PURE BENDING ANALYSIS OF THIN RECTANGULAR FLAT PLATES USING EULER-BERNOULLI RESIDUAL FORCE APPROACH

By

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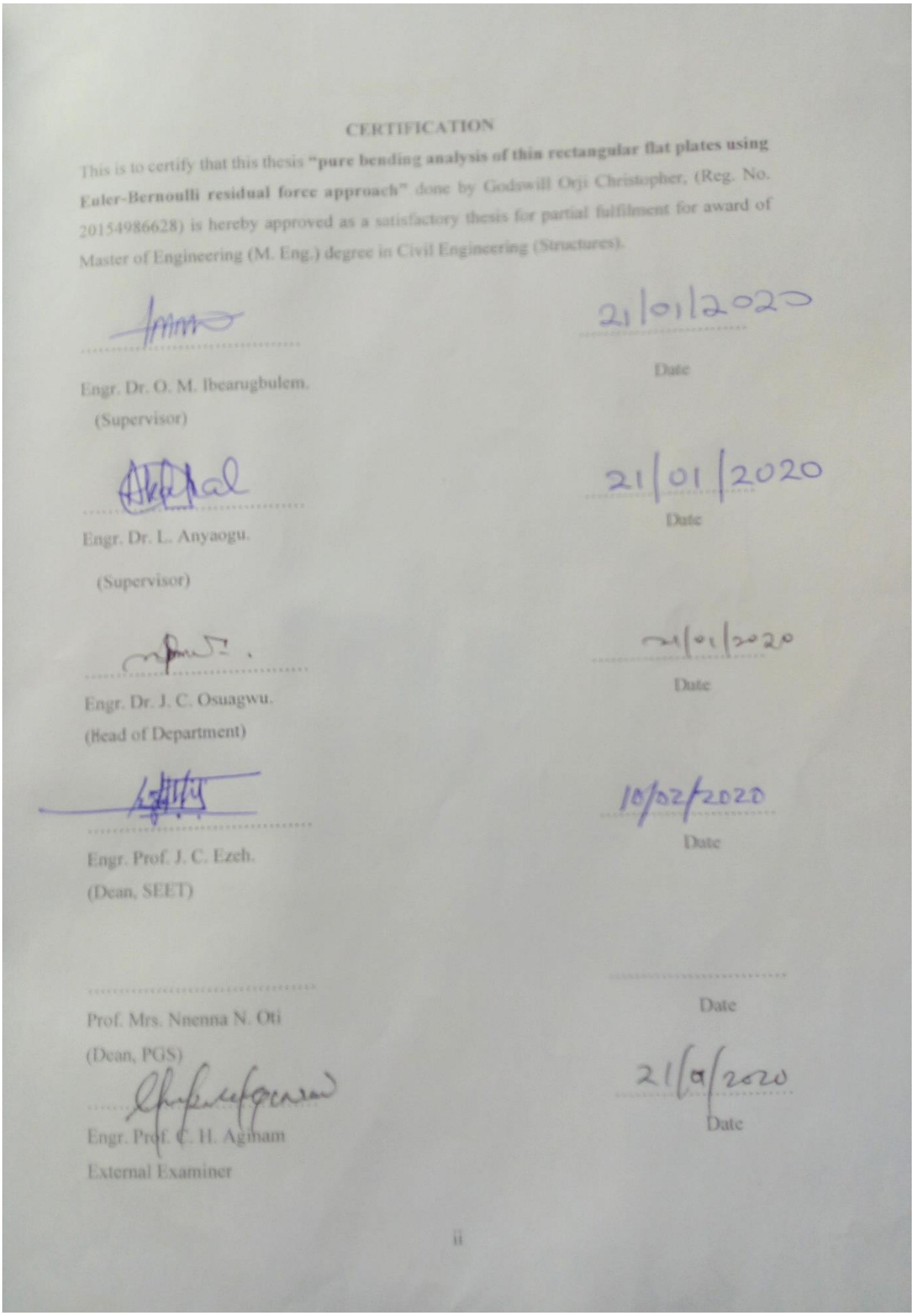

CERTIFICATION

This is to certify that this thesis **"pure bending analysis of thin rectangular flat plates using Euler-Bernoulli residual force approach"** done by Godswill Orji Christopher, (Reg. No. 20154986628) is hereby approved as a satisfactory thesis for partial fulfilment for award of Master of Engineering (M. Eng.) degree in Civil Engineering (Structures).

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DEDICATION

To God almighty.

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LIST OF SYMBOLS

A	:	Coefficient of deflection
a and b	:	Rectangular plate lateral dimensions
βx	:	Coefficient of maximum moment in x direction
βy	:	Coefficient of maximum moment in y direction
C	:	Clamped support
CCCC	:	For edges of plate are of clamped
CCCS	:	One edge of plate is simply supported and the other three are clamped
CSCS	:	Two opposite edges of plate are simply supported and the other two are clamped
CSSS	:	One edge of plate is clamped and the other three are simply supported
CCSS	:	Two adjacent edges of plate are clamped and the other two are simply supported
D	:	Modulus of flexural rigidity of the plate
E	:	Young’s modulus
ε	:	Normal strain
F	:	Euler-Bernoulli form of total equilibrium of forces
$\bar{F}$	:	Weighted residual form of total equilibrium of forces
G	:	Torsional modulus of elasticity of the plate
g	:	Euler-Bernoulli form of equilibrium of forces at an arbitrary point
h	:	Shape function of the plate under consideration
K_D	:	Coefficient of maximum deflection
Ksx	:	Coefficient of maximum shear force in x direction
Ksy	:	Coefficient of maximum shear force in y direction
M_x	:	Moment in x direction

M_y : Moment in y direction

M_{xy} : Moment in x-y direction

M_x : Moment in x direction

P : Aspect ratio of rectangular plate. That is $P = a/b$.

Q : Non dimensional axis (quantity) parallel to y axis. $Q = y/b$

q : Distributed load intensity

R : Non dimensional axis (quantity) parallel to x axis. $R = x/a$

S : Simple support

SSSS : For edges of plate are of simple support

t : Plate thickness

U : Internal (strain) energy

X : The primary axis of the plate. That is the shorter of the two axes of the major plane of the plate

Y : The secondary axis of the plate. That is the longer of the two axes of the major plane of the plate

Z : The tertiary axis of plate. That is the shortest of the three axes of the plate

$W = w(x, y)$: Plate displacement in z direction. It is a function of x and y.

Π : Potential energy functional of the plate

μ : Poisson’s ratio

σ : Normal stress of the plate

τ : Shear stress of the plate

γ : Shear strain of the plate

ABSTRACT

This study investigated pure bending analysis of thin rectangular flat plate using Euler-Bernoulli residual force equilibrium equation. The study derived from first principle; the total potential energy functional of a thin rectangular plate based on Kirchhoff's assumption. The study carried out direct differentiation of the equation with respect to the displacement function, w (x, y) to obtain the general Euler-Bernoulli residual force equilibrium equation for the plate. The study used direct integration to solve the Euler-Bernoulli residual force equilibrium equation of plates to obtain the exact general deflection equation with unknown coefficients. The boundary conditions (simple support designated with S and clamp support designated with C) of the plates were satisfied in the general solution to obtained particular solutions that are products of unknown coefficients and exact shape functions. The plates include SSSS (all edges simply supported), CCCC (all edges clamped), CSCS (two opposite edges simply supported and the other two edges clamped), CSSS (one edge simply supported and the other three edges clamped), CCCS (one edge simply supported and the other three edges clamped) and CCSS (two adjacent edges clamped the remaining two adjacent edges simply supported). The exact particular shape functions were substituted into the Euler-Bernoulli equation of equilibrium to obtain the exact coefficients of deflection of the plates for the boundary conditions considered. With the exact shape functions and their corresponding exact coefficients obtained, the study went on to determine the exact central deflection, exact maximum bending moments and exact maximum shear forces for the plates considered. To check the exactness of the approach used, the study obtained the values of residual forces from Euler-Bernoulli approach (the present study) and from other classical method considered (Ibearugbulem, 2014), for the selected plate types. This was done by substituting the integrands of the shape functions from the present study and Ibearugbulem, 2014, approaches into the Euler-Bernoulli governing partial differential equation determined from this study. The results of residual forces from the present study gave zero while the results from Ibearugbulem, 2014 did not give zero. This shows that the results from the present study are exact while the results from the other classical methods considered violate the law of equilibrium of forces which postulates that the forces must be as such that they cancel out. The results obtained herein showed that the average percentage differences between the present study and Ibearugbulem, 2014, recorded for SSSS, CCCC, CSCS, CSSS, CCSS and CCCS were 23.73%, 3.36%, 14.21%, 18.01%, 12.46% and 7.94% respectively. The method is simple and devoid of complexity.

Keywords: Euler-Bernoulli residual force, weighted residual force, shape function, coefficient of deflection, partial differential equation.

CHAPTER ONE

INTRODUCTION

1.1 Background Information

A plate is a solid that consists of two parallel plane surfaces separated by a small dimension, thickness - t (Ibearugbulem, 2014). If the forces applied on a plate are perpendicular to the plane of the plate, the plate resists the applied load by means of bending in two directions and twisting moment. In continuum mechanics (a branch of mechanics that deals with the analysis of the kinematics and the mechanical behaviour of materials modelled as a continuous mass rather than as discrete particles), plate theories are mathematical descriptions of the mechanics of flat plates that draws on the theory of beams. The typical thickness to width ratio of a plate structure is less than 0.1. (Timoshenko & Woinosky-Krieger, 1959). A plate theory takes advantage of this disparity in length scale to reduce the full three-dimensional solid mechanics problem to a two-dimensional problem. The aim of plate theory is to calculate the deformation and stresses in a plate subjected to loads. Of the numerous plate theories that have been developed since the late 19th century, two are widely accepted and used in engineering. These are; The Kirchhoff–Love theory of plates (Classical plate theory) and the Mindlin–Reissner theory of plates (First-order shear plate theory)

A flat plate, like a straight beam carries lateral load by bending. The analyses of plates are categorized into two types based on thickness to breadth ratio: Thick plate and Thin plate analyses. If the thickness to width ratio of the plate is less than 0.1 and the maximum deflection is less than one tenth of thickness, then the plate is classified as thin plate. The well-known Kirchhoff plate theory is used for the analysis of such thin plates. On the other hand, Mindlin plate theory is used for thick plate where the effect of shear deformation is included.

The Kirchhoff–Love theory is an extension of Euler–Bernoulli beam theory to thin plates. The classical beam theory was first applied to plates and shells by Love and Kirchhoff, (Reddy, 2007). Kirchhoff–Love plate theory is commonly known as Kirchhoff’s plate theory. Basically, three assumptions were used to reduce the equations of three-dimensional theory of elasticity to two dimensions (Ventsel & Krauthammer, 2001). The assumptions made by Kirchhoff-Love plate theory are;

i. The line normal to the neutral axis before bending remains straight after bending.

ii. The normal stress along the z - axis (thickness direction), σ_z is neglected. i.e., $\sigma_z = 0$. This assumption converts the three-dimensional problem into a two-dimensional problem.

iii. The transverse shearing strains were assumed to be zero. i.e., shear strains γ_{xz} and γ_{yz} were taken to be zero. Thus, the thickness of the plate does not change during bending.

The general Euler-Bernoulli theory for a continuum in equilibrium can be represented mathematically as given in Equation (1.1);

$$F = \frac{d\Pi}{dw} = \iint \left[\Delta w - \frac{q}{D}\right] dx\, dy = 0 \tag{1.1}$$

Where;

w = displacement function, Δw = derivative of the displacement function

Π = total potential energy functional of a plate

D = flexural rigidity of the plate, dx = elemental length in x − axis,

dy = elemetal distance in y − axis and q is the applied load on the plate.

The displacement function, w in Equation (1.1) is substituted with a deflection function, h which is often preselected such that the specified boundary conditions of the problem are satisfied. The preselected deflection (shape) h, has unknown coefficient, A. substituting the shape function, h and the coefficient, A, for the displacement function, w, in Equation (1.1) results to Equation (1.2).

$$F = \frac{d\Pi}{dw} = \iint \left[A\Delta h - \frac{q}{D}\right] dx\, dy = 0 \tag{1.2}$$

A = coefficient of deflection, Δh is derivative of deflecion (shape) function

Rearrangement of Equation (1.2), gave rise to Equation (1.3);

$$A = \frac{q/D}{\iint \Delta h dx dy} \tag{1.3}$$

The coefficient of the deflection function, A is obtained by using Equation (1.3) such that Equation (1.1) is satisfied in the exact sense. Therefore, the residual force from Equation (1.1) is zero. This agrees with Newton's third law of motion which states that 'For every action, there is an equal and opposite reaction'. This is the sole condition for static equilibrium. Any analysis that circumvents Equations (1.1) and (1.3) in its process will result in approximate solution.

In weighted residual method (WRM), an approximate solution of a boundary value problem is obtained by using the corresponding integral formulation. An approximate form for the solution (deflection function, h) is assumed in terms of a series containing known functions and unknown coefficients (Reddy, 2007). When this form is substituted in the integral formulation, a set of algebraic equations in terms of the unknown coefficients was obtained. Solution of the algebraic equations determines the coefficients.

Multiplying Equation (1.1) with any weighting function for instance the shape function h, yielded Equation (1.4).

$$\mathrm{Fh} = \frac{\mathrm{d}\Pi}{\mathrm{dw}}\mathrm{h} = \iint \left[\Delta \mathrm{w.h} - \frac{\mathrm{q}}{\mathrm{D}}\right] \mathrm{h\,dxdy} = \iint \left[\mathrm{A}\Delta \mathrm{h.h} - \frac{\mathrm{q}}{\mathrm{D}}\mathrm{h}\right] \mathrm{dxdy} = 0 \qquad (1.4)$$

Equation (1.4) is the weighted residual force equation of a plate. It is similar to Equation (1.1). Equation (1.1) is the true force equation, while Equation (1.4) is the quasi-force equation. Like Equation (1.3), the coefficient of the deflection function, A, can similarly be calculated by making A, the subject of the formula in Equation (1.4) as given in Equation (1.5).

$$\mathrm{A} = \mathrm{q}/\mathrm{D}\frac{\iint \mathrm{hdxdy}}{\iint \Delta \mathrm{h.hdxdy}} \qquad (1.5)$$

Equation (1.5) is characterized by its accelerated convergence. Hence, the sole reason why it is always employed in analysis, especially when dealing with approximate deflection functions with unknown coefficients. If the displacement function is approximate, it cannot be used in Euler-Bernoulli coefficient Equation (1.3). Most weighted residual force method (WRM) like Galerkin and Ritz employed in plate analysis assumed shape function whose boundary conditions satisfied equation (1.4) and went on to use equation (1.5) to determine the unknown coefficients of the shape function.

1.2 Problem statement

Most previous works on plate avoid satisfaction of the Euler-Bernoulli residual equation (Equation (1.1)) even though they satisfied the plate boundary conditions. This led most of them to also avoid the Euler-Bernoulli coefficient Equation (1.3), which is the exact coefficient and resort to using the WRM coefficient Equation (1.5), which is an approximate coefficient with high convergence rate.

In recent times, some previous works on plate satisfied both the Euler-Bernoulli residual equation (Equation (1.1)) as well as the plate boundary conditions. However, they went ahead to use the WRM coefficient equation (Equation (1.5)). This makes their work to have exact shape function with approximate coefficient. Thus, their deflection functions were all approximate functions.

1.3 Objectives

The main objective of this study is pure bending analysis of thin rectangular flat plate using Euler-Bernoulli residual force approach.

The specific objectives are:

1. To determine the total energy functional of the plate using Euler-Bernoulli approach based on Kirchhoff's assumptions
2. To determine the exact deflection function and exact coefficient of deflection that satisfied the Euler-Bernoulli residual equation for all the boundary conditions considered.
3. To determine the exact bending moments and shear forces of line continuum and thin plates of selected boundary conditions
4. To determine values of Euler-Bernoulli residual force of plates of selected boundary conditions.
5. To compare results of the present study with other classical methods.

1.4 Justification of study

When the objectives above are met, the following will be benefitted from this study;

i. The engineering analyst will have the displacement relationships, which are to be used in practical and theoretical plates' problem for various aspect ratios of SSSS, CCCC, CSCS, CSSS, CCSS, and CCCS thin rectangular plates by direct integration method.

ii. The engineering analyst can apply the stress functions for the plates considered in this study, in analyzing other plates' structural problems by the same direct integration method. Having obtained exact functions of displacement and stress functions in this study using the Euler-Bernoulli theory of plate boundary value problem, the engineering analyst can be confident in applying the method in other plate types.

iii. The engineering analyst would have the opportunity for obtaining exact numerical results of stresses of these plates for direct application and use in engineering designs of these plates.

iv. The engineering analyst would have information for exact pure bending analysis which have been substituted with approximate methods in literature.

1.5 Scope of study

This work is limited to pure bending analysis of isotropic rectangular plates for the following six (6) boundary conditions; SSSS, CCCC, CSCS, CSSS, CCSS, and CCCS rectangular plates. The plates were subjected to uniformly distributed lateral loads. To further delimit this work for basis of adequate analysis and improved accuracy of results, only rectangular plate is considered.

The equations established in this study were associated with the assumptions, equations, experiments and results of previous work done on elastic materials by Kirchhoff. No further attempts would be made in experimenting the properties of elastic materials now known as Hooke's materials. The plate is seen to be laterally loaded, thus neglecting the buckling effects. The plate here is thin plate. The analysis shall not go into numerical approaches.

CHAPTER TWO

LITERATURE REVIEW

2.1 Plate as a structural element

Plates are initially flat structural elements with thicknesses small compared with the remaining dimensions. Geometrically, plates are bounded either by straight or curved boundaries. A plate resists transverse loads by means of bending, exclusively. The flexural properties of a plate depend greatly upon its thickness in comparison with other dimensions. The distance between the plane faces is called the thickness, h of the plate (Ventsel & Krauthammer, 2001). It is usual to divide the thickness, t, into equal halves by a plane parallel to the faces. This plane is termed the mid-surface of the plate. The plate thickness is measured in a direction normal to the mid-surface at each point under consideration. Plates of technical significance are often defined as thin when the ratio of the thickness to the smaller span length is less than 1/20. This research is limited to small deflection theory of homogeneous, uniform thin plates. When, owing to external loading, deformation occurs, the mid-surface at any point suffers a deflection w. according to (Ventsel & Krauthammer, 2001), the fundamental assumptions of the small deflection theory of bending for isotropic, homogeneous, thin plates may be summarized as follows:

i. The deflection of the mid-surface is small in comparison with the thickness of the plate. The slope of the deflected surface is much less than unity.

ii. Straight lines initially normal to the mid-surface remain straight and normal to that surface subsequent to bending. This is equivalent to stating that the vertical shear strains γ_{xz} and γ_{yz} are negligible. The deflection of the plate is thus associated principally with bending strains, with the implication that the normal strain owing to vertical loading may also be neglected.

iii. No mid-surface straining or in-plane straining, stretching, or contracting occurs as a result of bending.

iv. The component of stress normal to the mid-surface, α_z is negligible.

These presuppositions are analogous to those associated with simple bending theory of beams.

2.2 Types of plates

Plate are classified based on several properties such as shape, thickness, and elastic behaviour. The following sections explains these classifications.

2.2.1 Classification of plates based on shape

Based on the shape, the following types of plate are identified; rectangular plates, circular plates, elliptical plates, sector-shaped plates, triangular plates, skew plates etc.

2.2.2 Classification of plates based on thickness

One of the properties that affect the bending of a plate is the ratio between the length of a side, a, and the thickness, t, of the material. Based on these ratios, (Ventsel and Krauthammer, 2001) identified the following classifications of plates: thick plates, thin plates and membrane plates. All of these different types of plates are characterized by the basic assumptions of their corresponding plate theory. Now a brief description of the assumptions will be discussed.

2.2.2.1 Thin plate theory

If deflection of a plate is small in comparison with its thickness one can satisfactorily use assumptions of Kirchhoff's thin plate theory as discussed in the previous section. Thin plates are usually characterized by the ratio a/t (the ratio between the length of a side, a, and the thickness of the material, t, falling between the values of 8 and 80 (Ventsel and Krauthammer, 2001). Depending on the value of the ratio of the maximum deflection, w of the plate to its thickness, t, w/t, the part of flexural and membrane forces here may be different. Therefore, Ventsel & Krauthammer (2001) said, this group, may also be subdivided into the following classes.

Thin plates with small deflection: This theory is satisfactory for plates with thickness less than 1 / 20 of its lateral dimension and having deflection less than 1 / 5 of its thickness. According to Ventsel & Krauthammer (2001), the following Kirchhoff assumptions were made:

1. Points on the plate lying initially on a normal to the middle surface of the plate remain on the normal to the middle surface of the plate even after bending.
2. The normal stresses in the direction transversal to the plate can be neglected i.e. Take σ_z, τ_{xz}, $\tau_{yz} = 0$.
3. There is no deformation in the middle surface of the plate. This plane remains neutral during bending.

Assumption I mean that shear deformations are neglected. This assumption is generally satisfactory, but in some cases e.g. in case of holes in the plate, the effect of shear becomes considerable and Hence corrections to the theory of thin plates are to be applied.

Assumption II is valid for thin plates, since the stresses are zero in z-direction at top and bottom of plates, as they are free edges. There may be small variation inside the plate at any depth z, but it is negligible.

Assumption III holds if the deflections are small. however, in actual structure when the plate bends, small forces may develop in the middle surface. This in-plane stress in the middle of plate reduces the bending moment at any other point. Hence neglecting this force is an assumption on safer side.

Thin plates with large deflection: Theory of thin plates with large deflections comes into play If the deflections are not small in comparison with its thickness, strains and stresses are introduced in the middle surface of the plate. These stresses are to be considered in deriving equilibrium equations. Inclusion of these stresses results into non-linear equations. This is called geometric nonlinearity. When this non-linearity is considered, the solution becomes more complicated.

