad (3.37)$$

It follows from (3.15) that

$$V_c(\bar{x}_h^*) = \left(\frac{1}{n_h} - \frac{1}{N_h}\right) S_{xh}^2 + \left(\frac{g_h - 1}{n_h}\right) W_{2h} S_{2xh}^2 \quad (3.38)$$

And from (3.1), (3.2) and (3.8), it also follows that

$$E_2(\bar{x}_h^*) = \bar{x}_h \quad (3.39)$$

and

$$\begin{aligned} C_2(\bar{y}_h^*, \bar{x}_h^*) &= C_2\left[\left(\frac{n_{1h}}{n_h}\right)\bar{y}_{1h} + \left(\frac{n_{2h}}{n_h}\right)\bar{y}_{2mh}, \left(\frac{n_{1h}}{n_h}\right)\bar{x}_{1h} + \left(\frac{n_{2h}}{n_h}\right)\bar{x}_{2mh}\right] \\ &= \left(\frac{n_{2h}}{n_h}\right)^2 C_2(\bar{y}_{2mh}, \bar{x}_{2mh}) = \left(\frac{n_{2h}}{n_h}\right)^2 \left(\frac{1}{m_h} - \frac{1}{n_{2h}}\right) S_{2yxh}^2 = \left(\frac{g_h - 1}{n_h}\right) \left(\frac{n_{2h}}{n_h}\right) S_{2yxh}^2 \end{aligned} \quad (3.40)$$

since other subsampling covariances vanished. Applying the theory of double or two-phase

sampling, the conditional covariance of the estimators, $\bar{y}_h^*$ and $\bar{x}_h^*$, can be written as:

$$C_c(\bar{y}_h^*, \bar{x}_h^*) = C_1[E_2(\bar{y}_h^*), E_2(\bar{x}_h^*)] + E_1 C_2(\bar{y}_h^*, \bar{x}_h^*) \quad (3.41)$$

Substituting (3.8), (3.39) and (3.40) in (3.41) gives

$$C_c(\bar{y}_h^*, \bar{x}_h^*) = C_1(\bar{y}_h, \bar{x}_h) + \left(\frac{g_h - 1}{n_h}\right) E_1\left(\frac{n_{2h}}{n_h}\right) E_1(S_{2y_{xh}}) = \left(\frac{1}{n_h} - \frac{1}{N_h}\right) S_{yxh} + \left(\frac{g_h - 1}{n_h}\right) W_{2h} S_{2y_{xh}} \quad (3.42)$$

Consequently, substituting (3.15), (3.38) and (3.42) in (3.37) gives

$$V_c(\bar{y}_{nd1h}) = \left(\frac{1}{n_h} - \frac{1}{N_h}\right) (S_{yh}^2 + k_h^2 S_{xh}^2 - 2k_h S_{yxh}) + \left(\frac{g_h - 1}{n_h}\right) W_{2h} (S_{2yh}^2 + k_h^2 S_{2xh}^2 - 2k_h S_{2y_{xh}}) \quad (3.43)$$

From (3.31) and noting that covariance between sample means from different strata vanishes, the conditional variance of the proposed difference-type estimator, $\bar{y}_{ns1}$, can be written as

$$V_c(\bar{y}_{nd1}) = V_c\left(\sum_{h=1}^L W_h \bar{y}_{nd1h}\right) = \sum_{h=1}^L W_h^2 V_c(\bar{y}_{nd1h}) \quad (3.44)$$

and substituting (3.43) in (3.44) we obtain the conditional variance of the proposed difference-type estimator, $\bar{y}_{nd1}$, when there is nonresponse on both the study and auxiliary variables, as:

$$V_c(\bar{y}_{nd1}) = \sum_{h=1}^L W_h^2 \left(\frac{1}{n_h} - \frac{1}{N_h}\right) (S_{yh}^2 + k_h^2 S_{xh}^2 - 2k_h S_{yxh}) + \sum_{h=1}^L W_h^2 \left(\frac{g_h - 1}{n_h}\right) W_{2h} (S_{2yh}^2 + k_h^2 S_{2xh}^2 - 2k_h S_{2y_{xh}}) \quad (3.45)$$

as stated in the theorem. Thus, this proves the first part of the theorem that bothers on the conditional properties of the proposed difference-type estimator, $\bar{y}_{nd1}$, of the population mean ($\bar{Y}$) of the study variable, y . To obtain the unconditional variance of the proposed estimator, $\bar{y}_{nd1}$, we take the unconditional expectation of (3.45) and use (3.25) to make the necessary substitutions. This gives

$$V(\bar{y}_{nd1}) = \left(\frac{1-f}{n}\right) \sum_{h=1}^L W_h (S_{yh}^2 + k_h^2 S_{xh}^2 - 2k_h S_{yxh}) + \left(\frac{1}{n}\right) \sum_{h=1}^L W_h (g_h - 1) W_{2h} (S_{2yh}^2 + k_h^2 S_{2xh}^2 - 2k_h S_{2y_{xh}}) \quad (3.46)$$

as stated in the theorem. And this completes the proof.

Estimator 3: Class of estimators, $\bar{y}_{nc1}$, when there is nonresponse on both the study and auxiliary variables under the Poststratified Sampling Scheme.

The proposed class of estimators, $\bar{y}_{nc1}$, given in (3.5), utilizes auxiliary information with known values of some population parameters of the auxiliary variable, x , in the h^{th} stratum. We give the conditional and unconditional properties of the estimator, $\bar{y}_{nc1}$, in theorem 3.3.

Theorem 3.3: Under the conditional argument or for an achieved sample configuration, say $\underline{n} = (n_1, n_2, \dots, n_L)$, the proposed class of estimators, $\bar{y}_{nc1}$, is biased for the population mean ($\bar{Y}$) of the study variable, y , with its conditional bias and mean square error obtained up to first order of approximations as:

$$B_c(\bar{y}_{nc1}) = \sum_{h=1}^L \frac{W_h \alpha_h \lambda_h}{2\bar{X}_h} \left(\frac{1}{n_h} - \frac{1}{N_h} \right) [(\alpha_h + 1) \lambda_h R_h S_{xh}^2 - 2S_{yxh}] \\ + \sum_{h=1}^L \frac{W_h \alpha_h \lambda_h}{2\bar{X}_h} \left(\frac{g_h - 1}{n_h} \right) W_{2h} [(\alpha_h + 1) \lambda_h R_h S_{2xh}^2 - 2S_{2yxh}] \quad (3.47)$$

$$MS E_c(\bar{y}_{nc1}) = \sum_{h=1}^L W_h^2 \left(\frac{1}{n_h} - \frac{1}{N_h} \right) (S_{yh}^2 + \alpha_h^2 \lambda_h^2 R_h^2 S_{xh}^2 - 2\alpha_h \lambda_h R_h S_{yxh}) \\ + \sum_{h=1}^L W_h^2 \left(\frac{g_h - 1}{n_h} \right) W_{2h} (S_{2yh}^2 + \alpha_h^2 \lambda_h^2 R_h^2 S_{2xh}^2 - 2\alpha_h \lambda_h R_h S_{2yxh}) \quad (3.48)$$

Furthermore, under the unconditional argument or for repeated samples of fixed size, n , the proposed class of estimators, $\bar{y}_{nc1}$, is biased for the population mean ($\bar{Y}$) of the study variable y , with its unconditional bias and mean square error obtained up to first order of approximations as:

$$B(\bar{y}_{nc1}) = \left(\frac{1-f}{n} \right) \sum_{h=1}^L \frac{\alpha_h \lambda_h}{2\bar{X}_h} [(\alpha_h + 1) \lambda_h R_h S_{xh}^2 - 2S_{yxh}] \\ + \left(\frac{1}{n} \right) \sum_{h=1}^L \frac{\alpha_h \lambda_h}{2\bar{X}_h} (g_h - 1) W_{2h} [(\alpha_h + 1) \lambda_h R_h S_{2xh}^2 - 2S_{2yxh}] \quad (3.49)$$

and

$$\begin{aligned}
 M S E(\bar{y}_{nc1}) &= \left(\frac{1-f}{n}\right) \sum_{h=1}^L W_h (S_{yh}^2 + \alpha_h^2 \lambda_h^2 R_h^2 S_{xh}^2 - 2\alpha_h \lambda_h R_h S_{yxh}) \\
 &+ \left(\frac{1}{n}\right) \sum_{h=1}^L W_h (g_h - 1) W_{2h} (S_{2yh}^2 + \alpha_h^2 \lambda_h^2 R_h^2 S_{2xh}^2 - 2\alpha_h \lambda_h R_h S_{2yxh})
 \end{aligned} \tag{3.50}$$

where

$$\lambda_h = \left(\frac{a_h \bar{X}_h}{a_h \bar{X}_h + b_h} \right) \tag{3.51}$$

Proof: The proposed class of estimators, $\bar{y}_{nc1}$, given in (3.5), can be written as:

$$\bar{y}_{nc1} = \sum_{h=1}^L W_h \bar{y}_h^* \left(\frac{a_h \bar{X}_h + b_h}{a_h \bar{x}_h^* + b_h} \right)^{\alpha_h} = \sum_{h=1}^L W_h \bar{y}_{nc1h} \tag{3.52}$$

where

$$\bar{y}_{nc1h} = \bar{y}_h^* \left(\frac{a_h \bar{X}_h + b_h}{a_h \bar{x}_h^* + b_h} \right)^{\alpha_h} \tag{3.53}$$

Consider adjusted mean deviations of the stratum mean estimators, $\bar{y}_h^*$ and $\bar{x}_h^*$, respectively given by

$$d_{2h} = \frac{\bar{y}_h^* - \bar{Y}_h}{\bar{Y}_h} \quad \text{or} \quad \bar{y}_h^* = \bar{Y}_h (1 + d_{2h}) \tag{3.54}$$

and

$$e_{2h} = \frac{\bar{x}_h^* - \bar{X}_h}{\bar{X}_h} \quad \text{or} \quad \bar{x}_h^* = \bar{X}_h (1 + e_{2h}) \tag{3.55}$$

so that under the conditional argument and using (3.11), (3.15), (3.34), (3.38) and (3.42), we have

$$\begin{aligned}
 E_c(d_{2h}) &= \frac{E_c(\bar{y}_h^*) - \bar{Y}_h}{\bar{Y}_h} = 0 \\
 E_c(e_{2h}) &= \frac{E_c(\bar{x}_h^*) - \bar{X}_h}{\bar{X}_h} = 0 \\
 E_c(d_{2h}^2) &= \frac{E_c(\bar{y}_h^* - \bar{Y}_h)^2}{\bar{Y}_h^2} = \frac{V_c(\bar{y}_h^*)}{\bar{Y}_h^2} = \frac{1}{\bar{Y}_h^2} \left[\left(\frac{1}{n_h} - \frac{1}{N_h} \right) S_{yh}^2 + \left(\frac{g_h - 1}{n_h} \right) W_{2h} S_{2yh}^2 \right] \\
 E_c(e_{2h}^2) &= \frac{E_c(\bar{x}_h^* - \bar{X}_h)^2}{\bar{X}_h^2} = \frac{V_c(\bar{x}_h^*)}{\bar{X}_h^2} = \frac{1}{\bar{X}_h^2} \left[\left(\frac{1}{n_h} - \frac{1}{N_h} \right) S_{xh}^2 + \left(\frac{g_h - 1}{n_h} \right) W_{2h} S_{2xh}^2 \right] \\
 E_c(d_{2h} e_{2h}) &= \frac{C_c(\bar{y}_h^*, \bar{x}_h^*)}{\bar{Y}_h \bar{X}_h} = \frac{1}{\bar{Y}_h \bar{X}_h} \left[\left(\frac{1}{n_h} - \frac{1}{N_h} \right) S_{yxh} + \left(\frac{g_h - 1}{n_h} \right) W_{2h} S_{2yxh} \right]
 \end{aligned} \tag{3.56}$$

Using (3.54) and (3.55), we can rewrite (3.53) as

$$\bar{y}_{nc1h} = \bar{Y}_h(1 + d_{2h})(1 + \lambda_h e_{2h})^{-\alpha_h} \quad (3.57)$$

Using Taylor's series to expand (3.57) up to first order of approximations in expected values, it follows that

$$(\bar{y}_{nc1h} - \bar{Y}_h) = \bar{Y}_h \left[d_{2h} - \alpha_h \lambda_h e_{2h} - \alpha_h \lambda_h d_{2h} e_{2h} + (1/2) \alpha_h (\alpha_h + 1) \lambda_h^2 e_{2h}^2 \right] \quad (3.58)$$

$$(\bar{y}_{nc1h} - \bar{Y}_h)^2 = \bar{Y}_h^2 \left[d_{2h}^2 + \alpha_h^2 \lambda_h^2 e_{2h}^2 - 2 \alpha_h \lambda_h d_{2h} e_{2h} \right] \quad (3.59)$$

Taking the conditional expectation of (3.58), and using (3.56) to make the necessary substitutions, gives the conditional bias of the stratum mean estimator, $\bar{y}_{nc1h}$, as

$$\begin{aligned} B_c(\bar{y}_{nc1h}) = E_c(\bar{y}_{nc1h} - \bar{Y}_h) &= \frac{\alpha_h \lambda_h}{2 \bar{X}_h} \left(\frac{1}{n_h} - \frac{1}{N_h} \right) \left[(\alpha_h + 1) \lambda_h R_h S_{xh}^2 - 2 S_{yxh} \right] \\ &+ \frac{\alpha_h \lambda_h}{2 \bar{X}_h} \left(\frac{g_h - 1}{n_h} \right) W_{2h} \left[(\alpha_h + 1) \lambda_h R_h S_{2xh}^2 - 2 S_{2yxh} \right] \end{aligned} \quad (3.60)$$

From (3.52), the conditional bias of the proposed class of estimators, $\bar{y}_{nc1}$, is

$$B_c(\bar{y}_{nc1}) = \sum_{h=1}^L W_h B_c(\bar{y}_{nc1h}) \quad (3.61)$$

and substituting (3.60) in (3.61) gives

$$\begin{aligned} B_c(\bar{y}_{nc1}) &= \sum_{h=1}^L \frac{W_h \alpha_h \lambda_h}{2 \bar{X}_h} \left(\frac{1}{n_h} - \frac{1}{N_h} \right) \left[(\alpha_h + 1) \lambda_h R_h S_{xh}^2 - 2 S_{yxh} \right] \\ &+ \sum_{h=1}^L \frac{W_h \alpha_h \lambda_h}{2 \bar{X}_h} \left(\frac{g_h - 1}{n_h} \right) W_{2h} \left[(\alpha_h + 1) \lambda_h R_h S_{2xh}^2 - 2 S_{2yxh} \right] \end{aligned} \quad (3.62)$$

as stated in the theorem. To obtain the conditional mean square error of the proposed class of estimators, $\bar{y}_{nc1}$, we first obtain the conditional mean square error of the stratum mean class of estimators, $\bar{y}_{nc1h}$, by taking the conditional expectation of (3.59) and using (3.56) to make the necessary substitutions. Thus, we have:

$$\begin{aligned}
MS E_c(\bar{y}_{nc1h}) &= E_c(\bar{y}_{nc1h} - \bar{Y}_h)^2 = \left(\frac{1}{n_h} - \frac{1}{N_h}\right)(S_{yh}^2 + \alpha_h^2 \lambda_h^2 R_h^2 S_{xh}^2 - 2\alpha_h \lambda_h R_h S_{yhx}) \\
&\quad + \left(\frac{g_h - 1}{n_h}\right)W_{2h}(S_{2yh}^2 + \alpha_h^2 \lambda_h^2 R_h^2 S_{2xh}^2 - 2\alpha_h \lambda_h R_h S_{2yhx})
\end{aligned} \tag{3.63}$$

Then from (3.52) and noting that covariance of sample means from different strata vanishes, the conditional mean square error of the proposed class of estimators, $\bar{y}_{nc1}$, is given by

$$MS E_c(\bar{y}_{nc1}) = \sum_{h=1}^L W_h^2 MS E_c(\bar{y}_{nc1h}) \tag{3.64}$$

and substituting (3.63) in (3.64) gives

$$\begin{aligned}
MS E_c(\bar{y}_{nc1}) &= \sum_{h=1}^L W_h^2 \left(\frac{1}{n_h} - \frac{1}{N_h}\right)(S_{yh}^2 + \alpha_h^2 \lambda_h^2 R_h^2 S_{xh}^2 - 2\alpha_h \lambda_h R_h S_{yhx}) \\
&\quad + \sum_{h=1}^L W_h^2 \left(\frac{g_h - 1}{n_h}\right)W_{2h}(S_{2yh}^2 + \alpha_h^2 \lambda_h^2 R_h^2 S_{2xh}^2 - 2\alpha_h \lambda_h R_h S_{2yhx})
\end{aligned} \tag{3.65}$$

as stated in the theorem. This gives the proof of the first part of the theorem on the conditional bias and mean square error of the proposed class of estimators, $\bar{y}_{nc1}$. To obtain the unconditional bias of the proposed class of estimators, $\bar{y}_{nc1}$, we take the unconditional expectation of (3.62) and use (3.25) to make the necessary substitutions. This gives

$$\begin{aligned}
B(\bar{y}_{nc1}) &= \left(\frac{1-f}{n}\right) \sum_{h=1}^L \frac{\alpha_h \lambda_h}{2\bar{X}_h} [(\alpha_h + 1)\lambda_h R_h S_{xh}^2 - 2S_{yhx}] \\
&\quad + \left(\frac{1}{n}\right) \sum_{h=1}^L \frac{\alpha_h \lambda_h}{2\bar{X}_h} (g_h - 1)W_{2h} [(\alpha_h + 1)\lambda_h R_h S_{2xh}^2 - 2S_{2yhx}]
\end{aligned} \tag{3.66}$$

as stated in the theorem. We take the unconditional expectation of (3.65) and use (3.25) to make the necessary substitutions to obtain the unconditional mean square error of $\bar{y}_{nc1}$ as:

$$\begin{aligned}
MS E(\bar{y}_{nc1}) &= \left(\frac{1-f}{n}\right) \sum_{h=1}^L W_h (S_{yh}^2 + \alpha_h^2 \lambda_h^2 R_h^2 S_{xh}^2 - 2\alpha_h \lambda_h R_h S_{yhx}) \\
&\quad + \left(\frac{1}{n}\right) \sum_{h=1}^L W_h (g_h - 1)W_{2h} (S_{2yh}^2 + \alpha_h^2 \lambda_h^2 R_h^2 S_{2xh}^2 - 2\alpha_h \lambda_h R_h S_{2yhx})
\end{aligned} \tag{3.67}$$

as stated in the theorem. And this completes the proof of theorem 3.3.

The proposed class of estimators, $\bar{y}_{nc1}$, given in (3.5), is capable of generating a wide range of estimators of the population mean ($\bar{Y}$) of the study variable, y , when there is nonresponse on both the study and auxiliary variables. This is achieved by making appropriate choices of the constants, α_h , a_h and b_h . While the constant, α_h is strictly a real number, the constants a_h and b_h could be real numbers or any known values of some population parameters of the auxiliary variable, x , in the h^{th} stratum, like the coefficient of variation, standard deviation, skewness, kurtosis, etc. Some special cases of the proposed class of estimators, $\bar{y}_{nc1}$, obtained by making suitable choices of the three constants, α_h , a_h and b_h , in the h^{th} stratum, are shown in Table 3.1.

The estimator, $\bar{y}_{nc1}(1) = \sum_{h=1}^L W_h \bar{y}_h^* (\bar{X}_h / \bar{x}_h^*)$, is the Customary ratio-type estimator

The estimator, $\bar{y}_{nc1}(2) = \sum_{h=1}^L W_h \bar{y}_h^* (\bar{X}_h + C_{xh}) / (\bar{x}_h^* + C_{xh})$, is a ratio-type estimator, adopted after Sud and Srivastava (1981) estimator

The estimator, $\bar{y}_{nc1}(3) = \sum_{h=1}^L W_h \bar{y}_h^* (\bar{X}_h + \beta_{2xh}) / (\bar{x}_h^* + \beta_{2xh})$, is a ratio-type estimator, adopted after Singh-Kakran (1993) estimator I

The estimator, $\bar{y}_{nc1}(4) = \sum_{h=1}^L W_h \bar{y}_h^* (\bar{X}_h C_{xh} + \beta_{2xh}) / (\bar{x}_h^* C_{xh} + \beta_{2xh})$, is a ratio-type estimator, adopted after Upadhyaya and Singh (1999) estimator I

The estimator, $\bar{y}_{nc1}(5) = \sum_{h=1}^L W_h \bar{y}_h^* (\bar{X}_h \beta_{2xh} + C_{xh}) / (\bar{x}_h^* \beta_{2xh} + C_{xh})$, is a ratio-type estimator, adopted after Upadhyaya and Singh (1999) estimator II

The estimator, $\bar{y}_{nc1}(6) = \sum_{h=1}^L W_h \bar{y}_h^* (\bar{x}_h^* / \bar{X}_h)$, Customary product-type estimator

The estimator, $\bar{y}_{nc1}(7) = \sum_{h=1}^L W_h \bar{y}_h^* (\bar{x}_h^* + C_{xh}) / (\bar{X}_h + C_{xh})$, is a product-type estimator, adopted after Pandey and Dubey (1988) estimator

The estimator, $\bar{y}_{nc1}(8) = \sum_{h=1}^L W_h \bar{y}_h^* (\bar{x}_h^* + \beta_{2xh}) / (\bar{X}_h + \beta_{2xh})$, is a product-type estimator, adopted after Singh-Kakran (1993) estimator II

The estimator, $\bar{y}_{nc1}(9) = \sum_{h=1}^L W_h \bar{y}_h^* (\bar{x}_h^* C_{xh} + \beta_{2xh}) / (\bar{X}_h C_{xh} + \beta_{2xh})$, is a product-type estimator, adopted after Upadhyaya and Singh (1999) estimator III

The estimator, $\bar{y}_{nc1}(10) = \sum_{h=1}^L W_h \bar{y}_h^* (\bar{x}_h^* \beta_{2xh} + \sigma_{xh}) / (\bar{X}_h \beta_{2xh} + \sigma_{xh})$, is a product-type estimator, adopted after Singh (2003) estimator

Table 3.1: Special Cases of Proposed Class of Estimators

SN	Estimator	α_h	a_h	b_h
1	$\bar{y}_{nc1}(1) = \sum_{h=1}^L W_h \bar{y}_h^* (\bar{X}_h / \bar{x}_h^*)$, Customary ratio-type estimator	1	1	0
2	$\bar{y}_{nc1}(2) = \sum_{h=1}^L W_h \bar{y}_h^* (\bar{X}_h + C_{xh}) / (\bar{x}_h^* + C_{xh})$, Sud and Srivastava (1981) estimator	1	1	C_{xh}
3	$\bar{y}_{nc1}(3) = \sum_{h=1}^L W_h \bar{y}_h^* (\bar{X}_h + \beta_{2xh}) / (\bar{x}_h^* + \beta_{2xh})$, Singh-Kakran (1993) estimator I	1	1	β_{2xh}
4	$\bar{y}_{nc1}(4) = \sum_{h=1}^L W_h \bar{y}_h^* (\bar{X}_h C_{xh} + \beta_{2xh}) / (\bar{x}_h^* C_{xh} + \beta_{2xh})$, Upadhyaya and Singh (1999) estimator I	1	C_{xh}	β_{2xh}
5	$\bar{y}_{nc1}(5) = \sum_{h=1}^L W_h \bar{y}_h^* (\bar{X}_h \beta_{2xh} + C_{xh}) / (\bar{x}_h^* \beta_{2xh} + C_{xh})$, Upadhyaya and Singh (1999) estimator II	1	β_{2xh}	C_{xh}
6	$\bar{y}_{nc1}(6) = \sum_{h=1}^L W_h \bar{y}_h^* (\bar{x}_h^* / \bar{X}_h)$, Customary product-type estimator	-1	1	0
7	$\bar{y}_{nc1}(7) = \sum_{h=1}^L W_h \bar{y}_h^* (\bar{x}_h^* + C_{xh}) / (\bar{X}_h + C_{xh})$, Pandey and Dubey (1988) estimator	-1	1	C_{xh}
8	$\bar{y}_{nc1}(8) = \sum_{h=1}^L W_h \bar{y}_h^* (\bar{x}_h^* + \beta_{2xh}) / (\bar{X}_h + \beta_{2xh})$, Singh-Kakran (1993) estimator II	-1	1	β_{2xh}
9	$\bar{y}_{nc1}(9) = \sum_{h=1}^L W_h \bar{y}_h^* (\bar{x}_h^* C_{xh} + \beta_{2xh}) / (\bar{X}_h C_{xh} + \beta_{2xh})$, Upadhyaya and Singh (1999) estimator III	-1	C_{xh}	β_{2xh}
10	$\bar{y}_{nc1}(10) = \sum_{h=1}^L W_h \bar{y}_h^* (\bar{x}_h^* \beta_{2xh} + \sigma_{xh}) / (\bar{X}_h \beta_{2xh} + \sigma_{xh})$, Singh (2003) estimator	-1	β_{2xh}	σ_{xh}

The best estimators in the proposed class of estimators, $\bar{y}_{nc1}$, are those with the best choices of the three constants, α_h , a_h and b_h , in the h^{th} stratum. Accordingly, the best choices of the constants, α_h , a_h and b_h , are those that minimize the conditional and unconditional mean square errors of the proposed class of estimators, $\bar{y}_{nc1}$, as the case may be. The results we obtained are given in theorem 3.4.

Theorem 3.4: Under the conditional argument, the values of the constants, α_h , a_h and b_h , that minimize the conditional mean square error of the proposed class of estimators, $\bar{y}_{nc1}$, are obtained as

$$\frac{\alpha_h a_h \bar{X}_h R_h}{a_h \bar{X}_h + b_h} = \frac{\alpha_h a_h \bar{Y}_h}{a_h \bar{X}_h + b_h} = \frac{\left(\frac{1}{n_h} - \frac{1}{N_h}\right) S_{yxh} + \left(\frac{g_h - 1}{n_h}\right) S_{2yxh}}{\left(\frac{1}{n_h} - \frac{1}{N_h}\right) S_{xh}^2 + \left(\frac{g_h - 1}{n_h}\right) S_{2xh}^2} \quad (3.68)$$

Similarly, under the unconditional argument, the values of the constants, α_h , a_h and b_h , that minimize the unconditional mean square error of the proposed class of estimators, $\bar{y}_{nc1}$, are obtained as

$$\theta_{h1} = \frac{\alpha_h a_h \bar{X}_h R_h}{a_h \bar{X}_h + b_h} = \frac{\alpha_h a_h \bar{Y}_h}{a_h \bar{X}_h + b_h} = \frac{(1 - f) S_{yxh} + (g_h - 1) W_{2h} S_{2yxh}}{(1 - f) S_{xh}^2 + (g_h - 1) W_{2h} S_{2xh}^2} \quad (3.69)$$

Proof: The conditional and unconditional mean square errors of the proposed class of estimators, $\bar{y}_{nc1}$, respectively given as (3.48) and (3.50), in theorem 3.3, can be written as:

$$\begin{aligned} MS E_c(\bar{y}_{nc1}) &= \sum_{h=1}^L W_h^2 \left(\frac{1}{n_h} - \frac{1}{N_h} \right) (S_{yh}^2 + \theta_h^2 S_{xh}^2 - 2\theta_h S_{yxh}) \\ &\quad + \sum_{h=1}^L W_h^2 \left(\frac{g_h - 1}{n_h} \right) W_{2h} (S_{2yh}^2 + \theta_h^2 S_{2xh}^2 - 2\theta_h S_{2yxh}) \end{aligned} \quad (3.70)$$

$$\begin{aligned} MS E(\bar{y}_{nc1}) &= \left(\frac{1 - f}{n} \right) \sum_{h=1}^L W_h (S_{yh}^2 + \theta_h^2 S_{xh}^2 - 2\theta_h S_{yxh}) \\ &\quad + \left(\frac{1}{n} \right) \sum_{h=1}^L W_h (g_h - 1) W_{2h} (S_{2yh}^2 + \theta_h^2 S_{2xh}^2 - 2\theta_h S_{2yxh}) \end{aligned} \quad (3.71)$$

where

$$\theta_h = \frac{\alpha_h a_h \bar{X}_h R_h}{a_h \bar{X}_h + b_h} = \frac{\alpha_h a_h \bar{Y}_h}{a_h \bar{X}_h + b_h} \quad (3.72)$$

Using the method of least squares to minimize (3.70) and (3.71) with respect to θ_h , the optimum values, (3.68) and (3.69), are obtained as given in the theorem. This completes the proof.

3.1.2 Nonresponse on the study variable and complete response on the auxiliary variable under the Poststratified Sampling Scheme

The focus here is to develop sampling strategies for the handling of nonresponse in poststratified sampling scheme, when there is nonresponse on the study variable and complete response on the auxiliary variable. The sampling design we propose, is as follows:

Sampling Design 3.1.2: Consider a population, $\Omega = \{U_1, U_2, \dots, U_N\}$, of N units, divided into subpopulations or strata of sizes, N_h , $h = 1, 2, \dots, L$, such that $\sum_{h=1}^L N_h = N$. For the purpose of estimating the population mean of a study variable under the poststratified sampling scheme, we use the simple random sampling without replacement (SRSWOR) method to select a sample of n units from the N units in the population and obtain information on the study, auxiliary and stratification variables. We assume that there was complete response on the auxiliary variable, x , while only n'_1 units responded on the study variable, y , out of the selected n units in the sample. We assume that n'_2 units did not respond on the study variable, y . The complete n units that responded on the auxiliary variable, x , were allocated to their various strata where n_h units fall into stratum h such that $\sum_{h=1}^L n_h = n$ units. The n'_1 units that responded on the study variable, y , were allocated to their various strata where n_{1h} units fall into stratum h such that $\sum_{h=1}^L n_{1h} = n'_1$ units. To reduce the effect of the nonresponse on the study variable, y , a subsample of m_h units is selected independently from the nonresponse sample group of n_{2h} units, in stratum h , such that $\sum_{h=1}^L m_h = m$ units. Extra efforts were made to obtain information on the missing study variable. We assume that both n'_1 and m are large enough such that $P(n_{1h} = 0) = P(m_h = 0) = 0 \forall h$.

The Sampling Design 3.1.2 is constructed to reduce the effect of nonresponse when there are missing values on only the study variable and complete response on the auxiliary variable when using the poststratified sampling scheme. Under the proposed sampling design, the available sample data consist of the observed auxiliary variable (x) values on the complete sample of size,

n_h , in stratum h , and the observed study variable (y) values from the response sample group of n_{1h} units in stratum h , supplemented with the observed values of the study variable, y , from the nonresponse subsample group of m_h units in stratum h . Consequently, the associated sample means, $\bar{x}_h$, $\bar{y}_{1h}$ and $\bar{y}_{2mh}$, form the basis for constructing estimators of the population mean ($\bar{Y}$) of the study variable, y , when there is nonresponse on the study variable and complete response on the auxiliary variable in poststratified sampling scheme. For the h^{th} stratum, we give a sample mean of the study variable, y , as a linear function of the response and nonresponse subsample means, given by:

$$\bar{y}_h^* = \left(\frac{n_{1h}}{n_h} \right) \bar{y}_{1h} + \left(\frac{n_{2h}}{n_h} \right) \bar{y}_{2mh} \quad (3.73)$$

while the sample mean of the auxiliary variable, x , based on the complete sample data of n_h units in stratum h , is given by:

$$\bar{x}_h = \frac{1}{n_h} \sum_{i=1}^{n_h} x_{hi} \quad (3.74)$$

where

$$\bar{y}_{1h} = \frac{1}{n_{1h}} \sum_{i=1}^{n_{1h}} y_{hi} \quad \text{is the sample mean of } y \text{ in stratum } h, \text{ based on the}$$

response group sample size, n_{1h}

and

$$\bar{y}_{2mh} = \frac{1}{m_h} \sum_{i=1}^{m_h} y_{hi} \quad \text{is the sample mean of } y \text{ in stratum } h, \text{ based on the}$$

nonresponse sub-sample group sample size, m_h

Based on the Sampling Design 3.1.2 and using the sample means (3.73) and (3.74), we propose three estimators of the population mean ($\bar{Y}$) of the study variable, y , when there is nonresponse only on the study variable and complete response on the auxiliary variable under the poststratified sampling scheme. The three estimators we suggest are as follows:

1. The customary nonresponse sample mean estimator given by

$$\bar{y}_{ns2} = \sum_{h=1}^L W_h \bar{y}_h^* \quad (3.75)$$

2. A nonresponse difference-type estimator given by

$$\bar{y}_{nd2} = \sum_{h=1}^L W_h [\bar{y}_h^* - k_h(\bar{x}_h - \bar{X}_h)] \quad (3.76)$$

3. A general class of estimators that makes use of known values of some population parameters of the auxiliary variable in the h^{th} stratum, given by

$$\bar{y}_{nc2} = \sum_{h=1}^L W_h \bar{y}_h^* \left(\frac{a_h \bar{X}_h + b_h}{a_h \bar{x}_h + b_h} \right)^{\alpha_h} \quad (3.77)$$

where $k_h(\alpha_h)$ are any real numbers and $a_h(b_h)$ are also any real numbers or known values of population parameters of an auxiliary variable, x , in the h^{th} stratum, like coefficient of variation, skewness, kurtosis, etc. Properties of the above proposed estimators shall be obtained both under the conditional argument or given an achieved sample configuration, say $\underline{n} = (n_1, n_2, \dots, n_L)$ and under the unconditional argument or for repeated samples of fixed size, n .

Estimator 4: Customary nonresponse sample mean estimator, $\bar{y}_{ns2}$, when there is nonresponse only on the study variable and complete response on the auxiliary variable under the Poststratified Sampling Scheme.

The proposed customary nonresponse sample mean estimator, $\bar{y}_{ns2}$, when there is nonresponse only on the study variable and complete response on the auxiliary variable, given in (3.75) is exactly the same with the proposed customary nonresponse sample mean estimator, $\bar{y}_{ns1}$, when there is nonresponse on both the study and auxiliary variables, given in (3.3). Consequently, the conditional and unconditional properties of the estimator, $\bar{y}_{ns2}$, are the same as those stated in theorem 3.1 for the estimator, $\bar{y}_{ns1}$.

Estimator 5: Difference-type estimator, $\bar{y}_{nd2}$, when there is nonresponse only on the study variable and complete response on the auxiliary variable under the Poststratified Sampling Scheme.

The proposed difference-type estimator, $\bar{y}_{nd2}$, given in (3.76), utilizes auxiliary information with known population mean ($\bar{X}_h$) of the auxiliary variable, x, in the h^{th} stratum. We give the conditional and unconditional properties of the estimator, $\bar{y}_{nd2}$, in theorem 3.5.

Theorem 3.5: Under the conditional argument or for an achieved sample configuration, say $\underline{n} = (n_1, n_2, \dots, n_L)$, the proposed difference-type estimator, $\bar{y}_{nd2}$, is unbiased for the population mean ($\bar{Y}$) of the study variable, y, with its conditional variance obtained as:

$$V_c(\bar{y}_{nd2}) = \sum_{h=1}^L W_h^2 \left(\frac{1}{n_h} - \frac{1}{N_h} \right) (S_{yh}^2 + k_h^2 S_{xh}^2 - 2k_h S_{yxh}) + \sum_{h=1}^L W_h^2 \left(\frac{g_h - 1}{n_h} \right) W_{2h} S_{2yh}^2 \quad (3.78)$$

Furthermore, under the unconditional argument or for repeated samples of fixed size, n, the difference-type estimator, $\bar{y}_{nd2}$, is an unbiased estimator of the population mean ($\bar{Y}$) of the study variable y, with its unconditional variance obtained up to first order of approximations as:

$$V(\bar{y}_{nd2}) = \left(\frac{1-f}{n} \right) \sum_{h=1}^L W_h (S_{yh}^2 + k_h^2 S_{xh}^2 - 2k_h S_{yxh}) + \left(\frac{1}{n} \right) \sum_{h=1}^L W_h (g_h - 1) W_{2h} S_{2yh}^2 \quad (3.79)$$

Proof: The conditional expectation of the proposed difference-type estimator, $\bar{y}_{nd2}$, given in (3.76), can be written as:

$$E_c(\bar{y}_{nd2}) = \sum_{h=1}^L W_h E_c[\bar{y}_h^* - k_h(\bar{x}_h - \bar{X}_h)] = \sum_{h=1}^L W_h [E_c(\bar{y}_h^*) - k_h\{E_c(\bar{x}_h) - \bar{X}_h\}] \quad (3.80)$$

Applying the theory of SRSWOR method, the conditional expectation and variance of the sample mean, $\bar{x}_h$, based on complete response on the auxiliary variable, x, in stratum h, are respectively given by:

$$E_c(\bar{x}_h) = \bar{X}_h \quad (3.81)$$

$$V_c(\bar{x}_h) = \left(\frac{1}{n_h} - \frac{1}{N_h} \right) S_{xh}^2 \quad (3.82)$$

Substituting (3.11) and (3.81) into (3.80), we obtain

$$E_c(\bar{y}_{nd2}) = \sum_{h=1}^L W_h [\bar{Y}_h - k_h(\bar{X}_h - \bar{X}_h)] = \sum_{h=1}^L W_h \bar{Y}_h = \bar{Y} \quad (3.83)$$

Hence, the proposed difference-type estimator, $\bar{y}_{nd2}$, is unbiased for $\bar{Y}$ under the conditional argument. Using (3.76) and recalling that covariance of sample means from different strata vanishes, we obtain the conditional variance of $\bar{y}_{nd2}$ as

$$V_c(\bar{y}_{nd2}) = \sum_{h=1}^L W_h^2 V_c[\bar{y}_h^* - k_h(\bar{x}_h - \bar{X}_h)] = \sum_{h=1}^L W_h^2 [V_c(\bar{y}_h^*) + k_h^2 V_c(\bar{x}_h) - 2k_h C_c(\bar{y}_h^*, \bar{x}_h)] \quad (3.84)$$

Applying the theory of double sampling and using (3.1) and (3.8), we can write

$$\begin{aligned} C_c(\bar{y}_h^*, \bar{x}_h) &= C_1[E_2(\bar{y}_h^*), E_2(\bar{x}_h)] + E_1 C_2(\bar{y}_h^*, \bar{x}_h) \\ &= C_1(\bar{y}_h, \bar{x}_h) + E_1 C_2\left[\left(\frac{n_{1h}}{n_h}\right)\bar{y}_{1h} + \left(\frac{n_{2h}}{n_h}\right)\bar{y}_{2mh}, \bar{x}_h\right] \\ &= \left(\frac{1}{n_h} - \frac{1}{N_h}\right)S_{yxh} + E_1(0) = \left(\frac{1}{n_h} - \frac{1}{N_h}\right)S_{yxh} \end{aligned} \quad (3.85)$$

Substituting (3.15), (3.82) and (3.85) in (3.84) gives

$$\begin{aligned} V_c(\bar{y}_{nd2}) &= \sum_{h=1}^L W_h^2 [V_c(\bar{y}_h^*) + k_h^2 V_c(\bar{x}_h) - 2k_h C_c(\bar{y}_h^*, \bar{x}_h)] \\ &= \sum_{h=1}^L W_h^2 \left[\left(\frac{1}{n_h} - \frac{1}{N_h}\right)S_{yh}^2 + \left(\frac{g_h - 1}{n_h}\right)W_{2h}S_{2yh}^2 + k_h^2 \left(\frac{1}{n_h} - \frac{1}{N_h}\right)S_{xh}^2 - 2k_h \left(\frac{1}{n_h} - \frac{1}{N_h}\right)S_{yxh} \right] \\ &= \sum_{h=1}^L W_h^2 \left(\frac{1}{n_h} - \frac{1}{N_h}\right)(S_{yh}^2 + k_h^2 S_{xh}^2 - 2k_h S_{yxh}) + \sum_{h=1}^L W_h^2 \left(\frac{g_h - 1}{n_h}\right)W_{2h}S_{2yh}^2 \end{aligned} \quad (3.86)$$

as stated in the theorem. This proves the first part of the theorem. Furthermore, we take the unconditional expectation of (3.86) and use (3.25) to make the necessary substitutions to obtain

$$V(\bar{y}_{nd2}) = \left(\frac{1-f}{n}\right) \sum_{h=1}^L W_h(S_{yh}^2 + k_h^2 S_{xh}^2 - 2k_h S_{yxh}) + \left(\frac{1}{n}\right) \sum_{h=1}^L W_h(g_h - 1)W_{2h}S_{2yh}^2 \quad (3.87)$$

as stated in the theorem. And this completes the proof.

Estimator 6: Class of estimators, $\bar{y}_{nc2}$, when there is nonresponse only on the study variable and complete response on the auxiliary variable under the Poststratified Sampling Scheme.

The proposed class of estimators, $\bar{y}_{nc2}$, given in (3.77), utilizes auxiliary information with known values of some population parameters of the auxiliary variable, x , in the h^{th} stratum. We give the conditional and unconditional properties of the estimator, $\bar{y}_{nc2}$, in theorem 3.6.

Theorem 3.6: Under the conditional argument or for an achieved sample configuration, say $\underline{n} = (n_1, n_2, \dots, n_L)$, the proposed class of estimators, $\bar{y}_{nc2}$, is biased for the population mean ($\bar{Y}$) of the study variable, y , with its conditional bias and mean square error obtained up to first order of approximations as:

$$B_c(\bar{y}_{nc2}) = \sum_{h=1}^L \frac{W_h \alpha_h \lambda_h}{2\bar{X}_h} \left(\frac{1}{n_h} - \frac{1}{N_h} \right) [(\alpha_h + 1) \lambda_h R_h S_{xh}^2 - 2S_{yhx}] \quad (3.88)$$

$$\begin{aligned} MS E_c(\bar{y}_{nc2}) = & \sum_{h=1}^L W_h^2 \left(\frac{1}{n_h} - \frac{1}{N_h} \right) (S_{yh}^2 + \alpha_h^2 \lambda_h^2 R_h^2 S_{xh}^2 - 2\alpha_h \lambda_h R_h S_{yhx}) \\ & + \sum_{h=1}^L W_h^2 \left(\frac{g_h - 1}{n_h} \right) W_{2h} S_{2yh}^2 \end{aligned} \quad (3.89)$$

Furthermore, under the unconditional argument or for repeated samples of fixed size, n , the proposed class of estimators, $\bar{y}_{nc2}$, is biased for the population mean ($\bar{Y}$) of the study variable y , with its unconditional bias and mean square error obtained up to first order of approximations as:

$$B(\bar{y}_{nc2}) = \left(\frac{1-f}{n} \right) \sum_{h=1}^L \frac{\alpha_h \lambda_h}{2\bar{X}_h} [(\alpha_h + 1) \lambda_h R_h S_{xh}^2 - 2S_{yhx}] \quad (3.90)$$

and

$$\begin{aligned} MS E(\bar{y}_{nc2}) = & \left(\frac{1-f}{n} \right) \sum_{h=1}^L W_h (S_{yh}^2 + \alpha_h^2 \lambda_h^2 R_h^2 S_{xh}^2 - 2\alpha_h \lambda_h R_h S_{yhx}) \\ & + \left(\frac{1}{n} \right) \sum_{h=1}^L W_h (g_h - 1) W_{2h} S_{2yh}^2 \end{aligned} \quad (3.91)$$

Proof: Following a similar procedure as in the proof of theorem 3.3, it can be shown that the conditional bias and mean square error of the proposed class of estimators, $\bar{y}_{nc2}$, given in (3.77), can be respectively written as:

$$B_c(\bar{y}_{nc2}) = \sum_{h=1}^L \frac{W_h \alpha_h \lambda_h}{2\bar{X}_h} [(\alpha_h + 1) \lambda_h R_h V_c(\bar{x}_h) - 2C_c(\bar{y}_h^*, \bar{x}_h)] \quad (3.92)$$

and

$$MS E_c(\bar{y}_{nc2}) = \sum_{h=1}^L W_h^2 [V_c(\bar{y}_h^*) + \alpha_h^2 \lambda_h^2 R_h^2 V_c(\bar{x}_h) - 2\alpha_h \lambda_h R_h C_c(\bar{y}_h^*, \bar{x}_h)] \quad (3.93)$$

Using (3.15), (3.82) and (3.85) to make the necessary substitutions in (3.92) and (3.93) gives the bias and mean square error of the proposed estimator, $\bar{y}_{nc2}$, as stated in the theorem. To obtain the unconditional bias and mean square error of the proposed class of estimators, $\bar{y}_{nc2}$, we take the unconditional expectations of (3.88) and (3.89) respectively, and use (3.25) to make the necessary substitutions. This completes the proof of theorem 3.6.

3.1.3 Nonresponse on the auxiliary variable and complete response on the study variable under the Poststratified Sampling Scheme

Here we develop sampling strategies for dealing with nonresponse in postratified sampling scheme, when there is complete response on the study variable and nonresponse on the auxiliary variable. The sampling design we propose is as follows:

Sampling Design 3.1.3: Consider a population, $\Omega = \{U_1, U_2, \dots, U_N\}$, of N units, divided into subpopulations or strata of sizes, N_h , $h = 1, 2, \dots, L$, such that $\sum_{h=1}^L N_h = N$. We use the simple random sampling without replacement (SRSWOR) method to select a sample of n units from the N units in the population when estimating the population mean of a study variable under the poststratified sampling scheme. We assume that there was complete response on the study variable, y , while only n'_1 units responded on the auxiliary variable, x , out of the selected n units in the sample. We also assume that n'_2 units did not respond on the auxiliary variable, x . The complete n units that responded on the study variable, y , were allocated to their various

strata where n_h units fall into stratum h such that $\sum_{h=1}^L n_h = n$ units. The n'_1 units that responded on the auxiliary variable, x , were also allocated to their various strata where n_{1h} units fall into stratum h such that $\sum_{h=1}^L n_{1h} = n'_1$ units. To reduce the effect of the nonresponse on the auxiliary variable, x , a subsample of m_h units is selected independently from the nonresponse sample group of n_{2h} units, in stratum h , such that $\sum_{h=1}^L m_h = m$ units. Extra efforts were made to obtain information on the missing auxiliary variable. We assume that both n'_1 and m are large enough such that $P(n_{1h} = 0) = P(m_h = 0) = 0 \forall h$.

The Sampling Design 3.1.3 is constructed to reduce the effect of nonresponse when there are missing values on only the auxiliary variable and complete response on the study variable when using the postratified sampling scheme. Under the proposed sampling design, the available sample data consist of the observed study variable (y) values on the complete sample of size, n_h , in stratum h , and the observed auxiliary variable (x) values from the response sample group of n_{1h} units in stratum h , supplemented with the observed values of the auxiliary variable, x , from the nonresponse subsample group of m_h units in stratum h . Hence, the associated sample means, $\bar{y}_h$, $\bar{x}_{1h}$ and $\bar{x}_{2mh}$, form the basis for constructing estimators of the population mean ($\bar{Y}$) of the study variable, y , when there is nonresponse on the auxiliary variable and complete response on the study variable in poststratified sampling scheme. The sample mean of the study variable, y , based on the complete sample data of n_h units in stratum h , is given by:

$$\bar{y}_h = \frac{1}{n_h} \sum_{i=1}^{n_h} y_{hi} \quad (3.94)$$

For the h^{th} stratum, we give a sample mean of the auxiliary variable, x , as a linear function of the response and nonresponse subsample means, given by:

$$\bar{x}_h^* = \left(\frac{n_{1h}}{n_h} \right) \bar{x}_{1h} + \left(\frac{n_{2h}}{n_h} \right) \bar{x}_{2mh} \quad (3.95)$$

where

$\bar{x}_{1h} = \frac{1}{n_{1h}} \sum_{i=1}^{n_{1h}} x_{hi}$ is the sample mean of x in stratum h, based on the response group sample size, n_{1h}

$\bar{x}_{2mh} = \frac{1}{m_h} \sum_{i=1}^{m_h} x_{hi}$ is the sample mean of x in stratum h, based on the nonresponse sub-sample group sample size, m_h

Based on the Sampling Design 3.1.3 and using the sample means (3.94) and (3.95), we propose three estimators of the population mean ($\bar{Y}$) of the study variable, y, when there is complete response on the study variable and nonresponse on the auxiliary variable under the poststratified sampling scheme. The three estimators we proposed are as follows:

1. The customary sample mean estimator given by

$$\bar{y}_{ns3} = \sum_{h=1}^L W_h \bar{y}_h \quad (3.96)$$

2. A nonresponse difference-type estimator given by

$$\bar{y}_{nd3} = \sum_{h=1}^L W_h [\bar{y}_h - k_h(\bar{x}_h^* - \bar{X}_h)] \quad (3.97)$$

3. A general class of estimators that makes use of known values of some population parameters of the auxiliary variable in the h^{th} stratum, given by

$$\bar{y}_{nc3} = \sum_{h=1}^L W_h \bar{y}_h \left(\frac{a_h \bar{X}_h + b_h}{a_h \bar{x}_h^* + b_h} \right)^{\alpha_h} \quad (3.98)$$

where $k_h(\alpha_h)$ are any real numbers and $a_h(b_h)$ are also any real numbers or known population parameters of an auxiliary variable, x, in the h^{th} stratum, like coefficient of variation, skewness, kurtosis, etc. The properties of the above proposed estimators shall be obtained both under the conditional argument or given an achieved sample configuration, say $\underline{n} = (n_1, n_2, \dots, n_L)$ and under the unconditional argument or for repeated samples of fixed size, n.

Estimator 7: Customary sample mean estimator, $\bar{y}_{ns3}$, when there is nonresponse on the auxiliary variable and complete response on the study variable under the Poststratified Sampling Scheme.

The proposed customary sample mean estimator, $\bar{y}_{ns3}$, when there is nonresponse on the auxiliary variable and complete response on the study variable, is exactly the same with the customary poststratified sample mean estimator, $\bar{y}_{ps}$, which is unbiased for $\bar{Y}$. Consequently, the conditional and unconditional variances of the estimator, $\bar{y}_{ns3}$, are the same as those of $\bar{y}_{ps}$. We can therefore write:

$$V_c(\bar{y}_{ns3}) = V_c(\bar{y}_{ps}) = \left(\frac{1}{n_h} - \frac{1}{N_h} \right) \sum_{h=1}^L W_h S_{yh}^2 \quad (3.99)$$

and

$$V(\bar{y}_{ns3}) = V(\bar{y}_{pst}) = \left(\frac{1-f}{n} \right) \sum_{h=1}^L W_h S_{yh}^2 \quad (3.100)$$

The results given in (3.99) and (3.100) follow from the theory of SRSWOR scheme.

Estimator 8: Difference-type estimator, $\bar{y}_{nd3}$, when there is nonresponse on the auxiliary variable and complete response on the study variable under the Poststratified Sampling Scheme.

The proposed difference-type estimator, $\bar{y}_{nd3}$, given in (3.97), utilizes auxiliary information with known population mean ($\bar{X}_h$) of the auxiliary variable, x , in the h^{th} stratum. We give the conditional and unconditional properties of the estimator, $\bar{y}_{nd3}$, in theorem 3.7.

Theorem 3.7: Under the conditional argument or for an achieved sample configuration, say $\underline{n} = (n_1, n_2, \dots, n_L)$, the proposed difference-type estimator, $\bar{y}_{nd3}$, is unbiased for the population mean ($\bar{Y}$) of the study variable, y , with its conditional variance obtained as:

$$V_c(\bar{y}_{nd3}) = \sum_{h=1}^L W_h^2 \left(\frac{1}{n_h} - \frac{1}{N_h} \right) (S_{yh}^2 + k_h^2 S_{xh}^2 - 2k_h S_{yxh}) + \sum_{h=1}^L W_h^2 \left(\frac{g_h - 1}{n_h} \right) W_{2h} k_h^2 S_{2xh}^2 \quad (3.101)$$

Furthermore, for repeated samples of fixed size, n , or under the unconditional argument, the difference-type estimator, $\bar{y}_{nd3}$, is an unbiased estimator of the population mean ($\bar{Y}$) of the study variable y , with its unconditional variance obtained up to first order of approximations as:

$$V(\bar{y}_{nd3}) = \left(\frac{1-f}{n}\right) \sum_{h=1}^L W_h (S_{yh}^2 + k_h^2 S_{xh}^2 - 2k_h S_{yxh}) + \left(\frac{1}{n}\right) \sum_{h=1}^L W_h (g_h - 1) W_{2h} k_h^2 S_{2xh}^2 \quad (3.102)$$

Proof: The conditional expectation of the proposed difference-type estimator, $\bar{y}_{nd3}$, given in (3.97), can be written as:

$$E_c(\bar{y}_{nd3}) = \sum_{h=1}^L W_h E_c[\bar{y}_h - k_h(\bar{x}_h^* - \bar{X}_h)] = \sum_{h=1}^L W_h [E_c(\bar{y}_h) - k_h\{E_c(\bar{x}_h^*) - \bar{X}_h\}] \quad (3.103)$$

Applying the theory of SRSWOR method, the conditional expectation and variance of the sample mean, $\bar{y}_h$, based on complete response on the study variable, y , in stratum h , are respectively given by:

$$E_c(\bar{y}_h) = \bar{Y}_h \quad (3.104)$$

and

$$V_c(\bar{y}_h) = \left(\frac{1}{n_h} - \frac{1}{N_h}\right) S_{yh}^2 \quad (3.105)$$

Substituting (3.34) and (3.104) into (3.103), we obtain

$$E_c(\bar{y}_{nd3}) = \sum_{h=1}^L W_h [\bar{Y}_h - k_h(\bar{X}_h - \bar{X}_h)] = \sum_{h=1}^L W_h \bar{Y}_h = \bar{Y} \quad (3.106)$$

Hence, the proposed difference-type estimator, $\bar{y}_{nd3}$, is unbiased for $\bar{Y}$ under the conditional argument. Using (3.97) and recalling that covariance of sample means from different strata vanishes, we obtain the conditional variance of $\bar{y}_{nd3}$ as

$$V_c(\bar{y}_{nd3}) = \sum_{h=1}^L W_h^2 V_c[\bar{y}_h - k_h(\bar{x}_h^* - \bar{X}_h)] = \sum_{h=1}^L W_h^2 [V_c(\bar{y}_h) + k_h^2 V_c(\bar{x}_h^*) - 2k_h C_c(\bar{y}_h, \bar{x}_h^*)] \quad (3.107)$$

Applying the theory of double sampling and using (3.39) and (3.95), we can write

$$\begin{aligned}
C_c(\bar{y}_h, \bar{x}_h^*) &= C_1[E_2(\bar{y}_h), E_2(\bar{x}_h^*)] + E_1 C_2(\bar{y}_h, \bar{x}_h^*) \\
&= C_1(\bar{y}_h, \bar{x}_h) + E_1 C_2\left[\bar{y}_h, \left(\frac{n_{1h}}{n_h}\right)\bar{x}_{1h} + \left(\frac{n_{2h}}{n_h}\right)\bar{x}_{2mh}\right] \\
&= \left(\frac{1}{n_h} - \frac{1}{N_h}\right)S_{yxh} + E_1(0) \\
&= \left(\frac{1}{n_h} - \frac{1}{N_h}\right)S_{yxh}
\end{aligned} \tag{3.108}$$

Substituting (3.38), (3.105) and (3.108) in (3.107) gives

$$\begin{aligned}
V_c(\bar{y}_{nd3}) &= \sum_{h=1}^L W_h^2 [V_c(\bar{y}_h) + k_h^2 V_c(\bar{x}_h^*) - 2k_h C_c(\bar{y}_h, \bar{x}_h^*)] \\
&= \sum_{h=1}^L W_h^2 \left[\left(\frac{1}{n_h} - \frac{1}{N_h}\right) S_{yh}^2 + k_h^2 \left(\frac{1}{n_h} - \frac{1}{N_h}\right) S_{xh}^2 + k_h^2 \left(\frac{g_h - 1}{n_h}\right) W_{2h} S_{2xh}^2 - 2k_h \left(\frac{1}{n_h} - \frac{1}{N_h}\right) S_{yxh} \right] \\
&= \sum_{h=1}^L W_h^2 \left(\frac{1}{n_h} - \frac{1}{N_h}\right) (S_{yh}^2 + k_h^2 S_{xh}^2 - 2k_h S_{yxh}) + \sum_{h=1}^L W_h^2 \left(\frac{g_h - 1}{n_h}\right) W_{2h} k_h^2 S_{2xh}^2
\end{aligned} \tag{3.109}$$

as stated in the theorem. Furthermore, we take the unconditional expectation of (3.109) and use (3.25) to make the necessary substitutions to obtain

$$V(\bar{y}_{nd3}) = \left(\frac{1-f}{n}\right) \sum_{h=1}^L W_h (S_{yh}^2 + k_h^2 S_{xh}^2 - 2k_h S_{yxh}) + \left(\frac{1}{n}\right) \sum_{h=1}^L W_h (g_h - 1) W_{2h} k_h^2 S_{2xh}^2 \tag{3.110}$$

as stated in the theorem. And this completes the proof.

Estimator 9: Class of estimators, $\bar{y}_{nc3}$, when there is nonresponse response on the auxiliary variable and complete response on the study variable under the Poststratified Sampling Scheme.

The proposed class of estimators, $\bar{y}_{nc3}$, given in (3.98), utilizes auxiliary information with known values of some population parameters of the auxiliary variable, x , in the h^{th} stratum. We give the conditional and unconditional properties of the estimator, $\bar{y}_{nc3}$, in theorem 3.8.

Theorem 3.8: Under the conditional argument or for an achieved sample configuration, say $\underline{n} = (n_1, n_2, \dots, n_L)$, the proposed class of estimators, $\bar{y}_{nc3}$, is biased for the population mean ($\bar{Y}$) of the study variable, y , with its conditional bias and mean square error obtained up to first order of approximations as:

$$B_c(\bar{y}_{nc3}) = \sum_{h=1}^L \frac{W_h \alpha_h \lambda_h}{2\bar{X}_h} \left[\left(\frac{1}{n_h} - \frac{1}{N_h} \right) \{ (\alpha_h + 1) \lambda_h R_h S_{xh}^2 - 2S_{yxh} \} + \left(\frac{g_h - 1}{n_h} \right) (\alpha_h + 1) W_{2h} \lambda_h R_h S_{2xh}^2 \right] \quad (3.111)$$

and

$$\begin{aligned} MS E_c(\bar{y}_{nc3}) &= \sum_{h=1}^L W_h^2 \left(\frac{1}{n_h} - \frac{1}{N_h} \right) (S_{yh}^2 + \alpha_h^2 \lambda_h^2 R_h^2 S_{xh}^2 - 2\alpha_h \lambda_h R_h S_{yxh}) \\ &\quad + \sum_{h=1}^L W_h^2 \left(\frac{g_h - 1}{n_h} \right) W_{2h} \alpha_h^2 \lambda_h^2 R_h^2 S_{2xh}^2 \end{aligned} \quad (3.112)$$

Furthermore, for repeated samples of fixed size, n , or under the unconditional argument, the proposed class of estimators, $\bar{y}_{nc3}$, is biased for the population mean ($\bar{Y}$) of the study variable y , with its unconditional bias and mean square error obtained up to first order of approximations as:

$$\begin{aligned} B(\bar{y}_{nc3}) &= \left(\frac{1-f}{n} \right) \sum_{h=1}^L \frac{\alpha_h \lambda_h}{2\bar{X}_h} [(\alpha_h + 1) \lambda_h R_h S_{xh}^2 - 2S_{yxh}] \\ &\quad + \left(\frac{1}{n} \right) \sum_{h=1}^L \frac{\alpha_h \lambda_h}{2\bar{X}_h} [(g_h - 1) (\alpha_h + 1) W_{2h} \lambda_h R_h S_{2xh}^2] \end{aligned} \quad (3.113)$$

and

$$\begin{aligned} MS E(\bar{y}_{nc3}) &= \left(\frac{1-f}{n} \right) \sum_{h=1}^L W_h (S_{yh}^2 + \alpha_h^2 \lambda_h^2 R_h^2 S_{xh}^2 - 2\alpha_h \lambda_h R_h S_{yxh}) \\ &\quad + \left(\frac{1}{n} \right) \sum_{h=1}^L W_h (g_h - 1) \alpha_h^2 \lambda_h^2 R_h^2 W_{2h} S_{2xh}^2 \end{aligned} \quad (3.114)$$

Proof: Following a similar procedure as in the proof of theorem 3.3, it can be shown that the conditional bias and mean square error of the proposed class of estimators, $\bar{y}_{nc3}$, given in (3.98), can be respectively written as:

$$B_c(\bar{y}_{nc3}) = \sum_{h=1}^L \frac{W_h \alpha_h \lambda_h}{2\bar{X}_h} [(\alpha_h + 1) \lambda_h R_h V_c(\bar{x}_h^*) - 2C_c(\bar{y}_h, \bar{x}_h^*)] \quad (3.115)$$

and

$$MS E_c(\bar{y}_{nc3}) = \sum_{h=1}^L W_h^2 [V_c(\bar{y}_h) + \alpha_h^2 \lambda_h^2 R_h^2 V_c(\bar{x}_h^*) - 2\alpha_h \lambda_h R_h C_c(\bar{y}_h, \bar{x}_h^*)] \quad (3.116)$$

Using (3.38), (3.105) and (3.108) to make the necessary substitutions in (3.115) and (3.116) gives the bias and mean square error of the proposed estimator, $\bar{y}_{nc3}$, as stated in the theorem. To obtain the unconditional bias and mean square error of the proposed class of estimators, $\bar{y}_{nc3}$, we take the unconditional expectations of (3.115) and (3.116) respectively, and use (3.25) to make the necessary substitutions. This completes the proof of theorem 3.8.

3.2 Measurement Errors in Poststratified Sampling Scheme

The previous section considered poststratified sampling schemes in the presence of nonresponse or missing data. Three cases were considered, namely, when there is nonresponse on both study and auxiliary variables, when there is nonresponse on the study variable alone and when there is nonresponse on the auxiliary variable alone. In this section, we consider poststratified sampling schemes involving measurement errors alone, without necessarily involving the occurrence of nonresponse too. We shall develop sampling strategies for estimating the population mean or other population parameters of the study variable when using poststratified sampling scheme in the presence of measurement errors. Just like in the case of nonresponse discussed in the previous section, three possible options in which measurement errors could occur shall be considered with respect to use of information on both the study and auxiliary variables. These are when measurement errors occur on both the study and auxiliary variables, when measurement errors occur only on the study variable and not on the auxiliary variable, and lastly, when measurement errors occur only on the auxiliary variable and not on the study variable. We shall discuss these three options, one after the other, starting with when there are measurement errors on both the study and auxiliary variables in poststratified sampling scheme. This involves developing sampling strategies for population mean in poststratified sampling scheme in the presence of measurement errors on both the study and auxiliary variables.

3.2.1 Measurement Errors on both the study and auxiliary variables under the Poststratified Sampling Scheme

In this section, we develop sampling strategies for handling measurement errors in poststratified sampling scheme, when there are measurement errors on both the study and auxiliary variables. First, we propose the following sampling design:

Sampling Design 3.2.1: Consider a population, $\Omega = \{U_1, U_2, \dots, U_N\}$, of N units, divided into subpopulations or strata of sizes, N_h , $h = 1, 2, \dots, L$, such that $\sum_{h=1}^L N_h = N$. For the purpose of estimating the population mean of a study variable under the poststratified sampling scheme, we use the simple random sampling without replacement (SRSWOR) method to select a sample of n units from the N units in the population and obtain information on the study, auxiliary and stratification variables. We assume that there are measurement errors on both the study and auxiliary variables. Let the pair of values, (y_{hi}, x_{hi}) denote the observed values of the study and auxiliary variables for the i^{th} unit in stratum h , while the pair (Y_{hi}, X_{hi}) denote the corresponding true values. We define “**measurement errors**” under poststratified sampling scheme as the differences between the observed values (y_{hi}, x_{hi}) and the true values (Y_{hi}, X_{hi}) . Thus, for the i^{th} population unit in stratum h ($i = 1, 2, \dots, N_h$) or the i^{th} sample unit in stratum h ($i = 1, 2, \dots, n_h$), the measurement errors on the study and auxiliary variables, under poststratified sampling scheme, are respectively given as

$$U_{hi} = y_{hi} - Y_{hi} \quad \text{and} \quad V_{hi} = x_{hi} - X_{hi} \quad (3.117)$$

The measurement errors in stratum h , U_{hi} and V_{hi} , are assumed to be stochastic in nature with zero means and stratum population variances, S_{uh}^2 and S_{vh}^2 , respectively, given by

$$S_{uh}^2 = \frac{1}{N_h - 1} \sum_{i=1}^{N_h} U_{hi}^2 \quad \text{and} \quad S_{vh}^2 = \frac{1}{N_h - 1} \sum_{i=1}^{N_h} V_{hi}^2 \quad (3.118)$$

To incorporate the measurement error in the observed value of the study variable, y , using (3.117), we write the sample mean of the observed values of y in stratum h , in the presence of measurement errors as

$$\bar{y}_h'' = \frac{1}{n_h} \sum_{h=1}^{n_h} y_{hi} = \frac{1}{n_h} \sum_{h=1}^{n_h} [Y_{hi} + (y_{hi} - Y_{hi})] = \frac{1}{n_h} \sum_{h=1}^{n_h} Y_{hi} + \frac{1}{n_h} \sum_{h=1}^{n_h} U_{hi} \quad (3.119)$$

Similarly, we write the sample mean of the observed values of x in stratum h, in the presence of measurement errors as

$$\bar{x}_h'' = \frac{1}{n_h} \sum_{h=1}^{n_h} x_{hi} = \frac{1}{n_h} \sum_{h=1}^{n_h} [X_{hi} + (x_{hi} - X_{hi})] = \frac{1}{n_h} \sum_{h=1}^{n_h} X_{hi} + \frac{1}{n_h} \sum_{h=1}^{n_h} V_{hi} \quad (3.120)$$

Based on the Sampling Design 3.2.1 and using (3.119) and (3.120), we propose three estimators of the population mean of the study variable, y, when there are measurement errors on both the study and auxiliary variables under poststratified sampling scheme. The proposed estimators are as follows:

1. The customary sample mean estimator in the presence of measurement errors given by

$$\bar{y}_{ms1} = \sum_{h=1}^L W_h \bar{y}_h'' \quad (3.121)$$

2. A measurement error difference-type estimator given by

$$\bar{y}_{md1} = \sum_{h=1}^L W_h [\bar{y}_h'' - k_h (\bar{x}_h'' - \bar{X}_h)] \quad (3.122)$$

3. A general class of estimators that makes use of known values of some population parameters of the auxiliary variable in the h^{th} stratum when there are measurement errors given by:

$$\bar{y}_{mc1} = \sum_{h=1}^L W_h \bar{y}_h'' \left(\frac{a_h \bar{X}_h + b_h}{a_h \bar{x}_h'' + b_h} \right)^{\alpha_h} \quad (3.123)$$

where $k_h(\alpha_h)$ are any real numbers and $a_h(b_h)$ are also any real numbers or known values of population parameters of an auxiliary variable, x, in the h^{th} stratum, like coefficient of variation, skewness, kurtosis, etc. The conditional and unconditional properties of the above proposed estimators shall be obtained accordingly.

Estimator 10: Customary measurement error sample mean estimator, $\bar{y}_{ms1}$, when there are measurement errors on both the study and auxiliary variables under the Poststratified Sampling Scheme

Theorem 3.9: Under the conditional argument or for an achieved sample configuration, say $\underline{n} = (n_1, n_2, \dots, n_L)$, the proposed customary measurement error sample mean estimator, $\bar{y}_{ms1}$, is unbiased for the population mean ($\bar{Y}$), of the study variable, y , with its conditional variance obtained as:

$$V_c(\bar{y}_{ms1}) = \sum_{h=1}^L W_h^2 \left(\frac{1}{n_h} - \frac{1}{N_h} \right) (S_{yh}^2 + S_{uh}^2) \quad (3.124)$$

Furthermore, under the unconditional argument or for repeated samples of fixed size, n , the proposed customary measurement error sample mean estimator, $\bar{y}_{ms1}$, is unbiased for the population mean ($\bar{Y}$) of the study variable y , with its unconditional variance obtained up to first order of approximations as:

$$V(\bar{y}_{ms1}) = \left(\frac{1-f}{n} \right) \sum_{h=1}^L W_h (S_{yh}^2 + S_{uh}^2) \quad (3.125)$$

Proof: Using (3.119), the proposed estimator, $\bar{y}_{ms1}$, given in (3.121), can be written as

$$\bar{y}_{ms1} = \sum_{h=1}^L W_h \left[\frac{1}{n_h} \sum_{i=1}^{n_h} Y_{hi} + \frac{1}{n_h} \sum_{i=1}^{n_h} U_{hi} \right] \quad (3.126)$$

Therefore, given an achieved sample configuration, say $\underline{n} = (n_1, n_2, \dots, n_L)$, and using (3.126), we obtain the conditional expectation of $\bar{y}_{ms1}$, as

$$\begin{aligned} E_c(\bar{y}_{ms1}) &= E_c \left\{ \sum_{h=1}^L W_h \left[\frac{1}{n_h} \sum_{i=1}^{n_h} Y_{hi} + \frac{1}{n_h} \sum_{i=1}^{n_h} U_{hi} \right] \right\} \\ &= \sum_{h=1}^L W_h \left[E_c \left(\frac{1}{n_h} \sum_{i=1}^{n_h} Y_{hi} \right) + E_c \left(\frac{1}{n_h} \sum_{i=1}^{n_h} U_{hi} \right) \right] = \sum_{h=1}^L W_h [\bar{Y}_h + \bar{U}_h] \\ &= \sum_{h=1}^L W_h [\bar{Y}_h + 0] = \sum_{h=1}^L W_h \bar{Y}_h = \bar{Y} \end{aligned} \quad (3.127)$$

The expression (3.127) shows that the estimator, $\bar{y}_{ms1}$ is unbiased for $\bar{Y}$ under the conditional argument. To obtain the conditional variance of the estimator using (3.126) we have

$$\begin{aligned}
V_c(\bar{y}_{ms1}) &= V_c\left\{\sum_{h=1}^L W_h \left[\frac{1}{n_h} \sum_{i=1}^{n_h} Y_{hi} + \frac{1}{n_h} \sum_{i=1}^{n_h} U_{hi} \right]\right\} \\
&= \sum_{h=1}^L W_h^2 \left[V_c\left(\frac{1}{n_h} \sum_{i=1}^{n_h} Y_{hi}\right) + V_c\left(\frac{1}{n_h} \sum_{i=1}^{n_h} U_{hi}\right) + 2C_c\left(\frac{1}{n_h} \sum_{i=1}^{n_h} Y_{hi}, \frac{1}{n_h} \sum_{i=1}^{n_h} U_{hi}\right) \right] \\
&= \sum_{h=1}^L W_h^2 \left[V_c\left(\frac{1}{n_h} \sum_{i=1}^{n_h} Y_{hi}\right) + V_c\left(\frac{1}{n_h} \sum_{i=1}^{n_h} U_{hi}\right) \right] \tag{3.128} \\
&= \sum_{h=1}^L W_h^2 \left[\left(\frac{1}{n_h} - \frac{1}{N_h}\right) S_{yh}^2 + \left(\frac{1}{n_h} - \frac{1}{N_h}\right) S_{uh}^2 \right] \\
&= \sum_{h=1}^L W_h^2 \left(\frac{1}{n_h} - \frac{1}{N_h}\right) [S_{yh}^2 + S_{uh}^2]
\end{aligned}$$

as stated in the theorem, under the conditional argument, noting that the true stratum values, Y_{hi} of the study variable, y , are uncorrelated with the associated measurement errors, U_{hi} . The unconditional expectation of the estimator, $\bar{y}_{ms1}$, is obtained by taking the expectation of (3.127) for repeated samples of fixed size n . This gives the unconditional expectation of $\bar{y}_{ms1}$ as

$$E(\bar{y}_{ms1}) = EE_c(\bar{y}_{ms1}) = E(\bar{Y}) = \bar{Y} \tag{3.129}$$

which shows the unbiasedness of the estimator, $\bar{y}_{ms1}$ under the unconditional argument, as stated in the theorem. The associated unconditional variance is obtained by taking the expectation of (3.128) for repeated samples of fixed size n , and this gives the unconditional variance of the estimator, $\bar{y}_{ms1}$ as

$$V(\bar{y}_{ms1}) = EV_c(\bar{y}_{ms1}) = \sum_{h=1}^L W_h^2 \left[E\left(\frac{1}{n_h} - \frac{1}{N_h}\right) \right] (S_{yh}^2 + S_{uh}^2) \tag{3.130}$$

Using (3.45) to make the necessary substitution in (3.130) gives, as stated in the theorem, the unconditional variance of the proposed simple sample mean estimator, $\bar{y}_{ms1}$ under poststratified sampling scheme in the presence of measurement errors. And this completes the proof.

Estimator 11: A difference-type estimator, $\bar{y}_{md1}$, when there are measurement errors on both the study and auxiliary variables under the Poststratified Sampling Scheme.

The proposed difference-type estimator, $\bar{y}_{md1}$, given in (3.122), utilizes auxiliary information with known population mean of the auxiliary variable, x , in the h^{th} stratum. We give the conditional and unconditional properties of the estimator, $\bar{y}_{md1}$ in Theorem 3.10.

Theorem 3.10: Under the conditional argument or for an achieved sample configuration, say $\underline{n} = (n_1, n_2, \dots, n_L)$, the proposed difference-type estimator, $\bar{y}_{md1}$, is unbiased for the population mean ($\bar{Y}$), of the study variable, y , with its conditional variance obtained as:

$$V_c(\bar{y}_{md1}) = \sum_{h=1}^L W_h^2 \left(\frac{1}{n_h} - \frac{1}{N_h} \right) (S_{yh}^2 + k_h^2 S_{xh}^2 - 2k_h S_{yxh}) + \sum_{h=1}^L W_h^2 \left(\frac{1}{n_h} - \frac{1}{N_h} \right) (S_{uh}^2 + k_h^2 S_{vh}^2) \quad (3.131)$$

Furthermore, under the unconditional argument or for repeated samples of fixed size, n , the proposed difference-type estimator, $\bar{y}_{md1}$, is unbiased for the population mean ($\bar{Y}$) of the study variable y , with its unconditional variance obtained up to first order of approximations as:

$$V(\bar{y}_{md1}) = \left(\frac{1-f}{n} \right) \sum_{h=1}^L W_h (S_{yh}^2 + k_h^2 S_{xh}^2 - 2k_h S_{yxh}) + \left(\frac{1-f}{n} \right) \sum_{h=1}^L W_h (S_{uh}^2 + k_h^2 S_{vh}^2) \quad (3.132)$$

Proof: Using (3.122) and the fact that covariance between two different strata vanishes, the conditional variance of the proposed estimator, $\bar{y}_{md1}$, can be written as:

$$\begin{aligned} V_c(\bar{y}_{md1}) &= V_c \left[\sum_{h=1}^L W_h \{ \bar{y}_h'' - k_h (\bar{x}_h'' - \bar{X}_h) \} \right] \\ &= \sum_{h=1}^L W_h^2 \left[V_c(\bar{y}_h'') + k_h^2 V_c(\bar{x}_h'') - 2k_h C_c(\bar{y}_h'', \bar{x}_h'') \right] \end{aligned} \quad (3.133)$$

Now, from (3.119) and the fact that the study variable (y) is uncorrelated with its measurement error U_{hi} , the conditional variance of the sample mean, $\bar{y}_h''$, can be written as

$$V_c(\bar{y}_h'') = V_c \left[\frac{1}{n_h} \sum_{i=1}^{n_h} Y_{hi} + \frac{1}{n_h} \sum_{i=1}^{n_h} U_{hi} \right] = V_c \left(\frac{1}{n_h} \sum_{i=1}^{n_h} Y_{hi} \right) + V_c \left(\frac{1}{n_h} \sum_{i=1}^{n_h} U_{hi} \right) \quad (3.134)$$

or

$$V_c(\bar{y}_h'') = \left(\frac{1}{n_h} - \frac{1}{N_h}\right)S_{yh}^2 + \left(\frac{1}{n_h} - \frac{1}{N_h}\right)S_{uh}^2 \quad (3.135)$$

Similarly, the conditional variance of the sample mean, $\bar{x}_h''$, is obtained as:

$$V_c(\bar{x}_h'') = \left(\frac{1}{n_h} - \frac{1}{N_h}\right)S_{xh}^2 + \left(\frac{1}{n_h} - \frac{1}{N_h}\right)S_{vh}^2 \quad (3.136)$$

To obtain the conditional covariance of $\bar{y}_h''$ and $\bar{x}_h''$, we write

$$\begin{aligned} C_c(\bar{y}_h'', \bar{x}_h'') &= C_c\left[\frac{1}{n_h} \sum_{h=1}^{n_h} Y_{hi} + \frac{1}{n_h} \sum_{h=1}^{n_h} U_{hi}, \frac{1}{n_h} \sum_{h=1}^{n_h} X_{hi} + \frac{1}{n_h} \sum_{h=1}^{n_h} V_{hi}\right] \\ &= C_c\left[\frac{1}{n_h} \sum_{h=1}^{n_h} Y_{hi}, \frac{1}{n_h} \sum_{h=1}^{n_h} X_{hi}\right] + C_c\left[\frac{1}{n_h} \sum_{h=1}^{n_h} Y_{hi}, \frac{1}{n_h} \sum_{h=1}^{n_h} V_{hi}\right] \\ &\quad + C_c\left[\frac{1}{n_h} \sum_{h=1}^{n_h} U_{hi}, \frac{1}{n_h} \sum_{h=1}^{n_h} X_{hi}\right] + C_c\left[\frac{1}{n_h} \sum_{h=1}^{n_h} U_{hi}, \frac{1}{n_h} \sum_{h=1}^{n_h} V_{hi}\right] \quad (3.137) \\ &= C_c\left[\frac{1}{n_h} \sum_{h=1}^{n_h} Y_{hi}, \frac{1}{n_h} \sum_{h=1}^{n_h} X_{hi}\right] + 0 + 0 + 0 \\ &= \left(\frac{1}{n_h} - \frac{1}{N_h}\right)S_{yxh} \end{aligned}$$

Using (3.135), (3.136) and (3.137) to make the necessary substitutions in (3.134) gives the conditional variance of the proposed difference-type estimator, $\bar{y}_{md1}$, in the presence of measurement errors in both study and auxiliary variables as stated in the theorem. To obtain the unconditional variance of the proposed difference-type estimators, $\bar{y}_{md1}$, we take the unconditional expectation of (3.131) and use (3.25) to make the necessary substitutions. This completes the proof of theorem 3.10.

Estimator 12: A class of estimators, $\bar{y}_{mc1}$, when there are measurement errors on both the study and auxiliary variables under the Poststratified Sampling Scheme.

The proposed class of estimators, $\bar{y}_{mc1}$, given in (3.123), utilizes auxiliary information with known population parameters of the auxiliary variable, x , in the h^{th} stratum. We give the conditional and unconditional properties of the estimator, $\bar{y}_{mc1}$ in Theorem 3.11.

Theorem 3.11: Under the conditional argument or for an achieved sample configuration, say $\underline{n} = (n_1, n_2, \dots, n_L)$, the proposed class of estimators, $\bar{y}_{mc1}$, is biased for the population mean ($\bar{Y}$), of the study variable, y , with its conditional bias and mean square error obtained as:

$$B_c(\bar{y}_{mc1}) = \sum_{h=1}^L W_h \left(\frac{\alpha_h \lambda_h}{2\bar{X}_h} \right) \left(\frac{1}{n_h} - \frac{1}{N_h} \right) [(\alpha_h + 1) \lambda_h R_h S_{xh}^2 - 2S_{yxh}]$$

$$+ \sum_{h=1}^L W_h \left(\frac{\alpha_h \lambda_h}{2\bar{X}_h} \right) \left(\frac{1}{n_h} - \frac{1}{N_h} \right) (\alpha_h + 1) \lambda_h R_h S_{vh}^2 \quad (3.138)$$

$$MSE_c(\bar{y}_{mc1}) = \sum_{h=1}^L W_h^2 \left(\frac{1}{n_h} - \frac{1}{N_h} \right) (S_{yh}^2 + \alpha_h^2 \lambda_h^2 R_h^2 S_{xh}^2 - 2\alpha_h \lambda_h R_h S_{yxh})$$

$$+ \sum_{h=1}^L W_h^2 \left(\frac{1}{n_h} - \frac{1}{N_h} \right) (S_{uh}^2 + \alpha_h^2 \lambda_h^2 R_h^2 S_{vh}^2) \quad (3.139)$$

Furthermore, for repeated samples of fixed size n or under the unconditional argument, the proposed class of estimators, $\bar{y}_{mc1}$, is biased for the population mean ($\bar{Y}$), with its unconditional bias and mean square error obtained up to first order of approximations as:

$$B(\bar{y}_{mc1}) = \left(\frac{1-f}{n} \right) \sum_{h=1}^L \frac{\alpha_h \lambda_h}{2\bar{X}_h} [(\alpha_h + 1) \lambda_h R_h S_{xh}^2 - 2S_{yxh}]$$

$$+ \left(\frac{1-f}{n} \right) \sum_{h=1}^L \frac{\alpha_h \lambda_h}{2\bar{X}_h} [(\alpha_h + 1) \lambda_h R_h S_{vh}^2] \quad (3.140)$$

$$MSE(\bar{y}_{mc1}) = \left(\frac{1-f}{n} \right) \sum_{h=1}^L W_h (S_{yh}^2 + \alpha_h^2 \lambda_h^2 R_h^2 S_{xh}^2 - 2\alpha_h \lambda_h R_h S_{yxh})$$

$$+ \left(\frac{1-f}{n} \right) \sum_{h=1}^L W_h (S_{uh}^2 + \alpha_h^2 \lambda_h^2 R_h^2 S_{vh}^2) \quad (3.141)$$

Proof: Consider the adjusted mean deviations of the stratum mean estimators, $\bar{y}_h''$ and $\bar{x}_h''$, respectively given by

$$d_{4h} = \frac{\bar{y}_h'' - \bar{Y}_h}{\bar{Y}_h} \quad \text{or} \quad \bar{y}_h'' = \bar{Y}_h(1 + d_{4h}) \quad (3.142)$$

and

$$e_{4h} = \frac{\bar{x}_h'' - \bar{X}_h}{\bar{X}_h} \quad \text{or} \quad \bar{x}_h'' = \bar{X}_h(1 + e_{4h}) \quad (3.143)$$

Under the conditional argument, it is trivial to show that

$$E_c(d_{4h}) = E_c(e_{4h}) = 0, E_c(d_{4h}^2) = \frac{V_c(\bar{y}_h'')}{\bar{Y}_h^2}, E_c(e_{4h}^2) = \frac{V_c(\bar{x}_h'')}{\bar{X}_h^2}, E_c(d_{4h}e_{4h}) = \frac{C_c(\bar{y}_h'', \bar{x}_h'')}{\bar{Y}_h\bar{X}_h} \quad (3.144)$$

Using (3.142) and (3.143), we can rewrite (3.123) as

$$\bar{y}_{mc1} = \sum_{h=1}^L W_h \bar{Y}_h (1 + d_{4h}) (1 + \lambda_h e_{4h})^{-\alpha_h} \quad (3.145)$$

Using Taylor's series to expand (3.145) up to first order of approximations in expected values, it follows that

$$(\bar{y}_{mc1} - \bar{Y}) = \sum_{h=1}^L W_h \bar{Y}_h \left[d_{4h} - \alpha_h \lambda_h e_{4h} - \alpha_h \lambda_h d_{4h} e_{4h} + (1/2) \alpha_h (\alpha_h + 1) \lambda_h^2 e_{4h}^2 \right] \quad (3.146)$$

$$(\bar{y}_{mc1} - \bar{Y})^2 = \sum_{h=1}^L W_h^2 \bar{Y}_h^2 \left[d_{4h}^2 + \alpha_h^2 \lambda_h^2 e_{4h}^2 - 2 \alpha_h \lambda_h d_{4h} e_{4h} \right] \quad (3.147)$$

Taking the conditional expectation of (3.146), and using (3.144), (3.136) and (3.137) to make the necessary substitutions, gives the conditional bias of the class of estimators, $\bar{y}_{mc1}$, as

$$\begin{aligned} B_c(\bar{y}_{mc1}) &= E_c(\bar{y}_{mc1} - \bar{Y}) \\ &= \sum_{h=1}^L W_h \bar{Y}_h \left[E_c(d_{4h}) - \alpha_h \lambda_h E_c(e_{4h}) - \alpha_h \lambda_h E_c(d_{4h} e_{4h}) + (1/2) \alpha_h (\alpha_h + 1) \lambda_h^2 E_c(e_{4h}^2) \right] \\ &= \sum_{h=1}^L W_h \bar{Y}_h \left[-\alpha_h \lambda_h \frac{C_c(\bar{y}_h'', \bar{x}_h'')}{\bar{Y}_h \bar{X}_h} + (1/2) \alpha_h (\alpha_h + 1) \lambda_h^2 \frac{V_c(\bar{x}_h'')}{\bar{X}_h^2} \right] \\ &= \sum_{h=1}^L W_h \left(\frac{\alpha_h \lambda_h}{2 \bar{X}_h} \right) \left(\frac{1}{n_h} - \frac{1}{N_h} \right) \left[(\alpha_h + 1) \lambda_h R_h S_{xh}^2 - 2 S_{yxh} \right] \\ &\quad + \sum_{h=1}^L W_h \left(\frac{\alpha_h \lambda_h}{2 \bar{X}_h} \right) \left(\frac{1}{n_h} - \frac{1}{N_h} \right) (\alpha_h + 1) \lambda_h R_h S_{vh}^2 \end{aligned} \quad (3.148)$$

as stated in the theorem. To obtain the conditional mean square error of the class of estimators, $\bar{y}_{mc1}$, we take the conditional expectation of (3.147), and use (3.144), (3.145), (3.136) and (3.137) to make the necessary substitutions, which gives:

$$\begin{aligned}
MS E_c(\bar{y}_{mc1}) &= E_c(\bar{y}_{mc1} - \bar{Y})^2 \\
&= \sum_{h=1}^L W_h^2 \bar{Y}_h^2 \left[E_c(d_{4h}^2) + \alpha_h^2 \lambda_h^2 E_c(e_{4h}^2) - 2\alpha_h \lambda_h E_c(d_{4h} e_{4h}) \right] \\
&= \sum_{h=1}^L W_h^2 \bar{Y}_h^2 \left[\frac{V_c(\bar{y}_h'')}{\bar{Y}_h^2} + \alpha_h^2 \lambda_h^2 \frac{V_c(\bar{x}_h'')}{\bar{X}_h^2} - 2\alpha_h \lambda_h \frac{C_c(\bar{y}_h'', \bar{x}_h'')}{\bar{Y}_h \bar{X}_h} \right] \\
&= \sum_{h=1}^L W_h^2 \left(\frac{1}{n_h} - \frac{1}{N_h} \right) (S_{yh}^2 + \alpha_h^2 \lambda_h^2 R_h^2 S_{xh}^2 - 2\alpha_h \lambda_h R_h S_{y xh}) \\
&\quad + \sum_{h=1}^L W_h^2 \left(\frac{1}{n_h} - \frac{1}{N_h} \right) (S_{uh}^2 + \alpha_h^2 \lambda_h^2 R_h^2 S_{vh}^2)
\end{aligned} \tag{3.149}$$

as stated in the theorem. And this gives the proof of the first part of the theorem on the conditional bias and mean square error of the proposed class of estimators, $\bar{y}_{mc1}$. To obtain the unconditional bias and mean square error of the proposed class of estimators, $\bar{y}_{mc1}$, we take the unconditional expectations of (3.148) and (3.149), and use (3.25) to make the necessary substitutions. This completes the proof.

3.2.2 Measurement Errors on the study variable and none on the Auxiliary variable under the Poststratified Sampling Scheme

The focus in this section is to develop sampling strategies for handling measurement errors in poststratified sampling scheme, when there are measurement errors on the study variable and none on the auxiliary variable. We suggest the following sampling design:

Sampling Design 3.2.2: The sampling design we propose when there are measurement errors on the study variable and none on the auxiliary variable under the poststratified sampling scheme is similar to Sampling Design 3.2.1. The only difference is that here we assume that there are measurement errors on the study variable and none on the auxiliary variable, unlike in Sampling Design 3.2.1 where we assumed the presence of measurement errors on both the study and auxiliary variables.

We observe, from (3.119), that the sample mean of the study variable in the presence of measurement errors is given by

$$\bar{y}_h'' = \frac{1}{n_h} \sum_{hi=1}^{n_h} y_{hi} = \frac{1}{n_h} \sum_{hi=1}^{n_h} [Y_{hi} + (y_{hi} - Y_{hi})] = \frac{1}{n_h} \sum_{hi=1}^{n_h} Y_{hi} + \frac{1}{n_h} \sum_{hi=1}^{n_h} U_{hi} \quad (3.150)$$

On the assumption that there are no measurement errors on the auxiliary variable, x , it means that the observed values of x are the same as its true values. Consequently, we write the sample mean of the observed values of x in stratum h as the sample mean of the true values, given as

$$\bar{x}_h' = \frac{1}{n_h} \sum_{hi=1}^{n_h} x_{hi} = \frac{1}{n_h} \sum_{hi=1}^{n_h} X_{hi} \quad (3.151)$$

Based on the Sampling Design 3.2.2 and using (3.150) and (3.151), we propose the following three estimators of the population mean ($\bar{Y}$), when there are measurement errors on the study variable and none on the auxiliary variable under poststratified sampling scheme.

1. The customary sample mean estimator in the presence of measurement errors given by

$$\bar{y}_{ms2} = \sum_{h=1}^L W_h \bar{y}_h'' \quad (3.152)$$

2. A measurement error difference-type estimator given by

$$\bar{y}_{md2} = \sum_{h=1}^L W_h [\bar{y}_h'' - k_h (\bar{x}_h' - \bar{X}_h)] \quad (3.153)$$

3. A general class of estimators that makes use of known values of some population parameters of the auxiliary variable in the h^{th} stratum given by:

$$\bar{y}_{mc2} = \sum_{h=1}^L W_h \bar{y}_h'' \left(\frac{a_h \bar{X}_h + b_h}{a_h \bar{x}_h' + b_h} \right)^{\alpha_h} \quad (3.154)$$

where $k_h(\alpha_h)$ are any real numbers and $a_h(b_h)$ are also any real numbers or known population parameters of an auxiliary variable, x , in the h^{th} stratum. We obtain the conditional and unconditional properties of the above proposed estimators accordingly.

Estimator 13: Customary measurement error sample mean estimator, $\bar{y}_{ms2}$, when there are measurement errors on the study variable and none on the auxiliary variable under the Post-stratified Sampling Scheme

We observe that the proposed estimator, $\bar{y}_{ms2}$, is the same as the estimator, $\bar{y}_{ms1}$, given (3.121). Accordingly, the properties of $\bar{y}_{ms2}$ are the same as those of $\bar{y}_{ms1}$, given in Theorem 3.9.

Estimator 14: A difference-type estimator, $\bar{y}_{md2}$, when there are measurement errors on the study variable and none on the auxiliary variable under the Poststratified Sampling Scheme

The proposed difference-type estimator, $\bar{y}_{md2}$, given in (3.153), utilizes auxiliary information with known population mean of the auxiliary variable, x , in the h^{th} stratum. Theorem 3.12 gives the conditional and unconditional properties of the estimator, $\bar{y}_{md2}$.

Theorem 3.12: Under the conditional argument or for an achieved sample configuration, say $\underline{n} = (n_1, n_2, \dots, n_L)$, the proposed difference-type estimator, $\bar{y}_{md2}$, is unbiased for the population mean ($\bar{Y}$), of the study variable, y , with its conditional variance obtained as:

$$V_c(\bar{y}_{md2}) = \sum_{h=1}^L W_h^2 \left(\frac{1}{n_h} - \frac{1}{N_h} \right) (S_{yh}^2 + k_h^2 S_{xh}^2 - 2k_h S_{yxh}) + \sum_{h=1}^L W_h^2 \left(\frac{1}{n_h} - \frac{1}{N_h} \right) S_{uh}^2 \quad (3.155)$$

Furthermore, under the unconditional argument or for repeated samples of fixed size, n , the proposed difference-type estimator, $\bar{y}_{md2}$, is unbiased for the population mean ($\bar{Y}$) of the study variable y , with its unconditional variance obtained up to first order of approximations as:

$$V(\bar{y}_{md2}) = \left(\frac{1-f}{n} \right) \sum_{h=1}^L W_h (S_{yh}^2 + k_h^2 S_{xh}^2 - 2k_h S_{yxh}) + \left(\frac{1-f}{n} \right) \sum_{h=1}^L W_h S_{uh}^2 \quad (3.156)$$

Proof: Using (3.153) and following similar procedure as in the proof of Theorem 3.10, the conditional variance of the proposed estimator, $\bar{y}_{md2}$, can be written as:

$$V_c(\bar{y}_{md2}) = \sum_{h=1}^L W_h^2 \left[V_c(\bar{y}_h'') + k_h^2 V_c(\bar{x}_h') - 2k_h C_c(\bar{y}_h'', \bar{x}_h') \right] \quad (3.157)$$

where

$$V_c(\bar{y}_h'') = \left(\frac{1}{n_h} - \frac{1}{N_h}\right)S_{yh}^2 + \left(\frac{1}{n_h} - \frac{1}{N_h}\right)S_{uh}^2 \quad (3.158)$$

$$V_c(\bar{x}_h') = \left(\frac{1}{n_h} - \frac{1}{N_h}\right)S_{xh}^2 \quad (3.159)$$

$$C_c(\bar{y}_h'', \bar{x}_h') = \left(\frac{1}{n_h} - \frac{1}{N_h}\right)S_{yxh} \quad (3.160)$$

Substituting (3.158), (3.159) and (3.160) in (3.157), we obtain the conditional variance of the proposed difference-type estimator, $\bar{y}_{md2}$, in the presence of measurement errors in the study variable and none in the auxiliary variable as stated in the theorem. To obtain the unconditional variance of the proposed difference-type estimators, $\bar{y}_{md2}$, we take the unconditional expectation of (3.155) and use (3.25) to make the necessary substitutions. This completes the proof of theorem 3.12.

Estimator 15: A class of estimators, $\bar{y}_{mc2}$, when there are measurement errors on the study variable and none on the auxiliary variable under the Poststratified Sampling Scheme

The proposed class of estimators, $\bar{y}_{mc2}$, given in (3.154), utilizes auxiliary information with known population parameters of the auxiliary variable, x , in the h^{th} stratum. The conditional and unconditional properties of the estimator, $\bar{y}_{mc2}$ are stated in Theorem 3.13.

Theorem 3.13: Under the conditional argument or for an achieved sample configuration, say $\underline{n} = (n_1, n_2, \dots, n_L)$, the proposed class of estimators, $\bar{y}_{mc2}$, is biased for the population mean ($\bar{Y}$), of the study variable, y , with its conditional bias and mean square error obtained as:

$$B_c(\bar{y}_{mc2}) = \sum_{h=1}^L W_h \left(\frac{\alpha_h \lambda_h}{2\bar{X}_h} \right) \left(\frac{1}{n_h} - \frac{1}{N_h} \right) [(\alpha_h + 1)\lambda_h R_h S_{xh}^2 - 2S_{yxh}] \quad (3.161)$$

$$MSE_c(\bar{y}_{mc2}) = \sum_{h=1}^L W_h^2 \left(\frac{1}{n_h} - \frac{1}{N_h} \right) (S_{yh}^2 + \alpha_h^2 \lambda_h^2 R_h^2 S_{xh}^2 - 2\alpha_h \lambda_h R_h S_{yxh}) + \sum_{h=1}^L W_h^2 \left(\frac{1}{n_h} - \frac{1}{N_h} \right) S_{uh}^2 \quad (3.162)$$

Furthermore, for repeated samples of fixed size n or under the unconditional argument, the proposed class of estimators, $\bar{y}_{mc2}$, is biased for the population mean ($\bar{Y}$), with its unconditional

bias and mean square error obtained up to first order of approximations as:

$$B(\bar{y}_{mc2}) = \left(\frac{1-f}{n}\right) \sum_{h=1}^L \frac{\alpha_h \lambda_h}{2\bar{X}_h} [(\alpha_h + 1)\lambda_h R_h S_{xh}^2 - 2S_{yxh}] \quad (3.163)$$

$$MSE(\bar{y}_{mc2}) = \left(\frac{1-f}{n}\right) \sum_{h=1}^L W_h (S_{yh}^2 + \alpha_h^2 \lambda_h^2 R_h^2 S_{xh}^2 - 2\alpha_h \lambda_h R_h S_{yxh}) + \left(\frac{1-f}{n}\right) \sum_{h=1}^L W_h S_{uh}^2 \quad (3.164)$$

Proof: Using (3.154) and following similar procedure as in the proof of Theorem 3.11, we obtain the conditional bias and mean square error of the class of estimators, $\bar{y}_{mc2}$, as

$$B_c(\bar{y}_{mc2}) = \sum_{h=1}^L W_h \bar{Y}_h \left[-\alpha_h \lambda_h \frac{C_c(\bar{y}_h'', \bar{x}_h')}{\bar{Y}_h \bar{X}_h} + (1/2)\alpha_h(\alpha_h + 1)\lambda_h^2 \frac{V_c(\bar{x}_h')}{\bar{X}_h^2} \right] \quad (3.165)$$

$$MSE_c(\bar{y}_{mc2}) = \sum_{h=1}^L W_h^2 \bar{Y}_h^2 \left[\frac{V_c(\bar{y}_h'')}{\bar{Y}_h^2} + \alpha_h^2 \lambda_h^2 \frac{V_c(\bar{x}_h')}{\bar{X}_h^2} - 2\alpha_h \lambda_h \frac{C_c(\bar{y}_h'', \bar{x}_h')}{\bar{Y}_h \bar{X}_h} \right] \quad (3.166)$$

Using (3.158), (3.159) and (3.160) to make necessary substitutions in (3.165) and (3.166), we obtain the conditional bias and mean square error of the proposed class of estimators, $\bar{y}_{mc2}$, as stated in the theorem. And this proves the first part of the theorem on the conditional bias and mean square error of the proposed class of estimators, $\bar{y}_{mc2}$. To obtain the unconditional bias and mean square error of the proposed class of estimators, $\bar{y}_{mc2}$, we take the unconditional expectations of (3.161) and (3.162), and use (3.25) to make the necessary substitutions. This completes the proof of Theorem 3.13.

3.2.3 Measurement Errors on the Auxiliary variable and none on the

Study variable under the Poststratified Sampling Scheme

In this section, we concentrate on developing sampling strategies for handling measurement errors in poststratified sampling scheme, when there are measurement errors on the auxiliary variable and none on the study variable. The sampling design is as follows:

Sampling Design 3.2.3: Just like Sampling Design 3.2.2, the sampling design we propose here when there are measurement errors on the auxiliary variable and none on the study variable

under the poststratified sampling scheme is similar to Sampling Design 3.2.1. The difference between Sampling Design 3.2.1 and Sampling Design 3.2.3 is that the former assumed the presence of measurement errors on both the study and auxiliary variables, while here we assume that there are measurement errors only on the auxiliary variable and none on the study variable.

Based on Sampling Design 3.2.3, the sample mean of the observed values of the study variable y in stratum h is the same as the sample mean of the true values of y , since there are no measurement errors in y . Hence we write the observed sample mean of y in stratum h as

$$\bar{y}'_h = \frac{1}{n_h} \sum_{h=1}^{n_h} y_{hi} = \frac{1}{n_h} \sum_{h=1}^{n_h} Y_{hi} \quad (3.167)$$

Again, Sampling Design 3.2.3 assumes there are measurement errors in the auxiliary variable, x , and we observe, from (3.120), that the h^{th} stratum sample mean of the auxiliary variable in the presence of measurement errors is given by

$$\bar{x}''_h = \frac{1}{n_h} \sum_{h=1}^{n_h} x_{hi} = \frac{1}{n_h} \sum_{h=1}^{n_h} [X_{hi} + (x_{hi} - X_{hi})] = \frac{1}{n_h} \sum_{h=1}^{n_h} X_{hi} + \frac{1}{n_h} \sum_{h=1}^{n_h} V_{hi} \quad (3.168)$$

Based on the Sampling Design 3.2.3 and using (3.167) and (3.168), we propose the following three estimators of the population mean ($\bar{Y}$), when there are measurement errors on the auxiliary variable and none on the study variable under poststratified sampling scheme.

1. The customary sample mean estimator when there are no measurement errors in the study variable, y , given by

$$\bar{y}_{ms3} = \sum_{h=1}^L W_h \bar{y}'_h \quad (3.169)$$

2. A measurement error difference-type estimator given by

$$\bar{y}_{md3} = \sum_{h=1}^L W_h [\bar{y}'_h - k_h (\bar{x}''_h - \bar{X}_h)] \quad (3.170)$$

3. A general class of estimators that makes use of known values of some population param-

eters of the auxiliary variable in the h^{th} stratum given by:

$$\bar{y}_{mc3} = \sum_{h=1}^L W_h \bar{y}'_h \left(\frac{a_h \bar{X}_h + b_h}{a_h \bar{X}_h'' + b_h} \right)^{\alpha_h} \quad (3.171)$$

where $k_h(\alpha_h)$ are any real numbers and $a_h(b_h)$ are also any real numbers or known values of population parameters of an auxiliary variable, x , in the h^{th} stratum, like coefficient of variation, skewness, kurtosis, etc. We obtain the conditional and unconditional properties of the above proposed estimators accordingly.

Estimator 16: Customary measurement error sample mean estimator, $\bar{y}_{ms3}$, when there are measurement errors on the auxiliary variable and none on the study variable under the Post-stratified Sampling Scheme

Using (3.167) and (3.169), and applying the theory of SRSWOR method in stratum h , it follows that the proposed estimator, $\bar{y}_{ms3}$, is unbiased for the population mean ($\bar{Y}$) with its conditional and unconditional variances respectively obtained as

$$V_c(\bar{y}_{ms3}) = V_c\left(\sum_{h=1}^L W_h \bar{y}'_h\right) = \sum_{h=1}^L W_h^2 \left(\frac{1}{n_h} - \frac{1}{N_h}\right) S_{yh}^2 \quad (3.172)$$

$$V(\bar{y}_{ms3}) = V\left(\sum_{h=1}^L W_h \bar{y}'_h\right) = \left(\frac{1-f}{n}\right) \sum_{h=1}^L W_h S_{yh}^2 \quad (3.173)$$

Estimator 17: A difference-type estimator, $\bar{y}_{md3}$, when there are measurement errors on the auxiliary variable and none on the study variable under the Poststratified Sampling Scheme.