2.2.2.2 Thick plate theory

The first two theories discussed above become unrealistic in the case of plates of larger thicknesses, especially in the case of highly concentrated loads. In such cases thick plate theory should be used. This theory considers analysis as a three-dimensional problem of elasticity. The analysis becomes lengthy and more complicated. Till today the problems are solved only for a few particular cases.

2.2.2.3 Membrane theory

The third group refers to plates with ratios a/h greater than 80…100. These plates are referred to as membranes and they are devoid of flexural rigidity. Membranes, according to Timoshenko and Woinosky – Krieger (1957), carry the lateral loads by axial tensile forces N and shear forces acting in the plate. These forces they called membrane forces; which produce projection on a vertical axis and thus balance a lateral load applied to the plate – membrane. Furthermore, the following conditions should be satisfied by true membrane plates;

The boundaries are free from transverse shear forces and moments. Loads applied to the boundaries must lie in planes tangent to the middle surface. The normal displacements and rotations at the edges are unconstrained: that is, these edges can displace freely in the direction of the normal to the middle surface. A membrane must have a smoothly varying, continuous surface.

2.2.3 Classification of plates based on elastic behaviour

Property of elasticity plays a vital role in the formulation of thin plate theory. Almost all engineering materials possess to a certain extent the property of elasticity. Hence, in this section, a brief description of elasticity and some additional assumptions will be discussed. A material is called perfectly elastic if it resumes its initial form completely after a removal of the external forces. Furthermore, a material can be considered as homogeneous if the matter of an elastic body is continuously distributed over its volume so that the smallest element cut from the body possess the same specific physical properties as the body. A material can also be taken as isotropic if the elastic properties are the same in all directions. On the other hand, anisotropy is the property of being directionally dependent, which implies different elastic properties in different directions, as opposed to isotropy.

Structural materials do not satisfy the above properties completely. They are found to contain crystals of various kinds and various orientations. These materials are far from being homogeneous, but experience shows that solutions of the theory of elasticity based on the assumptions of homogeneity and isotropy can be applied to structures with very great accuracy.

2.3 Boundary conditions of plates

The boundary conditions are the known conditions on the surfaces of the plate which must be prescribed in advance in order to obtain the solution of the governing equation of plate corresponding to a particular problem (Hsu, 2003).

Such conditions, include the load P (x, y) on the upper and lower faces of the plate; however, the load has been taken into account in the formulation of the general problem of bending of plates. For a plate, the solution of the governing differential equation requires that two boundary conditions be satisfied at each edge. These may be deflection and slope, or force and moment, or some combination of these (Ventsel & Krauthammer, 2001). Different boundary conditions of a rectangular plate are as shown in Figure 2.1.

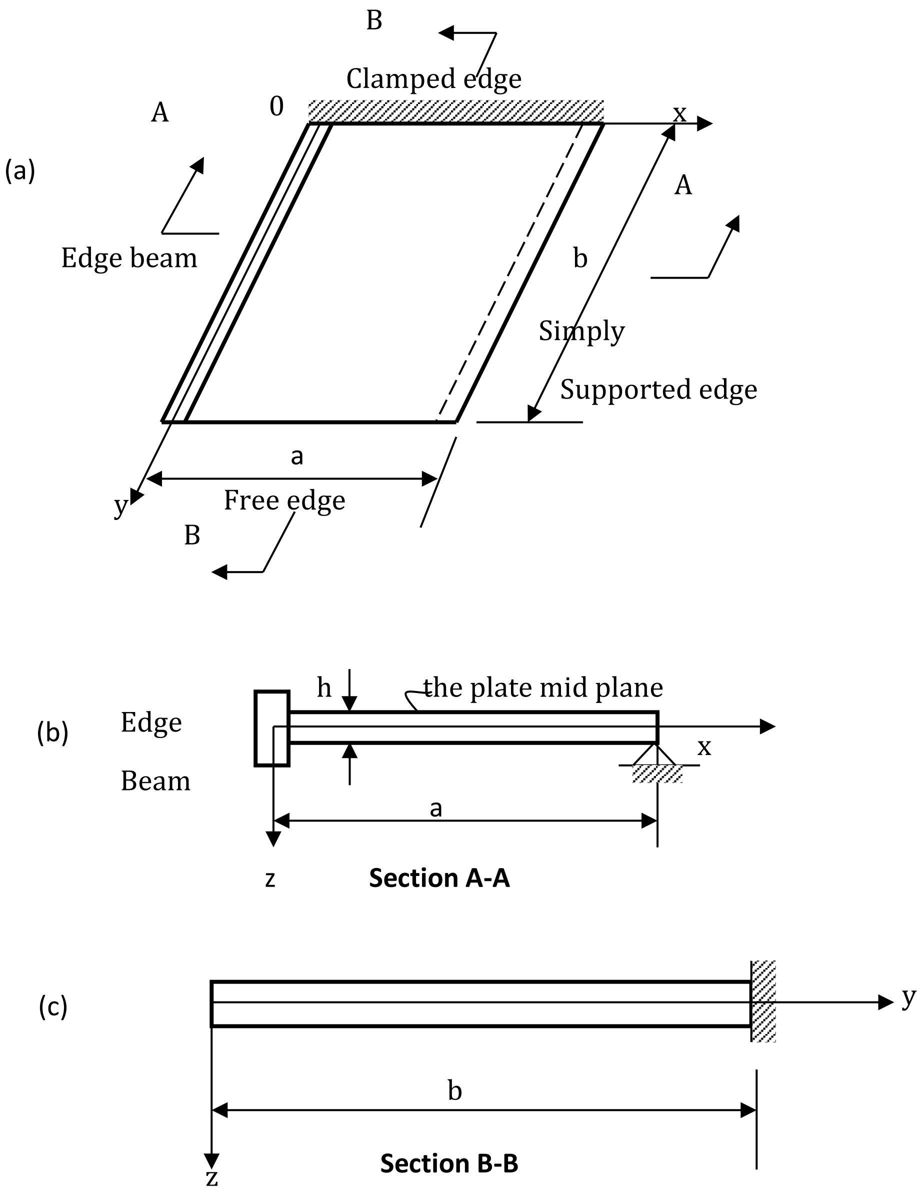

Figure 2.1: Rectangular Plate Showing Different Support Conditions

For the sake of simplicity, only rectangular plate whose edges are parallel to the axes 0x and 0y, as shown in Figure 2.1 are considered in this study

2.3.1 Clamped, or built in or fixed edge, y = 0

At the clamped edge, y = 0, the deflection, w and slope, $^{dw}/_{dy}$ are zero. This is as expressed in Equation (2.1).

$$w\,(x,y) = 0\left\{_{y\,=\,0}\right. , \quad \frac{\partial w}{\partial y} = \theta y = 0\left\{_{y\,=\,0}\right. \tag{2.1}$$

2.3.2 Simply supported edge, x = a

At this edge, the deflection, w and bending moment, Mx are both zero. This is as expressed in Equation (2.2).

$$\mathrm{w}\,(\mathrm{x},\mathrm{y}) = 0\left\{\mathrm{x} = \mathrm{a}\right., \qquad \mathrm{M_x} = -\mathrm{D}\left[\frac{\partial^2 \mathrm{w}}{\partial \mathrm{x}^2} + \mu\,\frac{\partial^2 \mathrm{w}}{\partial \mathrm{y}^2}\right] = 0\left\{\mathrm{x} = \mathrm{a}\right. \tag{2.2}$$

Equation (2.2) implies that along the edge x = a, all the derivatives of w with respect to y are zero. i.e.; if x = a and w = 0, then the first and second derivative of the deflection, w with respect to y – direction are as expressed in Equation (2.3)

$$\frac{\partial \mathrm{w}}{\partial \mathrm{y}} = \frac{\partial^2 \mathrm{w}}{\partial \mathrm{y}^2} = 0 \tag{2.3}$$

It follows that conditions expressed by Equation (2.3) may appear in the following equivalent form as expressed in Equation (2.4).

$$\mathrm{w}\,(\mathrm{x},\mathrm{y}) = 0\begin{cases}\mathrm{x} = 0\\ \mathrm{x} = \mathrm{a}\end{cases};\ \frac{\partial^2 \mathrm{w}}{\partial \mathrm{y}^2} = 0\begin{cases}\mathrm{x} = 0\\ \mathrm{x} = \mathrm{a}\end{cases} \tag{2.4}$$

2.3.3 Free edge, y = b

Suppose that the edge y = b is perfectly free. Since no stresses act over this edge, then it is reasonable to equate all the stress resultants and stress couples occurring at points of this edge to zero. i.e.

$$\mathrm{M_y} = 0\ |\mathrm{y} = \mathrm{b}; \qquad \mathrm{Q_y} = 0\ |\mathrm{y} = \mathrm{b}; \qquad \mathrm{M_{yx}} = 0\ |\mathrm{y} = \mathrm{b} \tag{2.5}$$

2.4 Solutions of plate bending problems

Bending of the plates could be studied, in general perspective, using the equilibrium approach or the energy approach (Iyengar, 1988). however, three approaches were identified in structural mechanics. They are equilibrium, energy and numeric approaches (Reddy, 1984). Equilibrium approach was also regarded as Euler approach. It sums all the forces acting on a continuum to zero. This summation was referred to the governing equation. It could either be ordinary differential equation or partial differential equation (Ugural, 1999).

Energy approach, on the other hand, sums all the work (strain energy and potential energy or external work) on the continuum to be equal to total potential energy (Iyengar, 1988). The numeric approach, depending on a particular method, will model the governing equation or the energy equation to approximate the solution of plate.

Generally, irrespective of the approach adopted, in pure bending analysis, stress resultants produced due to bending moments only are considered. In other words, deformation of the membrane due to external loads is ignored. Naturally, in this type of bending, middle surface remains neutral surface. In this research, some of the properties of bent surface are discussed and expressions are derived for stresses and moments in terms of single unknown deflection ‘w’. The following sections considers the method of plate bending solution in more details.

2.4.1 Equilibrium approach

According to Charles & Chad (2009), when a plate carries a static load, the plate must be in equilibrium, which means that the forces and moments acting on any arbitrary element of the plate must sum to zero. The equilibrium approach, commonly known as Euler approach, uses this equilibrium of forces to formulate the governing equation. This equation can be ordinary differential equation (for line continuum) or partial differential equation for plate. Using equilibrium approach, the formulation of ordinary differential equation for line continuum (e.g., beam) can be as illustrated below. The Euler-Bernoulli beam equation arises from a combination of four distinct subsets of beam theory: the kinematic, constitutive, force resultant, and equilibrium definition equations. The outcome of each of these segments is summarized in Equations (2.6a – 2.9).

The kinematic equation

$$\chi = -\theta = -\frac{dw}{dx} \qquad (2.6a)$$

$$\varepsilon(x, y) = -\frac{d\chi}{dx}.y = \frac{d^2w}{dx^2}.y \qquad (2.6b)$$

The constitutive equation

$$\sigma(x, y) = E.\varepsilon(x, y) \qquad (2.7)$$

The Resultant equations

$$M(x) = \iint y.\sigma(x, y).dydz \qquad (2.8)$$

$$\mathrm{V(x)} = \iint \sigma_{xy}\,(x,y).\,\mathrm{dydz} \tag{2.9}$$

The equilibrium equation is as expressed in Equation (2.10)

$$\frac{\mathrm{dM}}{\mathrm{dx}} = \mathrm{V} \text{ and } \frac{\mathrm{dV}}{\mathrm{dx}} = \mathrm{q} \tag{2.10}$$

To relate the beam out of plane displacement, ‘w’ to its pressure loading, p, we combine the results of the four beam sub-categories in the order shown,

Kinematics – Constitutive – Resultants – Equilibrium = Beam equation

This is demonstrated in equations (2.11). By working backwards, we first combine the two equilibrium equations to eliminate V.

$$\frac{\mathrm{d^2M}}{\mathrm{dx^2}} = \mathrm{q} \tag{2.11}$$

Substituting Equation (2.11) into Equation (2.8),

$$\frac{\mathrm{d^2}}{\mathrm{dx^2}} \iint \mathrm{y}.\,\sigma(\mathrm{x},\mathrm{y}).\,\mathrm{dydz} \quad = \mathrm{q} \tag{2.12}$$

Using the constitutive equation i.e. Equation (2.7) to eliminate the stress function, σ, in favour of strain, ε, and then using the kinematics to replace ε, in favour of the normal displacement, w, results in Equation (2.13).

$$\frac{\mathrm{d^2}}{\mathrm{dx^2}}\left[\mathrm{E} \iint \mathrm{y}.\,\varepsilon.\,\mathrm{dydz} \quad\right] = \frac{\mathrm{d^2}}{\mathrm{dx^2}}\left[\mathrm{E}\frac{\mathrm{d}\chi}{\mathrm{dx}} \iint \mathrm{y^2}.\,\mathrm{dydz} \quad\right] = \mathrm{q} \tag{2.13}$$

On simplification of Equation (2.13), we arrive at Equation (2.14)

$$\frac{\mathrm{d^2}}{\mathrm{dx^2}}\left[\mathrm{E}\frac{\mathrm{d^2w}}{\mathrm{dx^2}} \iint \mathrm{y^2}.\,\mathrm{dydz} \quad\right] = \mathrm{q} \tag{2.14}$$

Recognizing that the integral over y^2 is the definition of the beam’s area moment of inertia I, would be represented as shown in Equation (2.15).

$$\mathrm{I} = \iint \mathrm{y^2}.\,\mathrm{dydz} \tag{2.15}$$

Therefore, substituting Equation (2.15) into Equation (2.14), yielded Equation (2.16),

$$\frac{d^2}{dx^2}\left[EI\frac{d^2w}{dx^2}\right] = q \tag{2.16}$$

Further simplification of Equation (2.16), yielded Equation (2.17).

$$EI\frac{d^4w}{dx^4} - q = 0 \tag{2.17}$$

Equation (2.17) is the force equilibrium equation at any arbitrary point on the beam. To obtain equation of force equilibrium of the entire beam, both sides of Equation (2.17) was integrated with respect to x. the integration of Equation (2.17) gave rise Equation (2.18).

$$\int EI\frac{d^4w}{dx^4}dx - q\int dx = 0 \tag{2.18}$$

The Euler-Bernoulli residual force equation (Equation (2.19)) for a beam was obtained by summing the force components in Equation (2.18)

$$F = \int\left[EI\frac{d^4w}{dx^4}dx - q\right]dx = 0 \tag{2.19}$$

Similarly, the governing partial differential equation for plate which was given by Kirchhoff (1877) and Venant (1883) are as stated in Equations (2.20), (2.21) and (2.22) respectively.

$$p + N_x\frac{\partial^2 w}{\partial x^2} + 2N_{xy}\frac{\partial^2 w}{\partial x\,\partial y} + N_y\frac{\partial^2 w}{\partial y^2} - D\left[\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2\,\partial y^2} + \frac{\partial^4 w}{\partial y^4}\right] = 0 \tag{2.20}$$

For plate that is only under in-plane loads, p will be dropped from Equation (2.20) and the general equation becomes Equation (2.21);

$$N_x\frac{\partial^2 w}{\partial x^2} + 2N_{xy}\frac{\partial^2 w}{\partial x\,\partial y} + N_y\frac{\partial^2 w}{\partial y^2} - D\left[\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2\,\partial y^2} + \frac{\partial^4 w}{\partial y^4}\right] = 0 \tag{2.21}$$

For plate that is only in pure bending, the general equation becomes Equation (2.22);

$$g = D\left[\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2\,\partial y^2} + \frac{\partial^4 w}{\partial y^4}\right] = \frac{q}{D} \tag{2.22}$$

where g is the force at an arbitrary point on the plate.

Since the scope of this work is limited to thin rectangular plate in bending, we shall limit our discussion on Equation (2.22). It is important to note that Equation (2.22) is the force equilibrium equation (g = 0) at any arbitrary point on the plate see Equation (1.1). The whole plate equilibrium equation is obtained by integrating both sides of Equation (2.22). The whole plate equilibrium equation, F = 0, See Equation (1.1) is as stated in Equation (2.23).

$$F = \frac{\partial \Pi}{\partial w} = \frac{D}{2}\int_0^a \int_0^b \left(\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} \right) dx\, dy - q \int_0^a \int_0^b w\, dx\, dy \qquad (2.23)$$

Where F is whole force on plate.

This equation could be solved by direct integration. The integration of this type of differential equation would yield a number of constant terms. Determination of the true values of these constant terms would lead to the completion of the solution of the differential equation (Szilard, 1974; Vinson, 1974; Mansfield, 1964 and Donnell, 1976). The boundary conditions of the continuum would help in the determination of the constants. This approach of analysis of thin plates in elastic stability is not easy. For simple cases (like a rectangular plate simply supported all round, and loaded with in-plane load only along one axis) it could be employed to find exact solution of thin plates in elastic stability. however, for most practical cases in real time situations, this approach could be very complex and sometimes impossible (Reddy, 2002).

2.4.2 Energy approach

When the solution from using the governing differential equation is becoming intractable, approximate means would be adopted. Energy approach has the inherent characteristic to be used as an approximate method. As mentioned above, this approach uses the total potential energy of a system to determine the unknown force or displacement at a specified point along or on the continuum. Typical examples of energy method include virtual work, least work, Castigliano, Betti, Maxwell etc. (Richards, 1977; Shames and Dym, 1985; Davies, 1982). These energy methods yield approximate solution. The accuracy of these methods depends on the closeness between the approximate deflection function and exact deflection function. Hence, there is the need for variational principle. This principle makes use of total potential functional. In variational calculus methods, the functional was first reduced to the governing differential equation and its boundary condition. Then the solution of the problem was sought. This was an indirect way of solving a problem (Arthurs, 1975). What this meant was that, one begins with the total potential energy functional and goes down to governing differential equation and its boundary condition, and finally solves the problem. It was obvious here that one had gone back to initial equilibrium approach from energy approach and thus met the same old problem. This was the problem that led to the development of direct variational principle.

2.4.3 Weighted residual approach

Weighted residual method involves two major steps. In the first step, an approximate solution based on the general behavior of the dependent variable is assumed. The assumed solution is often selected so as to satisfy the boundary conditions. This assumed solution is then substituted in the differential equation. Since the assumed solution is only approximate, it does not in general satisfy the differential equation and Hence results in an error or what we call a residual. The residual is then made to vanish in some average sense over the entire solution domain to produce a system of algebraic equations. The second step is to solve the system of equations resulting from the first step subject to the prescribed boundary condition to yield the approximate solution sought. (Timoshenko & Woinosky Krieger, 2007).

2.4.4 Raleigh - Ritz approach

The Ritz method belongs among the so-called weighted residual methods that are commonly used as an approximate method for a solution of various boundary value problems of mechanics. These methods are based on variational principles of mechanics. The energy method developed by Ritz (Ritz, 1909) applied the principle of minimum potential energy.

In this method, a trial shape function is assumed. however, care must be taken in selecting the trial function. This shape function should as much as possible be close to the deformed shape function of the continuum. If the chosen shape function is far from being a good approximation of the exact deformed shape function, then the solution will be far from being an approximation of the exact solution. If, by accident, the exact shape function is assumed then the solution will correspond to exact solution. This shape function, $w(x) = \sum_{i=1}^{n} a_i w_i\,(x)$ is a function of generalized coordinate, a_i and coordinate, w_i.

Beam equilibrium equation can be obtained by the direct variational formulation by Ritz approach.

According to Ritz (1909), the total potential energy of the beam is given by Equation (2.24)

$$\Pi = \mathrm{U} + \mathrm{V} \qquad (2.24)$$

Where U and V denote the internal and external energies, respectively.

The internal energy, U can be expressed as shown in Equation (2.25)

$$U = \frac{EI}{2}\int \left(\frac{d^2w}{dx^2}\right)^2 dx \quad = \quad \frac{EI}{2}\int \frac{d^4w}{dx^4} dx \tag{2.25}$$

Similarly, the external energy due to applied load is expressed as shown in Equation (2.26)

$$V = \int qdx \tag{2.26}$$

Combining Equations (2.25) and (2.26) gives Equation (2.27a)

$$\Pi = \frac{EI}{2}\int \frac{d^4w}{dx^4} dx - q\int wdx \tag{2.27a}$$

Where Π is total potential energy functional of a beam

Equation (2.27a) is the Euler-Bernoulli equation of total potential energy functional of a beam. From elementary physics, Force, F is defined as a derivative of potential energy, Π. Considering the overall force equilibrium, Equation (2.27a) was differentiated with respect to the displacement, w, to yield Equation (2.27b).

$$F = \frac{\partial \Pi}{\partial w} = \frac{EI}{2}\int \frac{d^4w}{dx^4} dx - q\int wdx \tag{2.27b}$$

Assume a shape function for the deflected beam element as given on Equation (2.28)

$$w = Ah \tag{2.28}$$

Substituting the shape function into Equation (2.27a) and expanding the resulting equation, Equation (2.29) was obtained.

$$\Pi = \frac{EIA^2}{2}\int \left(\frac{d^2h}{dx^2}\right)^2 dx - qA\int hdx \tag{2.29}$$

Minimization of the total potential energy of the beam

To minimize Equation (2.29), it was differentiated with respect to the coefficient of the shape function, A, to yield Equation (2.30).

$$\frac{\partial \Pi}{\partial A} = EIA\int \left(\frac{d^2h}{dx^2}\right)^2 dx - q\int hdx \; = 0 \tag{2.30}$$

Making “A” the subject of Equation (2.30), yielded Equation (2.31);

$$A = \frac{q \int h dx}{EI \int \left(\frac{d^2h}{dx^2}\right)^2 dx} \tag{2.31}$$

Equation (2.31) is the coefficient of the shape function derived using Ritz approach for a beam.

It is obvious here that a function in terms of 'w' was assumed and substituted for displacement in Equation (2.27a) rather than solving directly the overall force equilibrium equation expressed in Equation (2.27b). This substitution successfully relaxed the equilibrium equation so that integration will be easy and devoid of complexity for solving. While the solution yields exact result for a one-dimensional element like a beam, it is not certain if the assumption stands for a two-dimensional element like a plate.