The proposed difference-type estimator, $\bar{y}_{md3}$, given in (3.170), utilizes auxiliary information with known population mean of the auxiliary variable, x , in the h^{th} stratum. Theorem 3.14 gives the conditional and unconditional properties of the estimator, $\bar{y}_{md3}$.

Theorem 3.14: Under the conditional argument or for an achieved sample configuration, say $\underline{n} = (n_1, n_2, \dots, n_L)$, the proposed difference-type estimator, $\bar{y}_{md3}$, is unbiased for the population mean ($\bar{Y}$), of the study variable, y , with its conditional variance obtained as:

$$V_c(\bar{y}_{md3}) = \sum_{h=1}^L W_h^2 \left(\frac{1}{n_h} - \frac{1}{N_h} \right) (S_{yh}^2 + k_h^2 S_{xh}^2 - 2k_h S_{yxh}) + \sum_{h=1}^L W_h^2 \left(\frac{1}{n_h} - \frac{1}{N_h} \right) S_{vh}^2 \quad (3.174)$$

Furthermore, under the unconditional argument or for repeated samples of fixed size, n , the proposed difference-type estimator, $\bar{y}_{md3}$, is unbiased for the population mean ($\bar{Y}$) of the study variable y , with its unconditional variance obtained up to first order of approximations as:

$$V(\bar{y}_{md3}) = \left(\frac{1-f}{n} \right) \sum_{h=1}^L W_h (S_{yh}^2 + k_h^2 S_{xh}^2 - 2k_h S_{yxh}) + \left(\frac{1-f}{n} \right) \sum_{h=1}^L W_h S_{vh}^2 \quad (3.175)$$

Proof: Using (3.170) and following similar procedure as in the proof of Theorem 3.12, the conditional variance of the proposed estimator, $\bar{y}_{md3}$, can be written as:

$$V_c(\bar{y}_{md3}) = \sum_{h=1}^L W_h^2 \left[V_c(\bar{y}'_h) + k_h^2 V_c(\bar{x}''_h) - 2k_h C_c(\bar{y}'_h, \bar{x}''_h) \right] \quad (3.176)$$

where

$$V_c(\bar{y}'_h) = \left(\frac{1}{n_h} - \frac{1}{N_h} \right) S_{yh}^2 \quad (3.177)$$

$$V_c(\bar{x}''_h) = \left(\frac{1}{n_h} - \frac{1}{N_h} \right) S_{xh}^2 + \left(\frac{1}{n_h} - \frac{1}{N_h} \right) S_{vh}^2 \quad (3.178)$$

$$C_c(\bar{y}'_h, \bar{x}''_h) = \left(\frac{1}{n_h} - \frac{1}{N_h} \right) S_{yxh} \quad (3.179)$$

Substituting (3.177), (3.178) and (3.179) in (3.176), we obtain the conditional variance of the proposed difference-type estimator, $\bar{y}_{md3}$, in the presence of measurement errors in the auxiliary variable and none in the study variable as stated in (3.174) of the theorem. To obtain the unconditional variance of the proposed difference-type estimators, $\bar{y}_{md3}$, we take the unconditional expectation of (3.174) and use (3.25) to make the necessary substitutions. This completes the proof of theorem 3.14.

Estimator 18: A class of estimators, $\bar{y}_{mc3}$, when there are measurement errors on the auxiliary variable and none on the study variable under the Poststratified Sampling Scheme.

The proposed class of estimators, $\bar{y}_{mc3}$, given in (3.171), utilizes auxiliary information with known population parameters of the auxiliary variable, x , in the h^{th} stratum. The conditional and unconditional properties of the estimator, $\bar{y}_{mc3}$ are stated in Theorem 3.15.

Theorem 3.15: Under the conditional argument or for an achieved sample configuration, say $\underline{n} = (n_1, n_2, \dots, n_L)$, the proposed class of estimators, $\bar{y}_{mc3}$, is biased for the population mean ($\bar{Y}$), of the study variable, y , with its conditional bias and mean square error obtained as:

$$B_c(\bar{y}_{mc3}) = \sum_{h=1}^L W_h \left(\frac{\alpha_h \lambda_h}{2\bar{X}_h} \right) \left(\frac{1}{n_h} - \frac{1}{N_h} \right) [(\alpha_h + 1)\lambda_h R_h S_{xh}^2 - 2S_{yxh}] \\ + \sum_{h=1}^L W_h \left(\frac{\alpha_h \lambda_h}{2\bar{X}_h} \right) \left(\frac{1}{n_h} - \frac{1}{N_h} \right) (\alpha_h + 1)\lambda_h R_h S_{vh}^2 \quad (3.180)$$

$$MSE_c(\bar{y}_{mc3}) = \sum_{h=1}^L W_h^2 \left(\frac{1}{n_h} - \frac{1}{N_h} \right) (S_{yh}^2 + \alpha_h^2 \lambda_h^2 R_h^2 S_{xh}^2 - 2\alpha_h \lambda_h R_h S_{yxh}) \\ + \sum_{h=1}^L W_h^2 \left(\frac{1}{n_h} - \frac{1}{N_h} \right) \alpha_h^2 \lambda_h^2 R_h^2 S_{vh}^2 \quad (3.181)$$

Furthermore, for repeated samples of fixed size n or under the unconditional argument, the proposed class of estimators, $\bar{y}_{mc3}$, is biased for the population mean ($\bar{Y}$), with its unconditional bias and mean square error obtained up to first order of approximations as:

$$B(\bar{y}_{mc3}) = \left(\frac{1-f}{n} \right) \sum_{h=1}^L \frac{\alpha_h \lambda_h}{2\bar{X}_h} [(\alpha_h + 1)\lambda_h R_h S_{xh}^2 - 2S_{yxh}] \\ + \left(\frac{1-f}{n} \right) \sum_{h=1}^L \frac{\alpha_h \lambda_h}{2\bar{X}_h} [(\alpha_h + 1)\lambda_h R_h S_{vh}^2] \quad (3.182)$$

$$MSE(\bar{y}_{mc3}) = \left(\frac{1-f}{n} \right) \sum_{h=1}^L W_h (S_{yh}^2 + \alpha_h^2 \lambda_h^2 R_h^2 S_{xh}^2 - 2\alpha_h \lambda_h R_h S_{yxh}) \\ + \left(\frac{1-f}{n} \right) \sum_{h=1}^L W_h \alpha_h^2 \lambda_h^2 R_h^2 S_{vh}^2 \quad (3.183)$$

Proof: Using (3.171) and following similar procedure as in the proof of Theorem 3.11, we obtain the conditional bias and mean square error of the class of estimators, $\bar{y}_{mc3}$, as

$$B_c(\bar{y}_{mc3}) = \sum_{h=1}^L W_h \bar{Y}_h \left[-\alpha_h \lambda_h \frac{C_c(\bar{y}'_h, \bar{x}''_h)}{\bar{Y}_h \bar{X}_h} + (1/2) \alpha_h (\alpha_h + 1) \lambda_h^2 \frac{V_c(\bar{x}''_h)}{\bar{X}_h^2} \right] \quad (3.184)$$

$$MS E_c(\bar{y}_{mc3}) = \sum_{h=1}^L W_h^2 \bar{Y}_h^2 \left[\frac{V_c(\bar{y}'_h)}{\bar{Y}_h^2} + \alpha_h^2 \lambda_h^2 \frac{V_c(\bar{x}''_h)}{\bar{X}_h^2} - 2 \alpha_h \lambda_h \frac{C_c(\bar{y}'_h, \bar{x}''_h)}{\bar{Y}_h \bar{X}_h} \right] \quad (3.185)$$

Using (3.177), (3.178) and (3.179) to make necessary substitutions in (3.184) and (3.185), we obtain the conditional bias and mean square error of the proposed class of estimators, $\bar{y}_{mc3}$, as stated in the theorem. Thus, the first part of the theorem on the conditional bias and mean square error of the proposed class of estimators, $\bar{y}_{mc3}$ is proved. To obtain the unconditional bias and mean square error of the proposed class of estimators, $\bar{y}_{mc3}$, we take the unconditional expectations of (3.180) and (3.181), and use (3.25) to make the necessary substitutions. This completes the proof of Theorem 3.15.

3.3 Joint Occurrence of Nonresponse and Measurement

Errors in Poststratified Sampling Scheme

In the previous sections, we considered separate occurrences of nonresponse alone and measurement errors alone under the poststratified sampling scheme. However, in the present section, we will consider joint occurrence of nonresponse and measurement errors in post stratified sampling scheme. The focus shall be on the development of sampling strategies for handling joint occurrence of nonresponse and measurement errors when using poststratified sampling scheme. We shall adopt the use of auxiliary information in developing the sampling strategies. Three possibilities or cases shall be considered. First we consider joint occurrence of nonresponse and measurement errors on both the study and auxiliary variables. Secondly we consider joint occurrence of nonresponse and measurement errors on only the study variable and not on the auxiliary variable. Thirdly we consider joint occurrence of nonresponse and measurement errors on only the auxiliary variable and not on the study variable.

3.3.1 Joint Occurrence of Nonresponse and Measurement Errors on both the study and auxiliary variables under the Poststratified Sampling Scheme

The focus here is to develop sampling strategies to facilitate the handling of joint occurrence of nonresponse and measurement errors on both the study and auxiliary variables in poststratified sampling scheme. We propose the following sampling design.

Sampling Design 3.3.1: The Sampling Design proposed here for handling the joint occurrence of nonresponse and measurement errors on both the study and auxiliary variables in poststratified sampling scheme is similar to Sampling Design 3.1.1. The only difference is that Sampling Design 3.1.1 only assumed the presence of nonresponse on both the study and auxiliary variables, while here, the proposed Sampling Design 3.3.1 assumes joint occurrence of nonresponse and measurement errors on both the study (y) and auxiliary (x) variables. This implies that the observed values of y and x from the response sample group of n_{1h} units in stratum h, supplemented with the observed values of y and x from the nonresponse subsample group of m_h units in stratum h, all have measurement errors. Consequently, for the h^{th} stratum, we give a sample mean of the study variable, y, when there is joint occurrence of nonresponse and measurement errors on y, as a linear function of the response and nonresponse subsample means, given by:

$$\begin{aligned}\bar{y}_h''' &= \left(\frac{n_{1h}}{n_h}\right)\bar{y}_{1h} + \left(\frac{n_{2h}}{n_h}\right)\bar{y}_{2mh} \\ &= \left(\frac{n_{1h}}{n_h}\right)\left[\frac{1}{n_{1h}} \sum_{i=1}^{n_{1h}} (Y_{hi} + U_{hi})\right] + \left(\frac{n_{2h}}{n_h}\right)\left[\frac{1}{m_h} \sum_{i=1}^{m_h} (Y_{hi} + U_{hi})\right] \\ &= \left(\frac{n_{1h}}{n_h}\right)\left(\frac{1}{n_{1h}} \sum_{i=1}^{n_{1h}} Y_{hi}\right) + \left(\frac{n_{1h}}{n_h}\right)\left(\frac{1}{n_{1h}} \sum_{i=1}^{n_{1h}} U_{hi}\right) + \left(\frac{n_{2h}}{n_h}\right)\left(\frac{1}{m_h} \sum_{i=1}^{m_h} Y_{hi}\right) + \left(\frac{n_{2h}}{n_h}\right)\left(\frac{1}{m_h} \sum_{i=1}^{m_h} U_{hi}\right)\end{aligned}\tag{3.186}$$

where $\bar{y}_{1h} = \frac{1}{n_{1h}} \sum_{i=1}^{n_{1h}} y_{hi}$ is the sample mean of y in stratum h, based on the response group sample size, n_{1h}

$\bar{y}_{2mh} = \frac{1}{m_h} \sum_{i=1}^{m_h} y_{hi}$ is the sample mean of y in stratum h, based on the nonresponse sub-sample group sample size, m_h

and the corresponding sample mean of the auxiliary variable, x, when there is joint occurrence of nonresponse and measurement errors on x, is given by:

$$\bar{x}_h''' = \left(\frac{n_{1h}}{n_h}\right)\left(\frac{1}{n_{1h}} \sum_{i=1}^{n_{1h}} X_{hi}\right) + \left(\frac{n_{1h}}{n_h}\right)\left(\frac{1}{n_{1h}} \sum_{i=1}^{n_{1h}} V_{hi}\right) + \left(\frac{n_{2h}}{n_h}\right)\left(\frac{1}{m_h} \sum_{i=1}^{m_h} X_{hi}\right) + \left(\frac{n_{2h}}{n_h}\right)\left(\frac{1}{m_h} \sum_{i=1}^{m_h} V_{hi}\right) \quad (3.187)$$

Based on the Sampling Design 3.3.1 and using the sample means (3.186) and (3.187), we propose three estimators of the population mean ($\bar{Y}$) of the study variable, y, when there is joint occurrence of nonresponse and measurement errors on both the study and auxiliary variables under the poststratified sampling scheme, as follows:

1. The customary sample mean estimator given by

$$\bar{y}_{js1} = \sum_{h=1}^L W_h \bar{y}_h''' \quad (3.188)$$

2. A difference-type estimator given by

$$\bar{y}_{jd1} = \sum_{h=1}^L W_h [\bar{y}_h''' - k_h (\bar{x}_h''' - \bar{X}_h)] \quad (3.189)$$

3. A general class of estimators that makes use of known values of some population parameters of the auxiliary variable in the h^{th} stratum, given by

$$\bar{y}_{jc1} = \sum_{h=1}^L W_h \bar{y}_h''' \left(\frac{a_h \bar{X}_h + b_h}{a_h \bar{x}_h''' + b_h} \right)^{\alpha_h} \quad (3.190)$$

where $k_h(\alpha_h)$ are any real numbers and $a_h(b_h)$ are also any real numbers or known values of population parameters of an auxiliary variable, x, in the h^{th} stratum, like coefficient of variation, skewness, kurtosis, etc. Properties of the above proposed estimators shall be obtained both under the conditional argument or given an achieved sample configuration, say $\underline{n} = (n_1, n_2, \dots, n_L)$ and under the unconditional argument or for repeated samples of fixed size, n.

Estimator 19: Customary sample mean estimator, $\bar{y}_{js1}$, when there is joint occurrence of non-response and measurement errors on both the study and auxiliary variables under the Poststratified Sampling Scheme.

The proposed sample mean estimator, $\bar{y}_{js1}$, does not utilize any auxiliary information. The conditional and unconditional properties of the estimator, $\bar{y}_{js1}$, are given in theorem 3.16.

Theorem 3.16: Under the conditional argument or for an achieved sample configuration, say $\underline{n} = (n_1, n_2, \dots, n_L)$, the proposed customary sample mean estimator, $\bar{y}_{js1}$, is unbiased for the population mean ($\bar{Y}$), of the study variable, y, with its conditional variance obtained as:

$$V_c(\bar{y}_{js1}) = \sum_{h=1}^L W_h^2 \left[\left(\frac{1}{n_h} - \frac{1}{N_h} \right) S_{yh}^2 + \left(\frac{g_h - 1}{n_h} \right) W_{2h} S_{2yh}^2 \right] + \sum_{h=1}^L W_h^2 \left[\left(\frac{1}{n_h} - \frac{1}{N_h} \right) S_{uh}^2 + \left(\frac{g_h - 1}{n_h} \right) W_{2h} S_{2uh}^2 \right] \quad (3.191)$$

Furthermore, under the unconditional argument or for repeated samples of fixed size, n, the nonresponse sample mean estimator, $\bar{y}_{js1}$, is unbiased for the population mean ($\bar{Y}$) of the study variable y, with its unconditional variance obtained up to first order of approximations as:

$$V(\bar{y}_{js1}) = \sum_{h=1}^L W_h \left[\left(\frac{1-f}{n} \right) S_{yh}^2 + \left(\frac{g_h - 1}{n} \right) W_{2h} S_{2yh}^2 \right] + \sum_{h=1}^L W_h \left[\left(\frac{1-f}{n} \right) S_{uh}^2 + \left(\frac{g_h - 1}{n} \right) W_{2h} S_{2uh}^2 \right] \quad (3.192)$$

Proof: Using (3.188), the conditional expectation and variance of the proposed estimator, $\bar{y}_{js1}$, are respectively given by

$$E_c(\bar{y}_{js1}) = \sum_{h=1}^L W_h E_c(\bar{y}_h''') \quad (3.193)$$

and

$$V_c(\bar{y}_{js1}) = \sum_{h=1}^L W_h^2 V_c(\bar{y}_h''') \quad (3.194)$$

The subsampling design used in addressing the nonresponse in the study variable, y, involved two phase or double sampling. Consequently, the conditional expectation and variance of the

stratum mean estimator, $\bar{y}_h'''$, are respectively given by

$$E_c(\bar{y}_h''') = E_1 E_2(\bar{y}_h''') \quad (3.195)$$

and

$$V_c(\bar{y}_h''') = V_1 E_2(\bar{y}_h''') + E_1 V_2(\bar{y}_h''') \quad (3.196)$$

where E_1 , V_1 and C_1 respectively denote expectation, variance and covariance over the initial or first phase sampling of n units out of N units in the population, while E_2 , V_2 and C_2 respectively denote expectation, variance and covariance over the subsample or second phase sampling of m units out of the n'_2 nonresponse units. Using (3.186), we have

$$\begin{aligned} E_2(\bar{y}_h''') &= E_2 \left[\left(\frac{n_{1h}}{n_h} \right) \left(\frac{1}{n_{1h}} \sum_{i=1}^{n_{1h}} Y_{hi} \right) + \left(\frac{n_{1h}}{n_h} \right) \left(\frac{1}{n_{1h}} \sum_{i=1}^{n_{1h}} U_{hi} \right) \right. \\ &\quad \left. + \left(\frac{n_{2h}}{n_h} \right) \left(\frac{1}{m_h} \sum_{i=1}^{m_h} Y_{hi} \right) + \left(\frac{n_{2h}}{n_h} \right) \left(\frac{1}{m_h} \sum_{i=1}^{m_h} U_{hi} \right) \right] \\ &= \left(\frac{n_{1h}}{n_h} \right) \left(\frac{1}{n_{1h}} \sum_{i=1}^{n_{1h}} Y_{hi} \right) + \left(\frac{n_{1h}}{n_h} \right) \left(\frac{1}{n_{1h}} \sum_{i=1}^{n_{1h}} U_{hi} \right) \\ &\quad + \left(\frac{n_{2h}}{n_h} \right) E_2 \left(\frac{1}{m_h} \sum_{i=1}^{m_h} Y_{hi} \right) + \left(\frac{n_{2h}}{n_h} \right) E_2 \left(\frac{1}{m_h} \sum_{i=1}^{m_h} U_{hi} \right) \\ &= \left(\frac{n_{1h}}{n_h} \right) \left(\frac{1}{n_{1h}} \sum_{i=1}^{n_{1h}} Y_{hi} \right) + \left(\frac{n_{1h}}{n_h} \right) \left(\frac{1}{n_{1h}} \sum_{i=1}^{n_{1h}} U_{hi} \right) \\ &\quad + \left(\frac{n_{2h}}{n_h} \right) \left(\frac{1}{n_{2h}} \sum_{i=1}^{n_{2h}} Y_{hi} \right) + \left(\frac{n_{2h}}{n_h} \right) \left(\frac{1}{n_{2h}} \sum_{i=1}^{n_{2h}} U_{hi} \right) \end{aligned} \quad (3.197)$$

or

$$E_2(\bar{y}_h''') = \frac{1}{n_h} \sum_{i=1}^{n_h} Y_{hi} + \frac{1}{n_h} \sum_{i=1}^{n_h} U_{hi} \quad (3.198)$$

Substituting (3.198) in (3.195) gives

$$E_c(\bar{y}_h''') = E_1 E_2(\bar{y}_h''') = E_1 \left[\frac{1}{n_h} \sum_{i=1}^{n_h} Y_{hi} + \frac{1}{n_h} \sum_{i=1}^{n_h} U_{hi} \right] = \frac{1}{N_h} \sum_{i=1}^{N_h} Y_{hi} + \frac{1}{N_h} \sum_{i=1}^{N_h} U_{hi} = \bar{Y}_h \quad (3.199)$$

since the study variable measurement error, U_{hi} , has zero population mean in stratum h .

Substituting (3.199) in (3.193) gives the conditional expectation of the proposed estimator, $\bar{y}_{js1}$,

as

$$E_c(\bar{y}_{js1}) = \sum_{h=1}^L W_h E_c(\bar{y}_h''') = \sum_{h=1}^L W_h \bar{Y}_h = \bar{Y} \quad (3.200)$$

Hence, the proposed estimator, $\bar{y}_{js1}$, is unbiased for the population mean, $\bar{Y}$, under the conditional argument. Using (3.186), we can write

$$\begin{aligned} V_2(\bar{y}_h''') &= V_2 \left[\left(\frac{n_{1h}}{n_h} \right) \left(\frac{1}{n_{1h}} \sum_{i=1}^{n_{1h}} Y_{hi} \right) + \left(\frac{n_{1h}}{n_h} \right) \left(\frac{1}{n_{1h}} \sum_{i=1}^{n_{1h}} U_{hi} \right) \right. \\ &\quad \left. + \left(\frac{n_{2h}}{n_h} \right) \left(\frac{1}{m_h} \sum_{i=1}^{m_h} Y_{hi} \right) + \left(\frac{n_{2h}}{n_h} \right) \left(\frac{1}{m_h} \sum_{i=1}^{m_h} U_{hi} \right) \right] \\ &= V_2 \left[\left(\frac{n_{2h}}{n_h} \right) \left(\frac{1}{m_h} \sum_{i=1}^{m_h} Y_{hi} \right) + \left(\frac{n_{2h}}{n_h} \right) \left(\frac{1}{m_h} \sum_{i=1}^{m_h} U_{hi} \right) \right] \\ &= \left(\frac{n_{2h}}{n_h} \right)^2 V_2 \left(\frac{1}{m_h} \sum_{i=1}^{m_h} Y_{hi} \right) + \left(\frac{n_{2h}}{n_h} \right)^2 V_2 \left(\frac{1}{m_h} \sum_{i=1}^{m_h} U_{hi} \right) \\ &\quad + 2 \left(\frac{n_{2h}}{n_h} \right)^2 C_2 \left[\left(\frac{1}{m_h} \sum_{i=1}^{m_h} Y_{hi} \right), \left(\frac{1}{m_h} \sum_{i=1}^{m_h} U_{hi} \right) \right] \\ &= \left(\frac{n_{2h}}{n_h} \right)^2 \left(\frac{1}{m_h} - \frac{1}{n_{2h}} \right) s_{2yh}^2 + \left(\frac{n_{2h}}{n_h} \right)^2 \left(\frac{1}{m_h} - \frac{1}{n_{2h}} \right) s_{2uh}^2 + 2 \left(\frac{n_{2h}}{n_h} \right)^2 \left(\frac{1}{m_h} - \frac{1}{n_{2h}} \right) s_{2yuh} \end{aligned} \quad (3.201)$$

or

$$V_2(\bar{y}_h''') = \left(\frac{n_{2h}}{n_h} \right) \left(\frac{g_h - 1}{n_h} \right) (s_{2yh}^2 + s_{2uh}^2 + 2s_{2yuh}) \quad (3.202)$$

so that

$$\begin{aligned} E_1 V_2(\bar{y}_h''') &= E_1 \left[\left(\frac{n_{2h}}{n_h} \right) \left(\frac{g_h - 1}{n_h} \right) (s_{2yh}^2 + s_{2uh}^2 + 2s_{2yuh}) \right] \\ &= \left(\frac{g_h - 1}{n_h} \right) E_1 \left(\frac{n_{2h}}{n_h} \right) E_1 (s_{2yh}^2 + s_{2uh}^2 + 2s_{2yuh}) \\ &= \left(\frac{g_h - 1}{n_h} \right) \left(\frac{N_{2h}}{N_h} \right) (S_{2yh}^2 + S_{2uh}^2 + 2S_{2yuh}) = \left(\frac{g_h - 1}{n_h} \right) W_{2h} (S_{2yh}^2 + S_{2uh}^2) \end{aligned} \quad (3.203)$$

since the study variable, y , is uncorrelated with measurement errors in stratum h . Using (3.198), we have

$$\begin{aligned} V_1 E_2(\bar{y}_h''') &= V_1 \left[\frac{1}{n_h} \sum_{i=1}^{n_h} Y_{hi} + \frac{1}{n_h} \sum_{i=1}^{n_h} U_{hi} \right] \\ &= V_1 \left(\frac{1}{n_h} \sum_{i=1}^{n_h} Y_{hi} \right) + V_1 \left(\frac{1}{n_h} \sum_{i=1}^{n_h} U_{hi} \right) + 2C_1 \left[\left(\frac{1}{n_h} \sum_{i=1}^{n_h} Y_{hi} \right), \left(\frac{1}{n_h} \sum_{i=1}^{n_h} U_{hi} \right) \right] \end{aligned} \quad (3.204)$$

or

$$V_1 E_2(\bar{y}_h''') = \left(\frac{1}{n_h} - \frac{1}{N_h}\right)(S_{yh}^2 + S_{uh}^2) \quad (3.205)$$

Substituting (3.203) and (3.205) in (3.196)

$$\begin{aligned} V_c(\bar{y}_h''') &= V_1 E_2(\bar{y}_h''') + E_1 V_2(\bar{y}_h''') \\ &= \left(\frac{1}{n_h} - \frac{1}{N_h}\right)(S_{yh}^2 + S_{uh}^2) + \left(\frac{g_h - 1}{n_h}\right)W_{2h}(S_{2yh}^2 + S_{2uh}^2) \end{aligned} \quad (3.206)$$

Substituting (3.206) in (3.194) gives the conditional variance of the proposed customary sample mean estimator, $\bar{y}_{js1}$, as

$$V_c(\bar{y}_{js1}) = \sum_{h=1}^L W_h^2 \left[\left(\frac{1}{n_h} - \frac{1}{N_h}\right)(S_{yh}^2 + S_{uh}^2) + \left(\frac{g_h - 1}{n_h}\right)W_{2h}(S_{2yh}^2 + S_{2uh}^2) \right] \quad (3.207)$$

or

$$\begin{aligned} V_c(\bar{y}_{js1}) &= \sum_{h=1}^L W_h^2 \left[\left(\frac{1}{n_h} - \frac{1}{N_h}\right)S_{yh}^2 + \left(\frac{g_h - 1}{n_h}\right)W_{2h}S_{2yh}^2 \right] \\ &\quad + \sum_{h=1}^L W_h^2 \left[\left(\frac{1}{n_h} - \frac{1}{N_h}\right)S_{uh}^2 + \left(\frac{g_h - 1}{n_h}\right)W_{2h}S_{2uh}^2 \right] \end{aligned} \quad (3.208)$$

as stated in the theorem. This proves the first part of the theorem on the conditional properties of the proposed customary sample mean estimator, $\bar{y}_{js1}$. To obtain the unconditional variance of the proposed estimator, $\bar{y}_{js1}$, we take the unconditional expectation of (3.208) and use (3.25) to make the necessary substitutions. And this completes the proof.

Estimator 20: Difference-type estimator, $\bar{y}_{jd1}$, when there is joint occurrence of nonresponse and measurement errors on both the study and auxiliary variables under the Poststratified Sampling Scheme.

The proposed difference-type estimator, $\bar{y}_{jd1}$, given in (3.189), utilizes auxiliary information with known population mean ($\bar{X}_h$) of the auxiliary variable, x , in the h^{th} stratum. We give the conditional and unconditional properties of the estimator, $\bar{y}_{jd1}$, in theorem 3.17.

Theorem 3.17: Under the conditional argument or for an achieved sample configuration, say $\underline{n} = (n_1, n_2, \dots, n_L)$, the proposed difference-type estimator, $\bar{y}_{jd1}$, is an unbiased estimator of the population mean ($\bar{Y}$), of the study variable, y, with its conditional variance obtained as:

$$\begin{aligned}
 V_c(\bar{y}_{jd1}) = & \sum_{h=1}^L W_h^2 \left(\frac{1}{n_h} - \frac{1}{N_h} \right) (S_{yh}^2 + k_h^2 S_{xh}^2 - 2k_h S_{yxh}) \\
 & + \sum_{h=1}^L W_h^2 \left(\frac{g_h - 1}{n_h} \right) W_{2h} (S_{2yh}^2 + k_h^2 S_{2xh}^2 - 2k_h S_{2yxh}) \\
 & + \sum_{h=1}^L W_h^2 \left(\frac{1}{n_h} - \frac{1}{N_h} \right) (S_{uh}^2 + k_h^2 S_{vh}^2) \\
 & + \sum_{h=1}^L W_h^2 \left(\frac{g_h - 1}{n_h} \right) W_{2h} (S_{2uh}^2 + k_h^2 S_{2vh}^2)
 \end{aligned} \tag{3.209}$$

Furthermore, under the unconditional argument or for repeated samples of fixed size, n, the difference-type estimator, $\bar{y}_{jd1}$, is an unbiased estimator of the population mean ($\bar{Y}$) of the study variable y, with its unconditional variance obtained up to first order of approximations as:

$$\begin{aligned}
 V(\bar{y}_{jd1}) = & \left(\frac{1-f}{n} \right) \sum_{h=1}^L W_h (S_{yh}^2 + k_h^2 S_{xh}^2 - 2k_h S_{yxh}) \\
 & + \left(\frac{1}{n} \right) \sum_{h=1}^L W_h (g_h - 1) W_{2h} (S_{2yh}^2 + k_h^2 S_{2xh}^2 - 2k_h S_{2yxh}) \\
 & + \left(\frac{1-f}{n} \right) \sum_{h=1}^L W_h (S_{uh}^2 + k_h^2 S_{vh}^2) \\
 & + \left(\frac{1}{n} \right) \sum_{h=1}^L W_h (g_h - 1) W_{2h} (S_{2uh}^2 + k_h^2 S_{2vh}^2)
 \end{aligned} \tag{3.210}$$

Proof: The proposed difference-type estimator, $\bar{y}_{jd1}$, given in (3.189), can be written as:

$$\bar{y}_{jd1} = \sum_{h=1}^L W_h [\bar{y}_h''' - k_h (\bar{x}_h''' - \bar{X}_h)] = \sum_{h=1}^L W_h \bar{y}_{jd1h} \tag{3.211}$$

where

$$\bar{y}_{jd1h} = \bar{y}_h''' - k_h (\bar{x}_h''' - \bar{X}_h) \tag{3.212}$$

with its conditional expectation given by

$$E_c(\bar{y}_{jd1h}) = E_c(\bar{y}_h''') - k_h[E_c(\bar{x}_h''') - \bar{X}_h] \quad (3.213)$$

It follows from (3.199) that

$$E_c(\bar{x}_h''') = \bar{X}_h \quad (3.214)$$

so that substituting (3.199) and (3.214) in (3.213) gives the conditional expectation of $\bar{y}_{jd1h}$ as

$$E_c(\bar{y}_{jd1h}) = \bar{Y}_h - k_h(\bar{X}_h - \bar{X}_h) = \bar{Y}_h \quad (3.215)$$

Using (3.211), the conditional expectation and variance of the proposed difference-type estimator, $\bar{y}_{jd1}$, can be respectively written as

$$E_c(\bar{y}_{jd1}) = \sum_{h=1}^L W_h E_c(\bar{y}_{jd1h}) \quad (3.216)$$

and

$$V_c(\bar{y}_{jd1}) = \sum_{h=1}^L W_h^2 V_c(\bar{y}_{jd1h}) \quad (3.217)$$

Consequently, substituting (3.215) in (3.216) gives the conditional expectation of the proposed difference-type estimator, $\bar{y}_{jd1}$, as

$$E_c(\bar{y}_{jd1}) = \sum_{h=1}^L W_h E_c(\bar{y}_{jd1h}) = \sum_{h=1}^L W_h \bar{Y}_h = \bar{Y} \quad (3.218)$$

Hence, the proposed difference-type estimator ($\bar{y}_{jd1}$), when there is joint occurrence of non-response and measurement errors on both the study and auxiliary variables, is unbiased for $\bar{Y}$ under the conditional argument. To obtain the associated conditional variance of $\bar{y}_{jd1}$, we note from (3.212) that the conditional variance of the stratum mean estimator, $\bar{y}_{jd1h}$, is given by:

$$V_c(\bar{y}_{jd1h}) = V_c[\bar{y}_h''' - k_h(\bar{x}_h''' - \bar{X}_h)] = V_c(\bar{y}_h''') + k_h^2 V_c(\bar{x}_h''') - 2C_c(\bar{y}_h''', \bar{x}_h''') \quad (3.219)$$

It follows from (3.206) that

$$V_c(\bar{x}_h''') = \left(\frac{1}{n_h} - \frac{1}{N_h}\right)(S_{xh}^2 + S_{vh}^2) + \left(\frac{g_h - 1}{n_h}\right)W_{2h}(S_{2xh}^2 + S_{2vh}^2) \quad (3.220)$$

To obtain the conditional covariance, $C_c(\bar{y}_h''', \bar{x}_h''')$, we write

$$C_c(\bar{y}_h''', \bar{x}_h''') = C_1[E_2(\bar{y}_h'''), E_2(\bar{x}_h''')] + E_1 C_2(\bar{y}_h''', \bar{x}_h''') \quad (3.221)$$

From (3.198), it follows that

$$E_2(\bar{x}_h''') = \frac{1}{n_h} \sum_{i=1}^{n_h} X_{hi} + \frac{1}{n_h} \sum_{i=1}^{n_h} V_{hi} \quad (3.222)$$

so that using (3.198) and (3.222) and noting that both the study and auxiliary variables are uncorrelated with the measurement errors, we obtain

$$\begin{aligned} C_1[E_2(\bar{y}_h'''), E_2(\bar{x}_h''')] &= C_1\left[\frac{1}{n_h} \sum_{i=1}^{n_h} Y_{hi} + \frac{1}{n_h} \sum_{i=1}^{n_h} U_{hi}, \frac{1}{n_h} \sum_{i=1}^{n_h} X_{hi} + \frac{1}{n_h} \sum_{i=1}^{n_h} V_{hi}\right] \\ &= C_1\left(\frac{1}{n_h} \sum_{i=1}^{n_h} Y_{hi}, \frac{1}{n_h} \sum_{i=1}^{n_h} X_{hi}\right) + C_1\left(\frac{1}{n_h} \sum_{i=1}^{n_h} Y_{hi}, \frac{1}{n_h} \sum_{i=1}^{n_h} V_{hi}\right) \\ &\quad + C_1\left(\frac{1}{n_h} \sum_{i=1}^{n_h} U_{hi}, \frac{1}{n_h} \sum_{i=1}^{n_h} X_{hi}\right) + C_1\left(\frac{1}{n_h} \sum_{i=1}^{n_h} U_{hi}, \frac{1}{n_h} \sum_{i=1}^{n_h} V_{hi}\right) \quad (3.223) \\ &= C_1\left(\frac{1}{n_h} \sum_{i=1}^{n_h} Y_{hi}, \frac{1}{n_h} \sum_{i=1}^{n_h} X_{hi}\right) \\ &= \left(\frac{1}{n_h} - \frac{1}{N_h}\right)S_{yxh} \end{aligned}$$

Furthermore, from (3.186) and (3.187), and noting that the subsample covariance (C_2) of samples based on the initial response sample size, n_{1h} , vanishes in stratum h, we have

$$\begin{aligned} C_2(\bar{y}_h''', \bar{x}_h''') &= C_2\left[\left(\frac{n_{2h}}{n_h}\right)\left(\frac{1}{m_h} \sum_{i=1}^{m_h} Y_{hi}\right) + \left(\frac{n_{2h}}{n_h}\right)\left(\frac{1}{m_h} \sum_{i=1}^{m_h} U_{hi}\right), \left(\frac{n_{2h}}{n_h}\right)\left(\frac{1}{m_h} \sum_{i=1}^{m_h} X_{hi}\right) + \left(\frac{n_{2h}}{n_h}\right)\left(\frac{1}{m_h} \sum_{i=1}^{m_h} V_{hi}\right)\right] \\ &= \left(\frac{n_{2h}}{n_h}\right)^2 C_2\left(\frac{1}{m_h} \sum_{i=1}^{m_h} Y_{hi}, \frac{1}{m_h} \sum_{i=1}^{m_h} X_{hi}\right) = \left(\frac{n_{2h}}{n_h}\right)^2 \left(\frac{1}{m_h} - \frac{1}{n_{2h}}\right) s_{2yxh} = \left(\frac{n_{2h}}{n_h}\right) \left(\frac{g_h - 1}{n_h}\right) s_{2yxh} \quad (3.224) \end{aligned}$$

so that

$$E_1 C_2(\bar{y}_h''', \bar{x}_h''') = \left(\frac{g_h - 1}{n_h}\right) E_1 \left(\frac{n_{2h}}{n_h}\right) E_1(s_{2yxh}) = \left(\frac{g_h - 1}{n_h}\right) W_{2h} S_{2yxh} \quad (3.225)$$

Thus, substituting (3.223) and (3.225) in (3.221) gives

$$C_c(\bar{y}_h''', \bar{x}_h''') = \left(\frac{1}{n_h} - \frac{1}{N_h}\right) S_{yxh} + \left(\frac{g_h - 1}{n_h}\right) W_{2h} S_{2yxh} \quad (3.226)$$

Now, substituting (3.206), (3.220) and (3.226) in (3.219) gives

$$\begin{aligned} V_c(\bar{y}_{jd1h}) &= \left(\frac{1}{n_h} - \frac{1}{N_h}\right) (S_{yh}^2 + k_h^2 S_{xh}^2 - 2k_h S_{yxh}) \\ &\quad + \left(\frac{g_h - 1}{n_h}\right) W_{2h} (S_{2yh}^2 + k_h^2 S_{2xh}^2 - 2k_h S_{2yxh}) \\ &\quad + \left(\frac{1}{n_h} - \frac{1}{N_h}\right) (S_{uh}^2 + k_h^2 S_{vh}^2) + \left(\frac{g_h - 1}{n_h}\right) W_{2h} (S_{2uh}^2 + k_h^2 S_{2vh}^2) \end{aligned} \quad (3.227)$$

And substituting (3.227) in (3.217) gives the conditional variance of the proposed difference-type estimator ($\bar{y}_{jd1}$), when there is joint occurrence of nonresponse and measurement errors on both the study and auxiliary variables, as

$$\begin{aligned} V_c(\bar{y}_{jd1}) &= \sum_{h=1}^L W_h^2 \left[\left(\frac{1}{n_h} - \frac{1}{N_h}\right) (S_{yh}^2 + k_h^2 S_{xh}^2 - 2k_h S_{yxh}) \right. \\ &\quad + \left(\frac{g_h - 1}{n_h}\right) W_{2h} (S_{2yh}^2 + k_h^2 S_{2xh}^2 - 2k_h S_{2yxh}) \\ &\quad \left. + \left(\frac{1}{n_h} - \frac{1}{N_h}\right) (S_{uh}^2 + k_h^2 S_{vh}^2) + \left(\frac{g_h - 1}{n_h}\right) W_{2h} (S_{2uh}^2 + k_h^2 S_{2vh}^2) \right] \end{aligned} \quad (3.228)$$

or

$$\begin{aligned} V_c(\bar{y}_{jd1}) &= \sum_{h=1}^L W_h^2 \left(\frac{1}{n_h} - \frac{1}{N_h} \right) (S_{yh}^2 + k_h^2 S_{xh}^2 - 2k_h S_{yxh}) \\ &\quad + \sum_{h=1}^L W_h^2 \left(\frac{g_h - 1}{n_h} \right) W_{2h} (S_{2yh}^2 + k_h^2 S_{2xh}^2 - 2k_h S_{2yxh}) \\ &\quad + \sum_{h=1}^L W_h^2 \left(\frac{1}{n_h} - \frac{1}{N_h} \right) (S_{uh}^2 + k_h^2 S_{vh}^2) \\ &\quad + \sum_{h=1}^L W_h^2 \left(\frac{g_h - 1}{n_h} \right) W_{2h} (S_{2uh}^2 + k_h^2 S_{2vh}^2) \end{aligned} \quad (3.229)$$

as stated in the theorem, and this proves the results on the conditional variance of the proposed difference-type estimator, $\bar{y}_{jd1}$. To obtain the unconditional variance of the proposed estimator, $\bar{y}_{jd1}$, we take the unconditional expectation of (3.229) and use (3.25) to make the necessary substitutions. This completes the proof of theorem 3.17.

Estimator 21: Class of estimators, $\bar{y}_{jc1}$, when there is joint occurrence of nonresponse and measurement errors on both the study and auxiliary variables under the Poststratified Sampling Scheme.

The proposed class of estimators, $\bar{y}_{jc1}$, given in (3.190), utilizes auxiliary information with known values of some population parameters of the auxiliary variable, x , in the h^{th} stratum. We give the conditional and unconditional properties of the estimator, $\bar{y}_{jc1}$, in theorem 3.18.

Theorem 3.18: Under the conditional argument or for an achieved sample configuration, say $\underline{n} = (n_1, n_2, \dots, n_L)$, the proposed class of estimators, $\bar{y}_{jc1}$, is biased for the population mean ($\bar{Y}$) of the study variable, y , with its conditional bias and mean square error obtained up to first order of approximations as:

$$\begin{aligned} B_c(\bar{y}_{jc1}) &= \sum_{h=1}^L \frac{W_h \alpha_h \lambda_h}{2\bar{X}_h} \left(\frac{1}{n_h} - \frac{1}{N_h} \right) [(\alpha_h + 1) \lambda_h R_h S_{xh}^2 - 2S_{yxh}] \\ &\quad + \sum_{h=1}^L \frac{W_h \alpha_h \lambda_h}{2\bar{X}_h} \left(\frac{g_h - 1}{n_h} \right) W_{2h} [(\alpha_h + 1) \lambda_h R_h S_{2xh}^2 - 2S_{2yxh}] \\ &\quad + \sum_{h=1}^L \frac{W_h \alpha_h \lambda_h}{2\bar{X}_h} \left[\left(\frac{1}{n_h} - \frac{1}{N_h} \right) (\alpha_h + 1) \lambda_h R_h S_{vh}^2 + \left(\frac{g_h - 1}{n_h} \right) W_{2h} (\alpha_h + 1) \lambda_h R_h S_{2vh}^2 \right] \end{aligned} \quad (3.230)$$

$$\begin{aligned} MS E_c(\bar{y}_{jc1}) &= \sum_{h=1}^L W_h^2 \left(\frac{1}{n_h} - \frac{1}{N_h} \right) (S_{yh}^2 + \alpha_h^2 \lambda_h^2 R_h^2 S_{xh}^2 - 2\alpha_h \lambda_h R_h S_{yxh}) \\ &\quad + \sum_{h=1}^L W_h^2 \left(\frac{g_h - 1}{n_h} \right) W_{2h} (S_{2yh}^2 + \alpha_h^2 \lambda_h^2 R_h^2 S_{2xh}^2 - 2\alpha_h \lambda_h R_h S_{2yxh}) \\ &\quad + \sum_{h=1}^L W_h^2 \left[\left(\frac{1}{n_h} - \frac{1}{N_h} \right) (S_{uh}^2 + \alpha_h^2 \lambda_h^2 R_h^2 S_{vh}^2) + \left(\frac{g_h - 1}{n_h} \right) W_{2h} (S_{2uh}^2 + \alpha_h^2 \lambda_h^2 R_h^2 S_{2vh}^2) \right] \end{aligned} \quad (3.231)$$

Furthermore, under the unconditional argument or for repeated samples of fixed size, n , the proposed class of estimators, $\bar{y}_{jc1}$, is biased for the population mean ($\bar{Y}$) of the study variable y , with its unconditional bias and mean square error obtained up to first order of approximations as:

$$\begin{aligned} B(\bar{y}_{jc1}) &= \left(\frac{1-f}{n}\right) \sum_{h=1}^L \frac{\alpha_h \lambda_h}{2\bar{X}_h} [(\alpha_h + 1)\lambda_h R_h S_{xh}^2 - 2S_{yxh}] \\ &+ \left(\frac{1}{n}\right) \sum_{h=1}^L \frac{\alpha_h \lambda_h}{2\bar{X}_h} (g_h - 1) W_{2h} [(\alpha_h + 1)\lambda_h R_h S_{2xh}^2 - 2S_{2yxh}] \\ &+ \sum_{h=1}^L \frac{\alpha_h (\alpha_h + 1) \lambda_h^2 R_h}{2\bar{X}_h} \left[\left(\frac{1-f}{n}\right) S_{vh}^2 + \left(\frac{g_h - 1}{n}\right) W_{2h} S_{2vh}^2 \right] \end{aligned} \quad (3.232)$$

and

$$\begin{aligned} MSE(\bar{y}_{jc1}) &= \left(\frac{1-f}{n}\right) \sum_{h=1}^L W_h (S_{yh}^2 + \alpha_h^2 \lambda_h^2 R_h^2 S_{xh}^2 - 2\alpha_h \lambda_h R_h S_{yxh}) \\ &+ \left(\frac{1}{n}\right) \sum_{h=1}^L W_h (g_h - 1) W_{2h} (S_{2yh}^2 + \alpha_h^2 \lambda_h^2 R_h^2 S_{2xh}^2 - 2\alpha_h \lambda_h R_h S_{2yxh}) \\ &+ \sum_{h=1}^L W_h \left[\left(\frac{1-f}{n}\right) (S_{uh}^2 + \alpha_h^2 \lambda_h^2 R_h^2 S_{vh}^2) + \left(\frac{g_h - 1}{n}\right) W_{2h} (S_{2uh}^2 + \alpha_h^2 \lambda_h^2 R_h^2 S_{2vh}^2) \right] \end{aligned} \quad (3.233)$$

where

$$\lambda_h = \left(\frac{a_h \bar{X}_h}{a_h \bar{X}_h + b_h} \right) \quad (3.234)$$

Proof: The proposed class of estimators, $\bar{y}_{jc1}$, given in (3.190), can be written as:

$$\bar{y}_{jc1} = \sum_{h=1}^L W_h \bar{y}_h''' \left(\frac{a_h \bar{X}_h + b_h}{a_h \bar{X}_h''' + b_h} \right)^{\alpha_h} = \sum_{h=1}^L W_h \bar{y}_{jc1h} \quad (3.235)$$

where

$$\bar{y}_{jc1h} = \bar{y}_h''' \left(\frac{a_h \bar{X}_h + b_h}{a_h \bar{X}_h''' + b_h} \right)^{\alpha_h} \quad (3.236)$$

From (3.235), the conditional bias and mean square error of the proposed class of estimators, $\bar{y}_{jc1}$, are respectively given by

$$B_c(\bar{y}_{jc1}) = \sum_{h=1}^L W_h B_c(\bar{y}_{jc1h}) \quad (3.237)$$

and

$$MS E_c(\bar{y}_{jc1}) = \sum_{h=1}^L W_h^2 MS E_c(\bar{y}_{jc1h}) \quad (3.238)$$

To obtain the properties of the stratum sample mean estimator, $\bar{y}_{jc1h}$, consider the adjusted mean deviations of the stratum mean estimators, $\bar{y}_h'''$ and $\bar{x}_h'''$, respectively given by

$$d_{5h} = \frac{\bar{y}_h''' - \bar{Y}_h}{\bar{Y}_h} \quad \text{or} \quad \bar{y}_h''' = \bar{Y}_h(1 + d_{5h}) \quad (3.239)$$

and

$$e_{5h} = \frac{\bar{x}_h''' - \bar{X}_h}{\bar{X}_h} \quad \text{or} \quad \bar{x}_h''' = \bar{X}_h(1 + e_{5h}) \quad (3.240)$$

Under the conditional argument, we have

$$\begin{aligned} E_c(d_{5h}) &= E_c(e_{5h}) = 0, \\ E_c(d_{5h}^2) &= \frac{V_c(\bar{y}_h''')}{\bar{Y}_h^2}, \quad E_c(e_{5h}^2) = \frac{V_c(\bar{x}_h''')}{\bar{X}_h^2}, \quad E_c(d_{5h}e_{5h}) = \frac{C_c(\bar{y}_h''', \bar{x}_h''')}{\bar{Y}_h\bar{X}_h} \end{aligned} \quad (3.241)$$

Using (3.239) and (3.240), we can rewrite (3.236) as

$$\bar{y}_{jc1h} = \bar{Y}_h(1 + d_{5h})(1 + \lambda_h e_{5h})^{-\alpha_h} \quad (3.242)$$

so that using Taylor's series to expand (3.242) up to first order of approximations in expected values, it follows that

$$(\bar{y}_{jc1h} - \bar{Y}_h) = \bar{Y}_h \left[d_{5h} - \alpha_h \lambda_h e_{5h} - \alpha_h \lambda_h d_{5h} e_{5h} + (1/2) \alpha_h (\alpha_h + 1) \lambda_h^2 e_{5h}^2 \right] \quad (3.243)$$

and

$$(\bar{y}_{jc1h} - \bar{Y}_h)^2 = \bar{Y}_h^2 \left[d_{5h}^2 + \alpha_h^2 \lambda_h^2 e_{5h}^2 - 2\alpha_h \lambda_h d_{5h} e_{5h} \right] \quad (3.244)$$

Taking the conditional expectation of (3.243) gives the conditional bias of $\bar{y}_{jc1h}$, as

$$\begin{aligned} B_c(\bar{y}_{jc1h}) &= E_c(\bar{y}_{jc1h} - \bar{Y}_h) \\ &= \bar{Y}_h \left[E_c(d_{5h}) - \alpha_h \lambda_h E_c(e_{5h}) - \alpha_h \lambda_h E_c(d_{5h} e_{5h}) + (1/2) \alpha_h (\alpha_h + 1) \lambda_h^2 E_c(e_{5h}^2) \right] \end{aligned} \quad (3.245)$$

and using (3.241), (3.220) and (3.226) to make the necessary substitutions in (3.245) gives

$$\begin{aligned}
B_c(\bar{y}_{jc1h}) &= \bar{Y}_h \left[-\alpha_h \lambda_h \frac{C_c(\bar{y}_h''', \bar{x}_h''')}{\bar{Y}_h \bar{X}_h} + (1/2) \alpha_h (\alpha_h + 1) \lambda_h^2 \frac{V_c(\bar{x}_h''')}{\bar{X}_h^2} \right] \\
&= \frac{\alpha_h \lambda_h}{2 \bar{X}_h} \left(\frac{1}{n_h} - \frac{1}{N_h} \right) [(\alpha_h + 1) \lambda_h R_h S_{xh}^2 - 2S_{yxh}] \\
&\quad + \frac{\alpha_h \lambda_h}{2 \bar{X}_h} \left(\frac{g_h - 1}{n_h} \right) W_{2h} [(\alpha_h + 1) \lambda_h R_h S_{2xh}^2 - 2S_{2yxh}] \\
&\quad + \frac{\alpha_h \lambda_h}{2 \bar{X}_h} \left[\left(\frac{1}{n_h} - \frac{1}{N_h} \right) (\alpha_h + 1) \lambda_h R_h S_{vh}^2 + \left(\frac{g_h - 1}{n_h} \right) W_{2h} (\alpha_h + 1) \lambda_h R_h S_{2vh}^2 \right]
\end{aligned} \tag{3.246}$$

Substituting (3.246) in (3.237) gives the conditional bias of the proposed class of estimators,

$\bar{y}_{jc1}$, as

$$\begin{aligned}
B_c(\bar{y}_{jc1}) &= \sum_{h=1}^L \frac{W_h \alpha_h \lambda_h}{2 \bar{X}_h} \left(\frac{1}{n_h} - \frac{1}{N_h} \right) [(\alpha_h + 1) \lambda_h R_h S_{xh}^2 - 2S_{yxh}] \\
&\quad + \sum_{h=1}^L \frac{W_h \alpha_h \lambda_h}{2 \bar{X}_h} \left(\frac{g_h - 1}{n_h} \right) W_{2h} [(\alpha_h + 1) \lambda_h R_h S_{2xh}^2 - 2S_{2yxh}] \\
&\quad + \sum_{h=1}^L \frac{W_h \alpha_h \lambda_h}{2 \bar{X}_h} \left[\left(\frac{1}{n_h} - \frac{1}{N_h} \right) (\alpha_h + 1) \lambda_h R_h S_{vh}^2 + \left(\frac{g_h - 1}{n_h} \right) W_{2h} (\alpha_h + 1) \lambda_h R_h S_{2vh}^2 \right]
\end{aligned} \tag{3.247}$$

as stated in the theorem. To obtain the conditional mean square error of the proposed class of estimators, $\bar{y}_{jc1}$, we first take the conditional expectation of (3.244), and use (3.241), (3.206), (3.220) and (3.226) to make the necessary substitutions, which gives:

$$\begin{aligned}
MSE_c(\bar{y}_{jc1h}) &= E_c(\bar{y}_{jc1h} - \bar{Y}_h)^2 \\
&= \bar{Y}_h^2 \left[E_c(d_{5h}^2) + \alpha_h^2 \lambda_h^2 E_c(e_{5h}^2) - 2\alpha_h \lambda_h E_c(d_{4h} e_{5h}) \right] \\
&= \bar{Y}_h^2 \left[\frac{V_c(\bar{y}_h''')}{\bar{Y}_h^2} + \alpha_h^2 \lambda_h^2 \frac{V_c(\bar{x}_h''')}{\bar{X}_h^2} - 2\alpha_h \lambda_h \frac{C_c(\bar{y}_h''', \bar{x}_h''')}{\bar{Y}_h \bar{X}_h} \right] \\
&= \left(\frac{1}{n_h} - \frac{1}{N_h} \right) (S_{yh}^2 + \alpha_h^2 \lambda_h^2 R_h^2 S_{xh}^2 - 2\alpha_h \lambda_h R_h S_{yxh}) \\
&\quad + \left(\frac{g_h - 1}{n_h} \right) W_{2h} (S_{2yh}^2 + \alpha_h^2 \lambda_h^2 R_h^2 S_{2xh}^2 - 2\alpha_h \lambda_h R_h S_{2yxh}) \\
&\quad + \left(\frac{1}{n_h} - \frac{1}{N_h} \right) (S_{uh}^2 + \alpha_h^2 \lambda_h^2 R_h^2 S_{vh}^2) + \left(\frac{g_h - 1}{n_h} \right) W_{2h} (S_{2uh}^2 + \alpha_h^2 \lambda_h^2 R_h^2 S_{2vh}^2)
\end{aligned} \tag{3.248}$$

Then substituting (3.248) in (3.238) gives the conditional mean square error of the proposed class of estimators, $\bar{y}_{jc1}$, as

$$\begin{aligned}
MS E_c(\bar{y}_{jc1}) = & \sum_{h=1}^L W_h^2 \left(\frac{1}{n_h} - \frac{1}{N_h} \right) (S_{yh}^2 + \alpha_h^2 \lambda_h^2 R_h^2 S_{xh}^2 - 2\alpha_h \lambda_h R_h S_{yxh}) \\
& + \sum_{h=1}^L W_h^2 \left(\frac{g_h - 1}{n_h} \right) W_{2h} (S_{2yh}^2 + \alpha_h^2 \lambda_h^2 R_h^2 S_{2xh}^2 - 2\alpha_h \lambda_h R_h S_{2yxh}) \\
& + \sum_{h=1}^L W_h^2 \left[\left(\frac{1}{n_h} - \frac{1}{N_h} \right) (S_{uh}^2 + \alpha_h^2 \lambda_h^2 R_h^2 S_{vh}^2) + \left(\frac{g_h - 1}{n_h} \right) W_{2h} (S_{2uh}^2 + \alpha_h^2 \lambda_h^2 R_h^2 S_{2vh}^2) \right]
\end{aligned} \tag{3.249}$$

as stated in the theorem, and this proves the first part of the theorem on the conditional bias and mean square error of the proposed class of estimators, $\bar{y}_{jc1}$. To obtain the unconditional bias and mean square error of the proposed class of estimators, $\bar{y}_{jc1}$, we take the unconditional expectations of (3.247) and (3.249), and use (3.25) to make the necessary substitutions. This completes the proof.

3.3.2 Joint Occurrence of Nonresponse and Measurement Errors on the study variable and not on the auxiliary variable under the Post-stratified Sampling Scheme

The focus here is to develop sampling strategies for situations where there is joint occurrence of nonresponse and measurement errors only on the study variable and not on the auxiliary variable in poststratified sampling scheme. We propose the following sampling design.

Sampling Design 3.3.2: The Sampling Design 3.3.2 proposed here for handling the joint occurrence of nonresponse and measurement errors on the study variable and not on the auxiliary variable in poststratified sampling scheme is similar to Sampling Design 3.3.1. The only difference is that in Sampling Design 3.3.1 we assumed the joint occurrence of nonresponse and measurement errors on both the study and auxiliary variables, while in the present Sampling Design 3.3.2, we assume joint occurrence of nonresponse and measurement errors on only the study variable and not on the auxiliary variable. Just like in the case of Sampling Design 3.3.1, there are measurement errors on the observed values of the study variable, y , from the response

sample group of n_{1h} units in stratum h , supplemented with the observed values of y from the nonresponse subsample group of m_h units in stratum h . Consequently, from (3.186), we give a sample mean of the study variable, y , in the h^{th} stratum when there is joint occurrence of non-response and measurement errors on y , as a linear function of the response and nonresponse subsample means, given by:

$$\bar{y}_h''' = \left(\frac{n_{1h}}{n_h}\right)\left(\frac{1}{n_{1h}} \sum_{i=1}^{n_{1h}} Y_{hi}\right) + \left(\frac{n_{1h}}{n_h}\right)\left(\frac{1}{n_{1h}} \sum_{i=1}^{n_{1h}} U_{hi}\right) + \left(\frac{n_{2h}}{n_h}\right)\left(\frac{1}{m_h} \sum_{i=1}^{m_h} Y_{hi}\right) + \left(\frac{n_{2h}}{n_h}\right)\left(\frac{1}{m_h} \sum_{i=1}^{m_h} U_{hi}\right) \quad (3.250)$$

Sampling Design 3.3.2 indicates that there is complete response and no measurement errors on the auxiliary variable, x . Hence, we give the sample mean of the auxiliary variable, x , when there is no nonresponse and measurement errors on x , as:

$$\bar{x}_h' = \frac{1}{n_h} \sum_{i=1}^{n_h} X_{hi} \quad (3.251)$$

Based on the Sampling Design 3.3.2 and using the sample means (3.250) and (3.251), we propose three estimators of the population mean ($\bar{Y}$) of the study variable, y , when there is joint occurrence of nonresponse and measurement errors on the study variable and not on the auxiliary variable under the poststratified sampling scheme, as follows:

1. The customary sample mean estimator given by

$$\bar{y}_{js2} = \sum_{h=1}^L W_h \bar{y}_h''' \quad (3.252)$$

2. A difference-type estimator given by

$$\bar{y}_{jd2} = \sum_{h=1}^L W_h [\bar{y}_h''' - k_h (\bar{x}_h' - \bar{X}_h)] \quad (3.253)$$

3. A general class of estimators that makes use of known values of some population parameters of the auxiliary variable in the h^{th} stratum, given by

$$\bar{y}_{jc2} = \sum_{h=1}^L W_h \bar{y}_h''' \left(\frac{a_h \bar{X}_h + b_h}{a_h \bar{x}_h' + b_h} \right)^{\alpha_h} \quad (3.254)$$

where $k_h(\alpha_h)$ are any real numbers and $a_h(b_h)$ are also any real numbers or known values of population parameters of an auxiliary variable, x , in the h^{th} stratum, like coefficient of variation, skewness, kurtosis, etc. Properties of the above proposed estimators shall be obtained both under the conditional argument or given an achieved sample configuration, say $\underline{n} = (n_1, n_2, \dots, n_L)$ and under the unconditional argument or for repeated samples of fixed size, n .

Estimator 22: Customary sample mean estimator, $\bar{y}_{js2}$, when there is joint occurrence of non-response and measurement errors on the study variable and not on the auxiliary variable under the Poststratified Sampling Scheme.

The proposed sample mean estimator, $\bar{y}_{js2}$, which does not utilize any auxiliary information, is the same as the estimator, $\bar{y}_{js2}$, given in (3.188). Consequently, the conditional and unconditional properties of the estimator, $\bar{y}_{js2}$, are as already given in theorem 3.16.

Estimator 23: Difference-type estimator, $\bar{y}_{jd2}$, when there is joint occurrence of nonresponse and measurement errors on the study variable and not on the auxiliary variable under the Post-stratified Sampling Scheme.

The proposed difference-type estimator, $\bar{y}_{jd2}$, given in (3.253), utilizes auxiliary information with known population mean ($\bar{X}_h$) of the auxiliary variable, x , in the h^{th} stratum. We give the conditional and unconditional properties of the estimator, $\bar{y}_{jd2}$, in theorem 3.19.

Theorem 3.19: Under the conditional argument or for an achieved sample configuration, say $\underline{n} = (n_1, n_2, \dots, n_L)$, the proposed difference-type estimator, $\bar{y}_{jd2}$, is an unbiased estimator of the population mean ($\bar{Y}$), of the study variable, y , with its conditional variance obtained as:

$$V_c(\bar{y}_{jd2}) = \sum_{h=1}^L W_h^2 \left(\frac{1}{n_h} - \frac{1}{N_h} \right) (S_{yh}^2 + k_h^2 S_{xh}^2 - 2k_h S_{yxh}) + \sum_{h=1}^L W_h^2 \left(\frac{g_h - 1}{n_h} \right) W_{2h} (S_{2yh}^2 + S_{2uh}^2) + \sum_{h=1}^L W_h^2 \left(\frac{1}{n_h} - \frac{1}{N_h} \right) S_{uh}^2 \quad (3.255)$$

Furthermore, under the unconditional argument or for repeated samples of fixed size, n , the difference-type estimator, $\bar{y}_{jd2}$, is an unbiased estimator of the population mean ($\bar{Y}$) of the study variable y , with its unconditional variance obtained up to first order of approximations as:

$$V(\bar{y}_{jd2}) = \left(\frac{1-f}{n}\right) \sum_{h=1}^L W_h (S_{yh}^2 + k_h^2 S_{xh}^2 - 2k_h S_{yxh}) + \left(\frac{1}{n}\right) \sum_{h=1}^L W_h (g_h - 1) W_{2h} (S_{2yh}^2 + S_{2uh}^2) + \left(\frac{1-f}{n}\right) \sum_{h=1}^L W_h S_{uh}^2 \quad (3.256)$$

Proof: The proposed difference-type estimator, $\bar{y}_{jd2}$, given in (3.253), can be written as

$$\bar{y}_{jd2} = \sum_{h=1}^L W_h [\bar{y}_h''' - k_h (\bar{x}_h' - \bar{X}_h)] = \sum_{h=1}^L W_h \bar{y}_{jd2h} \quad (3.257)$$

where

$$\bar{y}_{jd2h} = \bar{y}_h''' - k_h (\bar{x}_h' - \bar{X}_h) \quad (3.258)$$

From (3.257), the conditional expectation and variance of the proposed difference-type estimator, $\bar{y}_{jd2}$, can be written respectively as

$$E_c(\bar{y}_{jd2}) = \sum_{h=1}^L W_h E_c(\bar{y}_{jd2h}) \quad (3.259)$$

and

$$V_c(\bar{y}_{jd2}) = \sum_{h=1}^L W_h^2 V_c(\bar{y}_{jd2h}) \quad (3.260)$$

Using (3.258), we obtain

$$E_c(\bar{y}_{jd2h}) = E_c(\bar{y}_h''') - k_h [E_c(\bar{x}_h') - \bar{X}_h] \quad (3.261)$$

$$V_c(\bar{y}_{jd2h}) = V_c(\bar{y}_h''') + k_h^2 V_c(\bar{x}_h') - 2k_h C_c(\bar{y}_h''', \bar{x}_h') \quad (3.262)$$

Following similar procedure as in the proof of theorem 3.17, we obtain

$$E_c(\bar{y}_h''') = \bar{Y}_h \quad , \quad E_c(\bar{x}_h') = \bar{X}_h \quad (3.263)$$

and

$$\begin{aligned}
V_c(\bar{y}_h''') &= \left(\frac{1}{n_h} - \frac{1}{N_h}\right)(S_{yh}^2 + S_{uh}^2) + \left(\frac{g_h - 1}{n_h}\right)W_{2h}(S_{2yh}^2 + S_{2uh}^2) \\
V_c(\bar{x}_h') &= \left(\frac{1}{n_h} - \frac{1}{N_h}\right)S_{xh}^2 \\
C_c(\bar{y}_h''', \bar{x}_h') &= \left(\frac{1}{n_h} - \frac{1}{N_h}\right)S_{yxh}
\end{aligned} \tag{3.264}$$

Substituting (3.263) in (3.261) gives

$$E_c(\bar{y}_{jd2h}) = E_c(\bar{y}_h''') - k_h[E_c(\bar{x}_h') - \bar{X}_h] = \bar{Y}_h - k_h[\bar{X}_h - \bar{X}_h] = \bar{Y}_h \tag{3.265}$$

Substituting (3.265) in (3.259) gives the conditional expectation of the proposed difference-type estimator ($\bar{y}_{jd2}$) as

$$E_c(\bar{y}_{jd2}) = \sum_{h=1}^L W_h E_c(\bar{y}_{jd2h}) = \sum_{h=1}^L W_h \bar{Y}_h = \bar{Y} \tag{3.266}$$

Therefore, the proposed difference-type estimator ($\bar{y}_{jd2}$), is unbiased for $\bar{Y}$ when there is joint occurrence of nonresponse and measurement errors on the study variable and not on the auxiliary variable under the conditional argument. To obtain the associated conditional variance of the proposed difference-type estimator ($\bar{y}_{jd2}$), we substitute (3.264) in (3.262), to first obtain the conditional variance of the stratum mean estimator, ($\bar{y}_{jd2h}$), as:

$$\begin{aligned}
V_c(\bar{y}_{jd2h}) &= V_c(\bar{y}_h''') + k_h^2 V_c(\bar{x}_h') - 2k_h C_c(\bar{y}_h''', \bar{x}_h') \\
&= \left(\frac{1}{n_h} - \frac{1}{N_h}\right)(S_{yh}^2 + S_{uh}^2) + \left(\frac{g_h - 1}{n_h}\right)W_{2h}(S_{2yh}^2 + S_{2uh}^2) \\
&\quad + k_h^2 \left(\frac{1}{n_h} - \frac{1}{N_h}\right)S_{xh}^2 - 2k_h \left(\frac{1}{n_h} - \frac{1}{N_h}\right)S_{yxh} \\
&= \left(\frac{1}{n_h} - \frac{1}{N_h}\right)(S_{yh}^2 + k_h^2 S_{xh}^2 - 2k_h S_{yxh}) \\
&\quad + \left(\frac{g_h - 1}{n_h}\right)W_{2h}(S_{2yh}^2 + S_{2uh}^2) + \left(\frac{1}{n_h} - \frac{1}{N_h}\right)S_{uh}^2
\end{aligned} \tag{3.267}$$

Then, substituting (3.267) in (3.260), we obtain the conditional variance of the proposed difference-type estimator ($\bar{y}_{jd2}$), as

$$\begin{aligned}
V_c(\bar{y}_{jd2}) &= \sum_{h=1}^L W_h^2 V_c(\bar{y}_{jd2h}) \\
&= \sum_{h=1}^L W_h^2 \left[\left(\frac{1}{n_h} - \frac{1}{N_h} \right) (S_{yh}^2 + k_h^2 S_{xh}^2 - 2k_h S_{yxh}) \right. \\
&\quad \left. + \left(\frac{g_h - 1}{n_h} \right) W_{2h} (S_{2yh}^2 + S_{uh}^2) + \left(\frac{1}{n_h} - \frac{1}{N_h} \right) S_{uh}^2 \right] \\
&= \sum_{h=1}^L W_h^2 \left(\frac{1}{n_h} - \frac{1}{N_h} \right) (S_{yh}^2 + k_h^2 S_{xh}^2 - 2k_h S_{yxh}) \\
&\quad + \sum_{h=1}^L W_h^2 \left(\frac{g_h - 1}{n_h} \right) W_{2h} (S_{2yh}^2 + S_{uh}^2) + \sum_{h=1}^L W_h^2 \left(\frac{1}{n_h} - \frac{1}{N_h} \right) S_{uh}^2
\end{aligned} \tag{3.268}$$

as stated in the theorem. This proves the first part of the theorem on the conditional unbiasedness and variance of the proposed difference-type estimator, $\bar{y}_{jd2}$. To obtain the unconditional variance of the proposed difference-type estimator, $\bar{y}_{jd2}$, we take the unconditional expectation of (3.268) and use (3.25) to make the necessary substitutions. This completes the proof.

Estimator 24: Class of estimators, $\bar{y}_{jc2}$, when there is joint occurrence of nonresponse and measurement errors on the study variable and not on the auxiliary variable under the Poststratified Sampling Scheme.

The proposed class of estimators, $\bar{y}_{jc2}$, given in (3.254), utilizes auxiliary information with known values of some population parameters of the auxiliary variable, x , in the h^{th} stratum. We give the conditional and unconditional properties of the estimator, $\bar{y}_{jc2}$, in theorem 3.20.

Theorem 3.20: Under the conditional argument, the proposed class of estimators, $\bar{y}_{jc2}$, is biased for the population mean ($\bar{Y}$) of the study variable, y , with its conditional bias and mean square error obtained up to first order of approximations as:

$$B_c(\bar{y}_{jc2}) = \sum_{h=1}^L \frac{W_h \alpha_h \lambda_h}{2\bar{X}_h} \left(\frac{1}{n_h} - \frac{1}{N_h} \right) \left[(\alpha_h + 1) \lambda_h R_h S_{xh}^2 - 2S_{yxh} \right] \tag{3.269}$$

$$\begin{aligned}
MSE_c(\bar{y}_{jc2}) &= \sum_{h=1}^L W_h^2 \left(\frac{1}{n_h} - \frac{1}{N_h} \right) (S_{yh}^2 + \alpha_h^2 \lambda_h^2 R_h^2 S_{xh}^2 - 2\alpha_h \lambda_h R_h S_{yxh}) \\
&+ \sum_{h=1}^L W_h^2 \left(\frac{g_h - 1}{n_h} \right) W_{2h} S_{2yh}^2 + \sum_{h=1}^L W_h^2 \left[\left(\frac{1}{n_h} - \frac{1}{N_h} \right) S_{uh}^2 + \left(\frac{g_h - 1}{n_h} \right) W_{2h} S_{2uh}^2 \right]
\end{aligned} \tag{3.270}$$

Furthermore, under the unconditional argument or for repeated samples of fixed size, n , the proposed class of estimators, $\bar{y}_{jc2}$, is biased for the population mean ($\bar{Y}$) of the study variable y , with its unconditional bias and mean square error obtained up to first order of approximations as:

$$B(\bar{y}_{jc2}) = \left(\frac{1-f}{n} \right) \sum_{h=1}^L \frac{\alpha_h \lambda_h}{2\bar{X}_h} [(\alpha_h + 1) \lambda_h R_h S_{xh}^2 - 2S_{yxh}] \tag{3.271}$$

and

$$\begin{aligned}
MSE(\bar{y}_{jc2}) &= \left(\frac{1-f}{n} \right) \sum_{h=1}^L W_h (S_{yh}^2 + \alpha_h^2 \lambda_h^2 R_h^2 S_{xh}^2 - 2\alpha_h \lambda_h R_h S_{yxh}) \\
&+ \left(\frac{1}{n} \right) \sum_{h=1}^L W_h (g_h - 1) W_{2h} S_{2yh}^2 + \sum_{h=1}^L W_h \left[\left(\frac{1-f}{n} \right) S_{uh}^2 + \left(\frac{g_h - 1}{n} \right) W_{2h} S_{2uh}^2 \right]
\end{aligned} \tag{3.272}$$

where

$$\lambda_h = \left(\frac{a_h \bar{X}_h}{a_h \bar{X}_h + b_h} \right) \tag{3.273}$$

Proof: The proposed class of estimators, $\bar{y}_{jc2}$, given in (3.254), can be written as:

$$\bar{y}_{jc2} = \sum_{h=1}^L W_h \bar{y}_h''' \left(\frac{a_h \bar{X}_h + b_h}{a_h \bar{X}_h' + b_h} \right)^{\alpha_h} = \sum_{h=1}^L W_h \bar{y}_{jc2h} \tag{3.274}$$

where

$$\bar{y}_{jc2h} = \bar{y}_h''' \left(\frac{a_h \bar{X}_h + b_h}{a_h \bar{X}_h' + b_h} \right)^{\alpha_h} \tag{3.275}$$

From (3.274), the conditional bias and mean square error of the proposed class of estimators, $\bar{y}_{jc2}$, are respectively given by

$$B_c(\bar{y}_{jc2}) = \sum_{h=1}^L W_h B_c(\bar{y}_{jc2h}) \tag{3.276}$$

$$MSE_c(\bar{y}_{jc2}) = \sum_{h=1}^L W_h^2 MSE_c(\bar{y}_{jc2h}) \tag{3.277}$$

For the purpose of obtaining the properties of the stratum sample mean estimator, $\bar{y}_{jc2h}$, we consider the adjusted mean deviations of the stratum mean estimators, $\bar{y}_h'''$ and $\bar{x}_h'$, respectively given by

$$d_{5h} = \frac{\bar{y}_h''' - \bar{Y}_h}{\bar{Y}_h} \quad \text{or} \quad \bar{y}_h''' = \bar{Y}_h(1 + d_{5h}) \quad (3.278)$$

and

$$e_{3h} = \frac{\bar{x}_h' - \bar{X}_h}{\bar{X}_h} \quad \text{or} \quad \bar{x}_h' = \bar{X}_h(1 + e_{3h}) \quad (3.279)$$

Following similar procedure as in the proof of theorem 3.18 and using (3.264) to make the necessary substitutions, we obtain the conditional bias of the stratum mean estimator ($\bar{y}_{jc2h}$) as

$$\begin{aligned} B_c(\bar{y}_{jc2h}) &= \bar{Y}_h \left[-\alpha_h \lambda_h \frac{C_c(\bar{y}_h''', \bar{x}_h')}{\bar{Y}_h \bar{X}_h} + (1/2) \alpha_h (\alpha_h + 1) \lambda_h^2 \frac{V_c(\bar{x}_h')}{\bar{X}_h^2} \right] \\ &= \bar{Y}_h \left[-\frac{\alpha_h \lambda_h}{\bar{Y}_h \bar{X}_h} \left(\frac{1}{n_h} - \frac{1}{N_h} \right) S_{yxh} + \frac{(1/2) \alpha_h (\alpha_h + 1) \lambda_h^2}{\bar{X}_h^2} \left(\frac{1}{n_h} - \frac{1}{N_h} \right) S_{xh}^2 \right] \\ &= \frac{\alpha_h \lambda_h}{2 \bar{X}_h} \left(\frac{1}{n_h} - \frac{1}{N_h} \right) \left[(\alpha_h + 1) \lambda_h R_h S_{xh}^2 - 2 S_{yxh} \right] \end{aligned} \quad (3.280)$$

Substituting (3.280) in (3.276), we obtain the conditional bias of the proposed class of estimators, $\bar{y}_{jc2}$, as

$$B_c(\bar{y}_{jc2}) = \sum_{h=1}^L \frac{W_h \alpha_h \lambda_h}{2 \bar{X}_h} \left(\frac{1}{n_h} - \frac{1}{N_h} \right) \left[(\alpha_h + 1) \lambda_h R_h S_{xh}^2 - 2 S_{yxh} \right] \quad (3.281)$$

as stated in the theorem. Again, following similar procedure as in the proof of theorem 3.18, and using (3.264) to make the necessary substitutions, we obtain the conditional mean square error of the stratum mean estimator ($\bar{y}_{jc2h}$) as:

$$\begin{aligned} MS E_c(\bar{y}_{jc2h}) &= \bar{Y}_h^2 \left[\frac{V_c(\bar{y}_h''')}{\bar{Y}_h^2} + \alpha_h^2 \lambda_h^2 \frac{V_c(\bar{x}_h')}{\bar{X}_h^2} - 2 \alpha_h \lambda_h \frac{C_c(\bar{y}_h''', \bar{x}_h')}{\bar{Y}_h \bar{X}_h} \right] \\ &= \left(\frac{1}{n_h} - \frac{1}{N_h} \right) (S_{yh}^2 + \alpha_h^2 \lambda_h^2 R_h^2 S_{xh}^2 - 2 \alpha_h \lambda_h R_h S_{yxh}) \\ &\quad + \left(\frac{g_h - 1}{n_h} \right) W_{2h} S_{2yh}^2 + \left[\left(\frac{1}{n_h} - \frac{1}{N_h} \right) S_{uh}^2 + \left(\frac{g_h - 1}{n_h} \right) W_{2h} S_{2uh}^2 \right] \end{aligned} \quad (3.282)$$

Then substituting (3.282) in (3.277) gives the conditional mean square error of the proposed class of estimators, $\bar{y}_{jc2}$, as

$$\begin{aligned}
MS E_c(\bar{y}_{jc2}) &= \sum_{h=1}^L W_h^2 \left(\frac{1}{n_h} - \frac{1}{N_h} \right) (S_{yh}^2 + \alpha_h^2 \lambda_h^2 R_h^2 S_{xh}^2 - 2\alpha_h \lambda_h R_h S_{y.xh}) \\
&+ \sum_{h=1}^L W_h^2 \left(\frac{g_h - 1}{n_h} \right) W_{2h} S_{2yh}^2 + \sum_{h=1}^L W_h^2 \left[\left(\frac{1}{n_h} - \frac{1}{N_h} \right) S_{uh}^2 + \left(\frac{g_h - 1}{n_h} \right) W_{2h} S_{2uh}^2 \right]
\end{aligned} \tag{3.283}$$

as stated in the theorem. This proves the results on the conditional bias and mean square error of the proposed class of estimators, $\bar{y}_{jc2}$. To obtain the unconditional bias and mean square error of the proposed class of estimators, $\bar{y}_{jc2}$, we take the unconditional expectations of (3.281) and (3.283), and use (3.25) to make the necessary substitutions. This completes the proof.

3.3.3 Joint Occurrence of Nonresponse and Measurement Errors on the auxiliary variable and not on the study variable under the Poststratified Sampling Scheme

Here, we shall develop and focus on sampling strategies for handling situations where there is joint occurrence of nonresponse and measurement errors only on the auxiliary variable and not on the study variable under the poststratified sampling scheme. The associated sampling design is proposed as follows.

Sampling Design 3.3.3: Here, we propose the Sampling Design 3.3.3 for handling situations where there is joint occurrence of nonresponse and measurement errors on the auxiliary variable and not on the study variable in poststratified sampling scheme. The proposed Sampling Design 3.3.3 is similar to Sampling Design 3.3.1. While we assumed the joint occurrence of nonresponse and measurement errors on both the study and auxiliary variables in Sampling Design 3.3.1, in the present Sampling Design 3.3.3, we assume joint occurrence of nonresponse and measurement errors on only the auxiliary variable and not on the study variable. This means that there is complete response and no measurement errors on the study variable, y . Consequently, we give the sample mean of the study variable, y , when there is no nonresponse and measurement errors on y , as:

$$\bar{y}'_h = \frac{1}{n_h} \sum_{i=1}^{n_h} Y_{hi} \tag{3.284}$$

We observe also that the proposed Sampling Design 3.3.3 indicates that there are nonresponse and measurement errors on the auxiliary variable, x . This means that there are measurement errors on the observed values of the auxiliary variable, x , from the response sample group of n_{1h} units in stratum h , as well as the observed values of x from the nonresponse subsample group of m_h units in stratum h . Consequently, from (3.187), we give a sample mean of the auxiliary variable, x , in the h^{th} stratum when there is joint occurrence of nonresponse and measurement errors on x , as a linear function of the response and nonresponse subsample means, given by:

$$\bar{x}_h''' = \left(\frac{n_{1h}}{n_h}\right)\left(\frac{1}{n_{1h}} \sum_{i=1}^{n_{1h}} X_{hi}\right) + \left(\frac{n_{1h}}{n_h}\right)\left(\frac{1}{n_{1h}} \sum_{i=1}^{n_{1h}} V_{hi}\right) + \left(\frac{n_{2h}}{n_h}\right)\left(\frac{1}{m_h} \sum_{i=1}^{m_h} X_{hi}\right) + \left(\frac{n_{2h}}{n_h}\right)\left(\frac{1}{m_h} \sum_{i=1}^{m_h} V_{hi}\right) \quad (3.285)$$

Based on the Sampling Design 3.3.3 and using the sample means (3.284) and (3.285), we propose three estimators of the population mean ($\bar{Y}$) of the study variable, y , when there is joint occurrence of nonresponse and measurement errors on the auxiliary variable and not on the study variable under the poststratified sampling scheme, as follows:

1. The customary sample mean estimator given by

$$\bar{y}_{js3} = \sum_{h=1}^L W_h \bar{y}_h' \quad (3.286)$$

2. A difference-type estimator given by

$$\bar{y}_{jd3} = \sum_{h=1}^L W_h [\bar{y}_h' - k_h(\bar{x}_h''' - \bar{X}_h)] \quad (3.287)$$

3. A general class of estimators that makes use of known values of some population parameters of the auxiliary variable in the h^{th} stratum, given by

$$\bar{y}_{jc3} = \sum_{h=1}^L W_h \bar{y}_h' \left(\frac{a_h \bar{X}_h + b_h}{a_h \bar{x}_h''' + b_h} \right)^{\alpha_h} \quad (3.288)$$

where $k_h(\alpha_h)$ are any real numbers and $a_h(b_h)$ are also any real numbers or known values of population parameters of an auxiliary variable, x , in the h^{th} stratum, like coefficient of variation, skewness, kurtosis, etc. Properties of the above proposed estimators shall be obtained both

under the conditional argument or given an achieved sample configuration, say $\underline{n} = (n_1, n_2, \dots, n_L)$ and under the unconditional argument or for repeated samples of fixed size, n .

Estimator 25: Customary sample mean estimator, $\bar{y}_{js3}$, when there is joint occurrence of non-response and measurement errors on the auxiliary variable and not on the study variable under the Poststratified Sampling Scheme.

The proposed customary sample mean estimator, $\bar{y}_{js3}$, given in (3.286), does not utilize any auxiliary information, and is the same as the estimator, $\bar{y}_{ms3}$, given in (3.169), which is unbiased for the study variable population mean ($\bar{Y}$). Consequently, the conditional and unconditional variances of the proposed estimator, $\bar{y}_{js3}$, are the same as those of the estimator, $\bar{y}_{ms3}$, already given in (3.172) and (3.173) respectively. That is:

$$V_c(\bar{y}_{js3}) = V_c\left(\sum_{h=1}^L W_h \bar{y}'_h\right) = \sum_{h=1}^L W_h^2 \left(\frac{1}{n_h} - \frac{1}{N_h}\right) S_{yh}^2 \quad (3.289)$$

$$V(\bar{y}_{js3}) = V\left(\sum_{h=1}^L W_h \bar{y}'_h\right) = \left(\frac{1-f}{n}\right) \sum_{h=1}^L W_h S_{yh}^2 \quad (3.290)$$

Estimator 26: Difference-type estimator, $\bar{y}_{jd3}$, when there is joint occurrence of nonresponse and measurement errors on the auxiliary variable and not on the study variable under the Poststratified Sampling Scheme.

The proposed difference-type estimator, $\bar{y}_{jd3}$, given in (3.287), utilizes auxiliary information with known population mean ($\bar{X}_h$) of the auxiliary variable, x , in the h^{th} stratum. We give the conditional and unconditional properties of the estimator, $\bar{y}_{jd3}$, in theorem 3.21.

Theorem 3.21: Under the conditional argument or for an achieved sample configuration, say $\underline{n} = (n_1, n_2, \dots, n_L)$, the proposed difference-type estimator, $\bar{y}_{jd3}$, is an unbiased estimator of the population mean ($\bar{Y}$), of the study variable, y , with its conditional variance obtained as:

$$\begin{aligned}
V_c(\bar{y}_{jd3}) &= \sum_{h=1}^L W_h^2 \left(\frac{1}{n_h} - \frac{1}{N_h} \right) (S_{yh}^2 + k_h^2 S_{xh}^2 - 2k_h S_{yhxh}) \\
&\quad + \sum_{h=1}^L W_h^2 k_h^2 \left(\frac{g_h - 1}{n_h} \right) W_{2h} (S_{2xh}^2 + S_{2vh}^2) + \sum_{h=1}^L W_h^2 k_h^2 \left(\frac{1}{n_h} - \frac{1}{N_h} \right) S_{vh}^2
\end{aligned} \tag{3.291}$$

Furthermore, under the unconditional argument or for repeated samples of fixed size, n , the difference-type estimator, $\bar{y}_{jd3}$, is an unbiased estimator of the population mean ($\bar{Y}$) of the study variable y , with its unconditional variance obtained up to first order of approximations as:

$$\begin{aligned}
V(\bar{y}_{jd3}) &= \left(\frac{1-f}{n} \right) \sum_{h=1}^L W_h (S_{yh}^2 + k_h^2 S_{xh}^2 - 2k_h S_{yhxh}) \\
&\quad + \left(\frac{1}{n} \right) \sum_{h=1}^L W_h k_h^2 (g_h - 1) W_{2h} (S_{2xh}^2 + S_{2vh}^2) + \left(\frac{1-f}{n} \right) \sum_{h=1}^L W_h k_h^2 S_{vh}^2
\end{aligned} \tag{3.292}$$

Proof: We can rewrite the proposed difference-type estimator, $\bar{y}_{jd3}$, given in (3.287), as

$$\bar{y}_{jd3} = \sum_{h=1}^L W_h [\bar{y}'_h - k_h (\bar{x}'''_h - \bar{X}_h)] = \sum_{h=1}^L W_h \bar{y}_{jd3h} \tag{3.293}$$

where

$$\bar{y}_{jd3h} = \bar{y}'_h - k_h (\bar{x}'''_h - \bar{X}_h) \tag{3.294}$$

Then, the conditional expectation and variance of the proposed difference-type estimator, $\bar{y}_{jd3}$, can be written respectively, from (3.293), as

$$E_c(\bar{y}_{jd3}) = \sum_{h=1}^L W_h E_c(\bar{y}_{jd3h}) \tag{3.295}$$

and

$$V_c(\bar{y}_{jd3}) = \sum_{h=1}^L W_h^2 V_c(\bar{y}_{jd3h}) \tag{3.296}$$

Now, using (3.294), we obtain

$$E_c(\bar{y}_{jd3h}) = E_c(\bar{y}'_h) - k_h [E_c(\bar{x}'''_h) - \bar{X}_h] \tag{3.297}$$

$$V_c(\bar{y}_{jd3h}) = V_c(\bar{y}'_h) + k_h^2 V_c(\bar{x}'''_h) - 2k_h C_c(\bar{y}'_h, \bar{x}'''_h) \tag{3.298}$$

Using similar procedure as in the proof of theorem 3.17, we obtain

$$E_c(\bar{y}'_h) = \bar{Y}_h \quad , \quad E_c(\bar{x}'''_h) = \bar{X}_h \quad (3.299)$$

$$\begin{aligned} V_c(\bar{y}'_h) &= \left(\frac{1}{n_h} - \frac{1}{N_h}\right) S_{yh}^2 \\ V_c(\bar{x}'''_h) &= \left(\frac{1}{n_h} - \frac{1}{N_h}\right) (S_{xh}^2 + S_{vh}^2) + \left(\frac{g_h - 1}{n_h}\right) W_{2h} (S_{2xh}^2 + S_{2vh}^2) \\ C_c(\bar{y}'_h, \bar{x}'''_h) &= \left(\frac{1}{n_h} - \frac{1}{N_h}\right) S_{yxh} \end{aligned} \quad (3.300)$$

If we substitute (3.299) in (3.297) we obtain

$$E_c(\bar{y}_{jd3h}) = E_c(\bar{y}'_h) - k_h [E_c(\bar{x}'''_h) - \bar{X}_h] = \bar{Y}_h - k_h [\bar{X}_h - \bar{X}_h] = \bar{Y}_h \quad (3.301)$$

Also, substituting (3.301) in (3.295) gives the conditional expectation of the proposed difference-type estimator ($\bar{y}_{jd3}$) as

$$E_c(\bar{y}_{jd3}) = \sum_{h=1}^L W_h E_c(\bar{y}_{jd3h}) = \sum_{h=1}^L W_h \bar{Y}_h = \bar{Y} \quad (3.302)$$

Consequently, the proposed difference-type estimator ($\bar{y}_{jd3}$), is unbiased for $\bar{Y}$ when there is joint occurrence of nonresponse and measurement errors on the auxiliary variable and not on the study variable under the conditional argument. We substitute (3.300) in (3.298), to first obtain the conditional variance of the stratum mean estimator, ($\bar{y}_{jd3h}$), as:

$$\begin{aligned} V_c(\bar{y}_{jd3h}) &= V_c(\bar{y}'_h) + k_h^2 V_c(\bar{x}'''_h) - 2k_h C_c(\bar{y}'_h, \bar{x}'''_h) = \left(\frac{1}{n_h} - \frac{1}{N_h}\right) S_{yh}^2 - 2k_h \left(\frac{1}{n_h} - \frac{1}{N_h}\right) S_{yxh} \\ &\quad + k_h^2 \left[\left(\frac{1}{n_h} - \frac{1}{N_h}\right) (S_{xh}^2 + S_{vh}^2) + \left(\frac{g_h - 1}{n_h}\right) W_{2h} (S_{2xh}^2 + S_{2vh}^2) \right] \\ &= \left(\frac{1}{n_h} - \frac{1}{N_h}\right) (S_{yh}^2 + k_h^2 S_{xh}^2 - 2k_h S_{yxh}) \\ &\quad + k_h^2 \left[\left(\frac{g_h - 1}{n_h}\right) W_{2h} (S_{2xh}^2 + S_{2vh}^2) + \left(\frac{1}{n_h} - \frac{1}{N_h}\right) S_{vh}^2 \right] \end{aligned} \quad (3.303)$$

Then, substituting (3.303) in (3.296), we obtain the conditional variance of the proposed difference-type estimator ($\bar{y}_{jd3}$), as

$$\begin{aligned}
V_c(\bar{y}_{jd3}) &= \sum_{h=1}^L W_h^2 V_c(\bar{y}_{jd3h}) = \sum_{h=1}^L W_h^2 \left[\left(\frac{1}{n_h} - \frac{1}{N_h} \right) (S_{yh}^2 + k_h^2 S_{xh}^2 - 2k_h S_{yxh}) \right. \\
&\quad \left. + k_h^2 \left(\frac{g_h - 1}{n_h} \right) W_{2h} (S_{2xh}^2 + S_{vh}^2) + k_h^2 \left(\frac{1}{n_h} - \frac{1}{N_h} \right) S_{vh}^2 \right] \\
&= \sum_{h=1}^L W_h^2 \left(\frac{1}{n_h} - \frac{1}{N_h} \right) (S_{yh}^2 + k_h^2 S_{xh}^2 - 2k_h S_{yxh}) \\
&\quad + \sum_{h=1}^L W_h^2 k_h^2 \left(\frac{g_h - 1}{n_h} \right) W_{2h} (S_{2xh}^2 + S_{vh}^2) + \sum_{h=1}^L W_h^2 k_h^2 \left(\frac{1}{n_h} - \frac{1}{N_h} \right) S_{vh}^2
\end{aligned} \tag{3.304}$$

as stated in the theorem. This proves the first part of the theorem on the conditional unbiasedness and variance of the proposed difference-type estimator, $\bar{y}_{jd3}$. To obtain the unconditional variance of the proposed difference-type estimator, $\bar{y}_{jd3}$, we take the unconditional expectation of (3.304) and use (3.25) to make the necessary substitutions. This completes the proof.

Estimator 27: Class of estimators, $\bar{y}_{jc3}$, when there is joint occurrence of nonresponse and measurement errors on the auxiliary variable and not on the study variable under the Poststratified Sampling Scheme.

The proposed class of estimators, $\bar{y}_{jc3}$, given in (3.288), utilizes auxiliary information with known values of some population parameters of the auxiliary variable, x , in the h^{th} stratum. We give the conditional and unconditional properties of the estimator, $\bar{y}_{jc3}$, in theorem 3.22.

Theorem 3.22: Under the conditional argument, the proposed class of estimators, $\bar{y}_{jc3}$, is biased for the population mean ($\bar{Y}$) of the study variable, y , with its conditional bias and mean square error obtained up to first order of approximations as:

$$\begin{aligned}
B_c(\bar{y}_{jc3}) &= \sum_{h=1}^L \frac{W_h \alpha_h (\alpha_h + 1) \lambda_h^2 R_h}{2\bar{X}_h} \left[\left(\frac{1}{n_h} - \frac{1}{N_h} \right) S_{xh}^2 + \left(\frac{g_h - 1}{n_h} \right) W_{2h} S_{2xh}^2 \right] \\
&\quad + \sum_{h=1}^L \frac{W_h \alpha_h (\alpha_h + 1) \lambda_h^2 R_h}{2\bar{X}_h} \left[\left(\frac{1}{n_h} - \frac{1}{N_h} \right) S_{vh}^2 + \left(\frac{g_h - 1}{n_h} \right) W_{2h} S_{2vh}^2 \right] \\
&\quad - \sum_{h=1}^L \frac{W_h \alpha_h \lambda_h}{\bar{X}_h} \left(\frac{1}{n_h} - \frac{1}{N_h} \right) S_{yxh}
\end{aligned} \tag{3.305}$$

$$\begin{aligned}
MSE_c(\bar{y}_{jc3}) = & \sum_{h=1}^L W_h^2 \left(\frac{1}{n_h} - \frac{1}{N_h} \right) (S_{yh}^2 + \alpha_h^2 \lambda_h^2 R_h^2 S_{xh}^2 - 2\alpha_h \lambda_h R_h S_{yxh}) \\
& + \sum_{h=1}^L W_h^2 \alpha_h^2 \lambda_h^2 R_h^2 \left[\left(\frac{1}{n_h} - \frac{1}{N_h} \right) S_{vh}^2 + \left(\frac{g_h - 1}{n_h} \right) W_{2h} S_{2vh}^2 \right] \\
& + \sum_{h=1}^L W_h^2 \alpha_h^2 \lambda_h^2 R_h^2 \left(\frac{g_h - 1}{n_h} \right) W_{2h} S_{2xh}^2
\end{aligned} \tag{3.306}$$

Furthermore, under the unconditional argument or for repeated samples of fixed size, n , the proposed class of estimators, $\bar{y}_{jc3}$, is biased for the population mean ($\bar{Y}$) of the study variable y , with its unconditional bias and mean square error obtained up to first order of approximations as:

$$\begin{aligned}
B(\bar{y}_{jc3}) = & \sum_{h=1}^L \frac{\alpha_h(\alpha_h + 1)\lambda_h^2 R_h}{2\bar{X}_h} \left[\left(\frac{1-f}{n} \right) S_{xh}^2 + \left(\frac{g_h - 1}{n} \right) W_{2h} S_{2xh}^2 \right] \\
& + \sum_{h=1}^L \frac{\alpha_h(\alpha_h + 1)\lambda_h^2 R_h}{2\bar{X}_h} \left[\left(\frac{1-f}{n} \right) S_{vh}^2 + \left(\frac{g_h - 1}{n} \right) W_{2h} S_{2vh}^2 \right] \\
& - \sum_{h=1}^L \frac{W_h \alpha_h \lambda_h}{\bar{X}_h} \left(\frac{1-f}{n} \right) S_{yxh}
\end{aligned} \tag{3.307}$$

and

$$\begin{aligned}
MSE(\bar{y}_{jc3}) = & \sum_{h=1}^L W_h \left(\frac{1-f}{n} \right) (S_{yh}^2 + \alpha_h^2 \lambda_h^2 R_h^2 S_{xh}^2 - 2\alpha_h \lambda_h R_h S_{yxh}) \\
& + \sum_{h=1}^L W_h \alpha_h^2 \lambda_h^2 R_h^2 \left[\left(\frac{1-f}{n} \right) S_{vh}^2 + \left(\frac{g_h - 1}{n} \right) W_{2h} S_{2vh}^2 \right] \\
& + \sum_{h=1}^L W_h \alpha_h^2 \lambda_h^2 R_h^2 \left(\frac{g_h - 1}{n} \right) W_{2h} S_{2xh}^2
\end{aligned} \tag{3.308}$$

where

$$\lambda_h = \left(\frac{a_h \bar{X}_h}{a_h \bar{X}_h + b_h} \right) \tag{3.309}$$

Proof: The proposed class of estimators, $\bar{y}_{jc3}$, given in (3.288), can be rewritten as:

$$\bar{y}_{jc3} = \sum_{h=1}^L W_h \bar{y}'_h \left(\frac{a_h \bar{X}_h + b_h}{a_h \bar{X}_h'' + b_h} \right)^{\alpha_h} = \sum_{h=1}^L W_h \bar{y}_{jc3h} \tag{3.310}$$

where

$$\bar{y}_{jc3h} = \bar{y}'_h \left(\frac{a_h \bar{X}_h + b_h}{a_h \bar{X}_h''' + b_h} \right)^{\alpha_h} \quad (3.311)$$

From (3.310), the conditional bias and mean square error of the proposed class of estimators, $\bar{y}_{jc3}$, are respectively given by

$$B_c(\bar{y}_{jc3}) = \sum_{h=1}^L W_h B_c(\bar{y}_{jc3h}) \quad (3.312)$$

$$MSE_c(\bar{y}_{jc3}) = \sum_{h=1}^L W_h^2 MSE_c(\bar{y}_{jc3h}) \quad (3.313)$$

Consider the adjusted mean deviations of the stratum mean estimators, $\bar{y}'_h$ and $\bar{x}_h'''$, respectively given by

$$d_{3h} = \frac{\bar{y}'_h - \bar{Y}_h}{\bar{Y}_h} \quad \text{or} \quad \bar{y}'_h = \bar{Y}_h(1 + d_{3h}) \quad (3.314)$$

and

$$e_{5h} = \frac{\bar{x}_h''' - \bar{X}_h}{\bar{X}_h} \quad \text{or} \quad \bar{x}_h''' = \bar{X}_h(1 + e_{5h}) \quad (3.315)$$

The conditional bias of the stratum mean estimator ($\bar{y}_{jc3h}$) can be obtained by following similar procedure as in the proof of theorem 3.18 and using (3.300) to make the necessary substitutions.

This gives

$$\begin{aligned} B_c(\bar{y}_{jc3h}) &= \bar{Y}_h \left[-\alpha_h \lambda_h \frac{C_c(\bar{y}'_h, \bar{x}_h''')}{\bar{Y}_h \bar{X}_h} + (1/2) \alpha_h (\alpha_h + 1) \lambda_h^2 \frac{V_c(\bar{x}_h''')}{\bar{X}_h^2} \right] \\ &= \frac{\alpha_h (\alpha_h + 1) \lambda_h^2 R_h}{2 \bar{X}_h} \left[\left(\frac{1}{n_h} - \frac{1}{N_h} \right) (S_{xh}^2 + S_{vh}^2) + \left(\frac{g_h - 1}{n_h} \right) W_{2h} (S_{2xh}^2 + S_{2vh}^2) \right] \\ &\quad - \frac{\alpha_h \lambda_h}{\bar{X}_h} \left(\frac{1}{n_h} - \frac{1}{N_h} \right) S_{yxh} \\ &= \frac{\alpha_h (\alpha_h + 1) \lambda_h^2 R_h}{2 \bar{X}_h} \left[\left(\frac{1}{n_h} - \frac{1}{N_h} \right) S_{xh}^2 + \left(\frac{g_h - 1}{n_h} \right) W_{2h} S_{2xh}^2 \right] \\ &\quad + \frac{\alpha_h (\alpha_h + 1) \lambda_h^2 R_h}{2 \bar{X}_h} \left[\left(\frac{1}{n_h} - \frac{1}{N_h} \right) S_{vh}^2 + \left(\frac{g_h - 1}{n_h} \right) W_{2h} S_{2vh}^2 \right] \\ &\quad - \frac{\alpha_h \lambda_h}{\bar{X}_h} \left(\frac{1}{n_h} - \frac{1}{N_h} \right) S_{yxh} \end{aligned} \quad (3.316)$$

Substituting (3.316) in (3.312), we obtain the conditional bias of the proposed class of estimators, $\bar{y}_{jc3}$, as

$$\begin{aligned}
B_c(\bar{y}_{jc3}) &= \sum_{h=1}^L \frac{W_h \alpha_h (\alpha_h + 1) \lambda_h^2 R_h}{2 \bar{X}_h} \left[\left(\frac{1}{n_h} - \frac{1}{N_h} \right) S_{xh}^2 + \left(\frac{g_h - 1}{n_h} \right) W_{2h} S_{2xh}^2 \right] \\
&+ \sum_{h=1}^L \frac{W_h \alpha_h (\alpha_h + 1) \lambda_h^2 R_h}{2 \bar{X}_h} \left[\left(\frac{1}{n_h} - \frac{1}{N_h} \right) S_{vh}^2 + \left(\frac{g_h - 1}{n_h} \right) W_{2h} S_{2vh}^2 \right] \\
&- \sum_{h=1}^L \frac{W_h \alpha_h \lambda_h}{\bar{X}_h} \left(\frac{1}{n_h} - \frac{1}{N_h} \right) S_{yhx}
\end{aligned} \tag{3.317}$$

as stated in the theorem. In the same manner, following similar procedure as in the proof of theorem 3.18, and using (3.300) to make the necessary substitutions, we obtain the conditional mean square error of the stratum mean estimator ($\bar{y}_{jc3h}$) as:

$$\begin{aligned}
MS E_c(\bar{y}_{jc3h}) &= \bar{Y}_h^2 \left[\frac{V_c(\bar{y}'_h)}{\bar{Y}_h^2} + \alpha_h^2 \lambda_h^2 \frac{V_c(\bar{x}''_h)}{\bar{X}_h^2} - 2 \alpha_h \lambda_h \frac{C_c(\bar{y}'_h, \bar{x}''_h)}{\bar{Y}_h \bar{X}_h} \right] \\
&= \left(\frac{1}{n_h} - \frac{1}{N_h} \right) (S_{yh}^2 + \alpha_h^2 \lambda_h^2 R_h^2 S_{xh}^2 - 2 \alpha_h \lambda_h R_h S_{yhx}) \\
&+ \alpha_h^2 \lambda_h^2 R_h^2 \left[\left(\frac{1}{n_h} - \frac{1}{N_h} \right) S_{vh}^2 + \left(\frac{g_h - 1}{n_h} \right) W_{2h} S_{2vh}^2 \right] \\
&+ \alpha_h^2 \lambda_h^2 R_h^2 \left(\frac{g_h - 1}{n_h} \right) W_{2h} S_{2xh}^2
\end{aligned} \tag{3.318}$$

Then substituting (3.318) in (3.313) gives the conditional mean square error of the proposed class of estimators, $\bar{y}_{jc3}$, as

$$\begin{aligned}
MS E_c(\bar{y}_{jc3}) &= \sum_{h=1}^L W_h^2 MS E_c(\bar{y}_{jc3h}) \\
&= \sum_{h=1}^L W_h^2 \left(\frac{1}{n_h} - \frac{1}{N_h} \right) (S_{yh}^2 + \alpha_h^2 \lambda_h^2 R_h^2 S_{xh}^2 - 2 \alpha_h \lambda_h R_h S_{yhx}) \\
&+ \sum_{h=1}^L W_h^2 \alpha_h^2 \lambda_h^2 R_h^2 \left[\left(\frac{1}{n_h} - \frac{1}{N_h} \right) S_{vh}^2 + \left(\frac{g_h - 1}{n_h} \right) W_{2h} S_{2vh}^2 \right] \\
&+ \sum_{h=1}^L W_h^2 \alpha_h^2 \lambda_h^2 R_h^2 \left(\frac{g_h - 1}{n_h} \right) W_{2h} S_{2xh}^2
\end{aligned} \tag{3.319}$$

as stated in the theorem. This proves the results on the conditional bias and mean square error of the proposed class of estimators, $\bar{y}_{jc3}$. To obtain the unconditional bias and mean square error of the proposed class of estimators, $\bar{y}_{jc3}$, we take the unconditional expectations of (3.317) and (3.319), and use (3.25) to make the necessary substitutions. This completes the proof.