For rectangular thin plate, after selecting the shape function in terms of 'w', the boundary conditions are substituted into the total potential energy functional Equation (2.24), Π to reduce it to a peculiar shape function. Thereafter, the particular shape function is substituted into the total potential energy functional. This functional will be integrated over the domain to reduce it to a function $\prod$, of generalized coordinates. The resulting function is partially differentiated with respect to the generalized coordinates and equated to zero, as expressed in Equation (2.32). (Timoshenko & Woinosky Krieger, 2007).

$$\frac{\partial \prod}{\partial a_1} = \frac{\partial \prod}{\partial a_2} = \cdots = \frac{\partial \prod}{\partial a_n} = 0 \tag{2.32}$$

This minimization procedure yields n simultaneous algebraic equations in the undetermined coefficients 'a_1, a_2, a_3… a_n,' from which the unknown parameters a_i, are calculated. (Timoshenko & Woinosky Krieger, 2007). It should be noted that during the partial differentiation, all coefficients, except the specific 'a' under consideration, are taken as constant.

According to Szilard (2004), the strain energy, stored in a plate during deformation, is found by integrating (over the entire middle surface) the negative work of internal forces (U = -W). In general, the strain energy of the plate consists of bending (U_b) and membrane (U_m) parts, which are shown in Equations (2.33) and (2.34) respectively.

$$U_b = \frac{1}{2} \iint_0^1 \left(m_x k_x + m_y k_y + 2m_{xy}\chi\right) dxdy \tag{2.33}$$

$$U_m = \frac{1}{2}\iint_0^1 \left(n_x\varepsilon_x + n_y\varepsilon_y + 2n_{xy}\gamma\right)dxdy \tag{2.34}$$

Zero and 1 as used in the limits of the integrals represents the start and end points of a 1m length continuum. If we substitute Equations (2.33) and (2.34), into Equation (2.32), we will obtain the expressions for the internal moments in terms of the displacement components as expressed in Equations (2.35 – 2.36).

$$m_x = -D\left(\frac{\partial^2 w}{\partial x^2} + v\frac{\partial^2 w}{\partial y^2}\right) \tag{2.35}$$

$$m_y = -D\left(\frac{\partial^2 w}{\partial y^2} + v\frac{\partial^2 w}{\partial x^2}\right) \tag{2.36}$$

$$m_{xy} = -D(1 - v)\frac{\partial^2 w}{\partial x\,\partial y} \tag{2.37}$$

2.4.4.1 Advantages of Ritz method

The following are the basic advantages of Ritz method:

i. The basis lies in the fact that the coordinate functions f_i (x, y) must satisfy the kinematic (or geometrical) boundary conditions only. Therefore, the area of an application of the method to the plate bending problems is wider than that of the classical analytical methods. Therefore, the Ritz method is very efficient for the analysis of plate having free edges, for plates within openings.

ii. The method can also be applied successfully to rectangular plates of variable thickness, because there is no difference between the expressions of Π for plates of constant and variable thicknesses.

iii. The matrix of the linear algebraic equations is symmetrical. This results in stable and powerful logarithms for their numerical solution.

2.4.4.2 Disadvantages of Ritz method

The following are the basic disadvantages of the method:

i. The Ritz method can be applicable only on simple configuration of plates (rectangular, circular, etc.), because of the complexity of selecting the coordinate functions for domains of complex geometry.

ii. The Ritz method approximation results in algebraic equations that produces some difficulties in its numerical implementation.

It should be noted that the Ritz method can serve as a basis for such a popular and powerful method as the finite element method.

2.4.5 Galerkin's method

The method formulated by Galerkin (1915), can be applied successfully to diverse types of problems of applied elasticity including the plate bending problems. Although the mathematical theory behind the Galerkin's method is quite complicated, its physical interpretation is relatively simple. It should be noted that the Galerkin's method is more general than the Ritz method because no quadratic functional or virtual work principle is necessary. If the governing equations are derivable from variational principle, then the Galerkin's method reduces to the Ritz method and leads to an identical set of linear algebraic equations produced by the Ritz method.

Sometimes, the Galerkin's method may be preferable if it is more convenient to work with the governing differential equations rather than with the energy functional. Moreover, there are problems for which no satisfactory variational principles have been formulated, but for which a set of governing differential equation is available. Galerkin's method is as illustrated below;

Consider the general case of a differential equation shown in Equation (2.38);

$$\mathrm{Lu} = \mathrm{f} \tag{2.38}$$

Where u is an unknown field, L is the coefficient of the unknown field and f is the parameter represented by the expression.

Try an approximate solution to Equation (2.38) of the form shown in equation (2.39);

$$\tilde{\mathrm{u}} = \sum_{\mathrm{i}=1}^{\mathrm{N}} \mathrm{N_i}\, \mathrm{u_i} \tag{2.39}$$

Where;

N_i are the test functions (input) and u_i are the unknown quantities that we need to evaluate.

The solution must satisfy the boundary conditions. Since ũ is an approximation, substituting it into Equation (2.38) will result in an error; thus Equation (2.38) can be written as shown in Equation (2.40),

$$\mathrm{L\tilde{u} = f + \tilde{\epsilon}} \qquad (2.40)$$

Where, $\tilde{\epsilon}$, is the weighted average error resulting from the assumed approximate solution. At this point, the weighted average error will be minimized by equating Equation (2.40) to Zero to obtain the solution.

$$\mathrm{\tilde{\epsilon} = L\tilde{u} - f = 0} \qquad (2.41)$$

This is the basis of weighted residual methods of analysis. however, for a one-dimensional problem such as beams, this method yields the exact results but the result is not always exact for a two-dimensional problem such as plates.

2.5 Results from previous work

This section shows the results obtained from the works of previous scholars on the analysis of thin rectangular plates. The works considered here are the works done by Raleigh - Ritz, and Timoshenko & Krieger Woinosky on rectangular thin plates.

2.5.1 The general expressions of plate equilibrium

The general equilibrium equations for plates in pure bending are as stated in Equations (2.42) and (2.43). Equation (2.42) here is expressed in non-dimensional form. Equation (2.43) was obtained by Ibearugbulem (2014) in his work 'Energy methods in Theory of Rectangular Plates.' The equation is the general expression of total force plate equilibrium equation based on energy method.

$$\mathrm{g = \frac{\partial^4 w}{\partial R^4} + \frac{2}{p^2}\frac{\partial^4 w}{\partial R^2\,\partial Q^2} + \frac{1}{p^4}\frac{\partial^4 w}{\partial Q^4} - \frac{qa^4}{D} = 0} \qquad (2.42)$$

$$\mathrm{\bar{F} = \frac{d\Pi}{dA} = \int_0^1 \int_0^1 \left(\frac{\partial^4 h}{\partial R^4} + \frac{2}{p^2}.\frac{\partial^4 h}{\partial R^4 \partial Q^4} + \frac{1}{p^4}.\frac{\partial^4 h}{\partial R^4}\right) dR\, dQ = 0} \qquad (2.43)$$

2.5.2 Exact shape functions of the selected plates in the form of polynomial series

Equations (2.44) through Equation (2.49) are the results of shape function based on polynomial series obtained by Ibearugbulem (2014) using Raleigh – Ritz approach.

SSSS Plate

$$\mathrm{h = (R - 2R^3 + R^4)(Q - 2Q^3 + Q^4)} \qquad (2.44)$$

CCCC Plate

$$h = (R^2 - 2R^3 + R^4)(Q^2 - 2Q^3 + Q^4) \qquad (2.45)$$

CSCS Plate

$$h = (R - 2R^3 + R^4)(Q^2 - 2Q^3 + Q^4) \qquad (2.46)$$

CSSS Plate

$$h = (R - 2R^3 + R^4)(1.5Q^2 - 2.5Q^3 + Q^4) \qquad (2.47)$$

CCSS Plate

$$h = (1.5R - 2.5R^3 + R^4)(1.5Q^2 - 2.\,5Q^3 + Q^4) \qquad (2.48)$$

CCCS Plate

$$h = (1.5R^2 - 2.5R^3 + R^4)(Q^2 - 2Q^3 + Q^4) \qquad (2.49)$$

2.5.3 The coefficient of deflection based on polynomial shape functions

Equations (2.50) through Equation (2.55) are the results of coefficient of deflection, A based on polynomial shape function obtained by Ibearugbulem (2014) using Raleigh – Ritz approach.

SSSS Plate

$$A = \frac{0.04qa^4}{D\left[0.23621 + 0.47182\frac{1}{\alpha^2} + 0.23621\frac{1}{\alpha^4}\right]} \qquad (2.50)$$

CCCC Plate

$$A = \frac{0.00111111qa^4}{D\left[0.00127 + 0.00072\frac{1}{\alpha^2} + 0.00127\frac{1}{\alpha^4}\right]} \qquad (2.51)$$

CSCS Plate

$$A = \frac{0.00667qa^4}{D\left[0.00763 + 0.0185\frac{1}{\alpha^2} + 0.03937\frac{1}{\alpha^4}\right]} \qquad (2.52)$$

CSSS Plate

$$A = \frac{0.015qa^4}{D\left[0.036192 + 0.0832642\frac{1}{\alpha^2} + 0.088571\frac{1}{\alpha^4}\right]} \qquad (2.53)$$

CCSS Plate

$$\mathrm{A} = \frac{0.005625\mathrm{qa}^4}{\mathrm{D}\left[0.013572 + 0.0146938\frac{1}{\alpha^2} + 0.013572\frac{1}{\alpha^4}\right]} \tag{2.54}$$

CCCS Plate

$$\mathrm{A} = \frac{0.0025\mathrm{qa}^4}{\mathrm{D}\left[0.002857 + 0.00338536\frac{1}{\alpha^2} + 0.0060318\frac{1}{\alpha^4}\right]} \tag{2.55}$$

Where D is flexural rigidity, a is lateral dimension, q is load intensity and α is aspect ratio of the plate.

2.5.4 Deflection, bending moments and shear forces of selected plates

The results in Tables (2.1) and (2.2) were drawn from the works of Timoshenko & Woinosky Krieger (1959) and Ibearugbulem (2014). Timoshenko & Woinosky-Krieger used the equilibrium method while Ibearugbulem used Ritz energy method. Equation (2.50) through Equation (2.55) are the equations for the coefficient of deflection for a given aspect ratio for the selected plate types. The coefficients of maximum deflection, bending moments and shear forces of the selected rectangular plate types are summarized in the following tables.

Legend: b/a is the aspect ratio, k_D is the coefficient of maximum deflection, β_{xC} and β_{yC} are factors for maximum bending moment in x and y direction respectively. Similarly, K_{sxB} and K_{syA} are factors for maximum shear force in x and y direction respectively.

The coefficient of maximum deflection, bending moment and shear forces for SSSS as obtained by Timoshenko (1959) and Ibearugbulem (2014) are as shown in Tables (2.1) and (2.2)

Table 2.1 The coefficient of max. deflection, bending moment and shear forces for SSSS plate

Aspect ratio, P	k_D	β_{xC}	β_{yC}	K_{sxB}	K_{syA}
1.00	0.00406	0.0479	0.0479	0.4200	0.42000
1.10	0.00485	0.0554	0.0493	0.4400	0.44000
1.20	0.00564	0.0627	0.0501	0.4550	0.45300
1.30	0.00638	0.0694	0.0503	0.4680	0.46400
1.40	0.00705	0.0755	0.0502	0.4780	0.47100
1.50	0.00772	0.0812	0.0498	0.4860	0.48000
1.60	0.00830	0.0862	0.0492	0.4910	0.48500
1.70	0.00883	0.0908	0.0486	0.4960	0.48800
1.80	0.00931	0.0948	0.0479	0.4990	0.49100
1.90	0.00974	0.0985	0.0471	0.5020	0.49400
2.00	0.01013	0.1017	0.0464	0.5030	0.49600

Source - Timoshenko and Woinosky Krieger (1959)

Table 2.2 The coefficient of max. deflection, bending moment and shear forces for SSSS plate

Aspect ratio, P	k_D	β_{xC}	β_{yC}	K_{sxB}	K_{syA}
1.00	0.00414	0.051630	0.051630	0.374900	0.374900
1.10	0.00496	0.059430	0.053640	0.404580	0.378620
1.20	0.00576	0.066850	0.055020	0.430330	0.378890
1.30	0.00653	0.073830	0.055910	0.452620	0.376520
1.40	0.00725	0.080310	0.056430	0.471900	0.372160
1.50	0.00793	0.086280	0.056680	0.488600	0.366340
1.60	0.00856	0.091760	0.056730	0.503090	0.359480
1.70	0.00913	0.096770	0.056640	0.515690	0.351910
1.80	0.00966	0.101330	0.056450	0.526700	0.343910
1.90	0.01015	0.105490	0.056200	0.536340	0.335650
2.00	0.01059	0.109270	0.055910	0.544820	0.327300

Source - Ibearugbulem (2014)

The coefficient of maximum deflection, bending moment and shear forces for CCCC as obtained by Timoshenko (1959) and Ibearugbulem (2014) are as shown in Tables (2.3) and (2.4)

Table 2.3 The coefficient of max. deflection, bending moment and shear forces for CCCC plate

Aspect ratio, P	k_D	β_{xC}	β_{yC}
1.00	0.00126	0.0231	0.0231
1.10	0.00150	0.0264	0.0231
1.20	0.00172	0.0299	0.0228
1.30	0.00191	0.0327	0.0222
1.40	0.00207	0.0349	0.0212
1.50	0.00220	0.0368	0.0203
1.60	0.00230	0.0381	0.0193
1.70	0.00238	0.0392	0.0182
1.80	0.00245	0.0401	0.0174
1.90	0.00249	0.0407	0.0165
2.00	0.00254	0.0412	0.0158

Source - Timoshenko and Woinosky Krieger (1959)

Table 2.4 The coefficient of max. deflection, bending moment and shear forces for CCCC plate

Aspect ratio, P	k_D	β_{xC}	β_{yC}	K_{sxB}	K_{syA}
1.00	0.00133	0.027690	0.027690	0.468750	0.255620
1.10	0.00159	0.031720	0.028630	0.557040	0.229130
1.20	0.00182	0.035220	0.028990	0.632480	0.202420
1.30	0.00203	0.038200	0.028930	0.694380	0.177190
1.40	0.00221	0.040690	0.028590	0.743870	0.154320
1.50	0.00236	0.042750	0.028090	0.782860	0.134130
1.60	0.00249	0.044460	0.027490	0.813390	0.116590
1.70	0.00260	0.045870	0.026850	0.837260	0.101500
1.80	0.00269	0.047030	0.026200	0.855960	0.088580
1.90	0.00277	0.048000	0.025580	0.870690	0.077540
2.00	0.00284	0.048810	0.024980	0.882350	0.068110

Source - Ibearugbulem (2014)

The coefficient of maximum deflection, bending moment and shear forces for CSCS as obtained by Timoshenko (1959) and Ibearugbulem (2014) are as shown in Tables (2.5) and (2.6)

Table 2.5 The coefficient of max. deflection, bending moment and shear forces for CSCS plate

Aspect ratio, P	k_D	β_{xC}	β_{yC}
2.0	0.0026	0.0142	0.0420
1.5	0.0025	0.0179	0.0400
1.4	0.0024	0.0192	0.0399
1.3	0.0023	0.0203	0.0388
1.2	0.0022	0.0215	0.0375
1.1	0.0021	0.0230	0.0355

Source - Timoshenko and Woinosky Krieger (1959)

Table 2.6 The coefficient of max. deflection, bending moment and shear forces for CSCS plate

Aspect ratio, P	k_D	β_{xC}	β_{yC}	K_{sxB}	K_{syA}
1.000	0.002	0.029	0.038	0.249	0.382
1.100	0.003	0.035	0.042	0.289	0.377
1.200	0.003	0.043	0.046	0.326	0.367
1.300	0.004	0.050	0.050	0.362	0.352
1.400	0.005	0.057	0.053	0.395	0.334
1.500	0.006	0.065	0.055	0.425	0.314
1.600	0.006	0.072	0.057	0.452	0.293
1.700	0.007	0.078	0.058	0.476	0.272
1.800	0.008	0.084	0.060	0.497	0.251
1.900	0.008	0.090	0.060	0.516	0.231
2.000	0.009	0.096	0.061	0.533	0.212

Source - Ibearugbulem (2014)

The coefficient of maximum deflection, bending moment and shear forces for CSSS as obtained by Timoshenko (1959) and Ibearugbulem (2014) are as shown in Tables (2.7) and (2.8)

Table 2.7 The coefficient of max. deflection, bending moment and shear forces for CSSS plate

Aspect ratio, P	**k_D**	**β_{xC}**	**β_{yC}**
1.00	0.0028	0.034	0.039
1.10	0.0035	0.041	0.042
1.20	0.0043	0.049	0.044
1.30	0.0050	0.056	0.045
1.40	0.0058	0.063	0.047
1.50	0.0064	0.069	0.048
2.00	0.0093	0.094	0.047

Source - Timoshenko and Woinosky Krieger (1959)

Table 2.8 The coefficient of max. deflection, bending moment and shear forces for CSSS plate

Aspect ratio, P	**k_D**	**β_{xC}**	**β_{yC}**	**K_{sxB}**	**K_{syA}**
1.00	0.00282	0.03718	0.04191	0.29203	0.38667
1.10	0.00354	0.04452	0.04531	0.32696	0.40162
1.20	0.00429	0.05185	0.04805	0.35883	0.41169
1.30	0.00503	0.05901	0.05021	0.3875	0.41749
1.40	0.00576	0.06587	0.05185	0.41301	0.4197
1.50	0.00646	0.07236	0.05306	0.43553	0.41899
1.60	0.00713	0.07842	0.05392	0.45531	0.41597
1.70	0.00775	0.08405	0.05450	0.47264	0.41116
1.80	0.00833	0.08924	0.05485	0.48779	0.40501
1.90	0.00887	0.09400	0.05503	0.50104	0.3979
2.00	0.00937	0.09837	0.05509	0.51264	0.3901

Source - Ibearugbulem (2014)

The coefficient of maximum deflection, bending moment and shear forces for CCSS as obtained by Ibearugbulem (2014) are as shown in Table (2.9)

Table 2.9 The coefficient of max. deflection, bending moment and shear forces for CCSS plate

Aspect ratio, P	**k_D**	**β_{xC}**	**β_{yC}**	**K_{sxB}**	**K_{syA}**
1.00	0.00210	0.03277	0.03277	0.25209	0.25209
1.10	0.00251	0.03762	0.03396	0.30146	0.24552
1.20	0.00290	0.04203	0.03459	0.34784	0.23672
1.30	0.00325	0.04597	0.03481	0.39036	0.22672
1.40	0.00357	0.04943	0.03473	0.42870	0.21623
1.50	0.00386	0.05246	0.03446	0.46292	0.20574
1.60	0.00411	0.05510	0.03406	0.49324	0.19556
1.70	0.00433	0.05740	0.03359	0.52003	0.18587
1.80	0.00453	0.05940	0.03309	0.54365	0.17674
1.90	0.00470	0.06114	0.03257	0.56450	0.16822
2.00	0.00486	0.06266	0.03206	0.58290	0.16030

Source - Ibearugbulem (2014)

The coefficient of maximum deflection, bending moment and shear forces for CCCS as obtained by Ibearugbulem (2014) are as shown in Table (2.10)

Table 2.10 The coefficient of max. deflection, bending moment and shear forces for CCCS plate

Aspect ratio, P	**k_D**	**β_{xC}**	**β_{yC}**	**K_{sxB}**	**K_{syA}**
1.00	0.00159	0.02673	0.03119	0.19095	0.30552
1.10	0.00200	0.03190	0.03362	0.19816	0.28824
1.20	0.00241	0.03690	0.03540	0.20052	0.26736
1.30	0.00280	0.04157	0.03661	0.19891	0.24482
1.40	0.00317	0.04585	0.03733	0.19430	0.22206
1.50	0.00352	0.04971	0.03767	0.18758	0.20009
1.60	0.00383	0.05314	0.03772	0.17952	0.17952
1.70	0.00411	0.05616	0.03756	0.17071	0.16067
1.80	0.00436	0.05882	0.03725	0.16160	0.14364
1.90	0.00459	0.06115	0.03685	0.15249	0.12841
2.00	0.00479	0.06318	0.03638	0.14360	0.11488

Source - Ibearugbulem (2014)

2.5.5 Results of residual forces factors from Ibearugbulem (2014).

The factors in Table 2.11 were values obtained by Ibearugbulem (2014) using Raleigh-Ritz weighted residual method. For an exact solution, the factors which are integrands of the shape function of the selected plate types will give a value of zero residual force when substituted into the governing Euler-Bernoulli partial differential equation of isotropic thin rectangular flat plate.

Table 2.11. Residual force factors from Raleigh-Ritz weighted residual method

S/N	Plate type	$\mathbf{k_x}$	$\mathbf{k_{xy}}$	$\mathbf{k_y}$	$\mathbf{k_q}$
1	SSSS	0.23621	0.23591	0.23621	0.04
2	CCCC	0.00127	0.00036	0.00127	0.0011
3	CSCS	0.00763	0.00925	0.03937	0.007
4	CSSS	0.036192	0.0416321	0.088571	0.015
5	CCSS	0.013572	0.0073469	0.013572	0.006
6	CCCS	0.002857	0.00163268	0.0060318	0.0025

Source - Ibearugbulem (2014)

CHAPTER THREE

METHODOLOGY

3.1 General methodology

This work started by deriving basic elastic stress and strain analysis equations and relationships from first principles based on Kirchhoff's hypothesis for thin plate small deflection theory. The derived relationships were used to determine the governing differential equation for isotropic thin rectangular plate. This was followed by obtaining the shape functions associated with different boundary conditions using the powerful characteristic orthogonal polynomials. The shape functions were used with the overall equilibrium equation of plate to obtain the unknown coefficients. These were applied to obtain the equations of deflections, moments and shear forces for different types of plate.