3.4 Method of Data Collection and Analysis

In the preceding sections, we have obtained some analytical results for the handling of non-response and measurement errors under poststratified sampling scheme. There is need for numerical illustration and verification of these theoretical results. Consequently, this section discusses the method of data collection and analysis to be employed in carrying out some numerical illustration of the theoretical results obtained in this study. The classes of estimators proposed in this study utilize auxiliary information involving mainly ratio-type and product-type estimators. In the literature, ratio-type estimators are known to have high efficiency when the study and auxiliary variables are positively correlated, while the product-type estimators have high efficiency when there is negative correlation between the study and auxiliary variables. We shall use two datasets for the numerical illustration of the theoretical results obtained in this study. The first dataset (Dataset-1) involves study and auxiliary variables that are positively correlated, while the second dataset (Dataset-2) involves study and auxiliary variables that are negatively correlated.

For the numerical illustration involving positively correlated study and auxiliary variables, we shall use the Australian Survey data (Dataset-1), with sum of skin folds (ssf) as the study variable, y , while percentage Body fat (pcBfaT) is considered as the auxiliary variable, x . Here, the study and auxiliary variables are positively correlated. The respondents are stratified according to their sex with females as Stratum 1 and males as Stratum 2. It is assumed that a random sample of $n = 60$ respondents is taken from the population of $N = 202$ respondents. The summary statistics are shown in Table 3.2.

Similarly, for the numerical illustration involving negatively correlated study and auxiliary variables, we shall use the Fitness data set (Dataset-2), which contains measurements made on groups of men taking a physical fitness course at North Carolina State University. Here, we shall take the study and auxiliary variables to be oxygen (oxygen intake rate, in millilitres per kilogram of body weight per minute) and runtime (time taken to run 1.5 miles, in minutes)

respectively. The study and auxiliary variables are negatively correlated. We stratified the population into two strata, according to the age (in years) of the respondents with Stratum 1 as Younger (35-45yrs) and Stratum 2 as Older (46-60yrs). We assume that a sample of $n = 10$ respondents is randomly selected from the population of $N = 31$ respondents. The summary statistics are as shown in Table 3.3.

Table 3.2: Summary Statistics for Australian Survey data (Dataset-1)

Parameters	Stratum 1 (Females)	Stratum 2 (Males)
N_h	100	102
N_{2h}	40	42
W_h	0.495050	0.504951
W_{2h}	0.4	0.411765
$\bar{Y}_h$	17.849100	9.250882
$\bar{X}_h$	86.973000	51.422549
R_h	0.205226	0.179899
S_{yh}^2	29.734018	10.142270
S_{xh}^2	1145.860577	355.476219
S_{yxh}	178.960117	58.079227
S_{2yh}^2	12.696796	16.307434
S_{2xh}^2	517.849942	552.476986
S_{2yxh}	74.617189	92.682720
C_{xh}	0.389208	0.366650
β_{2xh}	0.775909	1.385599

Source: Australian Survey Data on sex, sport and body-size dependency of haematology in highly trained athletes by Telford and Cunningham (1991)

It should be noted that the values of the study and auxiliary variables, given in Dataset-1 and Dataset-2, are only observed values, which might not necessarily be the true values. Theoretically, the difference between the observed values and the true values constitutes the measurement errors. However, measurement errors are not usually observed in the field or included alongside the observed values. Thus, we shall commence the data analysis in the present study by first simulating the measurement errors associated with the observed values of the various study and auxiliary variables given in Dataset-1 and Dataset-2. The data simulation shall be carried out according to the assumptions made earlier, while deriving the theoretical results of the proposed estimators. The first assumption being that the measurement errors have zero means and they are uncorrelated with the associated study and auxiliary variables. Secondly, the measurement errors for the study variable are equally uncorrelated with the measurement errors for

the auxiliary variable. The required measurement errors, satisfying these assumptions, shall be simulated using the multivariate normal (mvrnorm) procedure in R software.

Table 3.3: Summary Statistics for Fitness data (Dataset-2)

Parameters	Stratum 1 (Younger)	Stratum 2 (Older)
N_h	12	19
N_{2h}	5	9
W_h	0.387097	0.612903
W_{2h}	0.416667	0.473684
$\bar{Y}_h$	48.384667	46.738632
$\bar{X}_h$	10.605833	10.573684
R_h	4.562081	4.420279
S_{yh}^2	49.884866	15.706699
S_{xh}^2	3.332517	1.171236
S_{yxh}	-11.937102	-3.347639
S_{2yh}^2	21.952347	10.297783
S_{2xh}^2	2.590580	0.530878
S_{2yxh}	-6.881608	-1.286200
C_{xh}	0.172124	0.102352
β_{2xh}	0.401876	0.515383

Source: Physical fitness data of athletics at North Carolina State University from SAS Software Version 2.9, SAS Institute, North Carolina

Furthermore, the simulated measurement errors together with the corresponding observed values of the study and auxiliary variables will provide the necessary data for the numerical verification of the theoretical results. Here, the theoretical results that are of major interest are the mean square errors of the proposed estimators. The smaller the mean square error, the more efficient the estimator. In this study, the efficiencies of the proposed estimators shall be compared to that of the usual poststratified sample mean estimator. Consequently, the concluding part of our data analysis shall involve the computation of percentage relative efficiencies of the proposed estimators over the usual poststratified sample mean estimator. The use of R software shall be employed in the data analysis. Specifically, we shall develop an R program for simulating the measurement errors, calculating the mean square errors of the proposed estimators and obtaining the associated percentage relative efficiencies of the proposed estimators over the usual poststratified sample mean estimator. The R codes we developed for the data analysis are shown in Appendix I.

CHAPTER FOUR

4.0 RESULTS AND DISCUSSIONS

For each of the three (3) events considered in the study, that is, when there is nonresponse, when there are measurement errors and when there is joint occurrence of nonresponse and measurement errors in poststratified sampling scheme, we also considered three (3) different cases, namely, the occurrence of an event (i) on both the study and auxiliary variables, (ii) on only the study variable, and (iii) on only the auxiliary variable. These, altogether, gave rise to nine (9) cases, namely:

1. Nonresponse on both the study and auxiliary variables under the poststratified sampling scheme
2. Nonresponse on only the study variable and not on the auxiliary variable under the post-stratified sampling scheme
3. Nonresponse on only the auxiliary variable and not on the study variable under the post-stratified sampling scheme
4. Measurement errors on both the study and auxiliary variables under the poststratified sampling scheme
5. Measurement errors on only the study variable and not on the auxiliary variable under the poststratified sampling scheme

6. Measurement errors on only the auxiliary variable and not on the study variable under the poststratified sampling scheme
7. Joint occurrence of Nonresponse and Measurement errors on both the study and auxiliary variables under the poststratified sampling scheme
8. Joint occurrence of Nonresponse and Measurement errors on only the study variable and not on the auxiliary variable under the poststratified sampling scheme
9. Joint occurrence of Nonresponse and Measurement errors on only the auxiliary variable and not on the study variable under the poststratified sampling scheme.

The present chapter discusses the theoretical results obtained under the above nine (9) cases, and also carries out the associated numerical illustrations, using Dataset-1 and Dataset-2, earlier presented in Section 3.4. We begin our discussion of results with the first case of when there is nonresponse on both the study and auxiliary variables under the poststratified sampling scheme.

4.1 Nonresponse on both the study and auxiliary variables under the Poststratified Sampling Scheme

Recall from (3.5), (3.50) and (3.51), that the proposed class of estimators of the population mean of the study variable (y) when there is nonresponse on both the study and auxiliary variables under the poststratified sampling scheme is given by

$$\bar{y}_{nc1} = \sum_{h=1}^L W_h \bar{y}_h^* \left(\frac{a_h \bar{X}_h + b_h}{a_h \bar{x}_h^* + b_h} \right)^{\alpha_h} \quad (4.1)$$

with its unconditional mean square error obtained as:

$$\begin{aligned} M S E(\bar{y}_{nc1}) = & \left(\frac{1-f}{n} \right) \sum_{h=1}^L W_h (S_{yh}^2 + \alpha_h^2 \lambda_h^2 R_h^2 S_{xh}^2 - 2\alpha_h \lambda_h R_h S_{yxh}) \\ & + \left(\frac{1}{n} \right) \sum_{h=1}^L W_h (g_h - 1) W_{2h} (S_{2yh}^2 + \alpha_h^2 \lambda_h^2 R_h^2 S_{2xh}^2 - 2\alpha_h \lambda_h R_h S_{2yxh}) \end{aligned} \quad (4.2)$$

where

$$\lambda_h = \left(\frac{a_h \bar{X}_h}{a_h \bar{X}_h + b_h} \right) \quad (4.3)$$

It has been noted earlier that the proposed class of estimators, $\bar{y}_{nc1}$, given in (4.1), can generate a wide range of estimators of the population mean ($\bar{Y}$), by making appropriate choices of the constants, α_h , a_h and b_h . Applying the least squares method, it had been earlier stated in (3.69) and Theorem 3.4 that the best choices of α_h , a_h and b_h that minimize the unconditional mean square error of $\bar{y}_{nc1}$ satisfy the expression:

$$\theta_{h1} = \frac{\alpha_h a_h \bar{X}_h R_h}{a_h \bar{X}_h + b_h} = \frac{\alpha_h a_h \bar{Y}_h}{a_h \bar{X}_h + b_h} = \frac{(1-f)S_{yxh} + (g_h - 1)W_{2h}S_{2yxh}}{(1-f)S_{xh}^2 + (g_h - 1)W_{2h}S_{2xh}^2} \quad (4.4)$$

with the associated optimum unconditional mean square error of $\bar{y}_{nc1}$ given by

$$\begin{aligned} MS E_0(\bar{y}_{nc1}) = & \left(\frac{1-f}{n} \right) \sum_{h=1}^L W_h (S_{yh}^2 + \theta_{h1}^2 S_{xh}^2 - 2\theta_{h1} S_{yxh}) \\ & + \left(\frac{1}{n} \right) \sum_{h=1}^L W_h (g_h - 1) W_{2h} (S_{2yh}^2 + \theta_{h1}^2 S_{2xh}^2 - 2\theta_{h1} S_{2yxh}) \end{aligned} \quad (4.5)$$

Notice that (4.4) could be satisfied by making different choices of α_h , a_h and b_h , thereby giving rise to a class of optimum estimators. However, all such optimum estimators would have the same optimum unconditional mean square error of $\bar{y}_{nc1}$ given in (4.5). Consequently, when there is nonresponse on both the study and auxiliary variables under the poststratified sampling scheme, the best estimators in the proposed class of estimators, $\bar{y}_{nc1}$, would all satisfy (4.4) and have the same optimum mean square error given in (4.5).

We shall carry out numerical illustrations of the theoretical results on the proposed class of estimators, $\bar{y}_{nc1}$, on the basis of some special cases of $\bar{y}_{nc1}$, earlier itemized in Table 3.1. Also to be utilized here for numerical illustrations are the proposed customary sample mean estimator, (3.3), and the proposed difference-type estimator, (3.4), when there is nonresponse on both the study and auxiliary variables. The Percentage Relative Efficiencies (PRE) of the proposed estimators over the usual poststratified mean estimator, ($\bar{y}_{ps}$), shall be obtained. The results for Dataset-1 and Dataset-2 are shown in Table 4.1 and Table 4.2 respectively.

Table 4.1: Percentage Relative Efficiency (PRE) of Some Proposed Estimators over the Usual Poststratified Mean Estimator ($\bar{y}_{ps}$) when there is nonresponse on both the study and auxiliary variables for positively correlated study and auxiliary variables (Dataset-1)

Estimators	Subsampling Fraction ($1/g_h$) ; where $g_h = n_{2h}/m_h > 1$ is the subsampling factor								
	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90
$\bar{y}_{ps} = \sum_{h=1}^L W_h \bar{y}_h$	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00
$\bar{y}_{ns1} = \sum_{h=1}^L W_h \bar{y}_h^*$	20.79	37.13	50.31	61.16	70.26	77.99	84.64	90.43	95.51
$\bar{y}_{nd1} = \sum_{h=1}^L W_h [\bar{y}_h^* - k_h(\bar{x}_h^* - \bar{X}_h)], k_h = S_{yxh}/S_{xh}^2 \forall h$	236.62	450.87	645.78	823.86	987.18	1137.53	1276.37	1404.99	1524.47
$\bar{y}_{nc1}(1) = \sum_{h=1}^L W_h \bar{y}_h^* (\bar{X}_h / \bar{x}_h^*)$	134.95	248.22	344.63	427.69	500.00	563.51	619.74	669.88	714.85
$\bar{y}_{nc1}(2) = \sum_{h=1}^L W_h \bar{y}_h^* (\bar{X}_h + C_{xh}) / (\bar{x}_h^* + C_{xh})$	137.25	252.69	351.13	436.07	510.11	575.22	632.92	684.41	730.65
$\bar{y}_{nc1}(3) = \sum_{h=1}^L W_h \bar{y}_h^* (\bar{X}_h + \beta_{2xh}) / (\bar{x}_h^* + \beta_{2xh})$	140.16	258.26	359.15	446.31	522.39	589.36	648.77	701.83	749.50
$\bar{y}_{nc1}(4) = \sum_{h=1}^L W_h \bar{y}_h^* (\bar{X}_h C_{xh} + \beta_{2xh}) / (\bar{x}_h^* C_{xh} + \beta_{2xh})$	147.13	272.10	379.58	472.98	554.92	627.37	691.89	749.72	801.85
$\bar{y}_{nc1}(5) = \sum_{h=1}^L W_h \bar{y}_h^* (\bar{X}_h \beta_{2xh} + C_{xh}) / (\bar{x}_h^* \beta_{2xh} + C_{xh})$	137.66	253.52	352.39	437.74	512.17	577.66	635.71	687.54	734.08
$\bar{y}_{nc1}(6) = \sum_{h=1}^L W_h \bar{y}_h^* (\bar{x}_h^* / \bar{X}_h)$	4.50	7.94	10.67	12.89	14.72	16.26	17.58	18.71	19.70
$\bar{y}_{nc1}(7) = \sum_{h=1}^L W_h \bar{y}_h^* (\bar{x}_h^* + C_{xh}) / (\bar{X}_h + C_{xh})$	4.52	7.99	10.74	12.96	14.81	16.35	17.68	18.82	19.81
$\bar{y}_{nc1}(8) = \sum_{h=1}^L W_h \bar{y}_h^* (\bar{x}_h^* + \beta_{2xh}) / (\bar{X}_h + \beta_{2xh})$	4.58	8.08	10.85	13.09	14.95	16.51	17.83	18.98	19.98
$\bar{y}_{nc1}(9) = \sum_{h=1}^L W_h \bar{y}_h^* (\bar{x}_h^* C_{xh} + \beta_{2xh}) / (\bar{X}_h C_{xh} + \beta_{2xh})$	4.71	8.30	11.13	13.42	15.31	16.90	18.25	19.41	20.42
$\bar{y}_{nc1}(10) = \sum_{h=1}^L W_h \bar{y}_h^* (\bar{x}_h^* \beta_{2xh} + \sigma_{xh}) / (\bar{X}_h \beta_{2xh} + \sigma_{xh})$	6.23	11.11	15.04	18.27	20.97	23.27	25.24	26.95	28.45
Optimum estimators	237.06	451.57	646.61	824.70	987.98	1138.21	1276.90	1405.33	1524.59

Table 4.2: Percentage Relative Efficiency (PRE) of Some Proposed Estimators over the Usual Poststratified Mean Estimator ($\bar{y}_{ps}$) when there is nonresponse on both the study and auxiliary variables for negatively correlated study and auxiliary variables (Dataset-2)

Estimators	Subsampling Fraction ($1/g_h$) ; where $g_h = n_{2h}/m_h > 1$ is the subsampling factor								
	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90
$\bar{y}_{ps} = \sum_{h=1}^L W_h \bar{y}_h$	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00
$\bar{y}_{ns1} = \sum_{h=1}^L W_h \bar{y}_h^*$	25.01	42.87	56.26	66.68	75.01	81.83	87.51	92.31	96.43
$\bar{y}_{nd1} = \sum_{h=1}^L W_h [\bar{y}_h^* - k_h (\bar{x}_h^* - \bar{X}_h)]$, $k_h = S_{yxh}/S_{xh}^2 \forall h$	61.26	117.56	169.46	217.47	262.00	303.43	342.07	378.18	412.01
$\bar{y}_{nc1}(1) = \sum_{h=1}^L W_h \bar{y}_h^* (\bar{X}_h / \bar{x}_h^*)$	5.25	9.13	12.10	14.45	16.36	17.94	19.27	20.41	21.39
$\bar{y}_{nc1}(2) = \sum_{h=1}^L W_h \bar{y}_h^* (\bar{X}_h + C_{xh}) / (\bar{x}_h^* + C_{xh})$	5.34	9.27	12.29	14.68	16.62	18.22	19.57	20.72	21.72
$\bar{y}_{nc1}(3) = \sum_{h=1}^L W_h \bar{y}_h^* (\bar{X}_h + \beta_{2xh}) / (\bar{x}_h^* + \beta_{2xh})$	5.50	9.55	12.65	15.11	17.10	18.75	20.14	21.33	22.35
$\bar{y}_{nc1}(4) = \sum_{h=1}^L W_h \bar{y}_h^* (\bar{X}_h C_{xh} + \beta_{2xh}) / (\bar{x}_h^* C_{xh} + \beta_{2xh})$	6.84	11.88	15.73	18.79	21.26	23.31	25.03	26.50	27.76
$\bar{y}_{nc1}(5) = \sum_{h=1}^L W_h \bar{y}_h^* (\bar{X}_h \beta_{2xh} + C_{xh}) / (\bar{x}_h^* \beta_{2xh} + C_{xh})$	5.46	9.48	12.56	14.99	16.97	18.60	19.98	21.15	22.16
$\bar{y}_{nc1}(6) = \sum_{h=1}^L W_h \bar{y}_h^* (\bar{x}_h^* / \bar{X}_h)$	39.42	76.33	110.96	143.53	174.20	203.14	230.49	256.39	280.93
$\bar{y}_{nc1}(7) = \sum_{h=1}^L W_h \bar{y}_h^* (\bar{x}_h^* + C_{xh}) / (\bar{X}_h + C_{xh})$	40.59	78.60	114.26	147.78	179.35	209.14	237.29	263.93	289.19
$\bar{y}_{nc1}(8) = \sum_{h=1}^L W_h \bar{y}_h^* (\bar{x}_h^* + \beta_{2xh}) / (\bar{X}_h + \beta_{2xh})$	42.79	82.94	120.70	156.26	189.82	221.55	251.58	280.05	307.08
$\bar{y}_{nc1}(9) = \sum_{h=1}^L W_h \bar{y}_h^* (\bar{x}_h^* C_{xh} + \beta_{2xh}) / (\bar{X}_h C_{xh} + \beta_{2xh})$	58.68	113.20	163.97	211.38	255.74	297.34	336.43	373.23	407.93
$\bar{y}_{nc1}(10) = \sum_{h=1}^L W_h \bar{y}_h^* (\bar{x}_h^* \beta_{2xh} + \sigma_{xh}) / (\bar{X}_h \beta_{2xh} + \sigma_{xh})$	60.72	114.89	163.52	207.42	247.25	283.54	316.76	347.27	375.39
Optimum estimators	61.91	118.60	170.71	218.77	263.24	304.50	342.90	378.71	412.19

Table 4.1 shows the percentage relative efficiencies of some proposed estimators over the usual poststratified mean estimator ($\bar{y}_{ps}$), when there is nonresponse on both the study and auxiliary variables for positively correlated study and auxiliary variables. Because of the positive correlation between the study and auxiliary variables, the proposed ratio-type estimators, as expected, performed better than the proposed product-type estimators, in terms of having higher percentage relative efficiencies. Also, the optimum estimators in the proposed class of estimators, ($\bar{y}_{nc1}$), as expected, gave the highest values of the percentage relative efficiencies, especially with increase in the size, m_h , of the subsample. Generally, we observe from Table 4.1 that the efficiencies of the estimators, in terms of having smaller mean square errors and higher percentage relative efficiencies, increase with an increase in the subsampling fraction ($1/g_h$). It could be seen from the table that with subsampling fractions of 0.10 (10%) and above, some of the proposed ratio-type estimators showed high percentage relative efficiencies (above 100%) over the usual poststratified sample mean estimator, $\bar{y}_{ps}$. However, the customary sample mean estimator, $\bar{y}_{ns1} = \sum_{h=1}^L W_h \bar{y}_h^*$, when there is nonresponse on both the study and auxiliary variables, could not provide a better estimate than the usual poststratified sample mean estimator, ($\bar{y}_{ps}$), when there is complete response, even with a subsampling fraction of 0.90 (90%). This necessitated the use of auxiliary information in this study as an attempt to achieving increased efficiency over the usual poststratified sample mean estimator, ($\bar{y}_{ps}$). Among the ten (10) special cases of the proposed class of estimators, ($\bar{y}_{nc1}$), when there is nonresponse on both the study and auxiliary variables, the proposed estimator (utilizing auxiliary information) that has the highest efficiency is the adopted Upadhyaya-Singh (1999) estimator, $\bar{y}_{nc1}(4) = \sum_{h=1}^L W_h \bar{y}_h^* (\bar{X}_h C_{xh} + \beta_{2xh}) / (\bar{x}_h^* C_{xh} + \beta_{2xh})$, followed by the adopted Singh-Kakran (1993) estimator, $\bar{y}_{nc1}(3) = \sum_{h=1}^L W_h \bar{y}_h^* (\bar{X}_h + \beta_{2xh}) / (\bar{x}_h^* + \beta_{2xh})$, while the estimator with the least efficiency among the ten (10) special cases under consideration is the customary product-type estimator, $\bar{y}_{nc1}(6) = \sum_{h=1}^L W_h \bar{y}_h^* (\bar{x}_h^* / \bar{X}_h)$.

We also observe from Table 4.1 that the efficiency of the proposed difference-type estimator, $\bar{y}_{nd1} = \sum_{h=1}^L W_h [\bar{y}_h^* - k_h (\bar{x}_h^* - \bar{X}_h)]$, $k_h = S_{yxh} / S_{xh}^2 \forall h$, comes very close to the efficiency of the optimum estimators in the proposed class of estimators, ($\bar{y}_{nc1}$). This is because our choice of the

constant, $k_h = S_{yxh}/S_{xh}^2 \forall h$, is very close to its optimum value that minimizes the variance of the difference-type estimator, $\bar{y}_{nd1}$. We had earlier observed that when there is nonresponse on both the study and auxiliary variables, the customary sample mean estimator, $\bar{y}_{ns1} = \sum_{h=1}^L W_h \bar{y}_h^*$ is less efficient than the usual poststratified sample mean estimator, $\bar{y}_{ps}$, hence, necessitating the use of auxiliary information to seek for estimators with increased efficiency. However, we observed from Table 4.1 that the use of auxiliary information does not always result in an improved efficiency over the customary sample mean estimators, which do not utilize auxiliary information. For Dataset-1, which involves positively correlated study and auxiliary variables, we observed that the customary sample mean estimator, $\bar{y}_{ns1} = \sum_{h=1}^L W_h \bar{y}_h^*$, was less efficient than the ratio-type estimators in the proposed class of estimators, $\bar{y}_{nc1}$, but more efficient than the product-type estimators, under considerations, in the proposed class. This means that the use of auxiliary information is most likely to result in an improved efficiency over the customary sample mean estimators when estimators that utilize auxiliary information are appropriately chosen, according to the nature and direction of the relationship between the study and auxiliary variables. Table 4.1 specifically reveals that the ratio-type estimators are preferable when there is a positive correlation between the study and auxiliary variables.

For negatively correlated study and auxiliary variables (Dataset-2), and when there is non-response on both the study and auxiliary variables, the percentage relative efficiencies of some proposed estimators over the usual poststratified mean estimator ($\bar{y}_{ps}$), are shown in Table 4.2. The table shows that the proposed product-type estimators, as expected, performed better than the proposed ratio-type estimators, in terms of having higher percentage relative efficiencies, since there is a negative correlation between the study and auxiliary variables. Furthermore, the optimum estimators in the proposed class of estimators, ($\bar{y}_{nc1}$), gave the highest values of the percentage relative efficiencies, especially as the subsampling fraction ($1/g_h$) increases. High percentage relative efficiencies (above 100%) over the usual poststratified sample mean estimator, $\bar{y}_{ps}$, are recorded with subsampling fractions of 0.20 (20%) and above. Just like in the case of positively correlated study and auxiliary variables (Dataset-1), it can be observed for Dataset-2 that the customary sample mean estimator, $\bar{y}_{ns1} = \sum_{h=1}^L W_h \bar{y}_h^*$, when there is nonre-

sponse on both the study and auxiliary variables, could not provide a better estimate than the usual poststratified sample mean estimator, $(\bar{y}_{ps})$, when there is complete response, even with a subsampling fraction of 0.90 (90%).

Table 4.2 also reveals that for the ten (10) special cases of the proposed class of estimators, $(\bar{y}_{nc1})$, when there is nonresponse on both the study and auxiliary variables, the proposed estimator (utilizing auxiliary information) that has the highest efficiency (for subsampling fractions of 30% and above) is the adopted Upadhyaya-Singh (1999) estimator III, $\bar{y}_{nc1}(9) = \sum_{h=1}^L W_h \bar{y}_h^* (\bar{x}_h^* C_{xh} + \beta_{2xh}) / (\bar{X}_h C_{xh} + \beta_{2xh})$, followed by the adopted Singh (2003) estimator, $\bar{y}_{nc1}(10) = \sum_{h=1}^L W_h \bar{y}_h^* (\bar{x}_h^* \beta_{2xh} + \sigma_{xh}) / (\bar{X}_h \beta_{2xh} + \sigma_{xh})$, while the estimator with the least efficiency among the ten (10) special cases under consideration is the customary ratio-type estimator, $\bar{y}_{nc1}(1) = \sum_{h=1}^L W_h \bar{y}_h^* (\bar{X}_h / \bar{x}_h^*)$, when there is nonresponse on both the study and auxiliary variables. The table also reveals that the proposed difference-type estimator, $\bar{y}_{nd1} = \sum_{h=1}^L W_h [\bar{y}_h^* - k_h (\bar{x}_h^* - \bar{X}_h)]$, $k_h = S_{yxh} / S_{xh}^2 \forall h$, is almost as efficient as the optimum estimators in the proposed class of estimators, $(\bar{y}_{nc1})$, since the choice of the constant, $k_h = S_{yxh} / S_{xh}^2 \forall h$, is very close to its optimum value that minimizes the variance of the difference-type estimator, $\bar{y}_{nd1}$. Table 4.2, just like Table 4.1, shows that the use of auxiliary information does not always result in an improved efficiency over the customary sample mean estimators, which do not utilize auxiliary information. For Dataset-2, which involves negatively correlated study and auxiliary variables, we observed from Table 4.2 that the customary sample mean estimator, $\bar{y}_{ns1} = \sum_{h=1}^L W_h \bar{y}_h^*$, was less efficient than the product-type estimators in the proposed class of estimators, $\bar{y}_{nc1}$, but more efficient than the ratio-type estimators, under considerations, in the proposed class. This again indicates that the use of auxiliary information is most likely to result in an improved efficiency over the customary sample mean estimators when estimators that utilize auxiliary information are appropriately chosen, according to the nature and direction of the relationship between the study and auxiliary variables. Table 4.2 specifically reveals that the product-type estimators are preferable when there is a negative correlation between the study and auxiliary variables.

4.2 Nonresponse on only the study variable and not on the auxiliary variable under the poststratified sampling scheme

Recall from (3.77), (3.91) and (3.51), that the proposed class of estimators of the population mean of the study variable (y) when there is nonresponse on only the study variable and not on the auxiliary variable under the poststratified sampling scheme is given by

$$\bar{y}_{nc2} = \sum_{h=1}^L W_h \bar{y}_h^* \left(\frac{a_h \bar{X}_h + b_h}{a_h \bar{x}_h + b_h} \right)^{\alpha_h} \quad (4.6)$$

with its unconditional mean square error obtained as:

$$MSE(\bar{y}_{nc2}) = \left(\frac{1-f}{n} \right) \sum_{h=1}^L W_h (S_{yh}^2 + \alpha_h^2 \lambda_h^2 R_h^2 S_{xh}^2 - 2\alpha_h \lambda_h R_h S_{yxh}) + \left(\frac{1}{n} \right) \sum_{h=1}^L W_h (g_h - 1) W_{2h} S_{2yh}^2 \quad (4.7)$$

where

$$\lambda_h = \left(\frac{a_h \bar{X}_h}{a_h \bar{X}_h + b_h} \right) \quad (4.8)$$

Applying the least squares method, it follows that the best choices of α_h , a_h and b_h that minimize the unconditional mean square error of $\bar{y}_{nc2}$ satisfy the expression:

$$\theta_{h2} = \frac{\alpha_h a_h \bar{X}_h R_h}{a_h \bar{X}_h + b_h} = \frac{\alpha_h a_h \bar{Y}_h}{a_h \bar{X}_h + b_h} = \frac{S_{yxh}}{S_{xh}^2} \quad (4.9)$$

with the associated optimum unconditional mean square error of $\bar{y}_{nc2}$ given by

$$MSE_0(\bar{y}_{nc2}) = \left(\frac{1-f}{n} \right) \sum_{h=1}^L W_h (S_{yh}^2 + \theta_{h2}^2 S_{xh}^2 - 2\theta_{h2} S_{yxh}) + \left(\frac{1}{n} \right) \sum_{h=1}^L W_h (g_h - 1) W_{2h} S_{2yh}^2 \quad (4.10)$$

Table 4.3 shows the Percentage Relative Efficiencies (PRE) of some special cases of the proposed estimators over the usual poststratified mean estimator, $(\bar{y}_{ps})$, using Dataset-1 (positively correlated study and auxiliary variables), when there is nonresponse on only the study variable and not on the auxiliary variable. Similarly, Table 4.4 shows the PREs, using Dataset-2 (negatively correlated study and auxiliary variables), when there is nonresponse on only the study variable and not on the auxiliary variable.

Table 4.3: Percentage Relative Efficiency (PRE) of Some Proposed Estimators over the Usual Poststratified Mean Estimator ($\bar{y}_{ps}$) when there is nonresponse on only the study variable for positively correlated study and auxiliary variables (Dataset-1)

Estimators	Subsampling Fraction ($1/g_h$) ; where $g_h = n_{2h}/m_h > 1$ is the subsampling factor								
	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90
$\bar{y}_{ps} = \sum_{h=1}^L W_h \bar{y}_h$	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00
$\bar{y}_{ns2} = \sum_{h=1}^L W_h \bar{y}_h^*$	20.79	37.13	50.31	61.16	70.26	77.99	84.64	90.43	95.51
$\bar{y}_{nd2} = \sum_{h=1}^L W_h [\bar{y}_h^* - k_h(\bar{x}_h - \bar{X}_h)]$, $k_h = S_{yxh}/S_{xh}^2 \forall h$	25.83	56.99	95.33	143.64	206.40	291.23	412.25	598.90	924.44
$\bar{y}_{nc2}(1) = \sum_{h=1}^L W_h \bar{y}_h^* (\bar{X}_h / \bar{x}_h)$	25.36	54.77	89.27	130.31	179.94	241.19	318.66	419.79	557.37
$\bar{y}_{nc2}(2) = \sum_{h=1}^L W_h \bar{y}_h^* (\bar{X}_h + C_{xh}) / (\bar{x}_h + C_{xh})$	25.38	54.86	89.50	130.80	180.89	242.89	321.64	424.98	566.55
$\bar{y}_{nc2}(3) = \sum_{h=1}^L W_h \bar{y}_h^* (\bar{X}_h + \beta_{2xh}) / (\bar{x}_h + \beta_{2xh})$	25.40	54.96	89.77	131.37	181.97	244.85	325.09	431.02	577.33
$\bar{y}_{nc2}(4) = \sum_{h=1}^L W_h \bar{y}_h^* (\bar{X}_h C_{xh} + \beta_{2xh}) / (\bar{x}_h C_{xh} + \beta_{2xh})$	25.46	55.21	90.45	132.84	184.80	249.99	334.21	447.20	606.74
$\bar{y}_{nc2}(5) = \sum_{h=1}^L W_h \bar{y}_h^* (\bar{X}_h \beta_{2xh} + C_{xh}) / (\bar{x}_h \beta_{2xh} + C_{xh})$	25.39	54.88	89.55	130.91	181.09	243.26	322.28	426.10	568.55
$\bar{y}_{nc2}(6) = \sum_{h=1}^L W_h \bar{y}_h^* (\bar{x}_h / \bar{X}_h)$	11.53	15.26	17.10	18.19	18.92	19.44	19.83	20.13	20.37
$\bar{y}_{nc2}(7) = \sum_{h=1}^L W_h \bar{y}_h^* (\bar{x}_h + C_{xh}) / (\bar{X}_h + C_{xh})$	11.57	15.32	17.17	18.28	19.02	19.54	19.94	20.24	20.48
$\bar{y}_{nc2}(8) = \sum_{h=1}^L W_h \bar{y}_h^* (\bar{x}_h + \beta_{2xh}) / (\bar{X}_h + \beta_{2xh})$	11.62	15.41	17.29	18.42	19.16	19.70	20.09	20.40	20.65
$\bar{y}_{nc2}(9) = \sum_{h=1}^L W_h \bar{y}_h^* (\bar{x}_h C_{xh} + \beta_{2xh}) / (\bar{X}_h C_{xh} + \beta_{2xh})$	11.76	15.66	17.61	18.77	19.55	20.10	20.52	20.84	21.10
$\bar{y}_{nc2}(10) = \sum_{h=1}^L W_h \bar{y}_h^* (\bar{x}_h \beta_{2xh} + \sigma_{xh}) / (\bar{X}_h \beta_{2xh} + \sigma_{xh})$	13.95	19.80	23.01	25.04	26.44	27.47	28.25	28.87	29.37
Optimum estimators	25.83	56.99	95.33	143.64	206.40	291.23	412.25	598.90	924.44

Table 4.4: Percentage Relative Efficiency (PRE) of Some Proposed Estimators over the Usual Poststratified Mean Estimator ($\bar{y}_{ps}$) when there is nonresponse on only the study variable for negatively correlated study and auxiliary variables (Dataset-2)

Estimators	Subsampling Fraction ($1/g_h$) ; where $g_h = n_{2h}/m_h > 1$ is the subsampling factor								
	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90
$\bar{y}_{ps} = \sum_{h=1}^L W_h \bar{y}_h$	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00
$\bar{y}_{ns2} = \sum_{h=1}^L W_h \bar{y}_h^*$	25.01	42.87	56.26	66.68	75.01	81.83	87.51	92.31	96.43
$\bar{y}_{nd2} = \sum_{h=1}^L W_h [\bar{y}_h^* - k_h(\bar{x}_h - \bar{X}_h)]$, $k_h = S_{yxh}/S_{xh}^2 \forall h$	31.02	64.19	99.73	137.92	179.06	223.50	271.65	324.01	381.16
$\bar{y}_{nc2}(1) = \sum_{h=1}^L W_h \bar{y}_h^* (\bar{X}_h / \bar{x}_h)$	13.34	17.16	18.96	20.02	20.71	21.19	21.56	21.84	22.06
$\bar{y}_{nc2}(2) = \sum_{h=1}^L W_h \bar{y}_h^* (\bar{X}_h + C_{xh}) / (\bar{x}_h + C_{xh})$	13.46	17.36	19.21	20.29	21.00	21.50	21.88	22.16	22.39
$\bar{y}_{nc2}(3) = \sum_{h=1}^L W_h \bar{y}_h^* (\bar{X}_h + \beta_{2xh}) / (\bar{x}_h + \beta_{2xh})$	13.70	17.74	19.68	20.82	21.57	22.10	22.49	22.80	23.04
$\bar{y}_{nc2}(4) = \sum_{h=1}^L W_h \bar{y}_h^* (\bar{X}_h C_{xh} + \beta_{2xh}) / (\bar{x}_h C_{xh} + \beta_{2xh})$	15.47	20.85	23.58	25.23	26.33	27.13	27.72	28.19	28.56
$\bar{y}_{nc2}(5) = \sum_{h=1}^L W_h \bar{y}_h^* (\bar{X}_h \beta_{2xh} + C_{xh}) / (\bar{x}_h \beta_{2xh} + C_{xh})$	13.63	17.63	19.54	20.66	21.40	21.92	22.31	22.61	22.85
$\bar{y}_{nc2}(6) = \sum_{h=1}^L W_h \bar{y}_h^* (\bar{x}_h / \bar{X}_h)$	30.06	60.20	90.41	120.71	151.09	181.56	212.10	242.73	273.44
$\bar{y}_{nc2}(7) = \sum_{h=1}^L W_h \bar{y}_h^* (\bar{x}_h + C_{xh}) / (\bar{X}_h + C_{xh})$	30.14	60.54	91.19	122.09	153.26	184.70	216.40	248.38	280.63
$\bar{y}_{nc2}(8) = \sum_{h=1}^L W_h \bar{y}_h^* (\bar{x}_h + \beta_{2xh}) / (\bar{X}_h + \beta_{2xh})$	30.31	61.23	92.78	124.97	157.82	191.35	225.59	260.56	296.28
$\bar{y}_{nc2}(9) = \sum_{h=1}^L W_h \bar{y}_h^* (\bar{x}_h C_{xh} + \beta_{2xh}) / (\bar{X}_h C_{xh} + \beta_{2xh})$	31.01	64.12	99.58	137.62	178.56	222.72	270.51	322.39	378.91
$\bar{y}_{nc2}(10) = \sum_{h=1}^L W_h \bar{y}_h^* (\bar{x}_h \beta_{2xh} + \sigma_{xh}) / (\bar{X}_h \beta_{2xh} + \sigma_{xh})$	30.79	63.22	97.42	133.54	171.74	212.21	255.16	300.83	349.47
Optimum estimators	31.02	64.19	99.73	137.92	179.06	223.50	271.65	324.01	381.16

Table 4.3 shows the percentage relative efficiencies of some proposed estimators over the usual poststratified mean estimator ($\bar{y}_{ps}$), when there is nonresponse on only the study variable and not on the auxiliary variable for positively correlated study and auxiliary variables. Interestingly, the performances of the proposed estimators are similar to those of the proposed estimators when there is nonresponse on both the study and auxiliary variables. First, the proposed ratio-type estimators performed better than the proposed product-type estimators, in terms of having higher percentage relative efficiencies. This is to be expected since Dataset-1 consists of positively correlated study and auxiliary variables. The efficiencies of the estimators, in terms of having smaller mean square errors and higher percentage relative efficiencies, increase with an increase in the subsampling fraction ($1/g_h$). With subsampling fractions of 0.40 (40%) and above, some of the proposed ratio-type estimators showed high percentage relative efficiencies (above 100%) over the usual poststratified sample mean estimator, $\bar{y}_{ps}$. The customary sample mean estimator, $\bar{y}_{ns2} = \sum_{h=1}^L W_h \bar{y}_h^*$, when there is nonresponse on only the study variable and not on the auxiliary variable, could not provide a better estimate than the usual poststratified sample mean estimator, ($\bar{y}_{ps}$), where there is complete response, even with a subsampling fraction of 0.90 (90%). Given the ten (10) special cases of the proposed class of estimators, ($\bar{y}_{nc2}$), the proposed estimator (utilizing auxiliary information) that has the highest efficiency is the adopted Upadhyaya-Singh (1999) estimator, $\bar{y}_{nc2}(4) = \sum_{h=1}^L W_h \bar{y}_h^* (\bar{X}_h C_{xh} + \beta_{2xh}) / (\bar{x}_h C_{xh} + \beta_{2xh})$, followed by the adopted Singh-Kakran (1993) estimator, $\bar{y}_{nc2}(3) = \sum_{h=1}^L W_h \bar{y}_h^* (\bar{X}_h + \beta_{2xh}) / (\bar{x}_h + \beta_{2xh})$, while the estimator with the least efficiency among the ten (10) special cases under consideration is the customary product-type estimator, $\bar{y}_{nc2}(6) = \sum_{h=1}^L W_h \bar{y}_h^* (\bar{x}_h / \bar{X}_h)$, when there is nonresponse on only the study variable and not on the auxiliary variable.

From Table 4.3, it can also be seen that the efficiency of the proposed difference-type estimator, $\bar{y}_{nd2} = \sum_{h=1}^L W_h [\bar{y}_h^* - k_h (\bar{x}_h - \bar{X}_h)]$, $k_h = S_{yxh} / S_{xh}^2 \forall h$, is the same as the efficiency of the optimum estimators in the proposed class of estimators, ($\bar{y}_{nc2}$). This is because the choice of the constant, $k_h = S_{yxh} / S_{xh}^2 \forall h$, is the same optimum value that minimizes the mean square error of the proposed class of estimators, $\bar{y}_{nc2}$. Furthermore, it can be observed that when there is nonresponse on only the study variable and not on the auxiliary variable, the customary sample mean

estimator, $\bar{y}_{ns2} = \sum_{h=1}^L W_h \bar{y}_h^*$ is less efficient than the ratio-type estimators in the proposed class of estimators, $\bar{y}_{nc2}$, but more efficient than the suggested product-type estimators. Thus, the use of auxiliary information is most likely to result in an improved efficiency over the customary sample mean estimators when estimators that utilize auxiliary information are appropriately chosen, according to the nature and direction of the relationship between the study and auxiliary variables. Table 4.3 specifically reveals that the ratio-type estimators are preferable when there is a positive correlation between the study and auxiliary variables.

For negatively correlated study and auxiliary variables (Dataset-2), and when there is non-response on only the study variable and not on the auxiliary variable, the percentage relative efficiencies of some proposed estimators over the usual poststratified mean estimator ($\bar{y}_{ps}$), are shown in Table 4.4. The table shows that the proposed product-type estimators performed better than the proposed ratio-type estimators, in terms of having higher percentage relative efficiencies, since there is a negative correlation between the study and auxiliary variables. High percentage relative efficiencies (above 100%) over the usual poststratified sample mean estimator, $\bar{y}_{ps}$, are recorded with subsampling fractions of 0.40 (40%) and above. The customary sample mean estimator, $\bar{y}_{ns2} = \sum_{h=1}^L W_h \bar{y}_h^*$, when there is nonresponse on only the study variable and not on the auxiliary variable, could not provide a better estimate than the usual poststratified sample mean estimator, ($\bar{y}_{ps}$), when there is complete response, even with a subsampling fraction of 0.90 (90%).

Table 4.4 also reveals that for the ten (10) special cases of the proposed class of estimators, ($\bar{y}_{nc2}$), when there is nonresponse on only the study variable and not on the auxiliary variable, the proposed estimator that has the highest efficiency is the adopted Upadhyaya-Singh (1999) estimator III, $\bar{y}_{nc2}(9) = \sum_{h=1}^L W_h \bar{y}_h^* (\bar{x}_h C_{xh} + \beta_{2xh}) / (\bar{X}_h C_{xh} + \beta_{2xh})$, followed by the adopted Singh (2003) estimator, $\bar{y}_{nc2}(10) = \sum_{h=1}^L W_h \bar{y}_h^* (\bar{x}_h \beta_{2xh} + \sigma_{xh}) / (\bar{X}_h \beta_{2xh} + \sigma_{xh})$, while the proposed estimator with the least efficiency among the ten (10) special cases under consideration is the customary ratio-type estimator, $\bar{y}_{nc2}(1) = \sum_{h=1}^L W_h \bar{y}_h^* (\bar{X}_h / \bar{x}_h)$. The proposed difference-type estimator,

$\bar{y}_{nd2} = \sum_{h=1}^L W_h [\bar{y}_h^* - k_h(\bar{x}_h - \bar{X}_h)]$, $k_h = S_{yxh}/S_{xh}^2 \forall h$, is as efficient as the optimum estimators in the proposed class of estimators, $(\bar{y}_{nc2})$, since the choice of the constant, $k_h = S_{yxh}/S_{xh}^2 \forall h$, is the same optimum value of the proposed class, $\bar{y}_{nc2}$. The use of auxiliary information does not always result in an improved efficiency over the customary sample mean estimators, which do not utilize auxiliary information. This is notable we observed from Table 4.4 where the customary sample mean estimator, $\bar{y}_{ns2} = \sum_{h=1}^L W_h \bar{y}_h^*$, was less efficient than the product-type estimators in the proposed class of estimators, $\bar{y}_{nc2}$, but more efficient than the suggested ratio-type estimators. Consequently, the use of auxiliary information is most likely to result in an improved efficiency over the customary sample mean estimators when estimators that utilize auxiliary information are appropriately chosen, according to the nature and direction of the relationship between the study and auxiliary variables. Table 4.4 specifically reveals that the product-type estimators are preferable when there is a negative correlation between the study and auxiliary variables.

4.3 Nonresponse on only the auxiliary variable and not on the study variable under the poststratified sampling scheme

Recall from (3.98), (3.114) and (3.51), that the proposed class of estimators of the population mean of the study variable (y) when there is nonresponse on only the auxiliary variable and not on the study variable under the poststratified sampling scheme is given by

$$\bar{y}_{nc3} = \sum_{h=1}^L W_h \bar{y}_h \left(\frac{a_h \bar{X}_h + b_h}{a_h \bar{x}_h^* + b_h} \right)^{\alpha_h} \quad (4.11)$$

with its unconditional mean square error obtained as:

$$\begin{aligned}
MS E(\bar{y}_{nc3}) = & \left(\frac{1-f}{n} \right) \sum_{h=1}^L W_h (S_{yh}^2 + \alpha_h^2 \lambda_h^2 R_h^2 S_{xh}^2 - 2\alpha_h \lambda_h R_h S_{yxh}) \\
& + \left(\frac{1}{n} \right) \sum_{h=1}^L W_h (g_h - 1) \alpha_h^2 \lambda_h^2 R_h^2 W_{2h} S_{2xh}^2
\end{aligned} \tag{4.12}$$

where

$$\lambda_h = \left(\frac{a_h \bar{X}_h}{a_h \bar{X}_h + b_h} \right) \tag{4.13}$$

Using the method of least squares, it follows that the best choices of α_h , a_h and b_h that minimize the unconditional mean square error of $\bar{y}_{nc3}$ satisfy the expression:

$$\theta_{h3} = \frac{\alpha_h a_h \bar{X}_h R_h}{a_h \bar{X}_h + b_h} = \frac{\alpha_h a_h \bar{Y}_h}{a_h \bar{X}_h + b_h} = \frac{(1-f) S_{yxh}}{(1-f) S_{xh}^2 + (g_h - 1) W_{2h} S_{2xh}^2} \tag{4.14}$$

with the associated optimum unconditional mean square error of $\bar{y}_{nc3}$ given by

$$\begin{aligned}
MS E_0(\bar{y}_{nc3}) = & \left(\frac{1-f}{n} \right) \sum_{h=1}^L W_h (S_{yh}^2 + \theta_{h3}^2 S_{xh}^2 - 2\theta_{h3} S_{yxh}) \\
& + \left(\frac{1}{n} \right) \sum_{h=1}^L W_h (g_h - 1) W_{2h} \theta_{h3}^2 S_{2xh}^2
\end{aligned} \tag{4.15}$$

Table 4.5 shows the Percentage Relative Efficiencies (PRE) of some special cases of the proposed estimators over the usual poststratified mean estimator, $(\bar{y}_{ps})$, using Dataset-1 (positively correlated study and auxiliary variables), when there is nonresponse on only the auxiliary variable and not on the study variable, while, Table 4.6 shows similar results, using Dataset-2 (negatively correlated study and auxiliary variables).