3.2 Determination of the total energy functional of the plate

Principles of theory of elasticity coupled with the basic relationships established herein were employed to formulate the governing differential equation of the rectangular plate. For plane stress (x-y plane) analysis, x-y planer torsional modulus of elasticity, EE, was considered where torsional modulus of elasticity is needed.

3.2.1 Basic relationships

In the present work, only the theory of small deflection is considered. This small deflection theory is mainly governed by Kirchhoff's hypothesis. The basic relationships between elastic material parameters for rectangular plates are shown on Equations (3.1) to (3.4).

$$\mathrm{EE} = \sqrt{\mathrm{E_x \,.\, E_y}} \quad (\text{for } \mathrm{x - y} \text{ plane}) \tag{3.1}$$

$$\mathrm{EE} = \sqrt{\mathrm{E_x \,.\, E_z}} \quad (\text{for } \mathrm{x - z} \text{ plane}) \tag{3.2}$$

$$\mathrm{EE} = \sqrt{\mathrm{E_y \,.\, E_z}} \quad (\text{for } \mathrm{y - z} \text{ plane}) \tag{3.3}$$

$$\mathrm{E_{xy} = E_{yx} = \mu_x E_y = \mu_y E_x} \tag{3.4}$$

$$\mu_{xy}^2 = \mu_x \mu_y \tag{3.5}$$

$$\mu_{xy} = \sqrt{\mu_x \mu_y} \tag{3.6}$$

$$\mathrm{D_{xy} = D_{yx} = \mu_x D_y = \mu_y D_x} \tag{3.7}$$

From Hooke's law, stress, σ is directly proportional to strain, ε. This is as shown in Equation (3.8);

$$\sigma \propto \varepsilon \tag{3.8}$$

modifying Equation (3.8) yielded Equation (3.9).

$$\sigma = \mathrm{E}\varepsilon \tag{3.9}$$

Where E is the Young's Modulus of elasticity of the plate.

From Poisson's theorem, applied stresses, σ_A and lateral stresses, σ_L are related to strain, ε as expressed in Equation (3.10).

$$\sigma = \sigma_A - \mu\sigma_L = E_A\varepsilon_A \tag{3.10}$$

Where σ is the total stress. Considering directions in x, y and z axis, Equation (3.10) be written as shown in Equations (3.11) to (3.13).

$$\sigma_x - \mu_x\sigma_y - \mu_x\sigma_z = E_x\varepsilon_x \tag{3.11}$$

$$\sigma_y - \mu_y\sigma_x - \mu_y\sigma_z = E_y\varepsilon_y \tag{3.12}$$

$$\sigma_z - \mu_z\sigma_x - \mu_z\sigma_y = E_z\varepsilon_z \tag{3.13}$$

For plane stress, the parameters in Equation (3.13) are as defined in Equation (3.14)

$$\sigma_z = \sigma_{zx} = \sigma_{zy} = 0 \tag{3.14}$$

This is to say they are negligible.

The deformation of a rectangular plate under planar stresses are as shown in Figure 3.1.

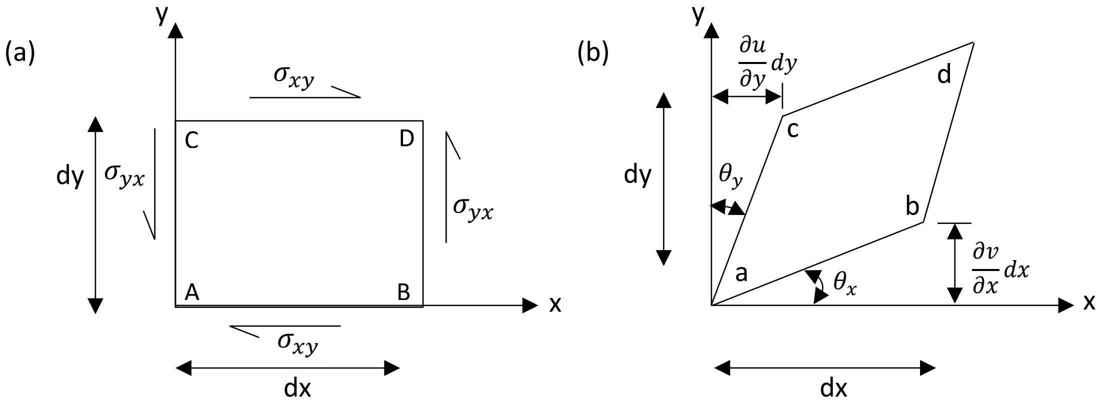

Figure 3.1: A rectangular element deformed under planer stresses, σ_{xy}

Consider a two-dimensional, infinitesimal, rectangular material element with dimensions dx and dy, shown in Figure 3.1a which, after deformation, takes the form of a rhombus as shown in Figure 3.1b. The element made angle zero with x and y axes before the shear stresses were introduced. After subjecting the element to shearing stresses, the element deformed and made angles θx and θy with x and y axes. From the geometry in Figure 3.1b we have;

$$\mathrm{Lenght(AB)} = \mathrm{dx} \tag{3.15}$$

And

$$\mathrm{Lenght(ab)} = \sqrt{\left(\mathrm{dx} + \frac{\partial \mathrm{u}}{\partial \mathrm{x}}\mathrm{dx}\right)^2 + \left(\frac{\partial \mathrm{v}}{\partial \mathrm{x}}\mathrm{dx}\right)^2} \tag{3.16}$$

$$\mathrm{Lenght(ab)} = \mathrm{dx}\sqrt{1 + 2\frac{\partial u}{\partial \mathrm{x}} + \left(\frac{\partial \mathrm{u}}{\partial \mathrm{x}}\right)^2 + \left(\frac{\partial \mathrm{v}}{\partial \mathrm{x}}\right)^2} \tag{3.17}$$

$\text{Where } \partial\mathrm{u} \text{ and } \partial\mathrm{v} \text{ are elemental displacements in x and y} - \text{axes respectivley}$

For very small displacement gradients, the squares of the derivatives are negligible and Equation (3.17) can be written as shown in Equation (3.18)

$$\mathrm{Lenght(ab)} = \mathrm{dx} + \frac{\partial u}{\partial \mathrm{x}}\mathrm{dx} \tag{3.18}$$

The normal strain in the *x*-direction of the rectangular element is defined by

$$\varepsilon_{\mathrm{xx}} = \frac{\text{extension}}{\text{original length}} = \frac{\mathrm{lenght(ab)} - \mathrm{length(AB)}}{\mathrm{length(AB)}} = \frac{\partial \mathrm{u}}{\partial \mathrm{x}} \tag{3.19}$$

Similarly, the normal strain in the *y*- and *z*-directions are expressed in Equation (3.20) and Equation (3.21).

$$\varepsilon_{\mathrm{yy}} = \frac{\partial \mathrm{v}}{\partial \mathrm{y}} \tag{3.20}$$

$$\varepsilon_{\mathrm{zz}} = \frac{\partial \mathrm{w}}{\partial \mathrm{z}} \tag{3.21}$$

If the shear stresses that brought about the deformation are equal, we shall expect that the angles of deformation, θx and θy are equal. Hence, the infinitesimal strain-displacement relationships can be summarized using the expression on Equation (3.22),

$$\varepsilon_{\mathrm{ij}} = \frac{1}{2}\left[\frac{\partial \mathrm{u_i}}{\partial \mathrm{x_j}} + \frac{\partial \mathrm{u_j}}{\partial \mathrm{x_i}}\right] \tag{3.22}$$

Where u is the displacement vector, *x* is coordinate, and the two indices *i* and *j* can range over the three coordinates {1, 2, 3} in three-dimensional space.

Expanding Equation (3.22) for each coordinate direction and simplifying the resulting equation gives Equations (3.23) to (3.25).

$$\varepsilon_{\mathrm{xy}} = \varepsilon_{\mathrm{yx}} = \frac{\partial \mathrm{u}}{\partial \mathrm{y}} = \frac{\partial \mathrm{v}}{\partial \mathrm{x}} \tag{3.23}$$

$$\varepsilon_{\mathrm{xz}} = \varepsilon_{\mathrm{zx}} = \frac{\partial \mathrm{u}}{\partial \mathrm{z}} = \frac{\partial \mathrm{w}}{\partial \mathrm{x}} \tag{3.24}$$

$$\varepsilon_{\mathrm{yz}} = \varepsilon_{\mathrm{zy}} = \frac{\partial \mathrm{v}}{\partial \mathrm{z}} = \frac{\partial \mathrm{w}}{\partial \mathrm{y}} \tag{3.25}$$

Equations (3.23) to (3.25) are the normal and shear strain equations for the element. The engineering shear strain, γ_{xy} is defined as the change in angle between lines AC and AB in Figure 3.1b. This is shown mathematically in Equation (3.26),

$$\gamma_{xy} = \theta_x + \theta_y \tag{3.26}$$

From the geometry of Figure 3.1b, the tangent of the angles θx and θy with respect to x and y axes are as shown in Equation (3.27) and Equation (3.28),

$$\tan\theta_x = \frac{\frac{\partial v}{\partial x}dx}{dx + \frac{\partial u}{\partial x}dx} = \frac{\frac{\partial v}{\partial x}}{1 + \frac{\partial u}{\partial x}} \tag{3.27}$$

$$\tan\theta_y = \frac{\frac{\partial w}{\partial y}dy}{dy + \frac{\partial v}{\partial y}dy} = \frac{\frac{\partial w}{\partial y}}{1 + \frac{\partial v}{\partial y}} \tag{3.28}$$

For small displacements gradients, we have

$$\frac{\partial u}{\partial x} \ll 1; \frac{\partial v}{\partial y} \ll 1$$

For small rotations, i.e.

$$\theta_x \text{and } \theta_y \text{ are } \ll 1, \text{and } \tan\theta_x \approx \theta_x, \tan\theta_y \approx \theta_y$$

Substituting the approximations into Equations (3.27) and (3.28), we have;

$$\theta_x \approx \frac{\partial v}{\partial x} \text{ and } \theta_y \approx \frac{\partial u}{\partial y}$$

Thus Equation (3.26) can be written as shown in equation (3.29),

$$\gamma_{xy} = \theta_x + \theta_y = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \tag{3.29}$$

By interchanging x and y and u and v, it can be shown that $\gamma_{xy} = \gamma_{yx}$. Similarly, for the xz and yz planes, total engineering shear strain expressions are as shown in Equations (3.30) to (3.31).

$$\gamma_{xz} = \gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \tag{3.30}$$

$$\gamma_{yz} = \gamma_{zy} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \tag{3.31}$$

It can be deduced from Equation (3.29) that,

$$\theta_x = \frac{\partial v}{\partial x}, \text{and } \theta_y = \frac{\partial u}{\partial y}$$

Also, from Equation (3.23) that

$$\varepsilon_{xy} = \varepsilon_{yx} = \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$$

Looking at the above extracts from Equations (3.29) and (3.23), we can figure out the relationship between the angle of rotation with the x and y axes and the normal strain.

$$\theta_x = \varepsilon_{xy} = \varepsilon_{yx}$$

$$\theta_y = \varepsilon_{xy} = \varepsilon_{yx}$$

From the ongoing, engineering shear strain can as well be expressed in terms of normal strain as shown in Equation (3.32)

$$\gamma_{xy} = \gamma_{yx} = 2\varepsilon_{xy} = 2\varepsilon_{yx} \quad (3.32)$$

Where, γ_{xy} is engineering shear strain in x – y plane.

Similarly, the expressions for engineering shear strains in x – z plane and y – z plane can be as shown in Equation (3.29) and Equation (3.30).

$$\gamma_{xz} = \gamma_{zx} = 2\varepsilon_{xz} = 2\varepsilon_{zx} \quad (3.33)$$

$$\gamma_{yz} = \gamma_{zy} = 2\varepsilon_{yz} = 2\varepsilon_{zy} \quad (3.34)$$

The total stress experienced by the element in Figure 3.1b is as given in Equation (3.35).

$$\sigma_p = \sigma_{xy} + \mu_{xy}\sigma_{yx} = \sigma_{yx} + \mu_{yx}\sigma_{xy} \quad (3.35)$$

Where, σ_p, is the total stress experienced by the plate element.

Equation (3.35) can be rewritten in the form of Equation (3.10) as shown in equation (3.36)

$$\sigma_{xy} + \mu_{xy}\sigma_{yx} = E\varepsilon_{xy} \quad (3.36)$$

Complimentary stresses are equal and opposite, therefore, from Equation (3.36)

$$\sigma_{xy} = \sigma_{yx} \quad (3.37)$$

And

$$\mu_{xy} = \mu_{yx} \quad (3.38)$$

Substituting Equation (3.37) into Equation (3.36) gives Equation (3.39):

$$\sigma_{xy}(1 + \mu_{xy}) = E\varepsilon_{xy} \quad (3.39)$$

Similarly, for x-z and y-z planes, Equation (3.39) can be written as shown in Equations (3.40) and (3.41).

$$\sigma_{xz}(1 + \mu_{xz}) = E\varepsilon_{xz} \quad (3.40)$$

$$\sigma_{yz}(1 + \mu_{yz}) = E\varepsilon_{yz} \quad (3.41)$$

Substituting Equations (3.32), (3.33), and (3.34) into Equations (3.39), (3.40) and (3.41) respectively gives Equations (3.42) to (3.44);

$$\sigma_{xy} = \sigma_{yx} = \frac{EE}{2(1 + \mu_{xy})}\gamma_{xy} \tag{3.42}$$

$$\sigma_{xz} = \sigma_{zx} = \frac{EE}{2(1 + \mu_{xz})}\gamma_{xz} \tag{3.43}$$

$$\sigma_{yz} = \sigma_{zy} = \frac{EE}{2(1 + \mu_{yz})}\gamma_{yz} \tag{3.44}$$

The torsional modulus of elasticity is defined mathematically as;

$$G = \frac{EE}{2(1 + \mu_{xy})} \tag{3.45}$$

Substituting Equation (3.14) into Equations (3.11), (3.12), (3.13), (3.43) and (3.44) respectively, gives Equations (3.46) to (3.50);

$$\sigma_x - \mu_x\sigma_y = E_x\varepsilon_x \tag{3.46}$$

$$\sigma_y - \mu_y\sigma_x = E_y\varepsilon_y \tag{3.47}$$

$$\mu_z\sigma_x + \mu_z\sigma_y = -E_z\varepsilon_z \tag{3.48}$$

$$\gamma_{xz} = 0 \tag{3.49}$$

$$\gamma_{yz} = 0 \tag{3.50}$$

Multiplying Equation (3.47) by μ_x gives Equation (3.51);

$$\mu_x\sigma_y - \mu_x\mu_y\sigma_x = \mu_x E_y\varepsilon_y \tag{3.51}$$

Adding Equations (3.46) and (3.51) together gives Equation (3.52);

$$\sigma_y(1 - \mu_x\mu_y) = E_x\varepsilon_x + \mu_x E_y\varepsilon_y \tag{3.52}$$

But $\mu_x E_y = \mu_y E_{x.}$ That is Equation (3.52) can be written as Equation (3.53)

$$\sigma_x(1 - \mu_x\mu_y) = E_x\varepsilon_x + \mu_y E_x\varepsilon_y = E_x(\varepsilon_x + \mu_y\varepsilon_y) \tag{3.53}$$

This mean that;

$$\sigma_x = \frac{E_x}{(1 - \mu_x\mu_y)}(\varepsilon_x + \mu_y\varepsilon_y) \tag{3.54}$$

Similarly, for total stress in y – direction, Equation (3.54) can be written as shown in Equation (3.55).

$$\sigma_y = \frac{E_y}{(1 - \mu_x\mu_y)}(\varepsilon_y + \mu_x\varepsilon_x) \tag{3.55}$$

Recall Equation (3.42);

$$\sigma_{xy} = \sigma_{yx} = \frac{EE}{2(1 + \mu_{xy})}\gamma_{xy}$$

Substituting Equation (3.49) into Equation (3.30) and rearranging the equation, the result will be as shown in Equation (3.56).

$$\partial \mathrm{u} = -\frac{\partial \mathrm{w}}{\partial \mathrm{x}} \partial \mathrm{z} \qquad (3.56)$$

Similarly, substituting Equation (3.50) into Equation (3.31) and rearranging the equation, the result will be as shown in Equation (3.57).

$$\partial \mathrm{v} = -\frac{\partial \mathrm{w}}{\partial \mathrm{y}} \partial \mathrm{z} \qquad (3.57)$$

Integrating Equations (3.56) and (3.57) with respect to z yields Equation (3.58) and Equation (3.54) respectively;

$$\mathrm{u} = -\mathrm{z}\frac{\partial \mathrm{w}}{\partial \mathrm{x}} + \mathrm{u}_0 \qquad (3.58)$$

$$\mathrm{v} = -\mathrm{z}\frac{\partial \mathrm{w}}{\partial \mathrm{y}} + \mathrm{v}_0 \qquad (3.59)$$

Where u_o and v_o are integration constants.

Substituting Equations (3.58) and (3.59) into Equation (3.29) gives;

$$\gamma_{xy} = \gamma_{yx} = \frac{\partial}{\partial \mathrm{x}}\left(-\mathrm{z}\frac{\partial \mathrm{w}}{\partial \mathrm{y}} + \mathrm{v}_0\right) + \frac{\partial}{\partial \mathrm{y}}\left(\mathrm{z}\frac{\partial \mathrm{w}}{\partial \mathrm{x}} + \mathrm{u}_0\right)$$

$$= \left(-\mathrm{z}\frac{\partial^2 \mathrm{w}}{\partial \mathrm{x}\,\partial \mathrm{y}} + 0\right) + \left(-\mathrm{z}\frac{\partial^2 \mathrm{w}}{\partial \mathrm{x}\,\partial \mathrm{y}} + 0\right)$$

Upon further simplification yields Equation (3.60).

$$\gamma_{xy} = \gamma_{yx} = -2\mathrm{z}\frac{\partial^2 \mathrm{w}}{\partial \mathrm{x}\,\partial \mathrm{y}} \qquad (3.60)$$

Substituting Equation (3.58) into Equation (3.19) yields Equation (3.61);

$$\varepsilon_{xx} = \frac{\partial}{\partial \mathrm{x}}\left(-\mathrm{z}\frac{\partial \mathrm{w}}{\partial \mathrm{x}} + \mathrm{u}_0\right) \qquad (3.61)$$

Upon further simplification Equation (3.61) in turn yields Equation (3.62);

$$\varepsilon_{xx} = -\mathrm{z}\frac{\partial^2 \mathrm{w}}{\partial \mathrm{x}^2} \qquad (3.62)$$

Similarly, for y – plane,

$$\varepsilon_{yy} = -\mathrm{z}\frac{\partial^2 \mathrm{w}}{\partial \mathrm{y}^2} \qquad (3.63)$$

Substituting Equation (3.61) and (3.62) into Equation (3.54) gives Equation (3.64);

$$\sigma_x = \frac{\mathrm{E}_x}{(1 - \mu_x\mu_y)}\left(-\mathrm{z}\frac{\partial^2 \mathrm{w}}{\partial \mathrm{x}^2} + \mu_y - \mathrm{z}\frac{\partial^2 \mathrm{w}}{\partial \mathrm{y}^2}\right) \qquad (3.64)$$

On further simplification, Equation (3.64) can be as shown in equation (3.65)

$$\sigma_x = \frac{-\mathrm{z}\mathrm{E}_x}{(1 - \mu_x\mu_y)}\left(\frac{\partial^2 \mathrm{w}}{\partial \mathrm{x}^2} + \mu_y\frac{\partial^2 \mathrm{w}}{\partial \mathrm{y}^2}\right) \qquad (3.65)$$

Similarly;

$$\sigma_y = \frac{-zE_y}{(1-\mu_x\mu_y)}\left(\mu_x\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right) \tag{3.66}$$

Substituting Equation (3.60) into Equation (3.42) gives Equation (3.67);

$$\sigma_{xy} = \sigma_{yx} = \frac{EE}{2(1+\mu_{xy})}\left(-2z\frac{\partial^2 w}{\partial x\,\partial y}\right) = G\left(-2z\frac{\partial^2 w}{\partial x\,\partial y}\right)$$

This is to say;

$$\sigma_{xy} = \sigma_{yx} = \frac{-2zEE}{2(1+\mu_{xy})}\left(\frac{\partial^2 w}{\partial x\,\partial y}\right) = -2Gz\left(\frac{\partial^2 w}{\partial x\,\partial y}\right) \tag{3.67}$$

The plane stresses acting on a plate element can be as shown in Figure 3.2.