Table 4.5: Percentage Relative Efficiency (PRE) of Some Proposed Estimators over the Usual Poststratified Mean Estimator ($\bar{y}_{ps}$) when there is nonresponse on only the auxiliary variable for positively correlated study and auxiliary variables (Dataset-1)

Estimators	Subsampling Fraction ($1/g_h$) ; where $g_h = n_{2h}/m_h > 1$ is the subsampling factor								
	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90
$\bar{y}_{ps} = \sum_{h=1}^L W_h \bar{y}_h$	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00
$\bar{y}_{ns3} = \sum_{h=1}^L W_h \bar{y}_h$	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00
$\bar{y}_{nd3} = \sum_{h=1}^L W_h [\bar{y}_h - k_h(\bar{x}_h^* - \bar{X}_h)]$, $k_h = S_{yxh}/S_{xh}^2 \forall h$	27.37	60.32	100.75	151.54	217.24	305.57	430.64	621.39	947.98
$\bar{y}_{nc3}(1) = \sum_{h=1}^L W_h \bar{y}_h (\bar{X}_h / \bar{x}_h^*)$	18.80	41.03	67.71	100.33	141.13	193.61	263.63	361.76	509.17
$\bar{y}_{nc3}(2) = \sum_{h=1}^L W_h \bar{y}_h (\bar{X}_h + C_{xh}) / (\bar{x}_h^* + C_{xh})$	19.02	41.52	68.55	101.63	143.03	196.37	267.65	367.80	518.76
$\bar{y}_{nc3}(3) = \sum_{h=1}^L W_h \bar{y}_h (\bar{X}_h + \beta_{2xh}) / (\bar{x}_h^* + \beta_{2xh})$	19.46	42.48	70.14	104.00	146.39	201.02	274.08	376.78	531.77
$\bar{y}_{nc3}(4) = \sum_{h=1}^L W_h \bar{y}_h (\bar{X}_h C_{xh} + \beta_{2xh}) / (\bar{x}_h^* C_{xh} + \beta_{2xh})$	20.56	44.91	74.19	110.06	155.04	213.10	290.93	400.67	567.04
$\bar{y}_{nc3}(5) = \sum_{h=1}^L W_h \bar{y}_h (\bar{X}_h \beta_{2xh} + C_{xh}) / (\bar{x}_h^* \beta_{2xh} + C_{xh})$	19.02	41.52	68.55	101.65	143.11	196.54	268.02	368.56	520.37
$\bar{y}_{nc3}(6) = \sum_{h=1}^L W_h \bar{y}_h (\bar{x}_h^* / \bar{X}_h)$	9.95	13.95	16.11	17.46	18.39	19.06	19.58	19.98	20.30
$\bar{y}_{nc3}(7) = \sum_{h=1}^L W_h \bar{y}_h (\bar{x}_h^* + C_{xh}) / (\bar{X}_h + C_{xh})$	10.04	14.06	16.22	17.58	18.50	19.18	19.69	20.09	20.42
$\bar{y}_{nc3}(8) = \sum_{h=1}^L W_h \bar{y}_h (\bar{x}_h^* + \beta_{2xh}) / (\bar{X}_h + \beta_{2xh})$	10.20	14.24	16.41	17.76	18.68	19.36	19.87	20.27	20.59
$\bar{y}_{nc3}(9) = \sum_{h=1}^L W_h \bar{y}_h (\bar{x}_h^* C_{xh} + \beta_{2xh}) / (\bar{X}_h C_{xh} + \beta_{2xh})$	10.60	14.70	16.89	18.24	19.16	19.83	20.33	20.73	21.05
$\bar{y}_{nc3}(10) = \sum_{h=1}^L W_h \bar{y}_h (\bar{x}_h^* \beta_{2xh} + \sigma_{xh}) / (\bar{X}_h \beta_{2xh} + \sigma_{xh})$	16.41	21.86	24.59	26.22	27.31	28.08	28.67	29.12	29.48
Optimum estimators	31.05	68.17	113.34	169.51	241.24	336.03	467.15	660.44	973.81

Table 4.6: Percentage Relative Efficiency (PRE) of Some Proposed Estimators over the Usual Poststratified Mean Estimator ($\bar{y}_{ps}$) when there is nonresponse on only the auxiliary variable for negatively correlated study and auxiliary variables (Dataset-2)

Estimators	Subsampling Fraction ($1/g_h$) ; where $g_h = n_{2h}/m_h > 1$ is the subsampling factor								
	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90
$\bar{y}_{ps} = \sum_{h=1}^L W_h \bar{y}_h$	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00
$\bar{y}_{ns3} = \sum_{h=1}^L W_h \bar{y}_h$	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00
$\bar{y}_{nd3} = \sum_{h=1}^L W_h [\bar{y}_h - k_h(\bar{x}_h^* - \bar{X}_h)]$, $k_h = S_{yxh}/S_{xh}^2 \forall h$	30.63	63.44	98.68	136.62	177.60	221.98	270.21	322.82	380.42
$\bar{y}_{nc3}(1) = \sum_{h=1}^L W_h \bar{y}_h (\bar{X}_h / \bar{x}_h^*)$	10.13	14.52	16.98	18.55	19.63	20.43	21.04	21.53	21.92
$\bar{y}_{nc3}(2) = \sum_{h=1}^L W_h \bar{y}_h (\bar{X}_h + C_{xh}) / (\bar{x}_h^* + C_{xh})$	10.36	14.82	17.29	18.87	19.97	20.77	21.38	21.87	22.26
$\bar{y}_{nc3}(3) = \sum_{h=1}^L W_h \bar{y}_h (\bar{X}_h + \beta_{2xh}) / (\bar{x}_h^* + \beta_{2xh})$	10.79	15.36	17.89	19.49	20.60	21.41	22.03	22.52	22.91
$\bar{y}_{nc3}(4) = \sum_{h=1}^L W_h \bar{y}_h (\bar{X}_h C_{xh} + \beta_{2xh}) / (\bar{x}_h^* C_{xh} + \beta_{2xh})$	14.75	20.25	23.13	24.89	26.09	26.95	27.61	28.12	28.53
$\bar{y}_{nc3}(5) = \sum_{h=1}^L W_h \bar{y}_h (\bar{X}_h \beta_{2xh} + C_{xh}) / (\bar{x}_h^* \beta_{2xh} + C_{xh})$	10.68	15.22	17.73	19.32	20.42	21.22	21.84	22.32	22.72
$\bar{y}_{nc3}(6) = \sum_{h=1}^L W_h \bar{y}_h (\bar{x}_h^* / \bar{X}_h)$	17.53	36.80	58.06	81.66	108.00	137.58	171.04	209.20	253.13
$\bar{y}_{nc3}(7) = \sum_{h=1}^L W_h \bar{y}_h (\bar{x}_h^* + C_{xh}) / (\bar{X}_h + C_{xh})$	18.04	37.87	59.76	84.05	111.16	141.61	176.05	215.33	260.55
$\bar{y}_{nc3}(8) = \sum_{h=1}^L W_h \bar{y}_h (\bar{x}_h^* + \beta_{2xh}) / (\bar{X}_h + \beta_{2xh})$	19.00	39.90	63.00	88.67	117.37	149.66	186.27	228.11	276.41
$\bar{y}_{nc3}(9) = \sum_{h=1}^L W_h \bar{y}_h (\bar{x}_h^* C_{xh} + \beta_{2xh}) / (\bar{X}_h C_{xh} + \beta_{2xh})$	28.22	58.79	92.02	128.27	167.96	211.61	259.86	313.46	373.35
$\bar{y}_{nc3}(10) = \sum_{h=1}^L W_h \bar{y}_h (\bar{x}_h^* \beta_{2xh} + \sigma_{xh}) / (\bar{X}_h \beta_{2xh} + \sigma_{xh})$	31.56	64.65	99.40	135.92	174.36	214.87	257.63	302.82	350.66
Optimum estimators	33.49	68.82	106.13	145.60	187.42	231.81	279.01	329.31	383.01

Table 4.5 shows the percentage relative efficiencies of some proposed estimators over the usual poststratified mean estimator ($\bar{y}_{ps}$), when there is nonresponse on only the auxiliary variable and not on the study variable for positively correlated study and auxiliary variables. It is worthy of note that the performances of the proposed estimators under this design are similar to those of the proposed estimators under the earlier discussed two situations, namely, when there is nonresponse on both the study and auxiliary variables and when there is nonresponse on only the study variable and not on the auxiliary variable. Just like in the earlier discussed situations, Table 4.5 shows that the proposed ratio-type estimators performed better than the proposed product-type estimators, in terms of having higher percentage relative efficiencies. This as earlier postulated is because Dataset-1 consists of positively correlated study and auxiliary variables. We also observe that the efficiencies of the estimators, in terms of having smaller mean square errors and higher percentage relative efficiencies, increase with an increase in the subsampling fraction ($1/g_h$). With subsampling fractions of 0.40 (40%) and above, some of the proposed ratio-type estimators showed high percentage relative efficiencies (above 100%) over the usual poststratified sample mean estimator, $\bar{y}_{ps}$. Interestingly, the customary sample mean estimator, $\bar{y}_{ns3} = \sum_{h=1}^L W_h \bar{y}_h$, under this design is the same as the usual poststratified sample mean estimator, ($\bar{y}_{ps}$), when there is complete response. Furthermore, considering the ten (10) special cases of the proposed class of estimators, ($\bar{y}_{nc3}$), when there is nonresponse on only the auxiliary variable and not on the study variable, Table 4.5 shows that the proposed estimator (utilizing auxiliary information) that has the highest efficiency is the adopted Upadhyaya-Singh (1999) estimator, $\bar{y}_{nc3}(4) = \sum_{h=1}^L W_h \bar{y}_h (\bar{X}_h C_{xh} + \beta_{2xh}) / (\bar{x}_h^* C_{xh} + \beta_{2xh})$, followed by the adopted Singh-Kakran (1993) estimator, $\bar{y}_{nc3}(3) = \sum_{h=1}^L W_h \bar{y}_h (\bar{X}_h + \beta_{2xh}) / (\bar{x}_h^* + \beta_{2xh})$, while the estimator with the least efficiency among the ten (10) special cases under consideration is the customary product-type estimator, $\bar{y}_{nc3}(6) = \sum_{h=1}^L W_h \bar{y}_h (\bar{x}_h^* / \bar{X}_h)$, when there is nonresponse on only the auxiliary variable and not on the study variable.

It was also observed from Table 4.5 that the efficiency of the proposed difference-type estimator, $\bar{y}_{nd3} = \sum_{h=1}^L W_h [\bar{y}_h - k_h (\bar{x}_h^* - \bar{X}_h)]$, $k_h = S_{yxh} / S_{xh}^2 \forall h$, is close to the efficiency of the optimum estimators in the proposed class of estimators, ($\bar{y}_{nc3}$), for reasons earlier discussed.

Furthermore, it was observed that under this design, the customary sample mean estimator, $\bar{y}_{ns3} = \sum_{h=1}^L W_h \bar{y}_h$, which happens to be the same as the usual poststratified sample mean estimator, $\bar{y}_{ps}$, is less efficient than the ratio-type estimators in the proposed class of estimators, $\bar{y}_{nc3}$, (for subsampling fractions of 0.40 or 40% and above) but more efficient than the suggested product-type estimators. This confirms that the use of auxiliary information is most likely to result in an improved efficiency over the customary sample mean estimators when estimators that utilize auxiliary information are appropriately chosen, according to the nature and direction of the relationship between the study and auxiliary variables. Table 4.5 specifically reveals that the ratio-type estimators are preferable when there is a positive correlation between the study and auxiliary variables.

For negatively correlated study and auxiliary variables (Dataset-2), and when there is non-response on only the auxiliary variable and not on the study variable, the percentage relative efficiencies of some proposed estimators over the usual poststratified mean estimator ($\bar{y}_{ps}$), are shown in Table 4.6. The table shows that the proposed product-type estimators, as expected, performed better than the proposed ratio-type estimators, in terms of having higher percentage relative efficiencies, since there is a negative correlation between the study and auxiliary variables. High percentage relative efficiencies (above 100%) were recorded over the usual poststratified sample mean estimator, $\bar{y}_{ps}$, for subsampling fractions of 0.40 (40%) and above. Table 4.6 also reveals that for the ten (10) special cases of the proposed class of estimators, ($\bar{y}_{nc3}$), under this design and for negatively correlated dataset, the proposed estimator that has the highest efficiency is the adopted Upadhyaya-Singh (1999) estimator III, $\bar{y}_{nc3}(9) = \sum_{h=1}^L W_h \bar{y}_h (\bar{x}_h^* C_{xh} + \beta_{2xh}) / (\bar{X}_h C_{xh} + \beta_{2xh})$, followed by the adopted Singh (2003) estimator, $\bar{y}_{nc3}(10) = \sum_{h=1}^L W_h \bar{y}_h (\bar{x}_h^* \beta_{2xh} + \sigma_{xh}) / (\bar{X}_h \beta_{2xh} + \sigma_{xh})$, while the proposed estimator with the least efficiency among the ten (10) special cases under consideration is the customary ratio-type estimator, $\bar{y}_{nc3}(1) = \sum_{h=1}^L W_h \bar{y}_h (\bar{X}_h / \bar{x}_h^*)$. The proposed difference-type estimator, $\bar{y}_{nd3} = \sum_{h=1}^L W_h [\bar{y}_h - k_h (\bar{x}_h^* - \bar{X}_h)]$, $k_h = S_{yxh} / S_{xh}^2 \forall h$, is nearly as efficient as the optimum estimators in the proposed class of estimators, ($\bar{y}_{nc3}$), because of the choice of the constant, $k_h = S_{yxh} / S_{xh}^2 \forall h$, as earlier pointed out for other designs. Further observations from Table

4.6 show that the customary sample mean estimator, $\bar{y}_{ns3} = \sum_{h=1}^L W_h \bar{y}_h$, was less efficient than the product-type estimators in the proposed class of estimators, $\bar{y}_{nc3}$, but more efficient than the suggested ratio-type estimators. This validates the fact that, the use of auxiliary information is most likely to result in an improved efficiency over the customary sample mean estimators when estimators that utilize auxiliary information are appropriately chosen, according to the nature and direction of the relationship between the study and auxiliary variables. Table 4.6 specifically reveals that the product-type estimators are preferable when there is a negative correlation between the study and auxiliary variables.

4.4 Measurement Errors on both the study and auxiliary variables under the poststratified sampling scheme

Recall from (3.123), (3.141) and (3.51), that the proposed class of estimators of the population mean of the study variable (y) when there are measurement errors on both the study and auxiliary variables under the poststratified sampling scheme is given by

$$\bar{y}_{mc1} = \sum_{h=1}^L W_h \bar{y}_h'' \left(\frac{a_h \bar{X}_h + b_h}{a_h \bar{x}_h'' + b_h} \right)^{\alpha_h} \quad (4.16)$$

with its unconditional mean square error obtained as:

$$\begin{aligned} M S E(\bar{y}_{mc1}) = & \left(\frac{1-f}{n} \right) \sum_{h=1}^L W_h (S_{yh}^2 + \alpha_h^2 \lambda_h^2 R_h^2 S_{xh}^2 - 2\alpha_h \lambda_h R_h S_{y x h}) \\ & + \left(\frac{1-f}{n} \right) \sum_{h=1}^L W_h (S_{uh}^2 + \alpha_h^2 \lambda_h^2 R_h^2 S_{vh}^2) \end{aligned} \quad (4.17)$$

where

$$\lambda_h = \left(\frac{a_h \bar{X}_h}{a_h \bar{X}_h + b_h} \right) \quad (4.18)$$

Applying the least squares method, it follows that the best choices of α_h , a_h and b_h that minimize the unconditional mean square error of $\bar{y}_{mc1}$ satisfy the expression:

$$\theta_{h4} = \frac{\alpha_h a_h \bar{X}_h R_h}{a_h \bar{X}_h + b_h} = \frac{\alpha_h a_h \bar{Y}_h}{a_h \bar{X}_h + b_h} = \frac{S_{yxh}}{S_{xh}^2 + S_{vh}^2} \quad (4.19)$$

with the associated optimum unconditional mean square error of $\bar{y}_{mc1}$ given by

$$MS E_0(\bar{y}_{mc1}) = \left(\frac{1-f}{n} \right) \sum_{h=1}^L W_h (S_{yh}^2 + \theta_{h4}^2 S_{xh}^2 - 2\theta_{h4} S_{yxh} + S_{uh}^2 + \theta_{h4}^2 S_{vh}^2) \quad (4.20)$$

Table 4.7 shows the Percentage Relative Efficiencies (PRE) of some special cases of the proposed estimators over the usual poststratified mean estimator, $(\bar{y}_{ps})$, using Dataset-1 (positively correlated study and auxiliary variables), when there are measurement errors on both the study and auxiliary variables. Similarly, Table 4.8 shows the PREs, using Dataset-2 (negatively correlated study and auxiliary variables), when there are measurement errors on both the study and auxiliary variables.

Table 4.7: Percentage Relative Efficiency (PRE) of Some Proposed Estimators over the Usual Poststratified Mean Estimator ($\bar{y}_{ps}$) with measurement errors on both the study and auxiliary variables for positively correlated study and auxiliary variables (Dataset-1)

Estimator	PRE
$\bar{y}_{ps} = \sum_{h=1}^L W_h \bar{y}_h$	100.00
$\bar{y}_{ms1} = \sum_{h=1}^L W_h \bar{y}_h''$	95.20
$\bar{y}_{md1} = \sum_{h=1}^L W_h [\bar{y}_h'' - k_h(\bar{x}_h'' - \bar{X}_h)], k_h = S_{yxh}/S_{xh}^2 \forall h$	886.35
$\bar{y}_{mc1}(1) = \sum_{h=1}^L W_h \bar{y}_h'' (\bar{X}_h / \bar{x}_h'')$	541.57
$\bar{y}_{mc1}(2) = \sum_{h=1}^L W_h \bar{y}_h'' (\bar{X}_h + C_{xh}) / (\bar{x}_h'' + C_{xh})$	550.29
$\bar{y}_{mc1}(3) = \sum_{h=1}^L W_h \bar{y}_h'' (\bar{X}_h + \beta_{2xh}) / (\bar{x}_h'' + \beta_{2xh})$	560.59
$\bar{y}_{mc1}(4) = \sum_{h=1}^L W_h \bar{y}_h'' (\bar{X}_h C_{xh} + \beta_{2xh}) / (\bar{x}_h'' C_{xh} + \beta_{2xh})$	588.60
$\bar{y}_{mc1}(5) = \sum_{h=1}^L W_h \bar{y}_h'' (\bar{X}_h \beta_{2xh} + C_{xh}) / (\bar{x}_h'' \beta_{2xh} + C_{xh})$	552.18
$\bar{y}_{mc1}(6) = \sum_{h=1}^L W_h \bar{y}_h'' (\bar{x}_h'' / \bar{X}_h)$	20.35
$\bar{y}_{mc1}(7) = \sum_{h=1}^L W_h \bar{y}_h'' (\bar{x}_h'' + C_{xh}) / (\bar{X}_h + C_{xh})$	20.46
$\bar{y}_{mc1}(8) = \sum_{h=1}^L W_h \bar{y}_h'' (\bar{x}_h'' + \beta_{2xh}) / (\bar{X}_h + \beta_{2xh})$	20.63
$\bar{y}_{mc1}(9) = \sum_{h=1}^L W_h \bar{y}_h'' (\bar{x}_h'' C_{xh} + \beta_{2xh}) / (\bar{X}_h C_{xh} + \beta_{2xh})$	21.08
$\bar{y}_{mc1}(10) = \sum_{h=1}^L W_h \bar{y}_h'' (\bar{x}_h'' \beta_{2xh} + \sigma_{xh}) / (\bar{X}_h \beta_{2xh} + \sigma_{xh})$	29.33
Optimum estimators	886.37

Table 4.7 shows the percentage relative efficiencies of some proposed estimators over the usual poststratified mean estimator ($\bar{y}_{ps}$), when there are measurement errors on both the study and auxiliary variables for positively correlated study and auxiliary variables (Dataset-1). The table shows that the proposed ratio-type estimators, as expected, performed better than the proposed product-type estimators, in terms of having higher percentage relative efficiencies, since Dataset-1 consists of positively correlated study and auxiliary variables. Considering the ten (10) special cases of the proposed class of estimators, ($\bar{y}_{mc1}$), when there are measurement errors on both the study and auxiliary variables, Table 4.7 indicates that the proposed estimator (utilizing auxiliary information) that has the highest efficiency is the adopted Upadhyaya-Singh

(1999) estimator, $\bar{y}_{mc1}(4) = \sum_{h=1}^L W_h \bar{y}_h'' (\bar{X}_h C_{xh} + \beta_{2xh}) / (\bar{x}_h'' C_{xh} + \beta_{2xh})$, followed by the adopted Singh-Kakran (1993) estimator, $\bar{y}_{mc1}(3) = \sum_{h=1}^L W_h \bar{y}_h'' (\bar{X}_h + \beta_{2xh}) / (\bar{x}_h'' + \beta_{2xh})$, while the estimator with the least efficiency among the ten (10) special cases under consideration is the customary product-type estimator, $\bar{y}_{mc1}(6) = \sum_{h=1}^L W_h \bar{y}_h'' (\bar{x}_h'' / \bar{X}_h)$, under this design.

Table 4.8: Percentage Relative Efficiency (PRE) of Some Proposed Estimators over the Usual Poststratified Mean Estimator ($\bar{y}_{ps}$) with measurement errors on both the study and auxiliary variables for negatively correlated study and auxiliary variables (Dataset-2)

Estimator	PRE
$\bar{y}_{ps} = \sum_{h=1}^L W_h \bar{y}_h$	100.00
$\bar{y}_{ms1} = \sum_{h=1}^L W_h \bar{y}_h''$	96.66
$\bar{y}_{md1} = \sum_{h=1}^L W_h [\bar{y}_h'' - k_h (\bar{x}_h'' - \bar{X}_h)], k_h = S_{yxh} / S_{xh}^2 \forall h$	165.41
$\bar{y}_{mc1}(1) = \sum_{h=1}^L W_h \bar{y}_h'' (\bar{X}_h / \bar{x}_h'')$	19.15
$\bar{y}_{mc1}(2) = \sum_{h=1}^L W_h \bar{y}_h'' (\bar{X}_h + C_{xh}) / (\bar{x}_h'' + C_{xh})$	19.46
$\bar{y}_{mc1}(3) = \sum_{h=1}^L W_h \bar{y}_h'' (\bar{X}_h + \beta_{2xh}) / (\bar{x}_h'' + \beta_{2xh})$	20.11
$\bar{y}_{mc1}(4) = \sum_{h=1}^L W_h \bar{y}_h'' (\bar{X}_h C_{xh} + \beta_{2xh}) / (\bar{x}_h'' C_{xh} + \beta_{2xh})$	25.80
$\bar{y}_{mc1}(5) = \sum_{h=1}^L W_h \bar{y}_h'' (\bar{X}_h \beta_{2xh} + C_{xh}) / (\bar{x}_h'' \beta_{2xh} + C_{xh})$	19.88
$\bar{y}_{mc1}(6) = \sum_{h=1}^L W_h \bar{y}_h'' (\bar{x}_h'' / \bar{X}_h)$	94.74
$\bar{y}_{mc1}(7) = \sum_{h=1}^L W_h \bar{y}_h'' (\bar{x}_h'' + C_{xh}) / (\bar{X}_h + C_{xh})$	97.14
$\bar{y}_{mc1}(8) = \sum_{h=1}^L W_h \bar{y}_h'' (\bar{x}_h'' + \beta_{2xh}) / (\bar{X}_h + \beta_{2xh})$	103.12
$\bar{y}_{mc1}(9) = \sum_{h=1}^L W_h \bar{y}_h'' (\bar{x}_h'' C_{xh} + \beta_{2xh}) / (\bar{X}_h C_{xh} + \beta_{2xh})$	156.65
$\bar{y}_{mc1}(10) = \sum_{h=1}^L W_h \bar{y}_h'' (\bar{x}_h'' \beta_{2xh} + \sigma_{xh}) / (\bar{X}_h \beta_{2xh} + \sigma_{xh})$	141.20
Optimum estimators	206.07

From Table 4.7 it can easily be seen that the efficiency of the proposed difference-type estimator, $\bar{y}_{md1} = \sum_{h=1}^L W_h [\bar{y}_h'' - k_h (\bar{x}_h'' - \bar{X}_h)], k_h = S_{yxh} / S_{xh}^2 \forall h$, is close to the efficiency of the optimum estimators in the proposed class of estimators, ($\bar{y}_{mc1}$), for the same reason as discussed for earlier designs. $k_h = S_{yxh} / S_{xh}^2 \forall h$, is close to the optimum value that minimizes the mean

square error of the proposed class of estimators, $\bar{y}_{mc1}$. Furthermore, for this design, the customary sample mean estimator, $\bar{y}_{ms1} = \sum_{h=1}^L W_h \bar{y}_h''$, has been observed to be less efficient than the ratio-type estimators in the proposed class of estimators, $\bar{y}_{mc1}$, but more efficient than the suggested product-type estimators. Table 4.7 specifically reveals that the ratio-type estimators are preferable when there is a positive correlation between the study and auxiliary variables.

For negatively correlated study and auxiliary variables (Dataset-2), under this design, the percentage relative efficiencies of some proposed estimators over the usual poststratified mean estimator ($\bar{y}_{ps}$), are shown in Table 4.8. The table shows that the proposed product-type estimators, as expected, performed better than the proposed ratio-type estimators, in terms of having higher percentage relative efficiencies, since there is a negative correlation between the study and auxiliary variables. Considering the ten (10) special cases of the proposed class of estimators, ($\bar{y}_{mc1}$), when there are measurement errors on both the study and auxiliary variables, we observe that the proposed estimator that has the highest efficiency is the adopted Upadhyaya-Singh (1999) estimator III, $\bar{y}_{mc1}(9) = \sum_{h=1}^L W_h \bar{y}_h'' (\bar{x}_h'' C_{xh} + \beta_{2xh}) / (\bar{X}_h C_{xh} + \beta_{2xh})$, followed by the adopted Singh (2003) estimator, $\bar{y}_{mc1}(10) = \sum_{h=1}^L W_h \bar{y}_h'' (\bar{x}_h'' \beta_{2xh} + \sigma_{xh}) / (\bar{X}_h \beta_{2xh} + \sigma_{xh})$. The proposed estimator with the least efficiency among the ten (10) special cases under consideration is the customary ratio-type estimator, $\bar{y}_{mc1}(1) = \sum_{h=1}^L W_h \bar{y}_h'' (\bar{X}_h / \bar{x}_h'')$. The efficiency of the proposed difference-type estimator, $\bar{y}_{md1} = \sum_{h=1}^L W_h [\bar{y}_h'' - k_h (\bar{x}_h'' - \bar{X}_h)]$, $k_h = S_{yxh} / S_{xh}^2 \forall h$, is again close to the efficiency of the optimum estimators in the proposed class of estimators, ($\bar{y}_{mc1}$), for the same reason stated in earlier result. Table 4.8 also show that the customary sample mean estimator, $\bar{y}_{ms1} = \sum_{h=1}^L W_h \bar{y}_h''$, was less efficient than the product-type estimators in the proposed class of estimators, $\bar{y}_{mc1}$, but more efficient than the suggested ratio-type estimators. This confirms that, the use of auxiliary information is most likely to result in an improved efficiency over the customary sample mean estimators when estimators that utilize auxiliary information are appropriately chosen, according to the nature and direction of the relationship between the study and auxiliary variables. Table 4.8 specifically reveals that the product-type estimators are preferable when there is a negative correlation between the study and auxiliary variables.

4.5 Measurement Errors on only the study variable and not on the auxiliary variable under the Poststratified Sampling Scheme

Recall from (3.154), (3.164) and (3.51), that the proposed class of estimators of the population mean of the study variable (y) when there are measurement errors on only the study variable and not on the auxiliary variable under the poststratified sampling scheme is given by

$$\bar{y}_{mc2} = \sum_{h=1}^L W_h \bar{y}_h'' \left(\frac{a_h \bar{X}_h + b_h}{a_h \bar{X}_h' + b_h} \right)^{\alpha_h} \quad (4.21)$$

with its unconditional mean square error obtained as:

$$MSE(\bar{y}_{mc2}) = \left(\frac{1-f}{n} \right) \sum_{h=1}^L W_h (S_{yh}^2 + \alpha_h^2 \lambda_h^2 R_h^2 S_{xh}^2 - 2\alpha_h \lambda_h R_h S_{yxh}) + \left(\frac{1-f}{n} \right) \sum_{h=1}^L W_h S_{uh}^2 \quad (4.22)$$

where

$$\lambda_h = \left(\frac{a_h \bar{X}_h}{a_h \bar{X}_h + b_h} \right) \quad (4.23)$$

Applying the least squares method, it follows that the best choices of α_h , a_h and b_h that minimize the unconditional mean square error of $\bar{y}_{mc2}$ satisfy the expression:

$$\theta_{h5} = \frac{\alpha_h a_h \bar{X}_h R_h}{a_h \bar{X}_h + b_h} = \frac{\alpha_h a_h \bar{Y}_h}{a_h \bar{X}_h + b_h} = \frac{S_{yxh}}{S_{xh}^2} \quad (4.24)$$

with the associated optimum unconditional mean square error of $\bar{y}_{mc2}$ given by

$$MSE_0(\bar{y}_{mc2}) = \left(\frac{1-f}{n} \right) \sum_{h=1}^L W_h (S_{yh}^2 + \theta_{h5}^2 S_{xh}^2 - 2\theta_{h5} S_{yxh} + S_{uh}^2) \quad (4.25)$$

Table 4.9 and Table 4.10 show the associated PREs using Dataset-1 and Dataset-2 respectively.

Table 4.9: Percentage Relative Efficiency (PRE) of Some Proposed Estimators over the Usual Poststratified Mean Estimator ($\bar{y}_{ps}$) when there are measurement errors on only the study

variable for positively correlated study and auxiliary variables (Dataset-1)

Estimator	PRE
$\bar{y}_{ps} = \sum_{h=1}^L W_h \bar{y}_h$	100.00
$\bar{y}_{ms2} = \sum_{h=1}^L W_h \bar{y}_h''$	95.20
$\bar{y}_{md2} = \sum_{h=1}^L W_h [\bar{y}_h'' - k_h(\bar{x}_h' - \bar{X}_h)], k_h = S_{yxh}/S_{xh}^2 \forall h$	896.59
$\bar{y}_{mc2}(1) = \sum_{h=1}^L W_h \bar{y}_h''(\bar{X}_h/\bar{x}_h')$	547.12
$\bar{y}_{mc2}(2) = \sum_{h=1}^L W_h \bar{y}_h''(\bar{X}_h + C_{xh})/(\bar{x}_h' + C_{xh})$	555.96
$\bar{y}_{mc2}(3) = \sum_{h=1}^L W_h \bar{y}_h''(\bar{X}_h + \beta_{2xh})/(\bar{x}_h' + \beta_{2xh})$	566.34
$\bar{y}_{mc2}(4) = \sum_{h=1}^L W_h \bar{y}_h''(\bar{X}_h C_{xh} + \beta_{2xh})/(\bar{x}_h' C_{xh} + \beta_{2xh})$	594.62
$\bar{y}_{mc2}(5) = \sum_{h=1}^L W_h \bar{y}_h''(\bar{X}_h \beta_{2xh} + C_{xh})/(\bar{x}_h' \beta_{2xh} + C_{xh})$	557.89
$\bar{y}_{mc2}(6) = \sum_{h=1}^L W_h \bar{y}_h''(\bar{x}_h'/\bar{X}_h)$	20.36
$\bar{y}_{mc2}(7) = \sum_{h=1}^L W_h \bar{y}_h''(\bar{x}_h' + C_{xh})/(\bar{X}_h + C_{xh})$	20.47
$\bar{y}_{mc2}(8) = \sum_{h=1}^L W_h \bar{y}_h''(\bar{x}_h' + \beta_{2xh})/(\bar{X}_h + \beta_{2xh})$	20.64
$\bar{y}_{mc2}(9) = \sum_{h=1}^L W_h \bar{y}_h''(\bar{x}_h' C_{xh} + \beta_{2xh})/(\bar{X}_h C_{xh} + \beta_{2xh})$	21.09
$\bar{y}_{mc2}(10) = \sum_{h=1}^L W_h \bar{y}_h''(\bar{x}_h' \beta_{2xh} + \sigma_{xh})/(\bar{X}_h \beta_{2xh} + \sigma_{xh})$	29.34
Optimum estimators	896.59

Table 4.10: Percentage Relative Efficiency (PRE) of Some Proposed Estimators over the Usual Poststratified Mean Estimator ($\bar{y}_{ps}$) when there are measurement errors on only the study variable for negatively correlated study and auxiliary variables (Dataset-2)

Estimator	PRE
$\bar{y}_{ps} = \sum_{h=1}^L W_h \bar{y}_h$	100.00
$\bar{y}_{ms2} = \sum_{h=1}^L W_h \bar{y}_h''$	96.66
$\bar{y}_{md2} = \sum_{h=1}^L W_h [\bar{y}_h'' - k_h(\bar{x}_h' - \bar{X}_h)], k_h = S_{yxh}/S_{xh}^2 \forall h$	384.76
$\bar{y}_{mc2}(1) = \sum_{h=1}^L W_h \bar{y}_h'' (\bar{X}_h / \bar{x}_h')$	22.07
$\bar{y}_{mc2}(2) = \sum_{h=1}^L W_h \bar{y}_h'' (\bar{X}_h + C_{xh}) / (\bar{x}_h' + C_{xh})$	22.41
$\bar{y}_{mc2}(3) = \sum_{h=1}^L W_h \bar{y}_h'' (\bar{X}_h + \beta_{2xh}) / (\bar{x}_h' + \beta_{2xh})$	23.05
$\bar{y}_{mc2}(4) = \sum_{h=1}^L W_h \bar{y}_h'' (\bar{X}_h C_{xh} + \beta_{2xh}) / (\bar{x}_h' C_{xh} + \beta_{2xh})$	28.58
$\bar{y}_{mc2}(5) = \sum_{h=1}^L W_h \bar{y}_h'' (\bar{X}_h \beta_{2xh} + C_{xh}) / (\bar{x}_h' \beta_{2xh} + C_{xh})$	22.86
$\bar{y}_{mc2}(6) = \sum_{h=1}^L W_h \bar{y}_h'' (\bar{x}_h' \bar{X}_h)$	275.29
$\bar{y}_{mc2}(7) = \sum_{h=1}^L W_h \bar{y}_h'' (\bar{x}_h' + C_{xh}) / (\bar{X}_h + C_{xh})$	282.58
$\bar{y}_{mc2}(8) = \sum_{h=1}^L W_h \bar{y}_h'' (\bar{x}_h' + \beta_{2xh}) / (\bar{X}_h + \beta_{2xh})$	298.45
$\bar{y}_{mc2}(9) = \sum_{h=1}^L W_h \bar{y}_h'' (\bar{x}_h' C_{xh} + \beta_{2xh}) / (\bar{X}_h C_{xh} + \beta_{2xh})$	382.47
$\bar{y}_{mc2}(10) = \sum_{h=1}^L W_h \bar{y}_h'' (\bar{x}_h' \beta_{2xh} + \sigma_{xh}) / (\bar{X}_h \beta_{2xh} + \sigma_{xh})$	352.50
Optimum estimators	384.76

Table 4.9 shows the percentage relative efficiencies of some proposed estimators over the usual poststratified mean estimator ($\bar{y}_{ps}$), when there are measurement errors on only the study variable and not on the auxiliary variable for positively correlated study and auxiliary variables (Dataset-1). The table shows that the proposed ratio-type estimators performed better than the proposed product-type estimators, in terms of efficiencies, since Dataset-1 consists of positively correlated study and auxiliary variables. For the ten (10) special cases of the proposed class of estimators, ($\bar{y}_{mc2}$), when there are measurement errors on only the study variable and not on the auxiliary variable, Table 4.9 indicates that the proposed estimator (utilizing auxiliary information) that has the highest efficiency is the adopted Upadhyaya-Singh (1999) estimator,

$\bar{y}_{mc2}(4) = \sum_{h=1}^L W_h \bar{y}_h'' (\bar{X}_h C_{xh} + \beta_{2xh}) / (\bar{x}_h' C_{xh} + \beta_{2xh})$, followed by the adopted Singh-Kakran (1993) estimator, $\bar{y}_{mc2}(3) = \sum_{h=1}^L W_h \bar{y}_h'' (\bar{X}_h + \beta_{2xh}) / (\bar{x}_h' + \beta_{2xh})$, while the estimator with the least efficiency among the ten (10) special cases under consideration is the customary product-type estimator, $\bar{y}_{mc2}(6) = \sum_{h=1}^L W_h \bar{y}_h'' (\bar{x}_h' \bar{X}_h)$, under this design.

We also observe from Table 4.9 that the efficiency of the proposed difference-type estimator, $\bar{y}_{md2} = \sum_{h=1}^L W_h [\bar{y}_h'' - k_h (\bar{x}_h' - \bar{X}_h)]$, $k_h = S_{yxh} / S_{xh}^2 \forall h$, is exactly the same as the efficiency of the optimum estimators in the proposed class of estimators, $(\bar{y}_{mc2})$, since the choice of the constant, $k_h = S_{yxh} / S_{xh}^2 \forall h$, is exactly the same as the optimum value that minimizes the mean square error of the proposed class of estimators, $\bar{y}_{mc2}$. Furthermore, it was observed that the customary sample mean estimator, $\bar{y}_{ms2} = \sum_{h=1}^L W_h \bar{y}_h''$, is less efficient than the ratio-type estimators in the proposed class of estimators, $\bar{y}_{mc2}$, but more efficient than the suggested product-type estimators. This attests to the fact that the use of auxiliary information is most likely to result in an improved efficiency over the customary sample mean estimators when estimators that utilize auxiliary information are appropriately chosen, according to the nature and direction of the relationship between the study and auxiliary variables. Table 4.9 specifically reveals that the ratio-type estimators are preferable when there is a positive correlation between the study and auxiliary variables.

For negatively correlated study and auxiliary variables (Dataset-2), and when there are measurement errors on only the study variable and not on the auxiliary variable, the percentage relative efficiencies of some proposed estimators over the usual poststratified mean estimator $(\bar{y}_{ps})$, are shown in Table 4.10. The table shows that the proposed product-type estimators performed better than the proposed ratio-type estimators, in terms of percentage relative efficiencies. This is because of the negative correlation between the study and auxiliary variables. For the ten (10) special cases of the proposed class of estimators, $(\bar{y}_{mc2})$, considered under this design, it can clearly be seen that the proposed estimator with the highest efficiency is the adopted Upadhyaya-Singh (1999) estimator III, $\bar{y}_{mc2}(9) = \sum_{h=1}^L W_h \bar{y}_h'' (\bar{x}_h' C_{xh} + \beta_{2xh}) / (\bar{X}_h C_{xh} + \beta_{2xh})$, fol-

lowed by the adopted Singh (2003) estimator, $\bar{y}_{mc2}(10) = \sum_{h=1}^L W_h \bar{y}_h'' (\bar{x}_h' \beta_{2xh} + \sigma_{xh}) / (\bar{X}_h \beta_{2xh} + \sigma_{xh})$, while the estimator with the least efficiency among the ten (10) special cases under consideration is the customary ratio-type estimator, $\bar{y}_{mc2}(1) = \sum_{h=1}^L W_h \bar{y}_h'' (\bar{X}_h / \bar{x}_h')$. The efficiency of the proposed difference-type estimator, $\bar{y}_{md2} = \sum_{h=1}^L W_h [\bar{y}_h'' - k_h (\bar{x}_h' - \bar{X}_h)]$, $k_h = S_{yxh} / S_{xh}^2 \forall h$, is exactly the same as the efficiency of the optimum estimators in the proposed class of estimators, $(\bar{y}_{mc2})$, since the choice of the constant, $k_h = S_{yxh} / S_{xh}^2 \forall h$, is exactly the same optimum value of the proposed class of estimators, $\bar{y}_{mc2}$. Table 4.10 also show that the use of auxiliary information does not always result in an improved efficiency over the customary sample mean estimators, which do not utilize auxiliary information, since the customary sample mean estimator, $\bar{y}_{ms2} = \sum_{h=1}^L W_h \bar{y}_h''$, was less efficient than the product-type estimators in the proposed class of estimators, $\bar{y}_{mc2}$, but more efficient than the suggested ratio-type estimators. Table 4.10 specifically reveals that the product-type estimators are preferable when there is a negative correlation between the study and auxiliary variables.

4.6 Measurement Errors on only the auxiliary variable and not on the study variable under the Poststratified Sampling Scheme

Recall from (3.171), (3.183) and (3.51), that the proposed class of estimators of the population mean of the study variable (y) when there are measurement errors on only the auxiliary variable and not on the study variable under the poststratified sampling scheme is given by

$$\bar{y}_{mc3} = \sum_{h=1}^L W_h \bar{y}_h' \left(\frac{a_h \bar{X}_h + b_h}{a_h \bar{x}_h' + b_h} \right)^{\alpha_h} \quad (4.26)$$

with its unconditional mean square error obtained as:

$$\begin{aligned} M S E(\bar{y}_{mc3}) = & \left(\frac{1-f}{n} \right) \sum_{h=1}^L W_h (S_{yh}^2 + \alpha_h^2 \lambda_h^2 R_h^2 S_{xh}^2 - 2\alpha_h \lambda_h R_h S_{yxh}) \\ & + \left(\frac{1-f}{n} \right) \sum_{h=1}^L W_h \alpha_h^2 \lambda_h^2 R_h^2 S_{vh}^2 \end{aligned} \quad (4.27)$$

where

$$\lambda_h = \left(\frac{a_h \bar{X}_h}{a_h \bar{X}_h + b_h} \right) \quad (4.28)$$

Applying the method of least squares, it follows that the best choices of α_h , a_h and b_h that minimize the unconditional mean square error of $\bar{y}_{mc3}$ satisfy the expression:

$$\theta_{h6} = \frac{\alpha_h a_h \bar{X}_h R_h}{a_h \bar{X}_h + b_h} = \frac{\alpha_h a_h \bar{Y}_h}{a_h \bar{X}_h + b_h} = \frac{S_{yxh}}{S_{xh}^2 + S_{vh}^2} \quad (4.29)$$

with the associated optimum unconditional mean square error of $\bar{y}_{mc3}$ given by

$$MS E_0(\bar{y}_{mc3}) = \left(\frac{1-f}{n} \right) \sum_{h=1}^L W_h (S_{yh}^2 + \theta_{h6}^2 S_{xh}^2 - 2\theta_{h6} S_{yxh} + \theta_{h6}^2 S_{vh}^2) \quad (4.30)$$

Table 4.11 shows the Percentage Relative Efficiencies (PRE) of some special cases of the proposed estimators over the usual poststratified mean estimator, $(\bar{y}_{ps})$, using Dataset-1 (positively correlated study and auxiliary variables), with measurement errors on only the auxiliary variable and not on the study variable. Table 4.12 gives similar results, using Dataset-2 for (negatively correlated study and auxiliary variables).

Table 4.11: Percentage Relative Efficiency (PRE) of Some Proposed Estimators over the Usual Poststratified Mean Estimator ($\bar{y}_{ps}$) when there are measurement errors on only the auxiliary variable for positively correlated study and auxiliary variables (Dataset-1)

Estimator	PRE
$\bar{y}_{ps} = \sum_{h=1}^L W_h \bar{y}_h$	100.00
$\bar{y}_{ms3} = \sum_{h=1}^L W_h \bar{y}'_h$	100.00
$\bar{y}_{md3} = \sum_{h=1}^L W_h [\bar{y}'_h - k_h(\bar{x}''_h - \bar{X}_h)], k_h = S_{yxh}/S_{xh}^2 \forall h$	1602.01
$\bar{y}_{mc3}(1) = \sum_{h=1}^L W_h \bar{y}'_h (\bar{X}_h / \bar{x}''_h)$	744.88
$\bar{y}_{mc3}(2) = \sum_{h=1}^L W_h \bar{y}'_h (\bar{X}_h + C_{xh}) / (\bar{x}''_h + C_{xh})$	761.49
$\bar{y}_{mc3}(3) = \sum_{h=1}^L W_h \bar{y}'_h (\bar{X}_h + \beta_{2xh}) / (\bar{x}''_h + \beta_{2xh})$	781.35
$\bar{y}_{mc3}(4) = \sum_{h=1}^L W_h \bar{y}'_h (\bar{X}_h C_{xh} + \beta_{2xh}) / (\bar{x}''_h C_{xh} + \beta_{2xh})$	836.86
$\bar{y}_{mc3}(5) = \sum_{h=1}^L W_h \bar{y}'_h (\bar{X}_h \beta_{2xh} + C_{xh}) / (\bar{x}''_h \beta_{2xh} + C_{xh})$	765.11
$\bar{y}_{mc3}(6) = \sum_{h=1}^L W_h \bar{y}'_h (\bar{x}''_h / \bar{X}_h)$	20.56
$\bar{y}_{mc3}(7) = \sum_{h=1}^L W_h \bar{y}'_h (\bar{x}''_h + C_{xh}) / (\bar{X}_h + C_{xh})$	20.68
$\bar{y}_{mc3}(8) = \sum_{h=1}^L W_h \bar{y}'_h (\bar{x}''_h + \beta_{2xh}) / (\bar{X}_h + \beta_{2xh})$	20.85
$\bar{y}_{mc3}(9) = \sum_{h=1}^L W_h \bar{y}'_h (\bar{x}''_h C_{xh} + \beta_{2xh}) / (\bar{X}_h C_{xh} + \beta_{2xh})$	21.31
$\bar{y}_{mc3}(10) = \sum_{h=1}^L W_h \bar{y}'_h (\bar{x}''_h \beta_{2xh} + \sigma_{xh}) / (\bar{X}_h \beta_{2xh} + \sigma_{xh})$	29.77
Optimum estimators	1602.07

Table 4.11 shows the percentage relative efficiencies of some proposed estimators over the usual poststratified mean estimator ($\bar{y}_{ps}$), when there are measurement errors on only the auxiliary variable and not on the study variable for positively correlated study and auxiliary variables (Dataset-1). The table shows that the proposed ratio-type estimators performed better than the proposed product-type estimators, in terms of having higher percentage relative efficiencies, since Dataset-1 consists of positively correlated study and auxiliary variables. For the ten (10) special cases of the proposed class of estimators, ($\bar{y}_{mc3}$), when there are measurement errors on only the auxiliary variable and not on the study variable, Table 4.11 shows that the proposed estimator (utilizing auxiliary information) that has the highest efficiency is the adopted

Upadhyaya-Singh (1999) estimator, $\bar{y}_{mc3}(4) = \sum_{h=1}^L W_h \bar{y}'_h (\bar{X}_h C_{xh} + \beta_{2xh}) / (\bar{x}''_h C_{xh} + \beta_{2xh})$, followed by the adopted Singh-Kakran (1993) estimator, $\bar{y}_{mc3}(3) = \sum_{h=1}^L W_h \bar{y}'_h (\bar{X}_h + \beta_{2xh}) / (\bar{x}''_h + \beta_{2xh})$, while the estimator with the least efficiency among the ten (10) special cases under consideration is the customary product-type estimator, $\bar{y}_{mc3}(6) = \sum_{h=1}^L W_h \bar{y}'_h (\bar{x}''_h / \bar{X}_h)$, when there are measurement errors on only the auxiliary variable and not on the study variable.

Table 4.12: Percentage Relative Efficiency (PRE) of Some Proposed Estimators over the Usual Poststratified Mean Estimator ($\bar{y}_{ps}$) when there are measurement errors on only the auxiliary variable for negatively correlated study and auxiliary variables (Dataset-2)

Estimator	PRE
$\bar{y}_{ps} = \sum_{h=1}^L W_h \bar{y}_h$	100.00
$\bar{y}_{ms3} = \sum_{h=1}^L W_h \bar{y}'_h$	100.00
$\bar{y}_{md3} = \sum_{h=1}^L W_h [\bar{y}'_h - k_h (\bar{x}''_h - \bar{X}_h)]$, $k_h = S_{yxh} / S_{xh}^2 \forall h$	175.43
$\bar{y}_{mc3}(1) = \sum_{h=1}^L W_h \bar{y}'_h (\bar{X}_h / \bar{x}''_h)$	19.27
$\bar{y}_{mc3}(2) = \sum_{h=1}^L W_h \bar{y}'_h (\bar{X}_h + C_{xh}) / (\bar{x}''_h + C_{xh})$	19.59
$\bar{y}_{mc3}(3) = \sum_{h=1}^L W_h \bar{y}'_h (\bar{X}_h + \beta_{2xh}) / (\bar{x}''_h + \beta_{2xh})$	20.25
$\bar{y}_{mc3}(4) = \sum_{h=1}^L W_h \bar{y}'_h (\bar{X}_h C_{xh} + \beta_{2xh}) / (\bar{x}''_h C_{xh} + \beta_{2xh})$	26.03
$\bar{y}_{mc3}(5) = \sum_{h=1}^L W_h \bar{y}'_h (\bar{X}_h \beta_{2xh} + C_{xh}) / (\bar{x}''_h \beta_{2xh} + C_{xh})$	20.02
$\bar{y}_{mc3}(6) = \sum_{h=1}^L W_h \bar{y}'_h (\bar{x}''_h / \bar{X}_h)$	97.95
$\bar{y}_{mc3}(7) = \sum_{h=1}^L W_h \bar{y}'_h (\bar{x}''_h + C_{xh}) / (\bar{X}_h + C_{xh})$	100.52
$\bar{y}_{mc3}(8) = \sum_{h=1}^L W_h \bar{y}'_h (\bar{x}''_h + \beta_{2xh}) / (\bar{X}_h + \beta_{2xh})$	106.93
$\bar{y}_{mc3}(9) = \sum_{h=1}^L W_h \bar{y}'_h (\bar{x}''_h C_{xh} + \beta_{2xh}) / (\bar{X}_h C_{xh} + \beta_{2xh})$	165.61
$\bar{y}_{mc3}(10) = \sum_{h=1}^L W_h \bar{y}'_h (\bar{x}''_h \beta_{2xh} + \sigma_{xh}) / (\bar{X}_h \beta_{2xh} + \sigma_{xh})$	148.44
Optimum estimators	221.87

We also observe from Table 4.11 that for Dataset-1, the proposed customary sample mean estimator, $\bar{y}_{ms3} = \sum_{h=1}^L W_h \bar{y}'_h$, when there are measurement errors on only the auxiliary variable and not on the study variable, is exactly the same as the usual poststratified mean estimator

($\bar{y}_{ps}$). Also, the efficiency of the proposed difference-type estimator, $\bar{y}_{md3} = \sum_{h=1}^L W_h [\bar{y}'_h - k_h(\bar{x}''_h - \bar{X}_h)]$, $k_h = S_{yxh}/S_{xh}^2 \forall h$, is the closest to the efficiency of the optimum estimators in the proposed class of estimators, ($\bar{y}_{mc3}$), among competing estimators since the choice of the constant, $k_h = S_{yxh}/S_{xh}^2 \forall h$, is very close to the optimum value of the proposed class of estimators, $\bar{y}_{mc3}$. Furthermore, we observe that under this design, the customary sample mean estimator, $\bar{y}_{ms3} = \sum_{h=1}^L W_h \bar{y}'_h$, is less efficient than the ratio-type estimators in the proposed class of estimators, $\bar{y}_{mc3}$, but more efficient than the suggested product-type estimators. This is in line with the use of auxiliary information to improve efficiency over the customary estimators. Table 4.11 specifically reveals that the ratio-type estimators are preferable when there is a positive correlation between the study and auxiliary variables.

For negatively correlated study and auxiliary variables (Dataset-2), and when there are measurement errors on only the auxiliary variable and not on the study variable, the percentage relative efficiencies of some proposed estimators over the usual poststratified mean estimator ($\bar{y}_{ps}$), are shown in Table 4.12. The table shows that the proposed product-type estimators performed better than the proposed ratio-type estimators, in terms of percentage relative efficiencies. This is because there is a negative correlation between the study and auxiliary variables. For the ten (10) special cases of the proposed class of estimators, ($\bar{y}_{mc3}$), under this design, it was observed that the proposed estimator that has the highest efficiency is the adopted Upadhyaya-Singh (1999) estimator III, $\bar{y}_{mc3}(9) = \sum_{h=1}^L W_h \bar{y}'_h (\bar{x}''_h C_{xh} + \beta_{2xh}) / (\bar{X}_h C_{xh} + \beta_{2xh})$, followed by the adopted Singh (2003) estimator, $\bar{y}_{mc3}(10) = \sum_{h=1}^L W_h \bar{y}'_h (\bar{x}''_h \beta_{2xh} + \sigma_{xh}) / (\bar{X}_h \beta_{2xh} + \sigma_{xh})$, while the proposed estimator with the least efficiency among the ten (10) special cases under consideration is the customary ratio-type estimator, $\bar{y}_{mc3}(1) = \sum_{h=1}^L W_h \bar{y}'_h (\bar{X}_h / \bar{x}''_h)$.

Table 4.12 also indicates that for Dataset-2, the efficiency of the proposed customary sample mean estimator, $\bar{y}_{ms3} = \sum_{h=1}^L W_h \bar{y}'_h$, under this design, is exactly the same as the efficiency of the usual poststratified mean estimator ($\bar{y}_{ps}$). Again, the efficiency of the proposed difference-type estimator, $\bar{y}_{md3} = \sum_{h=1}^L W_h [\bar{y}'_h - k_h(\bar{x}''_h - \bar{X}_h)]$, $k_h = S_{yxh}/S_{xh}^2 \forall h$, is the closest to that of the optimum estimators in the proposed class of estimators, ($\bar{y}_{mc3}$), since the choice of the constant,

$k_h = S_{y_{xh}}/S_{xh}^2 \forall h$, is close to the optimum value of the parameter that minimizes the mean square error of the proposed class of estimators, $\bar{y}_{mc3}$. It was also observed from Table 4.12 that the use of auxiliary information does not always result in an improved efficiency over the customary sample mean estimators, which do not utilize auxiliary information, since the customary sample mean estimator, $\bar{y}_{ms3} = \sum_{h=1}^L W_h \bar{y}'_h$, was less efficient than the product-type estimators in the proposed class of estimators, $\bar{y}_{mc3}$, but more efficient than the suggested ratio-type estimators. Therefore, the use of auxiliary information is most likely to result in an improved efficiency over the customary sample mean estimators when estimators that utilize auxiliary information are appropriately chosen, according to the nature and direction of the relationship between the study and auxiliary variables. Table 4.12 specifically reveals that the product-type estimators are preferable when there is a negative correlation between the study and auxiliary variables.

4.7 Joint Occurrence of Nonresponse and Measurement

Errors on both the study and auxiliary variables under the Poststratified Sampling Scheme

Recall from (3.190), (3.233) and (3.51), that the proposed class of estimators of the population mean of the study variable (y) when there is joint occurrence of nonresponse and measurement errors on both the study and auxiliary variables under the poststratified sampling scheme is

$$\bar{y}_{jc1} = \sum_{h=1}^L W_h \bar{y}'''_h \left(\frac{a_h \bar{X}_h + b_h}{a_h \bar{x}'''_h + b_h} \right)^{\alpha_h} \quad (4.31)$$

with its unconditional mean square error obtained as:

$$\begin{aligned}
MSE(\bar{y}_{jc1}) = & \left(\frac{1-f}{n} \right) \sum_{h=1}^L W_h (S_{yh}^2 + \alpha_h^2 \lambda_h^2 R_h^2 S_{xh}^2 - 2\alpha_h \lambda_h R_h S_{yxh}) \\
& + \left(\frac{1}{n} \right) \sum_{h=1}^L W_h (g_h - 1) W_{2h} (S_{2yh}^2 + \alpha_h^2 \lambda_h^2 R_h^2 S_{2xh}^2 - 2\alpha_h \lambda_h R_h S_{2yxh}) \\
& + \sum_{h=1}^L W_h \left[\left(\frac{1-f}{n} \right) (S_{uh}^2 + \alpha_h^2 \lambda_h^2 R_h^2 S_{vh}^2) + \left(\frac{g_h-1}{n} \right) W_{2h} (S_{2uh}^2 + \alpha_h^2 \lambda_h^2 R_h^2 S_{2vh}^2) \right]
\end{aligned} \tag{4.32}$$

where

$$\lambda_h = \left(\frac{a_h \bar{X}_h}{a_h \bar{X}_h + b_h} \right) \tag{4.33}$$

Applying the method of least squares, it follows that the best choices of α_h , a_h and b_h that minimize the unconditional mean square error of $\bar{y}_{jc1}$ satisfy the expression:

$$\theta_{h7} = \frac{\alpha_h a_h \bar{X}_h R_h}{a_h \bar{X}_h + b_h} = \frac{\alpha_h a_h \bar{Y}_h}{a_h \bar{X}_h + b_h} = \frac{(1-f)S_{yxh} + (g_h-1)W_{2h}S_{2yxh}}{(1-f)(S_{xh}^2 + S_{vh}^2) + (g_h-1)W_{2h}(S_{2xh}^2 + S_{2vh}^2)} \tag{4.34}$$

with the associated optimum unconditional mean square error of $\bar{y}_{jc1}$ given by

$$\begin{aligned}
MSE_0(\bar{y}_{jc1}) = & \left(\frac{1-f}{n} \right) \sum_{h=1}^L W_h (S_{yh}^2 + \theta_{h7}^2 S_{xh}^2 - 2\theta_{h7} S_{yxh} + S_{uh}^2 + \theta_{h7}^2 S_{vh}^2) \\
& + \left(\frac{1}{n} \right) \sum_{h=1}^L W_h (g_h - 1) W_{2h} (S_{2yh}^2 + \theta_{h7}^2 S_{2xh}^2 - 2\theta_{h7} S_{2yxh} + S_{2uh}^2 + \theta_{h7}^2 S_{2vh}^2)
\end{aligned} \tag{4.35}$$

Table 4.13 shows the Percentage Relative Efficiencies (PRE) of some special cases of the proposed estimators over the usual poststratified mean estimator, $(\bar{y}_{ps})$, using Dataset-1 (positively correlated study and auxiliary variables), when there is joint occurrence of nonresponse and measurement errors on both the study and auxiliary variables. Similarly, Table 4.14 shows the PREs, using Dataset-2 (negatively correlated study and auxiliary variables), under this study.

Table 4.13: Percentage Relative Efficiency (PRE) of Some Proposed Estimators over the Usual Poststratified Mean Estimator ($\bar{y}_{ps}$)
when there is joint occurrence of nonresponse and measurement errors on both the study and auxiliary variables
for positively correlated study and auxiliary variables (Dataset-1)

Estimators	Subsampling Fraction ($1/g_h$) ; where $g_h = n_{2h}/m_h > 1$ is the subsampling factor								
	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90
$\bar{y}_{ps} = \sum_{h=1}^L W_h \bar{y}_h$	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00
$\bar{y}_{js1} = \sum_{h=1}^L W_h \bar{y}_h'''$	19.52	34.96	47.48	57.83	66.54	73.96	80.37	85.95	90.85
$\bar{y}_{jd1} = \sum_{h=1}^L W_h [\bar{y}_h''' - k_h(\bar{x}_h''' - \bar{X}_h)]$, $k_h = S_{yxh}/S_{xh}^2 \forall h$	134.60	254.53	362.07	459.05	546.94	626.97	700.14	767.31	829.18
$\bar{y}_{jc1}(1) = \sum_{h=1}^L W_h \bar{y}_h''' (\bar{X}_h / \bar{x}_h''')$	93.90	173.64	242.21	301.78	354.03	400.23	441.37	478.24	511.46
$\bar{y}_{jc1}(2) = \sum_{h=1}^L W_h \bar{y}_h''' (\bar{X}_h + C_{xh}) / (\bar{x}_h''' + C_{xh})$	95.02	175.84	245.43	305.97	359.11	406.15	448.06	485.65	519.56
$\bar{y}_{jc1}(3) = \sum_{h=1}^L W_h \bar{y}_h''' (\bar{X}_h + \beta_{2xh}) / (\bar{x}_h''' + \beta_{2xh})$	96.43	178.57	249.38	311.05	365.24	413.24	456.05	494.47	529.14
$\bar{y}_{jc1}(4) = \sum_{h=1}^L W_h \bar{y}_h''' (\bar{X}_h C_{xh} + \beta_{2xh}) / (\bar{x}_h''' C_{xh} + \beta_{2xh})$	99.74	185.18	259.21	323.95	381.07	431.82	477.22	518.07	555.02
$\bar{y}_{jc1}(5) = \sum_{h=1}^L W_h \bar{y}_h''' (\bar{X}_h \beta_{2xh} + C_{xh}) / (\bar{x}_h''' \beta_{2xh} + C_{xh})$	95.21	176.24	246.04	306.79	360.13	407.36	449.46	487.22	521.29
$\bar{y}_{jc1}(6) = \sum_{h=1}^L W_h \bar{y}_h''' (\bar{x}_h''' / \bar{X}_h)$	4.43	7.84	10.54	12.73	14.54	16.07	17.38	18.50	19.49
$\bar{y}_{jc1}(7) = \sum_{h=1}^L W_h \bar{y}_h''' (\bar{x}_h''' + C_{xh}) / (\bar{X}_h + C_{xh})$	4.46	7.88	10.60	12.80	14.63	16.16	17.47	18.61	19.59
$\bar{y}_{jc1}(8) = \sum_{h=1}^L W_h \bar{y}_h''' (\bar{x}_h''' + \beta_{2xh}) / (\bar{X}_h + \beta_{2xh})$	4.51	7.97	10.71	12.93	14.77	16.31	17.63	18.77	19.76
$\bar{y}_{jc1}(9) = \sum_{h=1}^L W_h \bar{y}_h''' (\bar{x}_h''' C_{xh} + \beta_{2xh}) / (\bar{X}_h C_{xh} + \beta_{2xh})$	4.64	8.18	10.98	13.25	15.12	16.69	18.03	19.19	20.20
$\bar{y}_{jc1}(10) = \sum_{h=1}^L W_h \bar{y}_h''' (\bar{x}_h''' \beta_{2xh} + \sigma_{xh}) / (\bar{X}_h \beta_{2xh} + \sigma_{xh})$	6.11	10.91	14.77	17.96	20.62	22.89	24.84	26.53	28.02
Optimum estimators	134.73	254.75	362.32	459.30	547.18	627.18	700.31	767.42	829.23

Table 4.14: Percentage Relative Efficiency (PRE) of Some Proposed Estimators over the Usual Poststratified Mean Estimator ($\bar{y}_{ps}$)
when there is joint occurrence of nonresponse and measurement errors on both the study and auxiliary variables
for negatively correlated study and auxiliary variables (Dataset-2)

Estimators	Subsampling Fraction ($1/g_h$) ; where $g_h = n_{2h}/m_h > 1$ is the subsampling factor								
	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90
$\bar{y}_{ps} = \sum_{h=1}^L W_h \bar{y}_h$	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00
$\bar{y}_{js1} = \sum_{h=1}^L W_h \bar{y}_h'''$	23.58	40.66	53.60	63.74	71.90	78.62	84.23	89.00	93.10
$\bar{y}_{jd1} = \sum_{h=1}^L W_h [\bar{y}_h''' - k_h(\bar{x}_h''' - \bar{X}_h)]$, $k_h = S_{yxh}/S_{xh}^2 \forall h$	23.49	44.88	64.44	82.41	98.96	114.26	128.44	141.63	153.92
$\bar{y}_{jc1}(1) = \sum_{h=1}^L W_h \bar{y}_h''' (\bar{X}_h / \bar{x}_h''')$	4.15	7.34	9.88	11.94	13.66	15.10	16.33	17.40	18.33
$\bar{y}_{jc1}(2) = \sum_{h=1}^L W_h \bar{y}_h''' (\bar{X}_h + C_{xh}) / (\bar{x}_h''' + C_{xh})$	4.22	7.47	10.05	12.15	13.89	15.35	16.61	17.69	18.63
$\bar{y}_{jc1}(3) = \sum_{h=1}^L W_h \bar{y}_h''' (\bar{X}_h + \beta_{2xh}) / (\bar{x}_h''' + \beta_{2xh})$	4.37	7.74	10.40	12.57	14.37	15.88	17.17	18.28	19.25
$\bar{y}_{jc1}(4) = \sum_{h=1}^L W_h \bar{y}_h''' (\bar{X}_h C_{xh} + \beta_{2xh}) / (\bar{x}_h''' C_{xh} + \beta_{2xh})$	5.73	10.08	13.52	16.28	18.57	20.48	22.11	23.51	24.73
$\bar{y}_{jc1}(5) = \sum_{h=1}^L W_h \bar{y}_h''' (\bar{X}_h \beta_{2xh} + C_{xh}) / (\bar{x}_h''' \beta_{2xh} + C_{xh})$	4.32	7.64	10.28	12.42	14.20	15.69	16.97	18.07	19.03
$\bar{y}_{jc1}(6) = \sum_{h=1}^L W_h \bar{y}_h''' (\bar{x}_h''' / \bar{X}_h)$	13.13	25.18	36.28	46.53	56.04	64.87	73.10	80.79	87.99
$\bar{y}_{jc1}(7) = \sum_{h=1}^L W_h \bar{y}_h''' (\bar{x}_h''' + C_{xh}) / (\bar{X}_h + C_{xh})$	13.46	25.82	37.20	47.71	57.46	66.52	74.96	82.84	90.22
$\bar{y}_{jc1}(8) = \sum_{h=1}^L W_h \bar{y}_h''' (\bar{x}_h''' + \beta_{2xh}) / (\bar{X}_h + \beta_{2xh})$	14.26	27.36	39.43	50.58	60.93	70.55	79.52	87.90	95.75
$\bar{y}_{jc1}(9) = \sum_{h=1}^L W_h \bar{y}_h''' (\bar{x}_h''' C_{xh} + \beta_{2xh}) / (\bar{X}_h C_{xh} + \beta_{2xh})$	21.96	42.05	60.48	77.47	93.17	107.72	121.24	133.85	145.62
$\bar{y}_{jc1}(10) = \sum_{h=1}^L W_h \bar{y}_h''' (\bar{x}_h''' \beta_{2xh} + \sigma_{xh}) / (\bar{X}_h \beta_{2xh} + \sigma_{xh})$	20.48	39.01	55.86	71.23	85.33	98.29	110.26	121.34	131.62
Optimum estimators	35.36	65.50	91.51	114.17	134.09	151.75	167.50	181.65	194.41

Interestingly, the numerical results obtained when there is joint occurrence of nonresponse and measurement errors on both the study and auxiliary variables are similar to those obtained when there are separate occurrences of nonresponse and measurement errors. First, it is observed from Table 4.13 that the proposed ratio-type estimators performed better than the proposed product-type estimators, in terms of having higher percentage relative efficiencies. This is because of the positive correlation between the study and auxiliary variables, for Dataset-1. Results also show that the optimum estimators in the proposed class of estimators, $(\bar{y}_{jc1})$, gave the highest values of the percentage relative efficiencies, especially with increase in the sub-sample size, m_h , or the subsampling fraction $(1/g_h)$. Generally, observation from Table 4.13 show that the efficiencies of the estimators, in terms of having smaller mean square errors and higher percentage relative efficiencies, increase with an increase in the subsampling fraction $(1/g_h)$. The table shows that with subsampling fractions of 0.20 (20%) and above, some of the proposed estimators showed high percentage relative efficiencies (above 100%) over the usual poststratified sample mean estimator, $\bar{y}_{ps}$. However, the customary sample mean estimator, $\bar{y}_{js1} = \sum_{h=1}^L W_h \bar{y}_h''''$, under this design, failed to provide a better estimate than the usual poststratified sample mean estimator, $(\bar{y}_{ps})$, when there is complete response, even with a subsampling fraction of 0.90 (90%). It was also observed from Table 4.13 that among the ten (10) special cases of the proposed class of estimators, $(\bar{y}_{jc1})$, the proposed estimator (utilizing auxiliary information) that has the highest efficiency is the adopted Upadhyaya-Singh (1999) estimator, $\bar{y}_{jc1}(4) = \sum_{h=1}^L W_h \bar{y}_h''''(\bar{X}_h C_{xh} + \beta_{2xh}) / (\bar{x}_h'''' C_{xh} + \beta_{2xh})$, followed by the adopted Singh-Kakran (1993) estimator, $\bar{y}_{jc1}(3) = \sum_{h=1}^L W_h \bar{y}_h''''(\bar{X}_h + \beta_{2xh}) / (\bar{x}_h'''' + \beta_{2xh})$, while the estimator with the least efficiency among the ten (10) special cases under consideration is the customary product-type estimator, $\bar{y}_{jc1}(6) = \sum_{h=1}^L W_h \bar{y}_h''''(\bar{x}_h'''' / \bar{X}_h)$.

We also observe from Table 4.13 that the efficiency of the proposed difference-type estimator, $\bar{y}_{jd1} = \sum_{h=1}^L W_h [\bar{y}_h'''' - k_h(\bar{x}_h'''' - \bar{X}_h)]$, $k_h = S_{yxh}/S_{xh}^2 \forall h$, is very close to the efficiency of the optimum estimators in the proposed class of estimators, $(\bar{y}_{jc1})$, since our choice of the constant, $k_h = S_{yxh}/S_{xh}^2 \forall h$, is very close to the optimum value of the parameter that minimizes the mean square error of the proposed class of estimators, $\bar{y}_{jc1}$. Also to be observed from Table

4.13 is the fact that the use of auxiliary information does not always result in an improved efficiency over the customary sample mean estimators, which do not utilize auxiliary information. For Dataset-1, which involves positively correlated study and auxiliary variables, we observed that the customary sample mean estimator, $\bar{y}_{js1} = \sum_{h=1}^L W_h \bar{y}_h''''$, was less efficient than the ratio-type estimators in the proposed class of estimators, $\bar{y}_{jc1}$, but more efficient than the suggested product-type estimators in the proposed class. This means that the use of auxiliary information is most likely to result in an improved efficiency over the customary sample mean estimators when estimators that utilize auxiliary information are appropriately chosen, according to the nature and direction of the relationship between the study and auxiliary variables. Table 4.13 specifically reveals that the ratio-type estimators are preferable when there is a positive correlation between the study and auxiliary variables.

For negatively correlated study and auxiliary variables (Dataset-2), and when there is joint occurrence of nonresponse and measurement errors on both the study and auxiliary variables, the percentage relative efficiencies of some proposed estimators over the usual poststratified mean estimator ($\bar{y}_{ps}$), are shown in Table 4.14. The table shows that the proposed product-type estimators, as expected, performed better than the proposed ratio-type estimators, in terms of having higher percentage relative efficiencies, since there is a negative correlation between the study and auxiliary variables. Furthermore, the optimum estimators in the proposed class of estimators, ($\bar{y}_{jc1}$), gave the highest values of the percentage relative efficiencies, especially as the subsampling fraction ($1/g_h$) increases. High percentage relative efficiencies (above 100%) over the usual poststratified sample mean estimator, $\bar{y}_{ps}$, are recorded with subsampling fractions of 0.60 (60%) and above, which is really high and a bit worrisome. Also, Table 4.14 shows that for Dataset-2, the customary sample mean estimator, $\bar{y}_{js1} = \sum_{h=1}^L W_h \bar{y}_h''''$, could not provide a better estimate than the usual poststratified sample mean estimator, ($\bar{y}_{ps}$), where there is complete response, even with a subsampling fraction of 0.90 (90%).

Table 4.14 also reveals that for the ten (10) special cases of the proposed class of estimators, ($\bar{y}_{jc1}$), under this design, the proposed estimator (utilizing auxiliary information) that

has the highest efficiency is the adopted Upadhyaya-Singh (1999) estimator III, $\bar{y}_{jc1}(9) = \sum_{h=1}^L W_h \bar{y}_h''' (\bar{x}_h''' C_{xh} + \beta_{2xh}) / (\bar{X}_h C_{xh} + \beta_{2xh})$, followed by the adopted Singh (2003) estimator, $\bar{y}_{jc1}(10) = \sum_{h=1}^L W_h \bar{y}_h''' (\bar{x}_h''' \beta_{2xh} + \sigma_{xh}) / (\bar{X}_h \beta_{2xh} + \sigma_{xh})$, while the estimator with the least efficiency among the ten (10) special cases under consideration is the customary ratio-type estimator, $\bar{y}_{jc1}(1) = \sum_{h=1}^L W_h \bar{y}_h''' (\bar{X}_h / \bar{x}_h''')$. Table 4.14, also reveals that the proposed difference-type estimator, $\bar{y}_{jd1} = \sum_{h=1}^L W_h [\bar{y}_h''' - k_h (\bar{x}_h''' - \bar{X}_h)]$, $k_h = S_{yxh} / S_{xh}^2 \forall h$, is almost as efficient as the optimum estimators in the proposed class of estimators, $(\bar{y}_{jc1})$, since the choice of the constant, $k_h = S_{yxh} / S_{xh}^2 \forall h$, is very close to the optimum value of the parameter that minimizes the mean square error of the proposed class of estimators, $\bar{y}_{jc1}$. Table 4.14, just like Table 4.13, shows that the use of auxiliary information does not always result in an improved efficiency over the customary sample mean estimators, which do not utilize auxiliary information. For Dataset-2, which involves negatively correlated study and auxiliary variables, we observed from Table 4.14 that the customary sample mean estimator, $\bar{y}_{js1} = \sum_{h=1}^L W_h \bar{y}_h'''$, was less efficient than the product-type estimators in the proposed class of estimators, $\bar{y}_{jc1}$, but more efficient than the ratio-type estimators, under considerations, in the proposed class. This again indicates that the use of auxiliary information is most likely to result in an improved efficiency over the customary sample mean estimators when estimators that utilize auxiliary information are appropriately chosen, according to the nature and direction of the relationship between the study and auxiliary variables. Table 4.14 specifically reveals that the product-type estimators are preferable when there is a negative correlation between the study and auxiliary variables.

4.8 Joint Occurrence of Nonresponse and Measurement

Errors on only the study variable and not on the auxiliary variable under the Poststratified Sampling Scheme

Recall from (3.254), (3.272) and (3.51), that the proposed class of estimators of the population mean of the study variable (y) when there is joint occurrence of nonresponse and measurement errors on only the study variable and not on the auxiliary variable under the poststratified

sampling scheme is

$$\bar{y}_{jc2} = \sum_{h=1}^L W_h \bar{y}_h''' \left(\frac{a_h \bar{X}_h + b_h}{a_h \bar{X}_h' + b_h} \right)^{\alpha_h} \quad (4.36)$$

with its unconditional mean square error obtained as:

$$\begin{aligned} M S E(\bar{y}_{jc2}) &= \left(\frac{1-f}{n} \right) \sum_{h=1}^L W_h (S_{yh}^2 + \alpha_h^2 \lambda_h^2 R_h^2 S_{xh}^2 - 2\alpha_h \lambda_h R_h S_{yxh}) \\ &+ \left(\frac{1}{n} \right) \sum_{h=1}^L W_h (g_h - 1) W_{2h} S_{2yh}^2 + \sum_{h=1}^L W_h \left[\left(\frac{1-f}{n} \right) S_{uh}^2 + \left(\frac{g_h - 1}{n} \right) W_{2h} S_{2uh}^2 \right] \end{aligned} \quad (4.37)$$

where

$$\lambda_h = \left(\frac{a_h \bar{X}_h}{a_h \bar{X}_h + b_h} \right) \quad (4.38)$$

Applying the method of least squares, it follows that the best choices of α_h , a_h and b_h that minimize the unconditional mean square error of $\bar{y}_{jc2}$ satisfy the expression:

$$\theta_{h8} = \frac{\alpha_h a_h \bar{X}_h R_h}{a_h \bar{X}_h + b_h} = \frac{\alpha_h a_h \bar{Y}_h}{a_h \bar{X}_h + b_h} = \frac{S_{yxh}}{S_{xh}^2} \quad (4.39)$$

with the associated optimum unconditional mean square error of $\bar{y}_{jc2}$ given by

$$\begin{aligned} M S E_0(\bar{y}_{jc2}) &= \left(\frac{1-f}{n} \right) \sum_{h=1}^L W_h (S_{yh}^2 + \theta_{h8}^2 S_{xh}^2 - 2\theta_{h8} S_{yxh} + S_{uh}^2) \\ &+ \left(\frac{1}{n} \right) \sum_{h=1}^L W_h (g_h - 1) W_{2h} (S_{2yh}^2 + S_{2uh}^2) \end{aligned} \quad (4.40)$$

Table 4.15 and Table 4.16 show the associated PREs using Dataset-1 and Dataset-2 respectively.

Table 4.15: Percentage Relative Efficiency (PRE) of Some Proposed Estimators over the Usual Poststratified Mean Estimator ($\bar{y}_{ps}$)
when there is joint occurrence of nonresponse and measurement errors on only the study variable
for positively correlated study and auxiliary variables (Dataset-1)

Estimators	Subsampling Fraction ($1/g_h$) ; where $g_h = n_{2h}/m_h > 1$ is the subsampling factor								
	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90
$\bar{y}_{ps} = \sum_{h=1}^L W_h \bar{y}_h$	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00
$\bar{y}_{js2} = \sum_{h=1}^L W_h \bar{y}_h'''$	19.52	34.96	47.48	57.83	66.54	73.96	80.37	85.95	90.85
$\bar{y}_{jd2} = \sum_{h=1}^L W_h [\bar{y}_h''' - k_h(\bar{x}_h' - \bar{X}_h)]$, $k_h = S_{yxh}/S_{xh}^2 \forall h$	23.90	52.05	85.67	126.55	177.31	242.03	327.39	445.14	618.02
$\bar{y}_{jc2}(1) = \sum_{h=1}^L W_h \bar{y}_h''' (\bar{X}_h / \bar{x}_h')$	23.50	50.19	80.74	116.08	157.42	206.43	265.47	337.96	429.10
$\bar{y}_{jc2}(2) = \sum_{h=1}^L W_h \bar{y}_h''' (\bar{X}_h + C_{xh}) / (\bar{x}_h' + C_{xh})$	23.52	50.26	80.93	116.47	158.15	207.68	267.54	341.32	434.51
$\bar{y}_{jc2}(3) = \sum_{h=1}^L W_h \bar{y}_h''' (\bar{X}_h + \beta_{2xh}) / (\bar{x}_h' + \beta_{2xh})$	23.54	50.34	81.15	116.92	158.97	209.11	269.92	345.20	440.83
$\bar{y}_{jc2}(4) = \sum_{h=1}^L W_h \bar{y}_h''' (\bar{X}_h C_{xh} + \beta_{2xh}) / (\bar{x}_h' C_{xh} + \beta_{2xh})$	23.58	50.56	81.70	118.08	161.13	212.85	276.18	355.50	457.77
$\bar{y}_{jc2}(5) = \sum_{h=1}^L W_h \bar{y}_h''' (\bar{X}_h \beta_{2xh} + C_{xh}) / (\bar{x}_h' \beta_{2xh} + C_{xh})$	23.52	50.27	80.97	116.56	158.30	207.95	267.98	342.04	435.69
$\bar{y}_{jc2}(6) = \sum_{h=1}^L W_h \bar{y}_h''' (\bar{x}_h' / \bar{X}_h)$	11.13	14.88	16.76	17.89	18.64	19.18	19.59	19.90	20.15
$\bar{y}_{jc2}(7) = \sum_{h=1}^L W_h \bar{y}_h''' (\bar{x}_h' + C_{xh}) / (\bar{X}_h + C_{xh})$	11.16	14.94	16.83	17.97	18.74	19.28	19.69	20.01	20.26
$\bar{y}_{jc2}(8) = \sum_{h=1}^L W_h \bar{y}_h''' (\bar{x}_h' + \beta_{2xh}) / (\bar{X}_h + \beta_{2xh})$	11.21	15.03	16.95	18.10	18.88	19.43	19.84	20.17	20.43
$\bar{y}_{jc2}(9) = \sum_{h=1}^L W_h \bar{y}_h''' (\bar{x}_h' C_{xh} + \beta_{2xh}) / (\bar{X}_h C_{xh} + \beta_{2xh})$	11.34	15.26	17.25	18.45	19.25	19.83	20.26	20.60	20.87
$\bar{y}_{jc2}(10) = \sum_{h=1}^L W_h \bar{y}_h''' (\bar{x}_h' \beta_{2xh} + \sigma_{xh}) / (\bar{X}_h \beta_{2xh} + \sigma_{xh})$	13.37	19.16	22.40	24.47	25.90	26.95	27.76	28.40	28.91
Optimum estimators	23.90	52.05	85.67	126.55	177.31	242.03	327.39	445.14	618.02

Table 4.16: Percentage Relative Efficiency (PRE) of Some Proposed Estimators over the Usual Poststratified Mean Estimator ($\bar{y}_{ps}$)
when there is joint occurrence of nonresponse and measurement errors on only the study variable
for negatively correlated study and auxiliary variables (Dataset-2)

Estimators	Subsampling Fraction ($1/g_h$) ; where $g_h = n_{2h}/m_h > 1$ is the subsampling factor								
	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90
$\bar{y}_{ps} = \sum_{h=1}^L W_h \bar{y}_h$	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00
$\bar{y}_{js2} = \sum_{h=1}^L W_h \bar{y}_h'''$	23.58	40.66	53.60	63.74	71.90	78.62	84.23	89.00	93.10
$\bar{y}_{jd2} = \sum_{h=1}^L W_h [\bar{y}_h''' - k_h(\bar{x}_h' - \bar{X}_h)]$, $k_h = S_{yxh}/S_{xh}^2 \forall h$	28.86	59.36	91.66	125.92	162.32	201.06	242.40	286.58	333.92
$\bar{y}_{jc2}(1) = \sum_{h=1}^L W_h \bar{y}_h''' (\bar{X}_h / \bar{x}_h')$	12.93	16.79	18.65	19.74	20.46	20.97	21.35	21.65	21.88
$\bar{y}_{jc2}(2) = \sum_{h=1}^L W_h \bar{y}_h''' (\bar{X}_h + C_{xh}) / (\bar{x}_h' + C_{xh})$	13.04	16.98	18.89	20.01	20.75	21.27	21.67	21.97	22.21
$\bar{y}_{jc2}(3) = \sum_{h=1}^L W_h \bar{y}_h''' (\bar{X}_h + \beta_{2xh}) / (\bar{x}_h' + \beta_{2xh})$	13.26	17.35	19.35	20.52	21.30	21.86	22.27	22.59	22.84
$\bar{y}_{jc2}(4) = \sum_{h=1}^L W_h \bar{y}_h''' (\bar{X}_h C_{xh} + \beta_{2xh}) / (\bar{x}_h' C_{xh} + \beta_{2xh})$	14.92	20.31	23.09	24.79	25.94	26.76	27.39	27.87	28.26
$\bar{y}_{jc2}(5) = \sum_{h=1}^L W_h \bar{y}_h''' (\bar{X}_h \beta_{2xh} + C_{xh}) / (\bar{x}_h' \beta_{2xh} + C_{xh})$	13.19	17.24	19.21	20.37	21.14	21.68	22.09	22.40	22.65
$\bar{y}_{jc2}(6) = \sum_{h=1}^L W_h \bar{y}_h''' (\bar{x}_h' / \bar{X}_h)$	28.02	55.93	83.73	111.42	139.00	166.47	193.84	221.09	248.24
$\bar{y}_{jc2}(7) = \sum_{h=1}^L W_h \bar{y}_h''' (\bar{x}_h' + C_{xh}) / (\bar{X}_h + C_{xh})$	28.09	56.22	84.39	112.59	140.83	169.11	197.42	225.77	254.16
$\bar{y}_{jc2}(8) = \sum_{h=1}^L W_h \bar{y}_h''' (\bar{x}_h' + \beta_{2xh}) / (\bar{X}_h + \beta_{2xh})$	28.24	56.83	85.75	115.03	144.67	174.67	205.04	235.79	266.93
$\bar{y}_{jc2}(9) = \sum_{h=1}^L W_h \bar{y}_h''' (\bar{x}_h' C_{xh} + \beta_{2xh}) / (\bar{X}_h C_{xh} + \beta_{2xh})$	28.84	59.31	91.53	125.67	161.91	200.44	241.48	285.30	332.19
$\bar{y}_{jc2}(10) = \sum_{h=1}^L W_h \bar{y}_h''' (\bar{x}_h' \beta_{2xh} + \sigma_{xh}) / (\bar{X}_h \beta_{2xh} + \sigma_{xh})$	28.66	58.53	89.70	122.26	156.28	191.89	229.18	268.29	309.35
Optimum estimators	28.86	59.36	91.66	125.92	162.32	201.06	242.40	286.58	333.92

Table 4.15 shows the percentage relative efficiencies of some proposed estimators over the usual poststratified mean estimator ($\bar{y}_{ps}$), when there is joint occurrence of nonresponse and measurement errors on only the study variable and not on the auxiliary variable for positively correlated study and auxiliary variables (Dataset-1). It is observed from Table 4.15 that the proposed ratio-type estimators performed better than the proposed product-type estimators, in terms of percentage relative efficiencies. There is as a result of a positive correlation between the study and auxiliary variables for Dataset-1. Observation also show observe that the optimum estimators in the proposed class of estimators, ($\bar{y}_{jc2}$), gave the highest values of the percentage relative efficiencies, especially with increase in the subsampling fraction ($1/g_h$). The table shows that with subsampling fractions of 0.40 (40%) and above, some of the proposed estimators showed high percentage relative efficiencies (above 100%) over the usual poststratified sample mean estimator, $\bar{y}_{ps}$. However, the customary sample mean estimator, $\bar{y}_{js2} = \sum_{h=1}^L W_h \bar{y}_h''''$, under this design, failed to provide a better estimate than the usual poststratified sample mean estimator, ($\bar{y}_{ps}$), where there is complete response, even with a subsampling fraction of 0.90 (90%). It was also observed from Table 4.15 that among the ten (10) special cases of the proposed class of estimators, ($\bar{y}_{jc2}$), the proposed estimator that has the highest efficiency is the adopted Upadhyaya-Singh (1999) estimator, $\bar{y}_{jc2}(4) = \sum_{h=1}^L W_h \bar{y}_h'''' (\bar{X}_h C_{xh} + \beta_{2xh}) / (\bar{x}_h' C_{xh} + \beta_{2xh})$, followed by the adopted Singh-Kakran (1993) estimator, $\bar{y}_{jc2}(3) = \sum_{h=1}^L W_h \bar{y}_h'''' (\bar{X}_h + \beta_{2xh}) / (\bar{x}_h' + \beta_{2xh})$, while the estimator with the least efficiency among the ten (10) special cases under consideration is the customary product-type estimator, $\bar{y}_{jc2}(6) = \sum_{h=1}^L W_h \bar{y}_h'''' (\bar{x}_h' / \bar{X}_h)$, when there is joint occurrence of nonresponse and measurement errors on only the study variable and not on the auxiliary variable.

Observation from Table 4.15 also show that the efficiency of the proposed difference-type estimator, $\bar{y}_{jd2} = \sum_{h=1}^L W_h [\bar{y}_h'''' - k_h (\bar{x}_h' - \bar{X}_h)]$, $k_h = S_{yxh} / S_{xh}^2 \forall h$, is exactly the same as the efficiency of the optimum estimators in the proposed class of estimators, ($\bar{y}_{jc2}$), since our choice of the constant, $k_h = S_{yxh} / S_{xh}^2 \forall h$, is exactly the same as the optimum value of the parameter that minimizes the mean square error of the proposed class of estimators, $\bar{y}_{jc2}$. We also observe from Table 4.15 that for Dataset-1, the customary sample mean estimator, $\bar{y}_{js2} = \sum_{h=1}^L W_h \bar{y}_h''''$, was

less efficient than the ratio-type estimators in the proposed class of estimators, $\bar{y}_{jc2}$, but more efficient than the suggested product-type estimators in the proposed class. This means that the use of auxiliary information is most likely to result in an improved efficiency over the customary sample mean estimators when estimators that utilize auxiliary information are appropriately chosen, according to the nature and direction of the relationship between the study and auxiliary variables. Table 4.15 specifically reveals that the ratio-type estimators are preferable when there is a positive correlation between the study and auxiliary variables.

For negatively correlated study and auxiliary variables (Dataset-2), and when there is joint occurrence of nonresponse and measurement errors on only the study variable and not on the auxiliary variable, the percentage relative efficiencies of some proposed estimators over the usual poststratified mean estimator ($\bar{y}_{ps}$), are shown in Table 4.16. The table shows that the proposed product-type estimators performed better than the proposed ratio-type estimators, in terms of having higher percentage relative efficiencies, since there is a negative correlation between the study and auxiliary variables. Furthermore, the optimum estimators in the proposed class of estimators, ($\bar{y}_{jc2}$), gave the highest values of the percentage relative efficiencies, especially as the subsampling fraction ($1/g_h$) increases. High percentage relative efficiencies (above 100%) over the usual poststratified sample mean estimator, $\bar{y}_{ps}$, are recorded with subsampling fractions of 0.40 (40%) and above. Also, Table 4.16 shows that for Dataset-2, the customary sample mean estimator, $\bar{y}_{js2} = \sum_{h=1}^L W_h \bar{y}_h'''$, could not provide a better estimate than the usual poststratified sample mean estimator, ($\bar{y}_{ps}$), when there is complete response, even with a subsampling fraction of 0.90 (90%).

Table 4.16 also revealed that for the ten (10) special cases of the proposed class of estimators, ($\bar{y}_{jc2}$), when there is joint occurrence of nonresponse and measurement errors on only the study variable and not on the auxiliary variable, the proposed estimator (utilizing auxiliary information) that has the highest efficiency is the adopted Upadhyaya-Singh (1999) estimator III, $\bar{y}_{jc2}(9) = \sum_{h=1}^L W_h \bar{y}_h''' (\bar{x}_h' C_{xh} + \beta_{2xh}) / (\bar{X}_h C_{xh} + \beta_{2xh})$, followed by the adopted Singh (2003) estimator, $\bar{y}_{jc2}(10) = \sum_{h=1}^L W_h \bar{y}_h''' (\bar{x}_h' \beta_{2xh} + \sigma_{xh}) / (\bar{X}_h \beta_{2xh} + \sigma_{xh})$, while the estimator with the least efficiency

among the ten (10) special cases under consideration is the customary ratio-type estimator, $\bar{y}_{jc2}(1) = \sum_{h=1}^L W_h \bar{y}_h''' (\bar{X}_h / \bar{x}_h')$. The table also reveals that the proposed difference-type estimator, $\bar{y}_{jd2} = \sum_{h=1}^L W_h [\bar{y}_h''' - k_h (\bar{x}_h' - \bar{X}_h)]$, $k_h = S_{yxh} / S_{xh}^2 \forall h$, is as efficient as the optimum estimators in the proposed class of estimators, $(\bar{y}_{jc2})$, since the choice of the constant, $k_h = S_{yxh} / S_{xh}^2 \forall h$, is exactly the same as the optimum value of the parameter that minimizes the mean square error of the proposed class of estimators, $\bar{y}_{jc2}$. For Dataset-2, which involves negatively correlated study and auxiliary variables, we observed from Table 4.16 that the customary sample mean estimator, $\bar{y}_{js2} = \sum_{h=1}^L W_h \bar{y}_h'''$, was less efficient than the product-type estimators in the proposed class of estimators, $\bar{y}_{jc2}$, but more efficient than the ratio-type estimators, under considerations, in the proposed class. These agrees with its trend observed in earlier designs concerning the use of auxiliary information. Table 4.16 specifically reveals that the product-type estimators are preferable when there is a negative correlation between the study and auxiliary variables.

4.9 Joint Occurrence of Nonresponse and Measurement

Errors on only the auxiliary variable and not on the study variable under the Poststratified Sampling Scheme

Recall from (3.288), (3.308) and (3.51), that the proposed class of estimators of the population mean of the study variable (y) when there is joint occurrence of nonresponse and measurement errors on only the auxiliary variable and not on the study variable under the poststratified sampling scheme is

$$\bar{y}_{jc3} = \sum_{h=1}^L W_h \bar{y}_h' \left(\frac{a_h \bar{X}_h + b_h}{a_h \bar{x}_h''' + b_h} \right)^{\alpha_h} \quad (4.41)$$

with its unconditional mean square error obtained as:

$$\begin{aligned}
 MS E(\bar{y}_{jc3}) &= \sum_{h=1}^L W_h \left(\frac{1-f}{n} \right) (S_{yh}^2 + \alpha_h^2 \lambda_h^2 R_h^2 S_{xh}^2 - 2\alpha_h \lambda_h R_h S_{yxh}) \\
 &\quad + \sum_{h=1}^L W_h \alpha_h^2 \lambda_h^2 R_h^2 \left[\left(\frac{1-f}{n} \right) S_{vh}^2 + \left(\frac{g_h-1}{n} \right) W_{2h} S_{2vh}^2 \right] \\
 &\quad + \sum_{h=1}^L W_h \alpha_h^2 \lambda_h^2 R_h^2 \left(\frac{g_h-1}{n} \right) W_{2h} S_{2xh}^2
 \end{aligned} \tag{4.42}$$

where

$$\lambda_h = \left(\frac{a_h \bar{X}_h}{a_h \bar{X}_h + b_h} \right) \tag{4.43}$$

Applying the method of least squares, it follows that the best choices of α_h , a_h and b_h that minimize the unconditional mean square error of $\bar{y}_{jc3}$ satisfy the expression:

$$\theta_{h9} = \frac{\alpha_h a_h \bar{X}_h R_h}{a_h \bar{X}_h + b_h} = \frac{\alpha_h a_h \bar{Y}_h}{a_h \bar{X}_h + b_h} = \frac{(1-f) S_{yxh}}{(1-f)(S_{xh}^2 + S_{vh}^2) + (g_h-1) W_{2h} (S_{2xh}^2 + S_{2vh}^2)} \tag{4.44}$$

with the associated optimum unconditional mean square error of $\bar{y}_{jc3}$ given by

$$\begin{aligned}
 MS E_0(\bar{y}_{jc3}) &= \left(\frac{1-f}{n} \right) \sum_{h=1}^L W_h (S_{yh}^2 + \theta_{h9}^2 S_{xh}^2 - 2\theta_{h9} S_{yxh} + \theta_{h9}^2 S_{vh}^2) \\
 &\quad + \left(\frac{1}{n} \right) \sum_{h=1}^L W_h (g_h-1) W_{2h} \theta_{h9}^2 (S_{2xh}^2 + S_{2vh}^2)
 \end{aligned} \tag{4.45}$$

Table 4.17 shows the Percentage Relative Efficiencies (PRE) of some special cases of the proposed estimators over the usual poststratified mean estimator, $(\bar{y}_{ps})$, using Dataset-1 (positively correlated study and auxiliary variables), when there is joint occurrence of nonresponse and measurement errors on only the auxiliary variable and not on the study variable. Table 4.18 shows the PREs, using Dataset-2 (negatively correlated study and auxiliary variables).

Table 4.17: Percentage Relative Efficiency (PRE) of Some Proposed Estimators over the Usual Poststratified Mean Estimator ($\bar{y}_{ps}$)
when there is joint occurrence of nonresponse and measurement errors on only the auxiliary variable
for positively correlated study and auxiliary variables (Dataset-1)

Estimators	Subsampling Fraction ($1/g_h$) ; where $g_h = n_{2h}/m_h > 1$ is the subsampling factor								
	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90
$\bar{y}_{ps} = \sum_{h=1}^L W_h \bar{y}_h$	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00
$\bar{y}_{js3} = \sum_{h=1}^L W_h \bar{y}'_h$	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00
$\bar{y}_{jd3} = \sum_{h=1}^L W_h [\bar{y}'_h - k_h(\bar{x}'''_h - \bar{X}_h)]$, $k_h = S_{yxh}/S_{xh}^2 \forall h$	27.31	60.16	100.44	150.99	216.29	303.91	427.68	615.75	935.82
$\bar{y}_{jc3}(1) = \sum_{h=1}^L W_h \bar{y}'_h (\bar{X}_h / \bar{x}'''_h)$	18.76	40.93	67.51	99.98	140.54	192.64	262.02	358.98	504.06
$\bar{y}_{jc3}(2) = \sum_{h=1}^L W_h \bar{y}'_h (\bar{X}_h + C_{xh}) / (\bar{x}'''_h + C_{xh})$	18.98	41.42	68.35	101.27	142.44	195.38	266.01	364.95	513.51
$\bar{y}_{jc3}(3) = \sum_{h=1}^L W_h \bar{y}'_h (\bar{X}_h + \beta_{2xh}) / (\bar{x}'''_h + \beta_{2xh})$	19.42	42.37	69.93	103.63	145.78	200.01	272.39	373.86	526.36
$\bar{y}_{jc3}(4) = \sum_{h=1}^L W_h \bar{y}'_h (\bar{X}_h C_{xh} + \beta_{2xh}) / (\bar{x}'''_h C_{xh} + \beta_{2xh})$	20.52	44.80	73.96	109.67	154.39	212.03	289.12	397.54	561.22
$\bar{y}_{jc3}(5) = \sum_{h=1}^L W_h \bar{y}'_h (\bar{X}_h \beta_{2xh} + C_{xh}) / (\bar{x}'''_h \beta_{2xh} + C_{xh})$	18.98	41.41	68.35	101.30	142.51	195.55	266.37	365.70	515.08
$\bar{y}_{jc3}(6) = \sum_{h=1}^L W_h \bar{y}'_h (\bar{x}'''_h / \bar{X}_h)$	9.94	13.94	16.10	17.45	18.38	19.05	19.57	19.97	20.29
$\bar{y}_{jc3}(7) = \sum_{h=1}^L W_h \bar{y}'_h (\bar{x}'''_h + C_{xh}) / (\bar{X}_h + C_{xh})$	10.03	14.05	16.21	17.57	18.49	19.17	19.68	20.08	20.41
$\bar{y}_{jc3}(8) = \sum_{h=1}^L W_h \bar{y}'_h (\bar{x}'''_h + \beta_{2xh}) / (\bar{X}_h + \beta_{2xh})$	10.18	14.23	16.40	17.75	18.68	19.35	19.86	20.26	20.58
$\bar{y}_{jc3}(9) = \sum_{h=1}^L W_h \bar{y}'_h (\bar{x}'''_h C_{xh} + \beta_{2xh}) / (\bar{X}_h C_{xh} + \beta_{2xh})$	10.58	14.69	16.87	18.23	19.15	19.82	20.33	20.72	21.04
$\bar{y}_{jc3}(10) = \sum_{h=1}^L W_h \bar{y}'_h (\bar{x}'''_h \beta_{2xh} + \sigma_{xh}) / (\bar{X}_h \beta_{2xh} + \sigma_{xh})$	16.39	21.85	24.57	26.21	27.29	28.07	28.66	29.11	29.47
Optimum estimators	31.09	68.23	113.37	169.41	240.86	335.07	464.97	655.59	962.50

Table 4.18: Percentage Relative Efficiency (PRE) of Some Proposed Estimators over the Usual Poststratified Mean Estimator ($\bar{y}_{ps}$)
when there is joint occurrence of nonresponse and measurement errors on only the auxiliary variable
for negatively correlated study and auxiliary variables (Dataset-2)

Estimators	Subsampling Fraction ($1/g_h$) ; where $g_h = n_{2h}/m_h > 1$ is the subsampling factor								
	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90
$\bar{y}_{ps} = \sum_{h=1}^L W_h \bar{y}_h$	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00
$\bar{y}_{js3} = \sum_{h=1}^L W_h \bar{y}'_h$	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00
$\bar{y}_{jd3} = \sum_{h=1}^L W_h [\bar{y}'_h - k_h (\bar{x}'''_h - \bar{X}_h)]$, $k_h = S_{yxh}/S_{xh}^2 \forall h$	17.70	35.37	53.00	70.60	88.16	105.68	123.17	140.63	158.05
$\bar{y}_{jc3}(1) = \sum_{h=1}^L W_h \bar{y}'_h (\bar{X}_h / \bar{x}'''_h)$	6.80	10.62	13.06	14.76	16.01	16.97	17.72	18.34	18.85
$\bar{y}_{jc3}(2) = \sum_{h=1}^L W_h \bar{y}'_h (\bar{X}_h + C_{xh}) / (\bar{x}'''_h + C_{xh})$	6.96	10.84	13.32	15.04	16.30	17.27	18.03	18.65	19.16
$\bar{y}_{jc3}(3) = \sum_{h=1}^L W_h \bar{y}'_h (\bar{X}_h + \beta_{2xh}) / (\bar{x}'''_h + \beta_{2xh})$	7.30	11.32	13.87	15.63	16.92	17.90	18.67	19.30	19.82
$\bar{y}_{jc3}(4) = \sum_{h=1}^L W_h \bar{y}'_h (\bar{X}_h C_{xh} + \beta_{2xh}) / (\bar{x}'''_h C_{xh} + \beta_{2xh})$	10.65	15.86	18.94	20.98	22.43	23.52	24.36	25.03	25.58
$\bar{y}_{jc3}(5) = \sum_{h=1}^L W_h \bar{y}'_h (\bar{X}_h \beta_{2xh} + C_{xh}) / (\bar{x}'''_h \beta_{2xh} + C_{xh})$	7.17	11.14	13.67	15.41	16.69	17.67	18.44	19.07	19.58
$\bar{y}_{jc3}(6) = \sum_{h=1}^L W_h \bar{y}'_h (\bar{x}'''_h / \bar{X}_h)$	9.49	19.04	28.66	38.35	48.10	57.93	67.83	77.80	87.83
$\bar{y}_{jc3}(7) = \sum_{h=1}^L W_h \bar{y}'_h (\bar{x}'''_h + C_{xh}) / (\bar{X}_h + C_{xh})$	9.74	19.55	29.43	39.37	49.39	59.47	69.62	79.85	90.15
$\bar{y}_{jc3}(8) = \sum_{h=1}^L W_h \bar{y}'_h (\bar{x}'''_h + \beta_{2xh}) / (\bar{X}_h + \beta_{2xh})$	10.31	20.71	31.18	41.74	52.39	63.12	73.94	84.84	95.84
$\bar{y}_{jc3}(9) = \sum_{h=1}^L W_h \bar{y}'_h (\bar{x}'''_h C_{xh} + \beta_{2xh}) / (\bar{X}_h C_{xh} + \beta_{2xh})$	16.26	32.58	48.98	65.43	81.96	98.55	115.21	131.94	148.74
$\bar{y}_{jc3}(10) = \sum_{h=1}^L W_h \bar{y}'_h (\bar{x}'''_h \beta_{2xh} + \sigma_{xh}) / (\bar{X}_h \beta_{2xh} + \sigma_{xh})$	16.23	32.13	47.70	62.96	77.92	92.58	106.95	121.05	134.88
Optimum estimators	37.09	68.99	96.71	121.03	142.53	161.69	178.85	194.33	208.35

Table 4.17 shows that the proposed ratio-type estimators performed better than the proposed product-type estimators, in terms of having higher percentage relative efficiencies, since there is a positive correlation between the study and auxiliary variables for Dataset-1. It is observed from Table 4.17 that the optimum estimators in the proposed class of estimators, $(\bar{y}_{jc3})$, yielded the highest values of the percentage relative efficiencies, especially with increase in the subsampling fraction $(1/g_h)$. The Table further shows that with subsampling fractions of 0.40 (40%) and above, some of the proposed estimators showed high percentage relative efficiencies (above 100%) over the usual poststratified sample mean estimator, $\bar{y}_{ps}$. Also observation show that the customary sample mean estimator, $\bar{y}_{js3} = \sum_{h=1}^L W_h \bar{y}_h''''$, is as efficient as the usual poststratified sample mean estimator, $(\bar{y}_{ps})$, where there is complete response. Furthermore, Table 4.17 revealed that among the ten (10) special cases of the proposed class of estimators, $(\bar{y}_{jc3})$, when there is joint occurrence of nonresponse and measurement errors on only the auxiliary variable and not on the study variable, the proposed estimator that has the highest efficiency is the adopted Upadhyaya-Singh (1999) estimator, $\bar{y}_{jc3}(4) = \sum_{h=1}^L W_h \bar{y}_h' (\bar{X}_h C_{xh} + \beta_{2xh}) / (\bar{x}_h''' C_{xh} + \beta_{2xh})$, followed by the adopted Singh-Kakran (1993) estimator, $\bar{y}_{jc3}(3) = \sum_{h=1}^L W_h \bar{y}_h' (\bar{X}_h + \beta_{2xh}) / (\bar{x}_h''' + \beta_{2xh})$, while the estimator with the least efficiency among the ten (10) special cases under consideration is the customary product-type estimator, $\bar{y}_{jc3}(6) = \sum_{h=1}^L W_h \bar{y}_h' (\bar{x}_h''' / \bar{X}_h)$, when there is joint occurrence of nonresponse and measurement errors on only the auxiliary variable and not on the study variable.