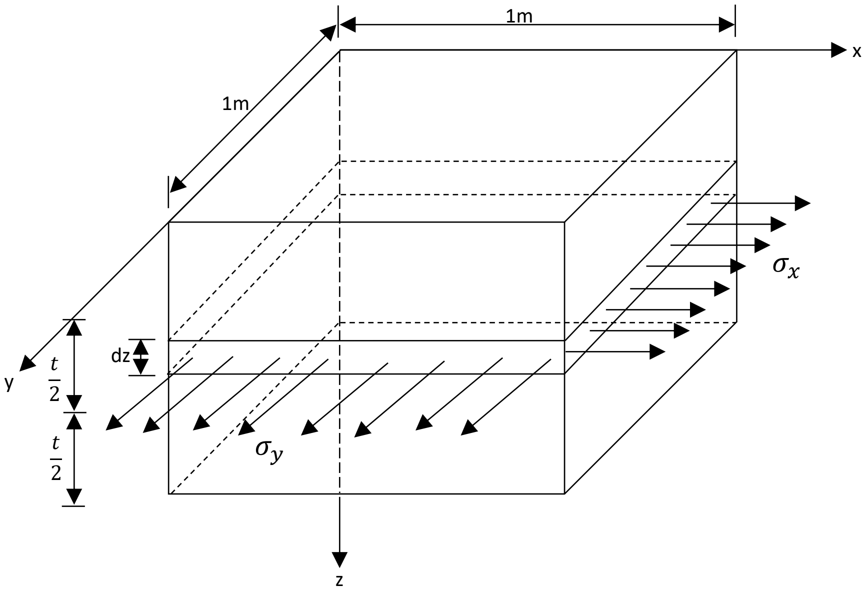

Figure 3.2: Plate element showing the three axes x, y and z and plane stresses, σ_x, and σ_y

Let the lengths of the plate to be considered in x and y axes be 1m each. Thickness of the plate is h. Hence, the cross-sectional area of the plate is;

$\text{In } x - z \text{ plane, Area, } A = 1 \times t = t\ m^2$

$\text{In } y - z \text{ plane, Area, } A = 1 \times t = t\ m^2$

The element of the plate has thickness of dz. Hence, the cross-sectional area of the plate element is;

$\text{In } x - z \text{ plane, Area, } A = 1 \times dz = dz\ m^2$

$\text{In } y - z \text{ plane, Area, } A = 1 \times dz = dz\ m^2$

From elementary physics, moment is defined mathematically as;

$\text{Moment} = \text{Force} \times \text{perpendincular distance}$

Stress is also defined as;

$$\text{Stress} = \frac{\text{Force}}{\text{Cross sectional area}}$$

Hence,

$$\text{Force} = \text{Stress} \times \text{cross sectional area}$$

Thus, the moment and the force on the elemental object are as shown in Equation (3.68) and Equation (3.64) respectively.

$$dM = dF.z \tag{3.68}$$

$$dF = \sigma.dz \tag{3.69}$$

Substituting Equation (3.69) into Equation (3.68) yields Equation (3.70);

$$dM = (\sigma.dz).z$$

That is;

$$dM = \sigma.z.dz \tag{3.70}$$

Integrating Equation (3.65) with respect to z yields Equation (3.66);

$$M = \int_{z_1}^{z_2} \sigma.z.dz \tag{3.71}$$

Equation (3.71) can be written in respect of x and y axes and x-y plane as shown in Equations (3.72) to (3.74);

$$M_x = \int_{1}^{z_2} \sigma_x.z.dz \tag{3.72}$$

$$M_y = \int_{z_1}^{z_2} \sigma_y.z.dz \tag{3.73}$$

$$M_{xy} = \int_{z_1}^{z_2} \sigma_{xy}.z.dz \tag{3.74}$$

In this case,

$$z_1 = \frac{-t}{2} \text{ and } z_2 = \frac{t}{2}$$

Substituting Equation (3.65) into Equation (3.72) yields Equation (3.75);

$$M_x = \int_{\frac{-t}{2}}^{\frac{t}{2}} \frac{-zE_x}{(1-\mu_x\mu_y)}\left(\frac{\partial^2 w}{\partial x^2} + \mu_y \frac{\partial^2 w}{\partial y^2}\right).z.dz$$

$$= \frac{-E_x}{(1-\mu_x\mu_y)}\left(\frac{\partial^2 w}{\partial x^2} + \mu_y \frac{\partial^2 w}{\partial y^2}\right)\int_{\frac{-t}{2}}^{\frac{t}{2}} z^2.dz$$

This means that;

$$M_x = \frac{-E_x t^3}{12\left(1 - \mu_x \mu_y\right)}\left(\frac{\partial^2 w}{\partial x^2} + \mu_y \frac{\partial^2 w}{\partial y^2}\right) \tag{3.75}$$

Similarly, for moment in y – direction, Equation (3.75) can be written as Equation (3.76)

$$M_y = \frac{-E_y t^3}{12\left(1 - \mu_x \mu_y\right)}\left(\mu_x \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) \tag{3.76}$$

Substituting Equation (3.67) into Equation (3.74) for biaxial moment in x and y directions yields Equation (3.77);

$$M_{xy} = \int_{\frac{-t}{2}}^{\frac{t}{2}} \left[\frac{-zEE}{\left(1 + \mu_{xy}\right)}\left(\frac{\partial^2 w}{\partial x\,\partial y}\right)\right].z.dz$$

$$= \left[\frac{-EE}{\left(1 + \mu_{xy}\right)}\left(\frac{\partial^2 w}{\partial x\,\partial y}\right)\right]\int_{\frac{-t}{2}}^{\frac{t}{2}}.z^2.dz = \frac{-EE.t^3}{12\left(1 + \mu_{xy}\right)}.\frac{\partial^2 w}{\partial x\,\partial y} \tag{3.77}$$

Further simplification of Equation (3.77) yields Equation (3.78) and Equation (3.79)

$$M_{xy} = \frac{-2EE}{2\left(1 + \mu_{xy}\right)}.\frac{\partial^2 w}{\partial x\,\partial y}.\frac{t^3}{12} = \frac{-2Gh^3}{12}.\frac{\partial^2 w}{\partial x\,\partial y} \tag{3.78}$$

$$M_{xy} = -2D_{uv}.\frac{\partial^2 w}{\partial x\,\partial y} \tag{3.79}$$

Where;

$$G = \frac{EE}{2(1 + \mu_{xy})} = \text{Shear modulus}$$

In summary:

$$M_x = -D_x\left(\frac{\partial^2 w}{\partial x^2} + \mu_y \frac{\partial^2 w}{\partial y^2}\right) \tag{3.80}$$

$$M_y = -D_y\left(\mu_x \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) \tag{3.81}$$

$$M_{xy} = -2D_{uv}.\frac{\partial^2 w}{\partial x\,\partial y} \tag{3.82}$$

Where;

$$D_x = \frac{-E_x t^3}{12\left(1 - \mu_x \mu_y\right)}\ ;\ D_y = \frac{-E_y t^3}{12\left(1 - \mu_x \mu_y\right)}\ ;\ D_{uv} = \frac{Gt^3}{12} = \frac{EE.t^3}{24\left(1 + \mu_{xy}\right)}$$

The forces acting on edges of the rectangular plate under consideration are as shown in Figure 3.3. The figure shows the lateral forces, bending moments and shear forces acting on the plate and their directions. Edges of the plate are shown on Figure 3.3b.

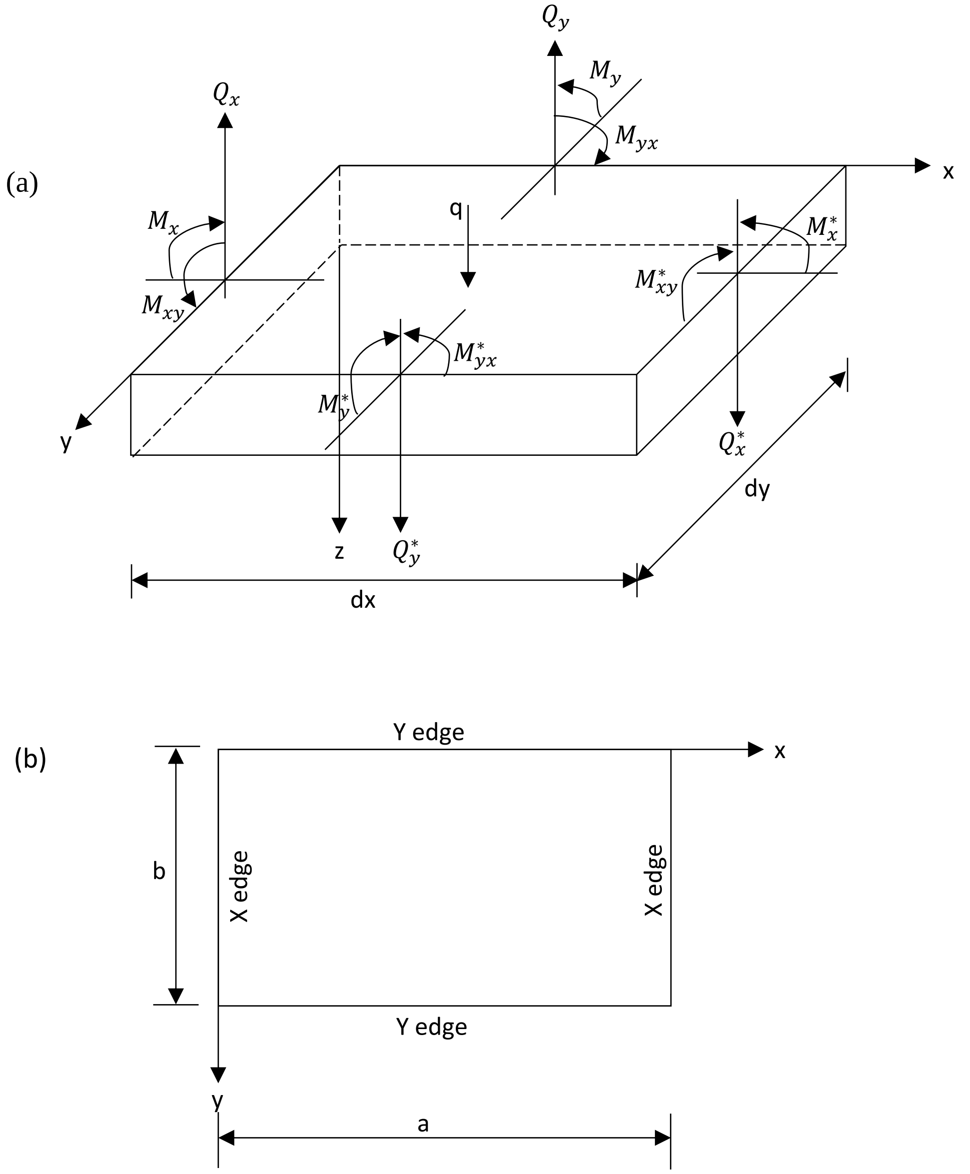

Figure 3.3: Rectangular plate element showing the lateral force, bending moments and shear forces on plate edges

The notations with asterisk shall be defined mathematically in terms of the notations without asterisk as expressed from Equation (3.83) to Equation (3.88).

$$Q_x^* = Q_x + \frac{\partial Q_x}{\partial x} dx \tag{3.83}$$

$$Q_y^* = Q_y + \frac{\partial Q_y}{\partial y} dy \tag{3.84}$$

$$M_x^* = M_x + \frac{\partial M_x}{\partial x} dx \tag{3.85}$$

$$M_y^* = M_y + \frac{\partial M_y}{\partial y} dy \tag{3.86}$$

$$M_{xy}^* = M_{xy} + \frac{\partial M_{xy}}{\partial x} dx \tag{3.87}$$

$$M_{yx}^* = M_{yx} + \frac{\partial M_{yx}}{\partial x} dy \tag{3.88}$$

The moments shown on Figure 3.3 are moments per unit length, m, and shear forces per unit length. The lateral load is force per unit area of the plate.

$\text{Total moment} = \text{moment} \times \text{length of plate}$

$\text{Total shear force} = \text{shear force} \times \text{length of plate}$

$\text{Total load} = \text{load} \times \text{area of the plate}$

3.2.2 Moment equilibrium

Equilibrium conditions require that summation of moments about any edge of the plate in Figure 3.3 be zero. The equilibrium equation is as shown in Equation (3.89):

$$\begin{aligned} M_x dy - M_x^* dy + M_{yx} dx - M_{yx}^* dx + (Q_x . dy). dx + \left(Q_y . dx\right). \frac{dx}{2} - \left(Q_y^* . dx\right). \frac{dx}{2} \\ - (q. dx. dy) \frac{dx}{2} = 0 \end{aligned} \tag{3.89}$$

Substituting Equations (3.84), (3.85) and (3.88) into Equation (3.89) and simplifying the resulting equation yields Equation (3.90)

$$-\frac{\partial M_x}{\partial x} dxdy - \frac{\partial M_{yx}}{\partial y} dxdy + Q_x dxdy - \frac{\partial Q_y}{2\,\partial y} dx^2 dy - \frac{1}{2} q dx^2 dy = 0 \tag{3.90}$$

After neglecting small quantity terms, Equation (3.90) can be written as Equation (3.91)

$$Q_x = \frac{\partial M_x}{\partial x} + \frac{\partial M_{yx}}{\partial y} \tag{3.91}$$

Similarly,

$$\mathrm{Q_y} = \frac{\partial \mathrm{M_y}}{\partial \mathrm{y}} + \frac{\partial \mathrm{M_{xy}}}{\partial \mathrm{x}} \tag{3.92}$$

The effective vertical force is as shown in Equation (3.93)

$$\mathrm{V_x} = \mathrm{Q_x} + \frac{\partial \mathrm{M_{yx}}}{\partial \mathrm{y}} \tag{3.93}$$

Substituting Equation (3.91) into Equation (3.93) gives Equation (3.94);

$$\mathrm{V_x} = \frac{\partial \mathrm{M_x}}{\partial \mathrm{x}} + \frac{2\,\partial \mathrm{M_{yx}}}{\partial \mathrm{y}} \tag{3.94}$$

Similarly, Equation (3.95) is obtained as;

$$\mathrm{V_y} = \frac{\partial \mathrm{M_y}}{\partial \mathrm{y}} + \frac{2\,\partial \mathrm{M_{xy}}}{\partial \mathrm{x}} \tag{3.95}$$

Substituting Equations (3.80) and (3.81) into (3.94) and (3.95) respectively, will give the shear force equations in x and y directions as shown in Equations (3.96) and (3.97) for x and y direction respectively;

$$\mathrm{Vx} = -\mathrm{D}\left(\frac{\partial^3 \mathrm{w}}{\partial \mathrm{x}^3} + (2-\mu)\frac{\partial^3 \mathrm{w}}{\partial \mathrm{x}\,\partial \mathrm{y}^2}\right) \tag{3.96}$$

And

$$\mathrm{Vy} = -\mathrm{D}\left(\frac{\partial^3 \mathrm{w}}{\partial \mathrm{y}^3} + (2-\mu)\frac{\partial^3 \mathrm{w}}{\partial \mathrm{x}^2\,\partial \mathrm{y}}\right) \tag{3.97}$$

Where;

$$\mathrm{D} = \frac{\mathrm{Et}^3}{12(1-\mu^2)} = \text{flexural rigidity of the plate element}$$

3.2.3 Vertical force equilibrium

Equilibrium of vertical forces requires that summation of all the vertical forces acting on the plate in Figure 3.3 be equal to zero. This is as represented in Equation (3.98):

$$\mathrm{Q_x dy} + \mathrm{Q_y dx} - \mathrm{Q_x^* dy} - \mathrm{Q_y^* dx} - \mathrm{qdxdy} = 0 \tag{3.98}$$

Substituting Equations (3.83) and (3.84) into Equation (3.98) yields Equation (3.99);

$$\frac{\partial \mathrm{Q_x}}{\partial \mathrm{x}} + \frac{\partial \mathrm{Q_y}}{\partial \mathrm{y}} + \mathrm{q} = 0 \tag{3.99}$$

Equation (3.100) was obtained by substituting Equations (3.91) and (3.92) into Equation (3.99).

$$\frac{\partial^2 M_x}{\partial x^2} + \frac{\partial^2 M_{yx}}{\partial x\,\partial y} + \frac{\partial^2 M_y}{\partial y^2} + \frac{\partial^2 M_{xy}}{\partial x\,\partial y} + q = 0 \tag{3.100}$$

Substituting Equations (3.80), (3.81) and (3.82) into Equation (3.100) yields equation (3.101);

$$-D_x\left(\frac{\partial^4 w}{\partial x^4} + \mu_y \frac{\partial^4 w}{\partial y^4}\right) - 2D_{uv}\frac{\partial^4 w}{\partial x^2\,\partial y^2} - D_y\left(\mu_x \frac{\partial^4 w}{\partial x^2\,\partial y^2} + \frac{\partial^4 w}{\partial y^4}\right) - 2D_{uv}\frac{\partial^4 w}{\partial x^2\,\partial y^2} + q = 0$$

$$D_x \frac{\partial^4 w}{\partial x^4} + \left(\mu_y D_x + 2D_{uv} + \mu_x D_y + 2D_{uv}\right)\frac{\partial^4 w}{\partial x^2\,\partial y^2} + D_y \frac{\partial^4 w}{\partial y^4} = q \tag{3.101}$$

Further simplification of equation (3.101) yields equation (3.102).

$$D_x \frac{\partial^4 w}{\partial x^4} + 2\left(D_{xy} + 2D_{uv}\right)\frac{\partial^4 w}{\partial x^2\,\partial y^2} + D_y \frac{\partial^4 w}{\partial y^4} = q \tag{3.102}$$

Let us define a new parameter, B and recall some parameters earlier defined:

$$B = D_{xy} + 2D_{uv} \tag{3.103}$$

$$D_x = \frac{E_x t^3}{12\left(1 - \mu_x \mu_y\right)} \tag{3.104}$$

$$D_y = \frac{E_y t^3}{12\left(1 - \mu_x \mu_y\right)} \tag{3.105}$$

$$D_{xy} = D_{yx} = \frac{\mu_y E_x t^3}{12\left(1 - \mu_x \mu_y\right)} = \frac{\mu_x E_y t^3}{12\left(1 - \mu_x \mu_y\right)} \tag{3.106}$$

$$D_{uv} = \frac{G t^3}{12} = \frac{EE.\,t^3}{24\left(1 + \mu_x \mu_y\right)} = \frac{EE.\,t^3}{24\left(1 + \mu_{xy}\right)} \tag{3.107}$$

Substituting Equations (3.103) into Equation (3.102) gives Equation (3.108).

$$D_x \frac{\partial^4 w}{\partial x^4} + 2B \frac{\partial^4 w}{\partial x^2\,\partial y^2} + D_y \frac{\partial^4 w}{\partial y^4} = q \tag{3.108}$$

Dividing Equation (3.108) by D_x gives Equation (3.109);

$$\frac{D_x}{D_x}\frac{\partial^4 w}{\partial x^4} + \frac{2B}{D_x}\frac{\partial^4 w}{\partial x^2\,\partial y^2} + \frac{D_y}{D_x}\frac{\partial^4 w}{\partial y^4} = \frac{q}{D_x} \tag{3.109}$$

$$\text{Let } \emptyset_1 = \frac{D_x}{D_x} = \emptyset_x \tag{3.110}$$

$$\emptyset_2 = \frac{B}{D_x} = \emptyset_{xy} \tag{3.111}$$

$$\emptyset_3 = \frac{D_y}{D_x} = \emptyset_y \tag{3.112}$$

Substituting Equations (3.110), (3.111) and (3.112) into Equation (3.109) gives Equation (3.113);

$$\emptyset_1 \frac{\partial^4 w}{\partial x^4} + 2\emptyset_2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + \emptyset_3 \frac{\partial^4 w}{\partial y^4} = \frac{q}{D_x} \qquad (3.113)$$

Equation (3.113) can be written as shown in Equation (3.114) using x and y axes notations.

$$\emptyset_x \frac{\partial^4 w}{\partial x^4} + 2\emptyset_{xy} \frac{\partial^4 w}{\partial x^2 \partial y^2} + \emptyset_y \frac{\partial^4 w}{\partial y^4} = \frac{q}{D_x} \qquad (3.114)$$

Equations (3.113) and (3.114) are the governing differential equations of orthotropic flat rectangular plate. Let non-dimensional axes be R and Q along x and y axes respectively.

$$R = \frac{x}{a} \qquad (3.115)$$

$$Q = \frac{y}{b} \qquad (3.116)$$

Where $0 \leq R \leq 1; 0 \leq Q \leq 1;$

a and b are lengths of the plate as shown on Figure 3.3. substituting Equations (3.115) and (3.116) into Equation (3.114) gives Equation (3.117).

$$\frac{\emptyset_x}{a^4} \frac{\partial^4 w}{\partial R^4} + \frac{2\emptyset_{xy}}{a^4 b^4} \frac{\partial^4 w}{\partial R^2 \partial Q^2} + \frac{\emptyset_y}{b^4} \frac{\partial^4 w}{\partial Q^4} = \frac{q}{D_x} \qquad (3.117)$$

3.2.4 Governing differential equation for isotropic flat rectangular plate

In isotropic case, the elastic parameters in both x and y axes are the same. Thus,

$$E_x = E_y = EE = E \qquad (3.104)$$

$$D_x = D_y = D \qquad (3.105)$$

$$\mu_x = \mu_y = \mu \qquad (3.106)$$

Substituting Equations (3.104) and (3.106) into Equation (3.2) gives;

$$E_{xy} = E_{yx} = \mu E \qquad (3.107)$$

Therefore,

$$D_{xy} = D_{yx} = \mu D \qquad (3.108)$$

Substituting Equations (3.104), and (3.106) into Equations (3.90), (3.91), (3.92) and (3.93) respectively yields Equations (3.109) to (3.112).

$$D_x = \frac{E_x t^3}{12(1 - \mu_x \mu_y)} = D \tag{3.109}$$

$$D_y = \frac{E_y t^3}{12(1 - \mu_x \mu_y)} = D \tag{3.110}$$

$$D_{xy} = D_{yx} = \frac{\mu_y E_x t^3}{12(1 - \mu_x \mu_y)} = \frac{\mu_x E_y t^3}{12(1 - \mu_x \mu_y)} = \mu D \tag{3.111}$$

$$D_{uv} = \frac{Gt^3}{12} = \frac{EE.t^3}{24(1 + \mu_{xy})} = \frac{EE.t^3}{24(1 + \mu_{yx})} = \frac{D(1 - \mu)}{2} \tag{3.112}$$

Equation (3.113) was obtained by substituting Equations (3.109), (3.110), (3.111) and (3.112) into Equation (3.88b) gives;

$$D\frac{\partial^4 w}{\partial x^4} + 2\left(\mu D + \frac{2D(1-\mu)}{2}\right)\frac{\partial^4 w}{\partial x^2\,\partial y^2} + D\frac{\partial^4 w}{\partial y^4} = q \tag{3.113}$$

Equation (3.113) was further simplified to yield Equation (3.114) which in turn yielded Equation (3.115) on simplification.

$$D\frac{\partial^4 w}{\partial x^4} + 2(\mu D + D - \mu D)\frac{\partial^4 w}{\partial x^2\,\partial y^2} + D\frac{\partial^4 w}{\partial y^4} = q$$

Therefore,

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2\,\partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{q}{D} \tag{3.114}$$

$$\frac{1}{a^4}\frac{\partial^4 w}{\partial R^4} + \frac{2}{a^2 b^2}\frac{\partial^4 w}{\partial R^2\,\partial Q^2} + \frac{1}{b^4}\frac{\partial^4 w}{\partial Q^4} = \frac{q}{D} \tag{3.115}$$

Equation (3.114) and (3.115) are the governing partial differential equation for the deflections of thin plate bending analysis based on Newton’s law of equilibrium of forces and Kirchhoff’s assumptions (Ventsel and Krauthammer, 2001). Mathematically, the partial differential Equation 3.115 is a linear partial differential equation of the fourth order having constant coefficients. Finally, they concluded that once a deflection function w (x, y) has been determined from Equation (3.115), the stress resultants and the stresses on the thin plate element are evaluated. It should be noted that Equation (3.115) is plate equilibrium equation at any arbitrary point on the plate (g = 0). To obtain the overall plate equilibrium equation, we integrate both sides of Equation (3.114). With that we obtained Equation (3.115b);

$$F = \iint \left(\frac{1}{a^4}\frac{\partial^4 w}{\partial x^4} + \frac{2}{a^2 b^2}\frac{\partial^4 w}{\partial R^2\,\partial Q^2} + \frac{1}{b^4}\frac{\partial^4 w}{\partial Q^4} - \frac{q}{D}\right) dxdy = 0 \tag{3.115b}$$

3.2.5 Governing differential equation by energy method

Considered here is the development of the partial differential equation of plate based on Bernoulli's principle of virtual work, which replaces the force vectors by work and potential energy, both of which are scalar quantities (Szilard, 2004). Without undermining other approaches of determining the strain energy of a plate, this work adopted the approach of Ibearugbulem et al. (2014). Hence, most of the basic relationships reproduced here are as derived by Ibearugbulem.

Let $\text{Area} = \text{A};\ \text{Length} = \text{L};\ \text{Displacement} = \Delta; \text{Energy} = \text{U};\ \text{Force} = \text{F}; \text{Strain} = \varepsilon;$ $\text{Stress} = \ \sigma$

From elementary physics, force is defined as the product of stress and area. That is:

$$\text{F} = \sigma.\text{A} \tag{3.116}$$

Energy (U) by definition is the ability to do work. Work and energy have the same unit. Mathematically, work is the product of force and displacement. Similarly, force is the product of stress and area.

$$\text{U} = \text{F}.\Delta \tag{3.117}$$

Similarly, displacement is the product of strain and length of a material. That is:

$$\Delta = \varepsilon\text{L} \tag{3.118}$$

Considering a finite element, Equation (3.118) can be rewritten as,

$$\text{d}\Delta = \varepsilon.\text{dL} \tag{3.119}$$

In the same manner, Equation (3.116) becomes:

$$\text{dF} = \sigma.\text{dA} \ = \sigma.\text{dy}.\text{dz} \tag{3.120}$$

Substituting Equation (3.119) and (3.120) into Equation (3.117) will give;

$$\text{dU} = \sigma.\text{dy}.\text{dz}\,.\,\varepsilon.\text{dx} = \ \sigma.\varepsilon.(\text{dx}.\text{dy}.\text{dz}\,) \tag{3.121}$$

Let strain energy before loading be zero and denoted as U_1 and the strain energy after loading be denoted as U_2. Then the average strain energy denoted as U becomes;

$$\text{U} = \frac{\text{U}_1 + \text{U}_2}{2} = \frac{\text{U}_2}{2} \tag{3.122}$$

$$\text{Therefore,}\quad \text{dU}_1 = 0 \tag{3.123}$$

$$\text{dU}_2 = \sigma.\varepsilon.(\text{dA}.\text{dL}) \tag{3.124}$$

$$\text{dU} = \frac{1}{2}[\sigma.\varepsilon.(\text{dA}.\text{dL})] \tag{3.125}$$

Let the finite area, dA be defined mathematically as:

$$\text{dA} = \text{dx}.\text{dy} \tag{3.126}$$

$\text{dA}, \text{dx and dx are the plate element area and dimensions shown on Figure 3.3a}$

Similarly,

$$\mathrm{dL} = \mathrm{dz}. \text{(That is the plate element thickness)} \tag{3.127}$$

substituting Equations (3.126) and (3.127) into Equation (3.125) gives;

$$\mathrm{dU} = \frac{1}{2}\sigma.\varepsilon.\,\mathrm{dx}.\mathrm{dy}.\mathrm{dz} \tag{3.128}$$

For a classical plate, only three stress components are used. That is:

$$\sigma = \sigma_x, \sigma_y \text{ and } \sigma_{xy} \tag{3.129}$$

Similarly, only three engineering strain components are used. That is:

$$\varepsilon = \varepsilon_x, \varepsilon_y \text{ and } \gamma_{xy} \tag{3.130}$$

Hence, the product of stress and strain for classical plate is:

$$\sigma\varepsilon = \sigma_x.\varepsilon_x + \sigma_{xy}\gamma_{xy} + \sigma_y.\varepsilon_y \tag{3.131}$$

Substituting Equation (3.131) Into Equation (3.128) gives:

$$\mathrm{dU} = \frac{1}{2}\left(\sigma_x.\varepsilon_x + \sigma_{xy}\gamma_{xy} + \sigma_y.\varepsilon_y\right)\mathrm{dxdydz} \tag{3.132}$$

From Equations (3.6), (3.39) and (3.40), assuming μ_x equals μ_y; E_x equals E_y; stress is defined as:

$$\sigma_x = \frac{E}{1-\mu^2}.\left(\varepsilon_x + \mu\varepsilon_y\right) \tag{3.133}$$

$$\sigma_y = \frac{E}{1-\mu^2}.\left(\varepsilon_y + \mu\varepsilon_x\right) \tag{3.134}$$

$$\sigma_{xy} = \frac{2E}{4(1-\mu^2)}.\left(\varepsilon_x + \mu\varepsilon_y\right).\gamma_{xy} \tag{3.135}$$

substituting Equations (3.57), (1.58), (3.59), (3.133), (3.134) and (3.135) into Equation (3.132) gives;

$$\mathrm{dU} = \frac{Ez^2}{2(1-\mu^2)}\left(\left[\frac{\partial^2 w}{\partial x^2}\right]^2 + 2\mu\frac{\partial^2 w}{\partial x^2}.\frac{\partial^2 w}{\partial y^2} + 2(1-\mu)\left[\frac{\partial^2 w}{\partial x\,\partial y}\right]^2 + \left[\frac{\partial^2 w}{\partial y^2}\right]^2\right)\mathrm{dxdydz} \tag{3.136}$$

Integrating Equation (3.136) w. r. t. (meaning “with respect to”) z;

$$\mathrm{U} = \frac{E}{2(1-\mu^2)}\frac{t^3}{12}\int_0^a\int_0^b\left(\left[\frac{\partial^2 w}{\partial x^2}\right]^2 + 2\mu\frac{\partial^2 w}{\partial x^2}.\frac{\partial^2 w}{\partial y^2} + 2(1-\mu)\left[\frac{\partial^2 w}{\partial x\,\partial y}\right]^2 + \left[\frac{\partial^2 w}{\partial y^2}\right]^2\right)\mathrm{dxdy} \tag{3.137}$$

Recall in Equation (3.90), $\mathrm{D} = \frac{Et^3}{12(1-\mu^2)}$

Therefore, substituting Equation (3.90) into Equation (3.137) gives,

$$U = \frac{D}{2}\int_0^a\int_0^b\left(\left[\frac{\partial^2 w}{\partial x^2}\right]^2 + 2\mu\frac{\partial^2 w}{\partial x^2}.\frac{\partial^2 w}{\partial y^2} + 2(1-\mu)\left[\frac{\partial^2 w}{\partial x\,\partial y}\right]^2 + \left[\frac{\partial^2 w}{\partial y^2}\right]^2\right)dxdy \qquad (3.137)$$

$$\text{Mathematically,} \int_0^a\int_0^b\left(\frac{\partial^2 w}{\partial x^2}.\frac{\partial^2 w}{\partial y^2}\right)dxdy = \int_0^a\int_0^b\left[\frac{\partial^2 w}{\partial x\,\partial y}\right]^2 dxdy \qquad (3.138)$$

Substituting Equation (3.138) into Equation (3.137), we have,

$$U = \frac{D}{2}\int_0^a\int_0^b\left(\left[\frac{\partial^2 w}{\partial x^2}\right]^2 + 2\left[\frac{\partial^2 w}{\partial x\partial y}\right]^2 + \left[\frac{\partial^2 w}{\partial y^2}\right]^2\right)dx\,dy \qquad (3.139)$$

Equation (3.139) is the strain energy equation of the thin rectangular plate. It shows the stored energy or the internal work done by the plate. We shall now determine the potential energy, V of external load acting on the plate.

The potential energy of external force is defined as the product of the external force and the displacement.

$$\text{Mathematically,} V = q.w \qquad (3.140)$$

When the load is a uniform lateral pressure q, Equation (3.140) can be written as;

$$dV = qw(x,y)dxdy \qquad (3.141)$$

Integrating both sides of Equation (3.141) we have,

$$V = q\int_0^a\int_0^b w\,dx\,dy \qquad (3.142)$$

The total potential energy, Π of a structural system consists of its strain energy and the potential energy of the load. (Rudolph Szilard, 2004). At this point, we can now state the total potential energy functional for the isotropic thin rectangular plate as;

$$\Pi = U - V \qquad (3.143)$$

Substituting Equations (3.139) and (3.142) into Equation (3.143) gives,

$$\Pi = \frac{D}{2}\int_0^a\int_0^b\left(\left[\frac{\partial^2 w}{\partial x^2}\right]^2 + 2\left[\frac{\partial^2 w}{\partial x\partial y}\right]^2 + \left[\frac{\partial^2 w}{\partial y^2}\right]^2\right)dx\,dy - q\int_0^a\int_0^b w\,dx\,dy \qquad (3.144)$$

Equation (3.144) is the total or overall potential energy functional for isotropic rectangular flat plate. In non-dimensional form, Equation (3.144) can be written as Equation (3.145). Equation (3.147) is obtained on further simplification of Equation (3.145).

$$\Pi = \frac{\mathrm{abD}}{2\mathrm{a}^4}\int_0^{\mathrm{a}}\int_0^{\mathrm{b}}\left(\left[\frac{\partial^2\mathrm{w}}{\partial\mathrm{R}^2}\right]^2 + \frac{2}{\mathrm{p}^2}\left[\frac{\partial^2\mathrm{w}}{\partial R\partial Q}\right]^2 + \left[\frac{\partial^2\mathrm{w}}{\partial\mathrm{Q}^2}\right]^2\right)\mathrm{dR\,dQ} - \mathrm{abq}\int_0^{\mathrm{a}}\int_0^{\mathrm{b}}\mathrm{w\,dR\,dQ} \quad (3.145)$$

$$\Pi = \frac{\mathrm{abD}}{2\mathrm{a}^4}\int_0^{\mathrm{a}}\int_0^{\mathrm{b}}\left(\frac{\partial^4\mathrm{w}^2}{\partial\mathrm{R}^4} + \frac{2}{\mathrm{p}^2}\frac{\partial^4\mathrm{w}^2}{\partial\mathrm{R}^2\,\partial\mathrm{Q}^2} + \frac{1}{\mathrm{p}^4}\frac{\partial^4\mathrm{w}^2}{\partial\mathrm{Q}^4}\right)\mathrm{dR\,dQ} - \mathrm{abq}\int_0^{\mathrm{a}}\int_0^{\mathrm{b}}\mathrm{w\,dR\,dQ} \quad (3.146)$$

$$\Pi = \frac{\mathrm{abD}}{2\mathrm{a}^4}\int_0^{1}\int_0^{1}\left(\frac{\partial^4\mathrm{w}}{\partial\mathrm{R}^4}\mathrm{w} + \frac{2}{\mathrm{p}^2}\frac{\partial^4\mathrm{w}}{\partial\mathrm{R}^2\,\partial\mathrm{Q}^2}\mathrm{w} + \frac{1}{\mathrm{p}^4}\frac{\partial^4\mathrm{w}}{\partial\mathrm{Q}^4}w\right)\mathrm{dR\,dQ}$$
$$- \mathrm{qab}\int_0^{1}\int_0^{1}\mathrm{w\,dR\,dQ} \quad (3.147)$$

3.3 Determination of the exact deflection shape functions of the selected plates

Attempt was made in this section to obtain the exact solution of the overall equilibrium equation plate. Following the solution is satisfying the boundary conditions, determining the derivatives of the shape functions and the determination of the integrands for the selected boundary conditions.

3.3.1 Exact solution plates

Equations (3.115) and (3.139) are the partial differential equations for thin rectangular plates derived from equilibrium method and energy methods respectively. In this section an exact solution of the differential equation based on energy method will be obtained. Thereafter, the solution will be used to determine the exact polynomial shape functions for selected plate types which will satisfy their boundary conditions.

The total potential energy of classical rectangular plate determined in Equation (3.147) can be written in the non-dimensional coordinates as:

$$\Pi = \frac{\mathrm{abD}}{2\mathrm{a}^4}\int_0^{\mathrm{a}}\int_0^{\mathrm{b}}\left(\frac{\partial^4\mathrm{w}^2}{\partial\mathrm{R}^4} + \frac{2}{\mathrm{p}^2}\frac{\partial^4\mathrm{w}^2}{\partial\mathrm{R}^2\,\partial\mathrm{Q}^2} + \frac{1}{\mathrm{p}^4}\frac{\partial^4\mathrm{w}^2}{\partial\mathrm{Q}^4}\right)\mathrm{dR\,dQ} - \mathrm{abq}\int_0^{\mathrm{a}}\int_0^{\mathrm{b}}\mathrm{w\,dR\,dQ} \quad (3.148)$$

Where $\mathrm{x} = \mathrm{aR}, \mathrm{y} = \mathrm{bQ}$ and $\mathrm{P} =$ is the aspect ratio of the plates

Energy is the product of force, F and distance, w (x). Mathematically, this is as expressed in Equation (3.149).

$$\mathrm{Energy}, \Pi = \mathrm{Force}, \mathrm{F} \times \mathrm{Distance}, \mathrm{w}(x) \quad (3.149)$$

Making force, F in Equation (3.149) the subject of the relationship yields Equation (3.150).

$$\mathrm{Therefore, Force, F} = \frac{\mathrm{dEnergy}}{\mathrm{dw}(x)} = \frac{\mathrm{d}\Pi}{\mathrm{dw}} \quad (3.150)$$

Differentiating (3.148) with respect to deflection, w gives resultant force as zero:

$$F = \frac{d\Pi}{dw} = \int_0^1 \int_0^1 \left(\frac{\partial^4 w}{\partial R^4} + \frac{2}{p^2} \frac{\partial^4 w}{\partial R^2\, \partial Q^2} + \frac{1}{p^4} \frac{\partial^4 w}{\partial Q^4} - \frac{qa^4}{D} \right) dR\, dQ = 0 \tag{3.151}$$

Let deflection be defined as:

$$w = Ah \tag{3.152}$$

Substituting Equation (3.152) into Equation (3.151) gives:

$$A \int_0^1 \int_0^1 \left(\frac{\partial^4 h}{\partial R^4} + \frac{2}{p^2} \frac{\partial^4 h}{\partial R^2\, \partial Q^2} + \frac{1}{p^4} \frac{\partial^4 h}{\partial Q^4} \right) dR\, dQ = \int_0^1 \int_0^1 \frac{qa^4}{D} dR\, dQ \tag{3.153}$$

That is:

$$A \int_0^1 \int_0^1 \left(\frac{\partial^4 h}{\partial R^4} + \frac{2}{p^2} \frac{\partial^4 h}{\partial R^2\, \partial Q^2} + \frac{1}{p^4} \frac{\partial^4 h}{\partial Q^4} \right) dR\, dQ = \frac{qa^4}{D} \tag{3.154}$$

That is:

$$A[k_1 + k_2 + k_3] = \frac{qa^4}{D} \tag{3.155}$$

Where:

$$k_1 = \int_0^1 \int_0^1 \frac{\partial^4 h}{\partial R^4} dR\, dQ \tag{3.156}$$

$$k_2 = \int_0^1 \int_0^1 \frac{2}{p^2} \frac{\partial^4 h}{\partial R^2\, \partial Q^2} dR\, dQ \tag{3.157}$$

$$k_3 = \int_0^1 \int_0^1 \frac{1}{p^4} \frac{\partial^4 h}{\partial Q^4} dR\, dQ \tag{3.158}$$

Thus, rearranging Equation (3.155) gives:

$$A = \frac{1}{[k_1 + k_2 + k_3]} \frac{qa^4}{D} \tag{3.159}$$

Equation (3.159) may be represented as shown in Equation (3.160).

$$A = k_T \frac{qa^4}{D} \tag{3.160}$$

Where:

$$k_T = \frac{1}{[k_1 + k_2 + k_3]} \tag{3.161}$$

Substituting Equation (3.160) into Equation (3.155) yields Equation (3.162).

$$k_T \frac{qa^4}{D} \; [k_1 + k_2 + k_3] = \frac{qa^4}{D} \tag{3.162}$$

On further simplification of equation (3.162), Equation (3.163) is obtained.

$$k_T \, [k_1 + k_2 + k_3] = 1 \tag{3.163}$$

Or

$$k_T\, k_1 + k_T k_2 + k_T k_3 = 1 \tag{3.164}$$

Since the integral of Equation (3.151) is zero, then the integrand is zero. This is shown in Equation (3.165).

$$\frac{\partial^4 w}{\partial R^4} + \frac{2}{p^2}\frac{\partial^4 w}{\partial R^2\, \partial Q^2} + \frac{1}{p^4}\frac{\partial^4 w}{\partial Q^4} - \frac{qa^4}{D} = 0 \tag{3.165}$$

Substituting Equation (3.164) into Equation (3.151) yields equation (3.166).

$$\int_0^1 \int_0^1 \left(\frac{\partial^4 w}{\partial R^4} + \frac{2}{p^2}\frac{\partial^4 w}{\partial R^2\, \partial Q^2} + \frac{1}{p^4}\frac{\partial^4 w}{\partial Q^4} - \frac{qa^4}{D}\left[k_T\, k_1 + k_T k_2 + k_T k_3\right]\right) dR\, dQ = 0 \tag{3.166}$$

This can be rearranged as shown in Equation (3.167).

$$\int_0^1 \int_0^1 \left(\left[\frac{\partial^4 w}{\partial R^4} - \frac{qa^4}{D} k_T\, k_1\right] + \left[\frac{2}{p^2}\frac{\partial^4 w}{\partial R^2\, \partial Q^2} - \frac{qa^4}{D} k_T k_2\right] + \left[\frac{1}{p^4}\frac{\partial^4 w}{\partial Q^4} - \frac{qa^4}{D} k_T k_3\right]\right) dR\, dQ = 0 \tag{3.167}$$

One of the conditions for which Equation (3.167) will be true is if each of these integrals is zero. That is if;

$$\int_0^1 \int_0^1 \left(\left[\frac{\partial^4 w}{\partial R^4} - \frac{qa^4}{D} k_T\, k_1\right]\right) dR\, dQ = 0 \tag{3.168}$$

$$\int_0^1 \int_0^1 \left(\left[\frac{2}{p^2}\frac{\partial^4 w}{\partial R^2\, \partial Q^2} - \frac{qa^4}{D} k_T k_2\right]\right) dR\, dQ = 0 \tag{3.169}$$

$$\int_0^1 \int_0^1 \left(\left[\frac{1}{p^4}\frac{\partial^4 w}{\partial Q^4} - \frac{qa^4}{D} k_T k_3\right]\right) dR\, dQ = 0 \tag{3.170}$$

Based on this assumption, let us split deflection into w_x and w_y as:

$$w = w_x . w_y \tag{3.171}$$

Substituting Equation (3.171) into Equations (3.168), (3.169) and (3.170) and rearranging the resulting equations gives respectively:

$$\int_0^1 \int_0^1 \left(\left[w_y \frac{\partial^4 w_x}{\partial R^4} - \frac{qa^4}{D} k_T\, k_1\right]\right) dR\, dQ = 0 \tag{3.172}$$

$$\int_0^1 \int_0^1 \left(\left[\frac{2}{p^2}\frac{\partial^2 w_x}{\partial R^2}\frac{\partial^2 w_y}{\partial Q^2} - \frac{qa^4}{D} k_T k_2\right]\right) dR\, dQ = 0 \tag{3.173}$$

$$\int_0^1 \int_0^1 \left(\left[\frac{w_x}{p^4}\frac{\partial^4 w_y}{\partial Q^4} - \frac{qa^4}{D} k_T k_3\right]\right) dR\, dQ = 0 \tag{3.174}$$

Carrying out the integration of Equations (3.172) with respect to Q and Equation (3.173) with respect to R gives respectively:

$$\int_0^1 \left(\left[w_x \frac{\partial^4 w_x}{\partial R^4} - \frac{qa^4}{D} k_T\, k_1\right]\right) dR \;= 0 \tag{3.175}$$

$$\int_0^1 \left(\left[\frac{w_y}{p^4} \frac{\partial^4 w_y}{\partial Q^4} - \frac{qa^4}{D} k_T k_3\right]\right)\; dQ = 0 \tag{3.176}$$

One of the conditions for which Equations (3.175) and (3.176) will be true is if their integrands is zero. Hence, the study assumes that;

$$\frac{\partial^4 w_x}{\partial R^4} = \frac{qa^4}{Dw_x} k_T\, k_1 \tag{3.177}$$

$$\frac{\partial^4 w_y}{\partial Q^4} = \frac{qa^4}{Dw_y} k_T k_3 p^4 \tag{3.178}$$

Let:

$$A_x = \frac{qa^4}{Dw_x} k_T\, k_1 \tag{3.179}$$

$$A_y = \frac{qa^4}{Dw_y} k_T k_3 p^4 \tag{3.180}$$

$$A = \frac{A_x}{24} . \frac{A_y}{24} \tag{3.181}$$

Substituting Equations (3.179) and (3.180) into Equations (3.177) and (3.178) shall respectively give:

$$\frac{\partial^4 w_x}{\partial R^4} = A_x \tag{3.182}$$

$$\frac{\partial^4 w_y}{\partial Q^4} = A_y \tag{3.183}$$

Solving Equations (3.182) and (3.183) by direct integration shall respectively give:

$$w_x = a_0 + a_1 R + a_2 \frac{R^2}{2} + a_3 \frac{R^3}{6} + A_x \frac{R^4}{24} \tag{3.184}$$

$$w_y = b_0 + b_1 Q + b_2 \frac{Q^2}{2} + b_3 \frac{Q^3}{6} + A_y \frac{Q^4}{24} \tag{3.185}$$

Multiplying Equations (3.184) and (3.185) gives:

$$w = w_x . w_y = \left(a_0 + a_1 R + a_2 \frac{R^2}{2} + a_3 \frac{R^3}{6} + A_x \frac{R^4}{24}\right) . \left(b_0 + b_1 Q + b_2 \frac{Q^2}{2} + b_3 \frac{Q^3}{6} + A_y \frac{Q^4}{24}\right) \tag{3.186}$$

3.3.2 Satisfying the boundary conditions for the plane continuums considered.

For the polynomial deflection equation derived in Equation (3.186) to be the solution of the governing differential equation, there is need to satisfy the boundary conditions of the selected boundary conditions of the plane continuums considered in this work to arrive at their exact polynomial shape functions. Hence, the subsections here are dedicated to satisfying boundary conditions for the plate types within the scope of this investigation. Many boundary conditions exist, but only two of them will be considered here. The two are simple (pin or roller) and clamp (fix) supports.