It is also observe from Table 4.17 that the efficiency of the proposed difference-type estimator, $\bar{y}_{jd3} = \sum_{h=1}^L W_h [\bar{y}_h' - k_h (\bar{x}_h''' - \bar{X}_h)]$, $k_h = S_{yxh} / S_{xh}^2 \forall h$, is close to the efficiency of the optimum estimators in the proposed class of estimators, $(\bar{y}_{jc3})$, since our choice of the constant, $k_h = S_{yxh} / S_{xh}^2 \forall h$, is closest to the optimum value of the parameter that minimizes the mean square error of the proposed class of estimators, $\bar{y}_{jc3}$. Result also show from Table 4.17 that for Dataset-1, the customary sample mean estimator, $\bar{y}_{js3} = \sum_{h=1}^L W_h \bar{y}_h'$, was less efficient than the ratio-type estimators in the proposed class of estimators, $\bar{y}_{jc2}$, as the subsampling fraction increased, but more efficient than the suggested product-type estimators in the proposed class. This implies that the use of auxiliary information is most likely to result in an improved

efficiency over the customary sample mean estimators when estimators that utilize auxiliary information are appropriately chosen, according to the nature and direction of the relationship between the study and auxiliary variables. Table 4.17 specifically reveals that the ratio-type estimators are preferable when there is a positive correlation between the study and auxiliary variables.

For negatively correlated study and auxiliary variables (Dataset-2), and when there is joint occurrence of nonresponse and measurement errors on only the auxiliary variable and not on the study variable, the percentage relative efficiencies of some proposed estimators over the usual poststratified mean estimator ($\bar{y}_{ps}$), are shown in Table 4.18. From the table the proposed product-type estimators, as expected, performed better than the proposed ratio-type estimators, in terms of having higher percentage relative efficiencies, since there is a negative correlation between the study and auxiliary variables. Furthermore, the optimum estimators in the proposed class of estimators, ($\bar{y}_{jc3}$), gave the highest values of the percentage relative efficiencies, especially as the subsampling fraction ($1/g_h$) increases. High percentage relative efficiencies of (other 100%) against the usual poststratified sample mean estimator, $\bar{y}_{ps}$, were observed with subsampling fractions of 0.60 (60%) and above, which is uncomfortably very high. However, Table 4.18 shows that for Dataset-2, the customary sample mean estimator, $\bar{y}_{js3} = \sum_{h=1}^L W_h \bar{y}'_h$, under this design, is as efficient as the usual poststratified sample mean estimator, ($\bar{y}_{ps}$), when there is complete response.

Table 4.18 also reveals that for the ten (10) special cases of the proposed class of estimators, ($\bar{y}_{jc3}$), when there is joint occurrence of nonresponse and measurement errors on only the auxiliary variable and not on the study variable, the proposed estimator (utilizing auxiliary information) that has the highest efficiency is the adopted Upadhyaya-Singh (1999) estimator III, $\bar{y}_{jc3}(9) = \sum_{h=1}^L W_h \bar{y}'_h (\bar{x}'''_h C_{xh} + \beta_{2xh}) / (\bar{X}_h C_{xh} + \beta_{2xh})$, followed by the adopted Singh (2003) estimator, $\bar{y}_{jc3}(10) = \sum_{h=1}^L W_h \bar{y}'_h (\bar{x}'''_h \beta_{2xh} + \sigma_{xh}) / (\bar{X}_h \beta_{2xh} + \sigma_{xh})$, while the estimator with the least efficiency among the ten (10) special cases under consideration is the customary ratio-type estimator, $\bar{y}_{jc3}(1) = \sum_{h=1}^L W_h \bar{y}'_h (\bar{X}_h / \bar{x}'''_h)$, when there is joint occurrence of nonresponse and measurement

errors on only the auxiliary variable and not on the study variable. The table also reveals that the proposed difference-type estimator, $\bar{y}_{jd3} = \sum_{h=1}^L W_h [\bar{y}'_h - k_h(\bar{x}'''_h - \bar{X}_h)]$, $k_h = S_{yxh}/S_{xh}^2 \forall h$, is not as efficient as the optimum estimators in the proposed class of estimators, $(\bar{y}_{jc3})$, since the choice of the constant, $k_h = S_{yxh}/S_{xh}^2 \forall h$, is not very close to the optimum value that minimizes the mean square error of the proposed class of estimators, $\bar{y}_{jc3}$. For Dataset-2, which involves negatively correlated study and auxiliary variables, It was also observed from Table 4.16 that the customary sample mean estimator, $\bar{y}_{js3} = \sum_{h=1}^L W_h \bar{y}'_h$, was less efficient than the product-type estimators in the proposed class of estimators, $\bar{y}_{jc3}$, as the subsampling fraction increased, but more efficient than the ratio-type estimators, under considerations, in the proposed class. This reveals that the use of auxiliary information is most likely to result in an improved efficiency over the customary sample mean estimators when estimators that utilize auxiliary information are appropriately chosen, according to the nature and direction of the relationship between the study and auxiliary variables. Table 4.18 specifically reveals that the product-type estimators are preferable when there is a negative correlation between the study and auxiliary variables.

CHAPTER FIVE

5.0 CONCLUSION AND RECOMMENDATIONS

5.1 Summary

This work bothered on the situation where nonresponse and measurement errors are encountered in the course of employing the postratified sampling scheme. Cases of separate occurrence of nonresponse or measurement error or joint occurrence of both were considered. The main aim was to construct sampling strategies for handling these three broad cases when using postratified sampling scheme. The proposed sampling scheme which included the utilization of auxiliary are for estimating the population mean of a study variable, as a parameter of interest. Three particular cases were considered for each of the three broad cases earlier mentioned after which three estimators namely; the customary, the difference and a class of estimators were proposed for each particular case thus giving rise to 27 sampling strategies for the postratified sampling scheme. For the case of separate occurrence of nonresponse in postratified sampling scheme, the three particular cases were (i) when there is nonresponse on both study and auxiliary variable (ii) where there is nonresponse on only the study variable (iii) where there is nonresponse on only the auxiliary variable. For each of these cases under nonresponse, sampling designs were developed along the line of Hansen Hurwitz (1946) who first utilized a subsampling design for dealing with nonresponse in simple random sampling scheme. Again, for each of the proposed sampling designs, three estimators were proposed including a class of estimators of the study variable population mean. The class of estimators proposed for the case when there is nonresponse on both the study and auxiliary variables under the poststratified

sampling scheme is given as

$$\bar{y}_{nc1} = \sum_{h=1}^L W_h \bar{y}_h^* \left(\frac{a_h \bar{X}_h + b_h}{a_h \bar{X}_h^* + b_h} \right)^{\alpha_h} \quad (5.1)$$

which was observed to be generally biased for the population mean of the study variable, y , with its bias and mean square errors, both for the conditional and unconditional arguments, obtained as shown in Theorem 3.3. Under the unconditional argument, the optimum mean square error up to the first order approximation was obtained as:

$$\begin{aligned} MS E_0(\bar{y}_{nc1}) &= \left(\frac{1-f}{n} \right) \sum_{h=1}^L W_h (S_{yh}^2 + \theta_{h1}^2 S_{xh}^2 - 2\theta_{h1} S_{yxh}) \\ &\quad + \left(\frac{1}{n} \right) \sum_{h=1}^L W_h (g_h - 1) W_{2h} (S_{2yh}^2 + \theta_{h1}^2 S_{2xh}^2 - 2\theta_{h1} S_{2yxh}) \end{aligned} \quad (5.2)$$

where

$$\theta_{h1} = \frac{\alpha_h a_h \bar{X}_h R_h}{a_h \bar{X}_h + b_h} = \frac{\alpha_h a_h \bar{Y}_h}{a_h \bar{X}_h + b_h} = \frac{(1-f) S_{yxh} + (g_h - 1) S_{2yxh}}{(1-f) S_{xh}^2 + (g_h - 1) S_{2xh}^2} \quad (5.3)$$

The summary of the proposed classes of estimators with the best estimators in the classes are given in Table 5.1. The optimum unconditional mean square errors of the classes are obtained up to first order of approximations by replacing θ_{h1} in (5.2) by θ_{hk} , $k = 2, 3, \dots, 9$ as shown in Table 5.1.

Table 5.1: Summary of Proposed Classes of Estimators

Description	Proposed Class of Estimators	Best Choices of $\theta_h = (\alpha_h a_h \bar{X}_h R_h) / (a_h \bar{X}_h + b_h)$
Nonresponse on both the study and auxiliary variables	$\bar{y}_{nc1} = \sum_{h=1}^L W_h \bar{y}_h^* [(a_h \bar{X}_h + b_h) / (a_h \bar{x}_h^* + b_h)]^{\alpha_h}$	$\theta_{h1} = [(1-f)S_{yxh} + (g_h - 1)S_{2yxh}] / [(1-f)S_{xh}^2 + (g_h - 1)S_{2xh}^2]$
Nonresponse on only the study variable	$\bar{y}_{nc2} = \sum_{h=1}^L W_h \bar{y}_h^* [(a_h \bar{X}_h + b_h) / (a_h \bar{x}_h + b_h)]^{\alpha_h}$	$\theta_{h2} = S_{yxh} / S_{xh}^2$
Nonresponse on only the auxiliary variable	$\bar{y}_{nc3} = \sum_{h=1}^L W_h \bar{y}_h [(a_h \bar{X}_h + b_h) / (a_h \bar{x}_h^* + b_h)]^{\alpha_h}$	$\theta_{h3} = (1-f)S_{yxh} / [(1-f)S_{xh}^2 + (g_h - 1)S_{2xh}^2]$
Measurement errors on both the study and auxiliary variables	$\bar{y}_{mc1} = \sum_{h=1}^L W_h \bar{y}_h'' [(a_h \bar{X}_h + b_h) / (a_h \bar{x}_h'' + b_h)]^{\alpha_h}$	$\theta_{h4} = S_{yxh} / (S_{xh}^2 + S_{vh}^2)$
Measurement errors on only the study variable	$\bar{y}_{mc2} = \sum_{h=1}^L W_h \bar{y}_h'' [(a_h \bar{X}_h + b_h) / (a_h \bar{x}_h' + b_h)]^{\alpha_h}$	$\theta_{h5} = S_{yxh} / S_{xh}^2$
Measurement errors on only the auxiliary variable	$\bar{y}_{mc3} = \sum_{h=1}^L W_h \bar{y}_h' [(a_h \bar{X}_h + b_h) / (a_h \bar{x}_h'' + b_h)]^{\alpha_h}$	$\theta_{h6} = S_{yxh} / (S_{xh}^2 + S_{vh}^2)$
Joint occurrence of non-response and measurement errors on both the study and auxiliary variables	$\bar{y}_{jc1} = \sum_{h=1}^L W_h \bar{y}_h''' [(a_h \bar{X}_h + b_h) / (a_h \bar{x}_h''' + b_h)]^{\alpha_h}$	$\theta_{h7} = [(1-f)S_{yxh} + (g_h - 1)S_{2yxh}] / [(1-f)(S_{xh}^2 + S_{vh}^2) + (g_h - 1)(S_{2xh}^2 + S_{2vh}^2)]$
Joint occurrence of non-response and measurement errors on only the study variable	$\bar{y}_{jc2} = \sum_{h=1}^L W_h \bar{y}_h''' [(a_h \bar{X}_h + b_h) / (a_h \bar{x}_h' + b_h)]^{\alpha_h}$	$\theta_{h8} = S_{yxh} / S_{xh}^2$
Joint occurrence of non-response and measurement errors on only the auxiliary variable	$\bar{y}_{jc3} = \sum_{h=1}^L W_h \bar{y}_h' [(a_h \bar{X}_h + b_h) / (a_h \bar{x}_h''' + b_h)]^{\alpha_h}$	$\theta_{h9} = (1-f)S_{yxh} / [(1-f)(S_{xh}^2 + S_{vh}^2) + (g_h - 1)(S_{2xh}^2 + S_{2vh}^2)]$

In summary, the study proposed three types of estimators for handling the problem of nonresponse and measurement errors in poststratified sampling scheme. These are the customary sample mean estimators that do not utilize auxiliary variables, the difference-type estimators and classes of ratio-type/product-type estimators. The results and discussions of the study revealed that the proposed ratio-type estimators performed better than the proposed product-type estimators, in terms of having higher percentage relative efficiencies, when there was a positive correlation between the study and auxiliary variables (Dataset-1), while the reverse was the case when dealing with negatively correlated study and auxiliary variables (Dataset-2). However, the empirical results also confirmed the theoretical results, which established that the highest efficiencies in the proposed classes were obtained with the use of the optimum estimators in the classes, with their efficiencies found to be higher than those of both the ratio-type and product-type estimators.

The performance of the difference-type estimators, on the other hand, were found to depend on the choice of the constant, k_h , used in defining the difference-type estimators. The efficiencies of the proposed difference-type estimators were found to be very close to the efficiencies of the optimum estimators in the proposed classes of estimators when the constant, k_h , was chosen to be very close to the optimum value that minimized the mean square errors of the proposed classes of estimators. However, when the chosen value of the constant, k_h , was not too close to the optimum value that minimized the mean square errors of the proposed classes of estimators, then the efficiencies of the difference-type estimators were found to be much lower than those of the optimum class of estimators. The difference-type estimators were found to be as much efficient as the optimum class of estimators only when the constant, k_h , is chosen to be exactly equal to the optimum value that minimized the mean square errors of the proposed classes of estimators.

For all cases involving the presence of nonresponse in the study variable or the auxiliary variable or both, we found out that the percentage relative efficiencies of the proposed estimators over the usual poststratified sample mean estimator, $\bar{y}_{ps}$, were increasing with increase in the

subsampling fraction ($1/g_h$). The empirical results revealed that with subsampling fractions of 0.40 (40%) and above, most of the proposed estimators showed high percentage relative efficiencies (above 100%) over the usual poststratified sample mean estimator, $\bar{y}_{ps}$.

Comparison of the efficiency of the usual poststratified sample mean estimator, $\bar{y}_{ps}$, with those of the various customary mean estimators of the different cases considered in the study, revealed that the efficiencies of both estimators were the same when there was complete response or no nonresponse on the study variable. Recall that the different proposed customary estimators, just like the usual poststratified sample mean estimator, $\bar{y}_{ps}$, do not involve the use of auxiliary information. The efficiencies of the usual poststratified sample mean estimator, $\bar{y}_{ps}$, and the proposed customary estimators in various cases, were also found to be the same when there were no measurement errors on the study variable. Similarly, both estimators also had the same efficiencies when there was no joint occurrence of nonresponse and measurement errors on the study variable. For all other cases, the efficiencies of the proposed customary estimators, based only on the study variable, were lower than that of the usual poststratified sample mean, $\bar{y}_{ps}$, even with subsampling fractions of 0.90 (90%) in the presence of nonresponse.

Furthermore, the empirical results revealed that among the ten (10) special cases of the proposed classes of estimators, when dealing with positively correlated study and auxiliary variables (Dataset-1), the proposed estimator that had the highest efficiency in all cases was the adopted Upadhyaya-Singh (1999) estimator I, followed by the adopted Singh-Kakran (1993) estimator, while the estimator with the least efficiency among the ten (10) special cases under consideration was the customary product-type estimator, for positively correlated study and auxiliary variables. For negatively correlated study and auxiliary variables (Dataset-2), the empirical results also revealed that for the ten (10) special cases of the proposed classes of estimators under consideration, the proposed estimator that had the highest efficiency was the adopted Upadhyaya-Singh (1999) estimator III, followed by the adopted Singh (2003) estimator, while the estimator with the least efficiency among the ten (10) special cases under consideration was the customary ratio-type estimator, for negatively correlated study and auxiliary variables.

We also observed that between the proposed customary sample mean estimators (without the use of auxiliary information) and the other proposed estimators that utilized auxiliary information, the use of auxiliary information did not always yield increased efficiency. For positively correlated study and auxiliary variables (Dataset-1), the proposed customary sample mean estimators were less efficient than the proposed ratio-type estimators but more efficient than the proposed product-type estimators. For negatively correlated study and auxiliary variables (Dataset-2), the proposed customary sample mean estimators were less efficient than the proposed product-type estimators but more efficient than the proposed ratio-type estimators. This revealed that the use of auxiliary information had greater chances to result in an improved efficiency over the customary sample mean estimators when estimators that utilized auxiliary information were appropriately chosen, according to the nature and direction of the relationship between the study and auxiliary variables. The empirical results specifically revealed that the ratio-type estimators were preferable over the customary sample mean estimators, for positively correlated study and auxiliary variables, while the product-type estimators were preferable over the customary sample mean estimators, for negatively correlated study and auxiliary variables.

5.2 Conclusions

The proposed customary sample mean estimators are as efficient as the usual poststratified sample mean estimator, $\bar{y}_{ps}$, when there are complete responses or no nonresponse on the study variable, when there are no measurement errors on the study variable and when there is no joint occurrence of nonresponse and measurement errors on the study variable. For all other cases, the proposed customary sample mean estimators are less efficient than the usual poststratified sample mean estimator, $\bar{y}_{ps}$, even with subsampling fractions of up to 0.90 (90%) in the presence of nonresponse. It can therefore be concluded that there is need to consider the use of auxiliary information in search of more efficient estimators than the usual poststratified sample mean estimator, $\bar{y}_{ps}$, in the handling of nonresponse and measurement errors under the poststratified sampling scheme.

The use of auxiliary information does not always result in improved estimators over the usual poststratified sample mean estimator, $\bar{y}_{ps}$, when handling nonresponse and measurement errors in poststratified sampling scheme. However, for positively correlated study and auxiliary variables, most of the proposed ratio-type estimators showed higher efficiency than the usual poststratified sample mean estimator, $\bar{y}_{ps}$. Similarly, for negatively correlated study and auxiliary variables, most of the proposed product-type estimators were more efficient than the usual poststratified sample mean estimator, $\bar{y}_{ps}$, in terms of having smaller mean square errors. It can therefore be concluded that the use of auxiliary information is more likely to result in increased efficiency over the usual poststratified sample mean estimator, $\bar{y}_{ps}$, when estimators that utilize auxiliary information are appropriately chosen, according to the nature and direction of the relationship between the study and auxiliary variables.

The best estimators or the most efficient estimators among the proposed classes of estimators are those shown in Table 5.1. Both theoretical and empirical results confirm that the obtained best estimators are more efficient than the usual poststratified sample mean estimator, $\bar{y}_{ps}$, the customary-type estimators and the proposed ratio-type/product-type estimators. The proposed difference-type estimators are equally less efficient than the best estimators in the proposed classes of estimators, except when the constant, k_h , is chosen to be exactly equal to its value that minimizes the mean square errors of the proposed classes of estimators. It can therefore be concluded that the obtained best estimators among the proposed classes of estimators are more efficient than the usual poststratified sample mean estimator, $\bar{y}_{ps}$, as well as more efficient than other ratio-type/product-type estimators in the proposed classes of estimators.

For positively correlated study and auxiliary variables (Dataset-1), the proposed estimator with the highest efficiency in all cases was the adopted Upadhyaya-Singh (1999) estimator I, while the estimator with the least efficiency among the ten (10) special cases under consideration was the customary product-type estimator. Similarly, for negatively correlated study and auxiliary variables (Dataset-2), the proposed estimator that had the highest efficiency was the adopted

Upadhyaya-Singh (1999) estimator III, while the estimator with the least efficiency among the ten (10) special cases under consideration was the customary ratio-type estimator. It can therefore be concluded that the Upadhyaya-Singh (1999) estimator I is the most efficient estimator for positively correlated study and auxiliary variables, while the Upadhyaya-Singh (1999) estimator III is the most efficient estimator for negatively correlated study and auxiliary variables among the considered ten (10) special cases of the proposed classes of estimators.

5.3 Recommendations

First, we would like to recommend the use of auxiliary information in the search of improved sampling strategies for the handling of the problem of nonresponse and measurement errors under the poststratified sampling scheme. This is because the study reveals that the use of customary-type estimators in handling nonresponse and measurement errors in poststratified sampling scheme does not produce estimators that are more efficient than the usual poststratified sample mean estimator, $\bar{y}_{ps}$.

Secondly, we recommend that the proposed ratio-type estimators are to be preferred when there is a positive correlation between the study and auxiliary variables when dealing with the problem of nonresponse and measurement errors in poststratified sampling scheme. Conversely, the proposed product-type estimators are preferable when dealing with negatively correlated study and auxiliary variables.

On the basis of the empirical results obtained in the study, we recommend the use of the proposed adopted (ratio-type) Upadhyaya-Singh (1999) estimator I, for positively correlated study and auxiliary variables. For negatively correlated study and auxiliary variables, we recommend the use of the proposed adopted (product-type) Upadhyaya-Singh (1999) estimator III. The use of the proposed difference-type estimators is recommended only when it is possible to choose the value of the constant, k_h , to be very close to the optimum values that minimize the mean square errors of the proposed classes of estimators.

5.4 Contribution to Knowledge

The study has contributed generally to the body of knowledge and literature on the use of poststratified sampling scheme in surveys, with emphasis on how to handle nonresponse and measurement errors in poststratified sampling scheme. Specifically, the study has contributed to the development of sampling strategies for handling the occurrence of nonresponse alone, the occurrence of measurement errors alone, and the joint occurrence of nonresponse and measurement errors when using the poststratified sampling scheme. Altogether, a total of twenty-seven (27) sampling strategies (sampling designs and estimators) has been proposed/provided under various situations. Further contribution of the study is in the area of providing information on estimators/sampling strategies that are more efficient than the usual poststratified sample mean estimator, $\bar{y}_{ps}$. Also, the study has contributed in the area of providing information on the best sampling strategies among the proposed sampling strategies. These contributions, no doubt, will go a long way to assisting researchers, survey statisticians and policy makers in making informed decisions when handling the problem of nonresponse and measurement errors under the poststratified sampling scheme.

5.5 Areas of Further Research

The present study has been restricted to the development of sampling strategies for the estimation of population mean of a study variable, y , in the presence of nonresponse and measurement errors under the poststratified sampling scheme. An area for further research could be on the estimation of other population parameters of interest, like variance, ratio, median, proportion, etc. in poststratified sample scheme when there are nonresponse and measurement errors. Again, the study only considered separate-type estimators when proposing sampling strategies for the handling of nonresponse and measurement errors in poststratified sampling scheme. Consequently, consideration of the combined-type estimators for dealing with the problem of nonresponse and measurement errors under the poststratified sampling scheme could be another area for further research.

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Appendix I

R-CODES FOR DATASET-1

```
library(MASS)
library(plyr)
library(dplyr)
library(tidyr)
library(stringr)
library(moments)
library(statsr)
library(cvcqv)
library(MBESS)

z<-
read.table("C:\\Users\\Aloy1\\Desktop\\austr
alian1.txt", header = TRUE)
z1<-filter(z, sex=="f")
z2<-filter(z, sex=="m")
z21<-filter(z1, sn<=40)
z22<-filter(z2, sn>=161)

y1=z1$pcBfat
y2=z2$pcBfat
y21=z21$pcBfat
y22=z22$pcBfat
x1=z1$ssf
x2=z2$ssf
x21=z21$ssf
x22=z22$ssf

N=202
n=60
N1=100
N2=102
N21=40
N22=42

set.seed(1)
m1<-
rbind(c(var(y1),cov(y1,x1),0,0),c(cov(y1,x1),var
(x1),0,0),c(0,0,1,0),c(0,0,0,1))
a1<-
mvrnorm(100,mu=c(mean(y1),mean(x1),0,0),Si
gma=m1,tol = 1e-6, empirical = TRUE)
u1=a1[,3]
v1=a1[,4]

set.seed(1)
m2<-
rbind(c(var(y2),cov(y2,x2),0,0),c(cov(y2,x2),var
(x2),0,0),c(0,0,1,0),c(0,0,0,1))
a2<-
mvrnorm(102,mu=c(mean(y2),mean(x2),0,0),Si
gma=m2,tol = 1e-6, empirical = TRUE)
u2=a2[,3]
v2=a2[,4]

set.seed(1)
```

```
m21<-
rbind(c(var(y21),cov(y21,x21),0,0),c(cov(y21,x
21),var(x21),0,0),c(0,0,1,0),c(0,0,0,1))
a21<-
mvrnorm(40,mu=c(mean(y21),mean(x21),0,0),
Sigma=m21,tol = 1e-6, empirical = TRUE)
u21=a21[,3]
v21=a21[,4]

set.seed(1)
m22<-
rbind(c(var(y22),cov(y22,x22),0,0),c(cov(y22,x
22),var(x22),0,0),c(0,0,1,0),c(0,0,0,1))
a22<-
mvrnorm(42,mu=c(mean(y22),mean(x22),0,0),
Sigma=m22,tol = 1e-6, empirical = TRUE)
u22=a22[,3]
v22=a22[,4]

my1=mean(y1)
mx1=mean(x1)
vy1=var(y1)
vx1=var(x1)
vy21=var(y21)
vx21=var(x21)
vu1=var(u1)
vv1=var(v1)
vu21=var(u21)
vv21=var(v21)
cy1x1=cov(y1,x1)
cy21x21=cov(y21,x21)

my2=mean(y2)
mx2=mean(x2)
vy2=var(y2)
vx2=var(x2)
vy22=var(y22)
vx22=var(x22)
vu2=var(u2)
vv2=var(v2)
vu22=var(u22)
vv22=var(v22)
cy2x2=cov(y2,x2)
cy22x22=cov(y22,x22)

W1=N1/N
W21=N21/N1
R1=my1/mx1

W2=N2/N
W22=N22/N2
R2=my2/mx2

b1=skewness(x1)
c1=sd(x1)/mx1

b2=skewness(x2)
c2=sd(x2)/mx2
```

```

vps_t0=m*(W1*vy1+W2*vy2)

# The index (j) is associated with the
subsampling fractions, 0.1, 0.2, ..., 0.9; while the
index (i) is associated with special
# cases of estimators, where i=1:7 involves
mainly ratio-type estimators and i=8:14
involves mainly product-type estimators.

m=(1/n-1/N)

g<-
c(9/n,4/n,7/(3*n),3/(2*n),1/n,2/(3*n),3/(7*n),
1/(4*n),1/(9*n))

d1<-
c(0,cy1x1/vx1,R1,my1/(mx1+c1),my1/(mx1+b
1),c1*my1/(c1*mx1+b1),b1*my1/(b1*mx1+c1),
-R1,-my1/(mx1+c1),-my1/(mx1+b1),-
c1*my1/(c1*mx1+b1),-
b1*my1/(b1*mx1+sd(x1)))
d2<-
c(0,cy2x2/vx2,R2,my2/(mx2+c2),my2/(mx2+b
2),c2*my2/(c2*mx2+b2),b2*my2/(b2*mx2+c2),
-R2,-my2/(mx2+c2),-my2/(mx2+b2),-
c2*my2/(c2*mx2+b2),-
b2*my2/(b2*mx2+sd(x2)))

t1<-
c((m*cy1x1+g[j]*W21*cy21x21)/(m*vx1+g[j]*
W21*vx21),cy1x1/vx1,m*cy1x1/(m*vx1+g[j]*W
21*vx21),cy1x1/(vx1+vv1),cy1x1/vx1,cy1x1/(v
x1+vv1),

(m*cy1x1+g[j]*W21*cy21x21)/(m*(vx1+vv1)+g
[j]*W21*(vx21+vv21)),cy1x1/vx1,m*cy1x1/(m*
(vx1+vv1)+g[j]*W21*(vx21+vv21)))
t2<-
c((m*cy2x2+g[j]*W22*cy22x22)/(m*vx2+g[j]*
W22*vx22),cy2x2/vx2,m*cy2x2/(m*vx2+g[j]*W
22*vx22),cy2x2/(vx2+vv2),cy2x2/vx2,cy2x2/(v
x2+vv2),

(m*cy2x2+g[j]*W22*cy22x22)/(m*(vx2+vv2)+g
[j]*W22*(vx22+vv22)),cy2x2/vx2,m*cy2x2/(m*
(vx2+vv2)+g[j]*W22*(vx22+vv22)))

# Case 1: When there is nonresponse on both
study and auxiliary variables.

for(j in 1:9){ for(i in 1:12) {
  mse_nc1=m*W1*(vy1+d1[i]*d1[i]*vx1-
2*d1[i]*cy1x1)+g[j]*W1*W21*(vy21+d1[i]*d1[i]
)*vx21-
2*d1[i]*cy21x21)+m*W2*(vy2+d2[i]*d2[i]*vx2-
2*d2[i]*cy2x2)+g[j]*W2*W22*(vy22+d2[i]*d2[i]
)*vx22-2*d2[i]*cy22x22)
  pre_nc1=vps_t0*100/mse_nc1
  print(pre_nc1)
}
}

```

```

}
}

for(j in 1:9) {
  mse_nc10=m*W1*(vy1+t1[1]*t1[1]*vx1-
2*t1[1]*cy1x1)+g[j]*W1*W21*(vy21+t1[1]*t1[1]
)*vx21-
2*t1[1]*cy21x21)+m*W2*(vy2+t2[1]*t2[1]*vx2
-
2*t2[1]*cy2x2)+g[j]*W2*W22*(vy22+t2[1]*t2[1]
)*vx22-2*t2[1]*cy22x22)
  pre_nc10=vps_t0*100/mse_nc10
  print(pre_nc10)
}

# Case 2: When there is nonresponse only on the
study variable alone and not on the auxiliary
variable.

for(j in 1:9){ for(i in 1:12) {
  mse_nc2=m*W1*(vy1+d1[i]*d1[i]*vx1-
2*d1[i]*cy1x1)+g[j]*W1*W21*vy21+m*W2*(vy
2+d2[i]*d2[i]*vx2-
2*d2[i]*cy2x2)+g[j]*W2*W22*vy22
  pre_nc2=vps_t0*100/mse_nc2
  print(pre_nc2)
}
}

for(j in 1:9) {
  mse_nc20=m*W1*(vy1+t1[2]*t1[2]*vx1-
2*t1[2]*cy1x1)+g[j]*W1*W21*vy21+m*W2*(vy
2+t2[2]*t2[2]*vx2-
2*t2[2]*cy2x2)+g[j]*W2*W22*vy22
  pre_nc20=vps_t0*100/mse_nc20
  print(pre_nc20)
}

# Case 3: When there is nonresponse only on the
auxiliary variable alone and not on the study
variable.

for(j in 1:9){ for(i in 1:12) {
  mse_nc3=m*W1*(vy1+d1[i]*d1[i]*vx1-
2*d1[i]*cy1x1)+g[j]*W1*W21*d1[i]*d1[i]*vx21
+m*W2*(vy2+d2[i]*d2[i]*vx2-
2*d2[i]*cy2x2)+g[j]*W2*W22*d2[i]*d2[i]*vx22
  pre_nc3=vps_t0*100/mse_nc3
  print(pre_nc3)
}
}

for(j in 1:9) {
  mse_nc30=m*W1*(vy1+t1[3]*t1[3]*vx1-
2*t1[3]*cy1x1)+g[j]*W1*W21*t1[3]*t1[3]*vx21
+m*W2*(vy2+t2[3]*t2[3]*vx2-
2*t2[3]*cy2x2)+g[j]*W2*W22*t2[3]*t2[3]*vx22
  pre_nc30=vps_t0*100/mse_nc30
}
}

```

```

print(pre_nc30)
}

# Case 4: When there are measurement errors
on both study and auxiliary variables.

for(i in 1:12) {
  mse_mc1<-m*W1*(vy1+d1[i]*d1[i]*vx1-
2*d1[i]*cy1x1+vu1+d1[i]*d1[i]*vv1)+m*W2*(v
y2+d2[i]*d2[i]*vx2-
2*d2[i]*cy2x2+vu2+d2[i]*d2[i]*vv2)
  pre_mc1<-vps_t0*100/mse_mc1
  print(pre_mc1)
}

for(j in 1:9) {
  mse_mc10<-m*W1*(vy1+t1[4]*t1[4]*vx1-
2*t1[4]*cy1x1+vu1+t1[4]*t1[4]*vv1)+m*W2*(v
y2+t2[4]*t2[4]*vx2-
2*t2[4]*cy2x2+vu2+t2[4]*t2[4]*vv2)
  pre_mc10<-vps_t0*100/mse_mc10
  print(pre_mc10)
}

# Case 5: When there measurement errors only
on the study variable alone and not on the
auxiliary variable.

for(i in 1:12) {
  mse_mc2<-m*W1*(vy1+d1[i]*d1[i]*vx1-
2*d1[i]*cy1x1+vu1)+m*W2*(vy2+d2[i]*d2[i]*v
x2-2*d2[i]*cy2x2+vu2)
  pre_mc2<-vps_t0*100/mse_mc2
  print(pre_mc2)
}

for(j in 1:9) {
  mse_mc20<-m*W1*(vy1+t1[5]*t1[5]*vx1-
2*t1[5]*cy1x1+vu1)+m*W2*(vy2+t2[5]*t2[5]*v
x2-2*t2[5]*cy2x2+vu2)
  pre_mc20<-vps_t0*100/mse_mc20
  print(pre_mc20)
}

# Case 6: When there measurement errors only
on the auxiliary variable alone and not on the
study variable.

for(i in 1:12) {
  mse_mc3<-m*W1*(vy1+d1[i]*d1[i]*vx1-
2*d1[i]*cy1x1+d1[i]*d1[i]*vv1)+m*W2*(vy2+d
2[i]*d2[i]*vx2-2*d2[i]*cy2x2+d2[i]*d2[i]*vv2)
  pre_mc3<-vps_t0*100/mse_mc3
  print(pre_mc3)
}

for(j in 1:9) {

```

```

  mse_mc30<-m*W1*(vy1+t1[6]*t1[6]*vx1-
2*t1[6]*cy1x1+t1[6]*t1[6]*vv1)+m*W2*(vy2+t
2[6]*t2[6]*vx2-2*t2[6]*cy2x2+t2[6]*t2[6]*vv2)
  pre_mc30<-vps_t0*100/mse_mc30
  print(pre_mc30)
}

# Case 7: Joint occurrence of nonresponse and
measurement errors on both study and auxiliary
variables.

for(j in 1:9) { for(i in 1:12) {
  mse_jc1=m*W1*(vy1+d1[i]*d1[i]*vx1-
2*d1[i]*cy1x1+vu1+d1[i]*d1[i]*vv1)+g[j]*W1*
W21*(vy21+d1[i]*d1[i]*vx21-
2*d1[i]*cy21x21+vu21+d1[i]*d1[i]*vv21)+m*W
2*(vy2+d2[i]*d2[i]*vx2-
2*d2[i]*cy2x2+vu2+d2[i]*d2[i]*vv2)+g[j]*W2*
W22*(vy22+d2[i]*d2[i]*vx22-
2*d2[i]*cy22x22+vu22+d2[i]*d2[i]*vv22)
  pre_jc1=vps_t0*100/mse_jc1
  print(pre_jc1)
}
}

for(j in 1:9) {
  mse_jc10=m*W1*(vy1+t1[7]*t1[7]*vx1-
2*t1[7]*cy1x1+vu1+t1[7]*t1[7]*vv1)+g[j]*W1*
W21*(vy21+t1[7]*t1[7]*vx21-
2*t1[7]*cy21x21+vu21+t1[7]*t1[7]*vv21)+m*
W2*(vy2+t2[7]*t2[7]*vx2-
2*t2[7]*cy2x2+vu2+t2[7]*t2[7]*vv2)+g[j]*W2*
W22*(vy22+t2[7]*t2[7]*vx22-
2*t2[7]*cy22x22+vu22+t2[7]*t2[7]*vv22)
  pre_jc10=vps_t0*100/mse_jc10
  print(pre_jc10)
}

# Case 8: Joint occurrence of nonresponse and
measurement errors only on the study variable
alone and not on the auxiliary variable.

for(j in 1:9) { for(i in 1:12) {
  mse_jc2=m*W1*(vy1+d1[i]*d1[i]*vx1-
2*d1[i]*cy1x1+vu1)+g[j]*W1*W21*(vy21+vu21
)+m*W2*(vy2+d2[i]*d2[i]*vx2-
2*d2[i]*cy2x2+vu2)+g[j]*W2*W22*(vy22+vu22
)
  pre_jc2=vps_t0*100/mse_jc2
  print(pre_jc2)
}
}

for(j in 1:9) {
  mse_jc20=m*W1*(vy1+t1[8]*t1[8]*vx1-
2*t1[8]*cy1x1+vu1)+g[j]*W1*W21*(vy21+vu21
)+m*W2*(vy2+t2[8]*t2[8]*vx2-
2*t2[8]*cy2x2+vu2)+g[j]*W2*W22*(vy22+vu22
)
}

```

```
pre_jc20=vps_t0*100/mse_jc20
print(pre_jc20)
}
```

Case 9: Joint occurrence of nonresponse and measurement errors only on the auxiliary variable alone and not on the study variable.

```
for(j in 1:9) { for(i in 1:12) {
  mse_jc3=m*W1*(vy1+d1[i]*d1[i]*vx1-
  2*d1[i]*cy1x1+d1[i]*d1[i]*vv1)+g[j]*W1*W21*
  d1[i]*d1[i]*(vx21+vv21)+m*W2*(vy2+d2[i]*d2[
  i]*vx2-
  2*d2[i]*cy2x2+d2[i]*d2[i]*vv2)+g[j]*W2*W22*
  d2[i]*d2[i]*(vx22+vv22)
  pre_jc3=vps_t0*100/mse_jc3
  print(pre_jc3)
}
}
```

```
for(j in 1:9) {
  mse_jc30=m*W1*(vy1+t1[9]*t1[9]*vx1-
  2*t1[9]*cy1x1+t1[9]*t1[9]*vv1)+g[j]*W1*W21*
  t1[9]*t1[9]*(vx21+vv21)+m*W2*(vy2+t2[9]*t2[
  9]*vx2-
  2*t2[9]*cy2x2+t2[9]*t2[9]*vv2)+g[j]*W2*W22*
  t2[9]*t2[9]*(vx22+vv22)
  pre_jc30=vps_t0*100/mse_jc30
  print(pre_jc30)
}
```

Summary Statistics

```
st1=c(N1,N21,W1,W21,my1,mx1,R1,vy1,vx1,cy1
x1,vy21,vx21,cy21x21,c1,b1)
st1
```

```
st2=c(N2,N22,W2,W22,my2,mx2,R2,vy2,vx2,cy2
x2,vy22,vx22,cy22x22,c2,b2)
st2
```

R-CODES FOR DATASET-2

```
library(MASS)
library(plyr)
library(dplyr)
library(tidyr)
library(stringr)
library(moments)
library(statsr)
library(cvcqv)
library(MBESS)
```

```
z<-
read.table("C:\\Users\\Aloy1\\Desktop\\fitnes
s1.txt", header = TRUE)
z1<-filter(z, agegrp=="1")
z2<-filter(z, agegrp=="2")
```

```
z21<-filter(z1, sn<=5)
z22<-filter(z2, sn>=23)
```

```
y1=z1$oxygen
y2=z2$oxygen
y21=z21$oxygen
y22=z22$oxygen
x1=z1$runtime
x2=z2$runtime
x21=z21$runtime
x22=z22$runtime
```

```
N=31
n=10
N1=12
N2=19
N21=5
N22=9
```

```
set.seed(1)
m1<-
rbind(c(var(y1),cov(y1,x1),0,0),c(cov(y1,x1),var
(x1),0,0),c(0,0,1,0),c(0,0,0,1))
a1<-
mvrnorm(12,mu=c(mean(y1),mean(x1),0,0),Sig
ma=m1,tol = 1e-6, empirical = TRUE)
u1=a1[,3]
v1=a1[,4]
```

```
set.seed(1)
m2<-
rbind(c(var(y2),cov(y2,x2),0,0),c(cov(y2,x2),var
(x2),0,0),c(0,0,1,0),c(0,0,0,1))
a2<-
mvrnorm(19,mu=c(mean(y2),mean(x2),0,0),Sig
ma=m2,tol = 1e-6, empirical = TRUE)
u2=a2[,3]
v2=a2[,4]
```

```
set.seed(1)
m21<-
rbind(c(var(y21),cov(y21,x21),0,0),c(cov(y21,x
21),var(x21),0,0),c(0,0,1,0),c(0,0,0,1))
a21<-
mvrnorm(5,mu=c(mean(y21),mean(x21),0,0),Si
gma=m21,tol = 1e-6, empirical = TRUE)
u21=a21[,3]
v21=a21[,4]
```

```
set.seed(1)
m22<-
rbind(c(var(y22),cov(y22,x22),0,0),c(cov(y22,x
22),var(x22),0,0),c(0,0,1,0),c(0,0,0,1))
a22<-
mvrnorm(9,mu=c(mean(y22),mean(x22),0,0),Si
gma=m22,tol = 1e-6, empirical = TRUE)
u22=a22[,3]
v22=a22[,4]
```

```

my1=mean(y1)
mx1=mean(x1)
vy1=var(y1)
vx1=var(x1)
vy21=var(y21)
vx21=var(x21)
vu1=var(u1)
vv1=var(v1)
vu21=var(u21)
vv21=var(v21)
cy1x1=cov(y1,x1)
cy21x21=cov(y21,x21)

my2=mean(y2)
mx2=mean(x2)
vy2=var(y2)
vx2=var(x2)
vy22=var(y22)
vx22=var(x22)
vu2=var(u2)
vv2=var(v2)
vu22=var(u22)
vv22=var(v22)
cy2x2=cov(y2,x2)
cy22x22=cov(y22,x22)

W1=N1/N
W21=N21/N1
R1=my1/mx1

W2=N2/N
W22=N22/N2
R2=my2/mx2

b1=skewness(x1)
c1=sd(x1)/mx1

b2=skewness(x2)
c2=sd(x2)/mx2

vps_t0=m*(W1*vy1+W2*vy2)

# The index (j) is associated with the
subsampling fractions, 0.1, 0.2, ..., 0.9; while the
index (i) is associated with special
# cases of estimators, where i=1:7 involves
mainly ratio-type estimators and i=8:14
involves mainly product-type estimators.

m=(1/n-1/N)

g<-
c(9/n,4/n,7/(3*n),3/(2*n),1/n,2/(3*n),3/(7*n),
1/(4*n),1/(9*n))

d1<-
c(0,cy1x1/vx1,R1,my1/(mx1+c1),my1/(mx1+b
1),c1*my1/(c1*mx1+b1),b1*my1/(b1*mx1+c1),

-R1,-my1/(mx1+c1),-my1/(mx1+b1),-
c1*my1/(c1*mx1+b1),-
b1*my1/(b1*mx1+sd(x1)))
d2<-
c(0,cy2x2/vx2,R2,my2/(mx2+c2),my2/(mx2+b
2),c2*my2/(c2*mx2+b2),b2*my2/(b2*mx2+c2),
-R2,-my2/(mx2+c2),-my2/(mx2+b2),-
c2*my2/(c2*mx2+b2),-
b2*my2/(b2*mx2+sd(x2)))

t1<-
c((m*cy1x1+g[j]*W21*cy21x21)/(m*vx1+g[j]*
W21*vx21),cy1x1/vx1,m*cy1x1/(m*vx1+g[j]*W
21*vx21),cy1x1/(vx1+vv1),cy1x1/vx1,cy1x1/(v
x1+vv1),

(m*cy1x1+g[j]*W21*cy21x21)/(m*(vx1+vv1)+g
[j]*W21*(vx21+vv21)),cy1x1/vx1,m*cy1x1/(m*
(vx1+vv1)+g[j]*W21*(vx21+vv21)))
t2<-
c((m*cy2x2+g[j]*W22*cy22x22)/(m*vx2+g[j]*
W22*vx22),cy2x2/vx2,m*cy2x2/(m*vx2+g[j]*W
22*vx22),cy2x2/(vx2+vv2),cy2x2/vx2,cy2x2/(v
x2+vv2),

(m*cy2x2+g[j]*W22*cy22x22)/(m*(vx2+vv2)+g
[j]*W22*(vx22+vv22)),cy2x2/vx2,m*cy2x2/(m*
(vx2+vv2)+g[j]*W22*(vx22+vv22)))

# Case 1: When there is nonresponse on both
study and auxiliary variables.

for(j in 1:9){ for(i in 1:12) {
  mse_nc1=m*W1*(vy1+d1[i]*d1[i]*vx1-
2*d1[i]*cy1x1+g[j]*W1*W21*(vy21+d1[i]*d1[i]
]*vx21-
2*d1[i]*cy21x21)+m*W2*(vy2+d2[i]*d2[i]*vx2-
2*d2[i]*cy2x2+g[j]*W2*W22*(vy22+d2[i]*d2[i]
]*vx22-2*d2[i]*cy22x22)
  pre_nc1=vps_t0*100/mse_nc1
  print(pre_nc1)
}
}

for(j in 1:9) {
  mse_nc10=m*W1*(vy1+t1[1]*t1[1]*vx1-
2*t1[1]*cy1x1+g[j]*W1*W21*(vy21+t1[1]*t1[1]
]*vx21-
2*t1[1]*cy21x21)+m*W2*(vy2+t2[1]*t2[1]*vx2
-
2*t2[1]*cy2x2+g[j]*W2*W22*(vy22+t2[1]*t2[1]
]*vx22-2*t2[1]*cy22x22)
  pre_nc10=vps_t0*100/mse_nc10
  print(pre_nc10)
}

# Case 2: When there is nonresponse only on the
study variable alone and not on the auxiliary
variable.

```

```

for(j in 1:9){ for(i in 1:12) {
  mse_nc2=m*W1*(vy1+d1[i]*d1[i]*vx1-
2*d1[i]*cy1x1)+g[j]*W1*W21*vy21+m*W2*(vy
2+d2[i]*d2[i]*vx2-
2*d2[i]*cy2x2)+g[j]*W2*W22*vy22
  pre_nc2=vps_t0*100/mse_nc2
  print(pre_nc2)
}
}

```

```

for(j in 1:9) {
  mse_nc20=m*W1*(vy1+t1[2]*t1[2]*vx1-
2*t1[2]*cy1x1)+g[j]*W1*W21*vy21+m*W2*(vy
2+t2[2]*t2[2]*vx2-
2*t2[2]*cy2x2)+g[j]*W2*W22*vy22
  pre_nc20=vps_t0*100/mse_nc20
  print(pre_nc20)
}

```

Case 3: When there is nonresponse only on the auxiliary variable alone and not on the study variable.

```

for(j in 1:9){ for(i in 1:12) {
  mse_nc3=m*W1*(vy1+d1[i]*d1[i]*vx1-
2*d1[i]*cy1x1)+g[j]*W1*W21*d1[i]*d1[i]*vx21
+m*W2*(vy2+d2[i]*d2[i]*vx2-
2*d2[i]*cy2x2)+g[j]*W2*W22*d2[i]*d2[i]*vx22
  pre_nc3=vps_t0*100/mse_nc3
  print(pre_nc3)
}
}

```

```

for(j in 1:9) {
  mse_nc30=m*W1*(vy1+t1[3]*t1[3]*vx1-
2*t1[3]*cy1x1)+g[j]*W1*W21*t1[3]*t1[3]*vx21
+m*W2*(vy2+t2[3]*t2[3]*vx2-
2*t2[3]*cy2x2)+g[j]*W2*W22*t2[3]*t2[3]*vx22
  pre_nc30=vps_t0*100/mse_nc30
  print(pre_nc30)
}

```

Case 4: When there are measurement errors on both study and auxiliary variables.

```

for(i in 1:12) {
  mse_mc1<-m*W1*(vy1+d1[i]*d1[i]*vx1-
2*d1[i]*cy1x1+vu1+d1[i]*d1[i]*vv1)+m*W2*(v
y2+d2[i]*d2[i]*vx2-
2*d2[i]*cy2x2+vu2+d2[i]*d2[i]*vv2)
  pre_mc1<-vps_t0*100/mse_mc1
  print(pre_mc1)
}

```

```

for(j in 1:9) {
  mse_mc10<-m*W1*(vy1+t1[4]*t1[4]*vx1-
2*t1[4]*cy1x1+vu1+t1[4]*t1[4]*vv1)+m*W2*(v

```

```

y2+t2[4]*t2[4]*vx2-
2*t2[4]*cy2x2+vu2+t2[4]*t2[4]*vv2)
  pre_mc10<-vps_t0*100/mse_mc10
  print(pre_mc10)
}

```

Case 5: When there measurement errors only on the study variable alone and not on the auxiliary variable.

```

for(i in 1:12) {
  mse_mc2<-m*W1*(vy1+d1[i]*d1[i]*vx1-
2*d1[i]*cy1x1+vu1)+m*W2*(vy2+d2[i]*d2[i]*v
x2-2*d2[i]*cy2x2+vu2)
  pre_mc2<-vps_t0*100/mse_mc2
  print(pre_mc2)
}

```

```

for(j in 1:9) {
  mse_mc20<-m*W1*(vy1+t1[5]*t1[5]*vx1-
2*t1[5]*cy1x1+vu1)+m*W2*(vy2+t2[5]*t2[5]*v
x2-2*t2[5]*cy2x2+vu2)
  pre_mc20<-vps_t0*100/mse_mc20
  print(pre_mc20)
}

```

Case 6: When there measurement errors only on the auxiliary variable alone and not on the study variable.

```

for(i in 1:12) {
  mse_mc3<-m*W1*(vy1+d1[i]*d1[i]*vx1-
2*d1[i]*cy1x1+d1[i]*d1[i]*vv1)+m*W2*(vy2+d
2[i]*d2[i]*vx2-2*d2[i]*cy2x2+d2[i]*d2[i]*vv2)
  pre_mc3<-vps_t0*100/mse_mc3
  print(pre_mc3)
}

```

```

for(j in 1:9) {
  mse_mc30<-m*W1*(vy1+t1[6]*t1[6]*vx1-
2*t1[6]*cy1x1+t1[6]*t1[6]*vv1)+m*W2*(vy2+t
2[6]*t2[6]*vx2-2*t2[6]*cy2x2+t2[6]*t2[6]*vv2)
  pre_mc30<-vps_t0*100/mse_mc30
  print(pre_mc30)
}

```

Case 7: Joint occurrence of nonresponse and measurement errors on both study and auxiliary variables.

```

for(j in 1:9) { for(i in 1:12) {
  mse_jc1=m*W1*(vy1+d1[i]*d1[i]*vx1-
2*d1[i]*cy1x1+vu1+d1[i]*d1[i]*vv1)+g[j]*W1*
W21*(vy21+d1[i]*d1[i]*vx21-
2*d1[i]*cy21x21+vu21+d1[i]*d1[i]*vv21)+m*W
2*(vy2+d2[i]*d2[i]*vx2-
2*d2[i]*cy2x2+vu2+d2[i]*d2[i]*vv2)+g[j]*W2*
W22*(vy22+d2[i]*d2[i]*vx22-
2*d2[i]*cy22x22+vu22+d2[i]*d2[i]*vv22)

```