3.3.2.1 Satisfying the boundary conditions for Simple support SS plane continuum

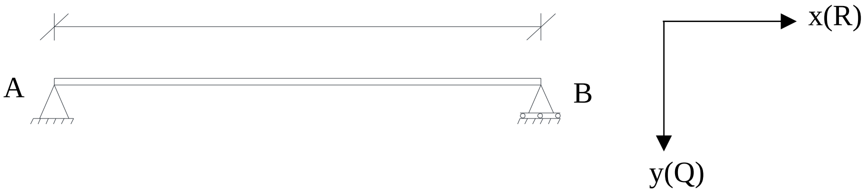

Figure 3.4 Two edges simply supported, SS line continuum

Recall, Equations (3.171), (3.184) and (3.185) stated here again respectively,

$$w = w_x . w_y$$

$$w_x = \left(a_0 + a_1 R + a_2 \frac{R^2}{2} + a_3 \frac{R^3}{6} + A_x \frac{R^4}{24}\right)$$

$$w_y = \left(b_0 + b_1 Q + b_2 \frac{Q^2}{2} + b_3 \frac{Q^3}{6} + A_y \frac{Q^4}{24}\right)$$

Only one direction, x, will be used since the condition is the same in the y direction.

$$\frac{dw}{dR} = \left(a_1 + a_2 R + a_3 \frac{R^2}{2} + A_x \frac{R^3}{6}\right) \qquad (3.187)$$

$$\frac{d^2w}{dR^2} = \left(a_2 + a_3 R + A_x \frac{R^2}{2}\right) \qquad (3.188)$$

$$\frac{d^3w}{dR^3} = (a_3 + A_x R)\,. \qquad (3.189)$$

$$\frac{d^4w}{dR^4} = A_x \qquad (3.190)$$

Boundary conditions;

$$\text{At } R = 0, w = \frac{d^2w}{dR^2} = 0, \text{and at } R = 1, w = \frac{d^2w}{dR^2} = 0$$

Substituting these boundary conditions into Equation (3.184) and solving gives;

$$a_0 = 0, a_1 = \frac{A_x}{24}, a_2 = 0, a_3 = -\frac{A_x}{2}$$

Substituting back these constants into Equation (3.184) yields Equations (3.191) and (3.192)

$$w_x = \frac{A_x}{24}(R - 2R^3 + R^4) \tag{3.191}$$

Similarly,

$$w_y = \frac{A_y}{24}(Q - 2Q^3 + Q^4) \tag{3.192}$$

3.3.2.2 Satisfying the boundary conditions for all edge clamp, CC plane continuum

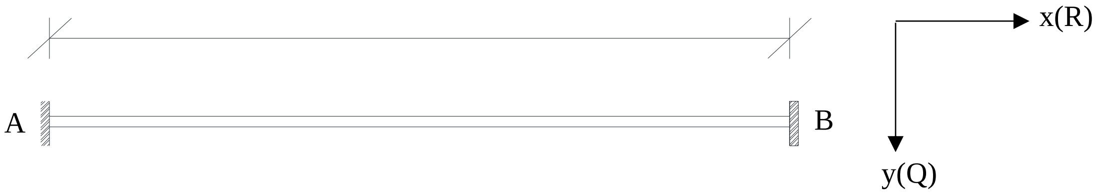

Figure 3.5 Two edges clamp, CC line continuum

Boundary conditions;

$$\text{At } R = 0, w = \frac{dw}{dR} = 0, \text{and at } R = 1, w = \frac{dw}{dR} = 0$$

Similarly,

$$\text{At } Q = 0, w = \frac{dw}{dQ} = 0, \text{and at } Q = 1, w = \frac{dw}{dQ} = 0$$

Substituting these boundary conditions into Equation (3.184) and solving gives;

$$a_0 = 0, a_1 = 0, a_2 = \frac{A_x}{12}, a_3 = -\frac{A_x}{2}, \text{and } b_0 = 0, b_1 = 0, b_2 = \frac{A_y}{12}, b_3 = -\frac{A_y}{2}$$

Substituting back these constants into Equation (3.184) yields Equations (3.193) and (3.194)

$$w_x = \frac{A_x}{24}(R^2 - 2R^3 + R^4) \tag{3.193}$$

Similarly,

$$w_y = \frac{A_y}{24}(Q^2 - 2Q^3 + Q^4) \tag{3.194}$$

3.3.2.3 Satisfying the boundary conditions for one end simple support and opposite end clamp, CS plane continuum

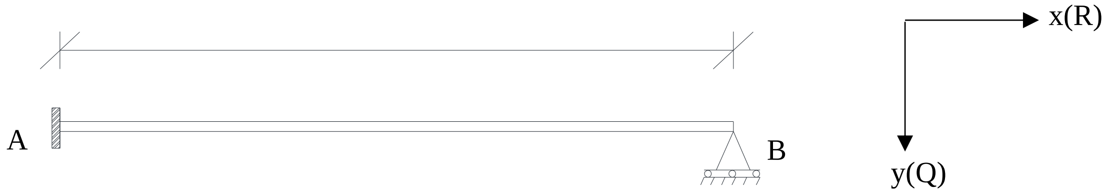

Figure 3.6 One end simply supported, opposite end clamp line continuum

Boundary condition;

$$\text{At } R = 0, w = \frac{dw}{dR} = 0, \text{and at } R = 1, w = \frac{d^2w}{dR^2} = 0$$

Similarly,

$$\text{At } Q = 0, w = \frac{dw}{dQ} = 0, \text{and at } Q = 1, w = \frac{dw}{dQ} = 0$$

Substituting these boundary conditions into Equation (3.184) and solving gives;

$$a_0 = 0, a_1 = \frac{A_x}{24}, a_2 = 0, a_3 = -\frac{A_x}{2}, \text{and } b_0 = 0, b_1 = 0, b_2 = \frac{A_y}{12}, b_3 = -\frac{A_y}{2}$$

Substituting back these constants into Equation (3.184) yields Equation (3.199).

$$w_x = \frac{A_x}{24}(1.5R^2 - 2.5R^3 + R^4) \quad (3.195)$$

Similarly,

$$w_y = \frac{A_y}{24}(1.5Q^2 - 2.5Q^3 + Q^4) \quad (3.196)$$

From the ongoing, we can now determine the actual deflection functions of the plates considered by combination of the appropriate boundary conditions in the x and y directions of the plate.

SSSS plate

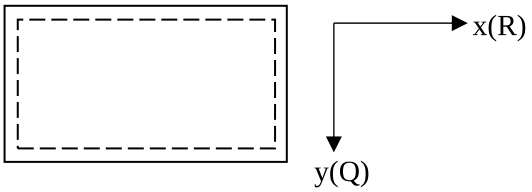

Figure 3.7: SSSS Rectangular Plate

The plate is SS in both directions, therefore, the deflection function, w, will the product of deflection functions of SS line continuum in x and y directions.

$w = \frac{A_x}{24}.\frac{A_y}{24}(R - 2R^3 + R^4).(Q - 2Q^3 + Q^4)$. That is:

$$w = A\,(R - 2R^3 + R^4).(Q - 2Q^3 + Q^4) \qquad (3.197)$$

CCCC plate

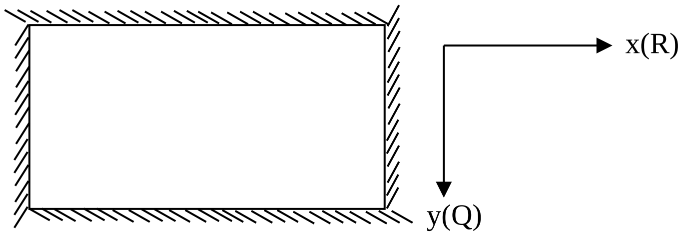

Figure 3.8; CCCC Rectangular Plate

The plate is CC in both directions, therefore, the deflection function, w, will the product of deflection functions of CC line continuum in x and y directions.

$w = \frac{A_x}{24}.\frac{A_y}{24}(R^2 - 2R^3 + R^4).(Q^2 - 2Q^3 + Q^4)$. That is:

$$w = A\,(R^2 - 2R^3 + R^4).(Q^2 - 2Q^3 + Q^4) \qquad (3.198)$$

CSCS plate

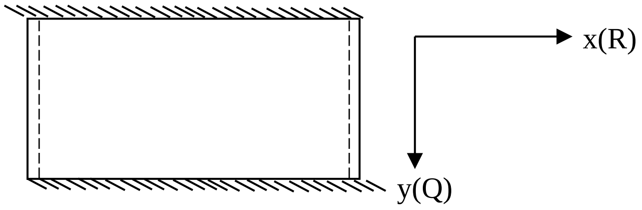

Figure 3.9: CSCS Rectangular Plate

The plate is SS in x – direction and CC in y - direction, therefore, the deflection function, w, will the product of deflection functions of SS and CC line continuums for x and y directions respectively.

$$w = \frac{A_x}{24}.\frac{A_y}{24}(R - 2R^3 + R^4).(Q^2 - 2Q^3 + Q^4)\text{. That is:}$$

$$w = A\,(R - 2R^3 + R^4).(Q^2 - 2Q^3 + Q^4) \tag{3.199}$$

CCSS plate

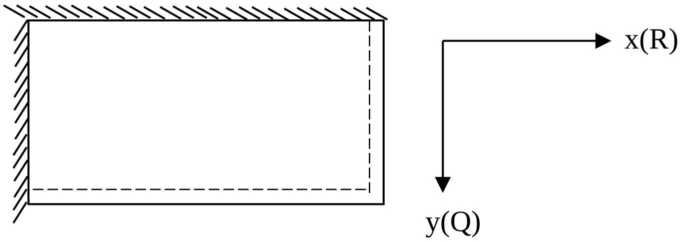

Figure 3.10: CCSS Rectangular Plate

The plate is CS in both directions, therefore, the deflection function, w, will the product of deflection functions of CS line continuum in x and y directions.

$$w = \frac{A_x}{24}.\frac{A_y}{24}(1.5R^2 - 2R^3 + R^4).(1.5Q^2 - 2Q^3 + Q^4)\text{. That is:}$$

$$w = A\,(1.5R^2 - 2R^3 + R^4).(1.5Q^2 - 2Q^3 + Q^4) \tag{3.200}$$

CSSS plate

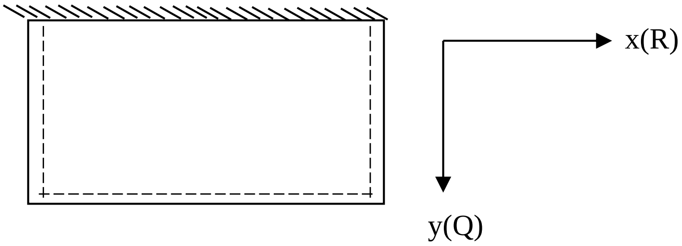

Figure 3.11: CSSS Rectangular Plate

The plate is SS in x – direction and CS in y - direction, therefore, the deflection function, w, will the product of deflection functions of SS and CS line continuums for x and y directions respectively.

$$w = \frac{A_x}{24}.\frac{A_y}{24}(R - 2R^3 + R^4).(1.5Q^2 - 2Q^3 + Q^4)\text{. That is:}$$

$$w = A\,(R - 2R^3 + R^4)\,.\,(1.5Q^2 - 2Q^3 + Q^4) \qquad (3.201)$$

CCCS plate

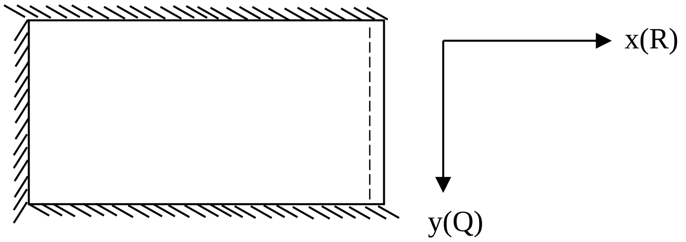

Figure 3.12: CCCS Rectangular Plate

The plate is CS in x – direction and CC in y - direction, therefore, the deflection function, w, will the product of deflection functions of CS and CC line continuums for x and y directions respectively.

$$w = \frac{A_x}{24}.\frac{A_y}{24}(1.5R^2 - 2R^3 + R^4)\,.\,(Q^2 - 2Q^3 + Q^4)\,.\,\text{That is:}$$

$$w = A\,(1.5R^2 - 2R^3 + R^4)\,.\,(Q^2 - 2Q^3 + Q^4) \qquad (3.202)$$

3.4 Pure bending analysis

The deflection function, w derived in section 3.3 above will be employed in this section to determine the exact bending moment and shear force for line continuum and thin plates of selected boundary conditions.

3.4.1 The effective span of plane continuum in pure bending under uniform load.

When line continuum buckles in first deformation mode, a region within the span takes bent configuration. Bending of continuum reaches the support (simple support or free support). The bending cannot reach the clamped support. The bent sections of the line continuums are shown on Figures 3.13, 3.14 and 3.15. The bent region is the effective span. Equation (3.203) is the general expression of effective span of a line continuum from Ibearugbulem, 2018.

$$Le = \sqrt[4]{\left(\frac{h_{max}}{24}\right) \times \left(\frac{384}{5}\right)} \times L \qquad (3.203)$$

Where;

Le is the effective length(span)of the line continuum.

L is the total length (span)of the line continuum.

$\mathrm{h_{max}}$ is a point on the span of the continuum where the slope is zero.

The plane continuum deflection equation, w as derived in section 3.3 of this work was splitted into the othorgonal deflection equations in x and y directions, i.e. w_x and w_y. As a result, an SSSS plate type is a product of SS (simply supported at both ends) line continuum in the x direction and SS line continuum in the y direction. Similarly, a CCCC plate type is a product of CC (clamped at both ends) line continuum in the x direction and CC line continuum in the y direction. The condition is the same for other plate types considered in this work. Thus, CSCS plate is a product of SS line continuum in one direction (in this work, x direction) and CC line continnum in the y direction. CSSS plate is a product of SS line continuum in one direction (in this work, x direction) and CS (simply supported at one end and clamped at opposite end) line continnum in the y direction. CCSS plate is a product of CS line continuum in the x and y directions and lastly, CCCS plate is a product of CS line continuum in one direction (in this work, x direction) and CC line continnum in the y direction. The effective span of the plane continuum for the boundary conditions considered are derived in the following section.

3.4.1.1 The effective span of a line continuum simply supported both ends, SS.

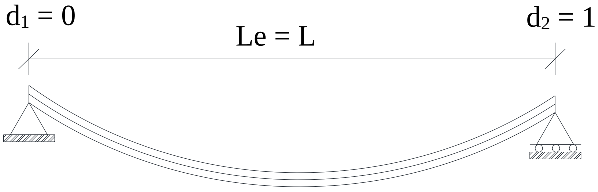

Figure 3.13 Two edges simply supported, SS line continuum

The polynomial shape function for SS line continuum is as was derived in Equation (3.191) is as shown:

$$\mathrm{h = R - 2R^3 + R^4}$$

Deflection will be maximum at a point within the span where slope is zero. This point is exactly at the mid-span (R = 0.5). The slope and the maximum value of shape function are as given in Equations (3.204) and (3.205).

$$\frac{\mathrm{dh(R = 0.5)}}{\mathrm{dR}} = 0 \tag{3.204}$$

$$\mathrm{h_{max} = R - 2R^3 + R^4 = 0.5 - 2 \times\ 0.5^3 + 0.5^4 = 0.3125} \tag{3.205}$$

Thus, substituting Equation (3.205) into Equation (3.203) will yield the effective span of SS line continuum as shown in Equation (3.206)

$$\mathrm{Le} = \sqrt[4]{\left(\frac{0.3125}{24}\right) \times \left(\frac{384}{5}\right) \times \mathrm{L}} = \mathrm{L} \tag{3.206}$$

3.4.1.2 The effective span of a line continuum clamped both ends, CC.

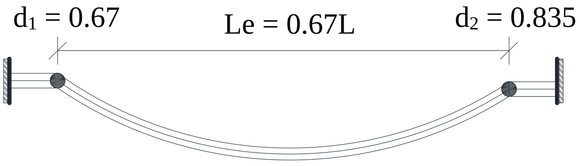

Figure 3.14 Two edges clamped, CC line continuum

The polynomial shape function for CC line continuum as derived previously in Equation (3.193) is as shown:

$$\mathrm{h} = \mathrm{R}^2 - 2\mathrm{R}^3 + \mathrm{R}^4$$

Deflection will be maximum at a point within the span where slope is zero. This point is exactly at the mid-span (R = 0.5). The slope and the maximum value of shape function are as given in Equations (3.207) and (3.208).

$$\frac{\mathrm{dh}(\mathrm{R} = 0.5)}{\mathrm{dR}} = 0 \tag{3.207}$$

$$\mathrm{h}_{\mathrm{max}} = \mathrm{R}^2 - 2\mathrm{R}^3 + \mathrm{R}^4 = 0.5^2 - 2 \times 0.5^3 + 0.5^4 = 0.0625 \tag{3.208}$$

Thus, substituting Equation (3.208) into Equation (3.203) will yield the effective span of CC line continuum as shown in Equation (3.209).

$$\mathrm{Le} = \sqrt[4]{\left(\frac{0.0625}{24}\right) \times \left(\frac{384}{5}\right) \times \mathrm{L}} = 0.67\mathrm{L} \tag{3.209}$$

3.4.1.3 The effective span of a line continuum clamped at one end and simply supported at the other end, CS.

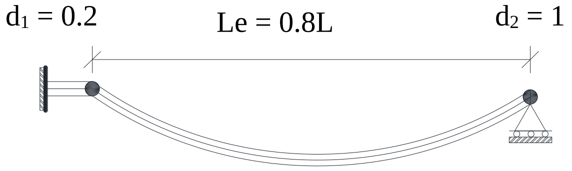

Figure 3.15 One edge clamped, opposite edge simply supported, CS line continuum

The polynomial shape functions for CS line continuum is as derived previously is as given below:

$$\mathrm{h} = 1.5\mathrm{R}^2 - 2.5\mathrm{R}^3 + \mathrm{R}^4$$

Deflection will be maximum at a point within the span where slope is zero. This point is exactly at the mid-span (R = 0.578464834591373 ≈ 0.5784). The slope and the maximum value of shape function are as given in Equations (3.210) and (3.211).

$$\frac{\mathrm{dh}(\mathrm{R} = 0.5784)}{\mathrm{dR}} = 0 \qquad (3.210)$$

$$\mathrm{h}_{\max} = \mathrm{R}^2 - 2\mathrm{R}^3 + \mathrm{R}^4 = 0.5784^2 - 2 \times 0.5784^3 + 0.5784^4 \approx 0.1299 \qquad (3.211)$$

Thus, substituting Equation (3.211) into Equation (3.203) will yield the effective span of CC line continuum as shown in Equation (3.212).

$$\mathrm{Le} = \sqrt[4]{\left(\frac{0.1299}{24}\right) \times \left(\frac{384}{5}\right) \times \mathrm{L}} = 0.803087\mathrm{L} \approx 0.8\mathrm{L} \qquad (3.212)$$

3.4.2 Pure bending analysis of line continuum.

Recall Equation (3.171), $\mathrm{w} = \mathrm{w_x}.\mathrm{w_y}$

The deflection function, w was obtained by decoupling the orthogonal deflection function into x and y directions. w_x and w_y are one dimensional deflection functions in x and y directions respectively. This section will use the deflection function, w_x in x – direction to carry out pure bending analysis of line continuum using the Euler-Bernoulli residual force (exact) approach and Ritz (weighted residual) method. The boundary conditions of the selected line continuum are as follows: all edges simply supported, SS, all edges clamped, CC, and one edge clamped, opposite edge simply supported, CS. Figures 3.13, 3.14 and 3.15 shows SS, CC and CS line continuums.

The study assumes deflection function, w to be as shown in Equation (3.213).

$$w = Ah \quad (3.213)$$

The corresponding shape function, h of two edges simply supported beam, SS is as given in Equation (3.214).

$$h = R - 2R^3 + R^4 \quad (3.214)$$

The corresponding shape function, h of two edges simply supported beam, CC is as given in Equation (3.215).

$$h = R^2 - 2R^3 + R^4 \quad (3.215)$$

Similarly, the corresponding shape function, h of one end simply supported and opposite end clamped beam, CS is as given in Equation (3.216).

$$h = 1.5R^2 - 2.5R^3 + R^4 \quad (3.216)$$

3.4.2.1 The derivatives of the shape functions of the line continuums.

The derivatives of all edge simply supported, SS line continuum.

$$w = Ah = A(R - 2R^3 + R^4) \quad (3.217)$$

$$h = (R - 2R^3 + R^4) \quad (3.218)$$

$$h^2 = (R^2 - 4R^4 + 2R^5 + 4R^6 - 4R^7 + R^8) \quad (3.219)$$

$$\frac{\partial h}{\partial R} = (1 - 6R^2 + 4R^3) \quad (3.220)$$

$$\left(\frac{\partial h}{\partial R}\right)^2 = (1 - 12R^2 + 8R^3 + 36R^4 - 48R^5 + 16R^6) \quad (3.221)$$

$$\frac{\partial^2 h}{\partial R^2} = 12(R^2 - R) \quad (3.222)$$

$$\left(\frac{\partial^2 h}{\partial R^2}\right)^2 = 144(R^4 - 2R^3 + R^2)dR \quad (3.223)$$

$$\frac{\partial^3 h}{\partial R^3} = 12(2R - 1) \quad (3.224)$$

$$\frac{\partial^4 h}{\partial R^4} = 24 \quad (3.225)$$

The derivatives of all edge clamped CC line continuum

$$w = Ah = A(R^2 - 2R^3 + R^4) \quad (3.226)$$

$$h = (R^2 - 2R^3 + R^4) \quad (3.227)$$

$$h^2 = (R^4 - 4R^5 + 6R^6 - 4R^7 + R^8) \quad (3.228)$$

$$\frac{dh}{dR} = 2R - 6R^2 + 4R^3 \quad (3.229)$$

$$\left(\frac{dh}{dR}\right)^2 = (4R^2 - 24R^3 + 52R^4 - 48R^5 + 16R^6) \quad (3.230)$$

$$\frac{d^2h}{dR^2} = 2 - 12R + 12R^2 \quad (3.231)$$

$$\left(\frac{\partial^2 h}{\partial R^2}\right)^2 = (4 - 48R + 192R^2 - 288R^3 + 144R^4)dR \quad (3.232)$$

$$\frac{d^3h}{dR^3} = 12(2R - 1) \quad (3.233)$$

$$\frac{d^4h}{dR^4} = 24 \quad (3.234)$$

The derivatives of one edge clamped and one edge simply supported CS line continuum

$$w = Ah = A(1.5R^2 - 2.5R^3 + R^4) \quad (3.235)$$

$$h = 1.5R^2 - 2.5R^3 + R^4$$

$$h^2 = 2.25R^4 - 7.5R^5 + 9.25R^6 - 5R^7 + R^8 \quad (3.236)$$

$$\frac{\partial h}{\partial R} = (3R - 7.5R^2 + 4R^3) \quad (3.237)$$

$$\left(\frac{dh}{dR}\right)^2 = (9R^2 - 45R^3 + 80.25R^4 - 60R^5 + 16R^6) \quad (3.238)$$

$$\frac{\partial^2 h}{\partial R^2} = 3(1 - 5R + 4R^2) \quad (3.239)$$

$$\left(\frac{\partial^2 h}{\partial R^2}\right)^2 = (9 - 90R + 297R^2 - 360R^3 + 144R^4)dR \quad (3.240)$$

$$\frac{\partial^3 h}{\partial R^3} = 3(8R - 5) \quad (3.241)$$

$$\frac{\partial^4 h}{\partial R^4} = 24 \quad (3.242)$$

3.4.2.2 The stiffness coefficients for the selected line continuums.

The shape functions derived from this study is exactly the same with those assumed in Ritz (weighted residual) method. The derivatives of the shape functions derived in section 3.4.2.1 will be used to determine the stiffness coefficients of the line continuums considered herein using the Euler-Bernoulli(exact) approach and Ritz (weighted residual) method for comparison of results. The results of the stiffness coefficients derived herein is tabulated on Table 3.1.

All edges simply supported, SS line continuum for Euler Bernoulli (Exact Method).

$$k_q = \int_{d_1}^{d_2} dR = \int_0^1 1dR = 1 \times \left[\frac{R}{1}\right]_0^1 = 1$$

$$k_{xx} = \int_{d_1}^{d_2} \frac{d^2w}{dR^2} dR = \int_0^1 12(R^2 - R)dR = 12 \times \left[\frac{R^3}{3} - \frac{R^2}{2}\right]_0^1 = -2$$

$$k_{xxx} = \int_{d_1}^{d_2} \frac{d^3w}{dR^3} dR = \int_0^1 12(2R - 1)dR = 12 \times \left[\frac{2R^2}{2} - \frac{R}{1}\right]_0^1 = 0$$

$$k_{xxxx} = \int_{d_1}^{d_2} \frac{d^4w}{dR^4} dR = \int_0^1 24dR = 24 \times \left[\frac{R}{1}\right]_0^1 = 24$$

All edges simply supported, SS line continuum for Ritz (Weighted Residual Method).

$$k_q = \int_{d_1}^{d_2} h\, dR = \int_0^1 (R - 2R^3 + R^4)dR = \frac{1}{2} - \frac{2}{4} + \frac{1}{5} = \frac{1}{5} = 0.2$$

$$k_{xx} = \int_{d_1}^{d_2} \left(\frac{dh}{dR}\right)^2 dR = \int_0^1 (1 - 12R^2 + 8R^3 + 36R^4 - 48R^5 + 16R^6)dR$$

$$= \left[R - 12\frac{R^3}{3} + 8\frac{R^4}{4} + 36\frac{R^5}{5} - 48\frac{R^6}{6} + 16\frac{R^7}{7}\right]_0^1$$

$$= 1 - \frac{12}{3} + \frac{8}{4} + \frac{36}{5} - \frac{48}{6} + \frac{16}{7} = \frac{17}{35} = 0.48571$$

$$k_{xxx} = \int_{d_1}^{d_2} h^2 dR = \int_0^1 (R^2 - 4R^4 + 2R^5 + 4R^6 - 4R^7 + R^8)dR$$

$$= \left[\frac{R^3}{3} - 4\frac{R^5}{5} + 2\frac{R^6}{6} + 4\frac{R^7}{7} - 4\frac{R^8}{8} + \frac{R^9}{9}\right]_0^1$$

$$= \frac{1}{3} - \frac{4}{5} + \frac{2}{6} + \frac{4}{7} - \frac{4}{8} + \frac{1}{9} = \frac{31}{630} = 0.04921$$

$$k_{xxxx} = \int_{d_1}^{d_2} \left(\frac{d^2h}{dR^2}\right)^2 dR = \int_0^1 144(R^4 - 2R^3 + R^2)dR = 144 \times \left[\frac{R^5}{5} - 2\frac{R^4}{4} + \frac{R^3}{3}\right]_0^1$$

$$= 144 \times \left[\frac{1}{5} - 2\frac{1}{4} + \frac{1}{3}\right]_0^1 = \frac{24}{5} = 4.8$$

All edges clamped line continuum, CC for Euler Bernoulli (Exact Method).

$$k_q = \int_{d_1}^{d_2} dR = \int_{0.165}^{0.835} 1dR = 1 \times \left[\frac{R}{1}\right]_{0.165}^{0.835} = 0.67$$

$$k_{xx} = \int_{d_1}^{d_2} \frac{d^2h}{dR^2} dR = \int_{0.165}^{0.835} (2 - 12R + 12R^2)dR = \left[\frac{2R}{1} - \frac{12R^2}{2} + \frac{12R^3}{3}\right]_{0.165}^{0.835}$$
$$= -0.369237$$
$$k_{xxx} = \int_{d_1}^{d_2} \frac{d^3h}{dR^3} dR = \int_{0.165}^{0.835} 12(2R - 1)dR = 12 \times \left[\frac{2R^2}{2} - \frac{R}{1}\right]_{0.165}^{0.835} = 0$$
$$k_{xxxx} = \int_{d_1}^{d_2} \frac{d^4h}{dR^4} dR = \int_{0.165}^{0.835} 24dR = 24 \times \left[\frac{R}{1}\right]_{0.165}^{0.835} = 16.08$$

All edges simply supported, CC line continuum for Ritz (Weighted Residual Method).

$$k_q = \int_{d_1}^{d_2} h\, dR = \int_0^1 (R^2 - 2R^3 + R^4)dR = \frac{1}{3} - \frac{2}{4} + \frac{1}{5} = \frac{1}{30} = 0.03333333333$$
$$k_{xx} = \int_{d_1}^{d_2} \left(\frac{dh}{dR}\right)^2 dR = \int_0^1 (4R^2 - 24R^3 + 52R^4 - 48R^5 + 16R^6)dR$$
$$= \left[4\frac{R^3}{3} - 24\frac{R^4}{4} + 52\frac{R^5}{5} - 48\frac{R^6}{6} + 16\frac{R^7}{7}\right]_0^1$$
$$= \frac{4}{3} - \frac{24}{4} + \frac{52}{5} - \frac{48}{6} + \frac{16}{7} = \frac{2}{105} = 0.01904761905$$

$$k_{xxx} = \int_{d_1}^{d_2} h^2 dR = \int_0^1 (R^4 - 4R^5 + 6R^6 - 4R^7 + R^8)dR$$
$$= \left[\frac{R^5}{5} - 4\frac{R^6}{6} + 6\frac{R^7}{7} - 4\frac{R^8}{8} + \frac{R^9}{9}\right]_0^1$$
$$= \frac{1}{5} - \frac{4}{6} + \frac{6}{7} - \frac{4}{8} + \frac{1}{9} = \frac{1}{630} = 0.0015873$$

$$k_{xxxx} = \int_{d_1}^{d_2} \left(\frac{d^2h}{dR^2}\right)^2 dR = \int_0^1 (4 - 48R + 192R^2 - 288R^3 + 144R^4)dR$$
$$= \left[4\frac{R}{1} - 48\frac{R^2}{2} + 192\frac{R^3}{3} - 288\frac{R^4}{4} + 144\frac{R^5}{5}\right]_0^1$$
$$= \frac{4}{1} - \frac{48}{2} + \frac{192}{3} - \frac{288}{4} + \frac{144}{5} = \frac{4}{5} = 0.8$$

One edge clamped, opposite edge simply supported, CS line continuum for Euler Bernoulli (Exact Method).

$$k_q = \int_{d_1}^{d_2} dR = \int_{0.165}^{0.835} 1dR = 1 \times \left[\frac{R}{1}\right]_{0.2}^{1} = 0.8$$

$$k_{xx} = \int_{d_1}^{d_2} \frac{d^2w}{dR^2} dR = \int_{0.2}^{1} 3(1 - 5R + 4R^2)dR = 3\left[\frac{R}{1} - \frac{5R^2}{2} + \frac{4R^3}{3}\right]_{0.2}^{1} = -0.832$$

$$k_{xxx} = \int_{d_1}^{d_2} \frac{d^3w}{dR^3} dR = \int_{0.2}^{1} 3(8R - 5)dR = 3 \times \left[\frac{8R^2}{2} - \frac{5R}{1}\right]_{0.2}^{1} = -0.48$$

$$k_{xxxx} = \int_{d_1}^{d_2} \frac{d^4w}{dR^4} dR = \int_{0.2}^{1} 24dR = 24 \times \left[\frac{R}{1}\right]_{0.2}^{1} = 19.2$$

One edge clamped, opposite edge simply supported, CS line continuum for Ritz (Weighted Residual Method).

$$k_q = \int_{d_1}^{d_2} h\, dR = \int_{0}^{1} (1.5R^2 - 2.5R^3 + R^4)dR = \frac{1.5}{3} - \frac{2.5}{4} + \frac{1}{5} = \frac{3}{40} = 0.075$$

$$k_{xx} = \int_{d_1}^{d_2} \left(\frac{dh}{dR}\right)^2 dR = \int_{0}^{1} (9R^2 - 45R^3 + 80.25R^4 - 60R^5 + 16R^6)dR$$

$$= \left[9\frac{R^3}{3} - 45\frac{R^4}{4} + 80.25\frac{R^5}{5} - 60\frac{R^6}{6} + 16\frac{R^7}{7}\right]_0^1$$

$$= \frac{9}{3} - \frac{45}{4} + \frac{80.25}{5} - \frac{60}{6} + \frac{16}{7} = \frac{3}{35} = 0.0857142$$

$$k_{xxx} = \int_{d_1}^{d_2} h^2 dR = \int_{0}^{1} (2.25R^4 - 7.5R^5 + 9.25R^6 - 5R^7 + R^8)dR$$

$$= \left[2.25\frac{R^5}{5} - 7.5\frac{R^6}{6} + 9.25\frac{R^7}{7} - 5\frac{R^8}{8} + \frac{R^9}{9}\right]_0^1$$

$$= \frac{2.25}{5} - \frac{7.5}{6} + \frac{9.25}{7} - \frac{5}{8} + \frac{1}{9} = \frac{19}{2520} = 0.007539682$$

$$k_{xxxx} = \int_{d_1}^{d_2} \left(\frac{d^2h}{dR^2}\right)^2 dR = \int_{0}^{1} (9 - 90R + 297R^2 - 360R^3 + 144R^4)dR$$

$$= \left[9\frac{R}{1} - 90\frac{R^2}{2} + 297\frac{R^3}{3} - 360\frac{R^4}{4} + 144\frac{R^5}{5}\right]_0^1$$

$$= \frac{9}{1} - \frac{90}{2} + \frac{297}{3} - \frac{360}{4} + \frac{144}{5} = \frac{9}{5} = 1.8$$

Table 3.1 Stiffness coefficient of the line continuums

Euler-Bernoulli Method				Ritz Method			
Kq	Kxx	Kxxx	Kxxxx	Kq	Kxx	Kxxx	Kxxxx
1	-2	0	24	$\frac{1}{5}$	$\frac{17}{35}$	$\frac{31}{630}$	$\frac{24}{5}$
0.67	-0.36924	0	16.08	$\frac{1}{30}$	$\frac{2}{105}$	$\frac{1}{630}$	$\frac{4}{5}$
0.8	-0.832	-0.48	19.2	$\frac{3}{40}$	$\frac{3}{35}$	$\frac{19}{2520}$	$\frac{9}{5}$

3.4.2.3 The coefficient of deflection, A

The equilibrium equation for a beam from first principle is given as in Equation (3.243). This is the Euler-Bernoulli equation of equilibrium and it is exact. The equation is as derived in in Equation (2.18) of section 2.4.1.

$$\int_{d_1}^{d_2} EI\frac{d^4w}{dx^4}\,dx - \int_{d_1}^{d_2} q\,dx = 0 \tag{3.243}$$

The exact general expression of the coefficient of deflection, A is as derived below.

Let R be a non-dimensional coordinate. R can be defined in terms of x and a as given in Equation (3.244). where 'a' is lateral dimension of the line continuum.

$$R = \frac{x}{a} \quad \text{and}\; x = aR \tag{3.244}$$

Substituting Equation (3.244) into Equation (3.243), yields Equation (3.245).

$$\int_{d_1}^{d_2} \frac{EI}{a^4}\frac{d^4w}{dR^4}\,dR - \int_{d_1}^{d_2} q\,dR = 0 \tag{3.245}$$

Recall, w = Ah

Therefore, Equation (3.245) can be re-written as in Equation (3.246)

$$\int_{d_1}^{d_2} \frac{EI}{a^4}\frac{d^4w}{dR^4}\,dR - \int_{d_1}^{d_2} q\,dR = 0 \tag{3.246}$$

Rearranging Equation (3.246), yields Equation (3.247),

$$\int_{d_1}^{d_2} A\frac{d^4h}{dR^4}\,dR = \int_{d_1}^{d_2} \frac{qa^4}{EI}\,dR \tag{3.247}$$

From Equation (3.247), the coefficient of deflection, A can be expressed as given in Equation (3.248).

$$A = \frac{\frac{qa^4}{EI}\int_{d_1}^{d_2} dR}{\int_{d_1}^{d_2} \frac{d^4h}{dR^4} dR} = \left(\frac{qa^4}{EI}\right) \times \frac{k_q}{k_{xxxx}} \tag{3.248}$$

The corresponding coefficient of deflection used by Ritz is as given in Equation (3.249). See Equation (2.31)

$$A = \frac{\frac{qa^4}{EI}\int_{d_1}^{d_2} hdR}{\int_{d_1}^{d_2} \left(\frac{\partial^2 h}{\partial R^2}\right)^2 R} = \frac{qa^4}{EI} \times \frac{k_q}{k_{xxxx}} \tag{3.249}$$

Equation (3.248) is the exact coefficient of deflection, A. While Equation (3.249) is an approximate coefficient of deflection, A, equation used by Ritz in weighted residual method. Equations (3.248) and (3.249) will be used respectively to determine the particular coefficient of deflection for the line continuums considered here.

Deflection coefficient, A, for an SS line continuum

Substituting the stiffness coefficients of SS line continuum from Table 3.1 for Euler-Bernoulli method and Ritz method will respectively yield the results in Equations (3.250) and (3.251)

Euler-Bernoulli method

$k_q = 1$ and $k_{xxxx} = 24$

$$\text{Therefore, } A = \frac{qa^4}{EI} \times \frac{k_q}{k_{xxxx}} = \frac{qa^4}{EI} \times \frac{1}{24} = \frac{qa^4}{24EI} \tag{3.250}$$

Ritz method (weighted residual method)

$k_q = 0.2$ and $k_{xxxx} = 4.8$

$$\text{Therefore, } A = \frac{qa^4}{EI} \times \frac{k_q}{k_{xxxx}} = \frac{qa^4}{EI} \times \frac{0.2}{4.8} = \frac{qa^4}{24EI} \tag{3.251}$$

Deflection coefficient, A for a CC line continuum

Substituting the stiffness coefficients of CC line continuum from Table 3.1 for Euler-Bernoulli method and Ritz method will respectively yield the results in Equations (3.252) and (3.253)

Euler-Bernoulli method

$\mathrm{k_q} = 0.67$ and $\mathrm{k_{xxxx}} = 16.08$

$$\mathrm{A} = \frac{\mathrm{qa^4}}{\mathrm{EI}} \times \frac{0.67}{16.08} = \frac{\mathrm{qa^4}}{24\mathrm{EI}} \tag{3.252}$$

Ritz method (weighted residual method)

$\mathrm{k_q} = \frac{1}{30}$ and $\mathrm{k_{xxxx}} = \frac{4}{5}$

$$\mathrm{A} = \frac{\mathrm{qa^4}}{\mathrm{EI}} \times \left(\frac{\frac{1}{30}}{\frac{4}{5}}\right) = \frac{\mathrm{qa^4}}{\mathrm{EI}} \times \left(\frac{1}{30} \times \frac{4}{5}\right) = \frac{\mathrm{qa^4}}{24\mathrm{EI}} \tag{3.253}$$

Deflection coefficient, A for a CS line continuum

Substituting the stiffness coefficients of CS line from Table 3.1 for Euler-Bernoulli method and Ritz method will respectively yield the results in Equations (3.254) and (3.255)

Euler-Bernoulli method

$\mathrm{k_q} = 0.8$ and $\mathrm{k_{xxxx}} = 19.2$

$$\mathrm{A} = \frac{\mathrm{qa^4}}{\mathrm{EI}} \times \frac{0.8}{19.2} = \frac{\mathrm{qa^4}}{24\mathrm{EI}} \tag{3.254}$$

Ritz method (weighted residual method)

$\mathrm{k_q} = 0.075$ and $\mathrm{k_{xxxx}} = 1.8$

$$\mathrm{A} = \frac{\mathrm{qa^4}}{\mathrm{EI}} \times \left(\frac{0.075}{1.8}\right) = \frac{\mathrm{qa^4}}{24\mathrm{EI}} \tag{3.255}$$

3.4.2.4 Bending moment and shear force of line continuums.

Using the shape functions, h and the deflection function coefficients, A for the various line continuums determined in the previous sections, pure bending analysis will be carried out in this section for the line continuums using both Euler-Bernoulli(exact) method and Ritz (weighted residual) method to determine the bending moments and shear forces acting at their end supports and mid-span.

SS line continuum

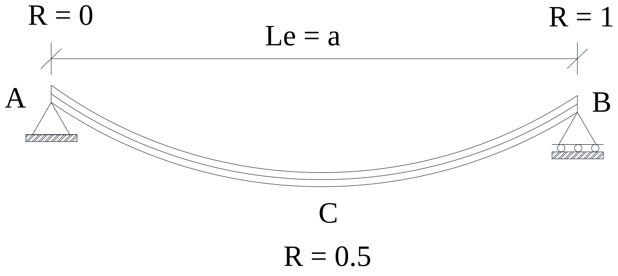

Figure 3.16 Two edges simply supported, SS line continuum

Recall,

Deflection function, $h = (R - 2R^3 + R^4)$

Bending moment function, $\frac{\partial^2 h}{\partial R^2} = 12(R^2 - R)$

Shear force function, $\frac{\partial^3 h}{\partial R^3} = 12(2R - 1)$

At point A, R = 0,

$h = (0 - 2(0)^3 + 0^4) = 0$

$\frac{\partial^2 h}{\partial R^2} = 12(R^2 - R) = 12 \times (0^2 - 0) = 0$

$\frac{\partial^3 h}{\partial R^3} = 12(2R - 1) = 12 \times (2 \times 0 - 1) = -12$

At point C, R = 0.5,

$h = (0.5 - 2(0.5)^3 + 0.5^4) = 0.3125$

$\frac{\partial^2 h}{\partial R^2} = 12(R^2 - R) = 12 \times (0.5^2 - 0.5) = -3.0$

$\frac{\partial^3 h}{\partial R^3} = 12(2R - 1) = 12 \times (2 \times 0.5 - 1) = 0$

At point B, R = 1,

$h = (1 - 2(1)^3 + 1^4) = 0$

$\frac{\partial^2 h}{\partial R^2} = 12(R^2 - R) = 12 \times (1^2 - 1) = 0$

$\frac{\partial^3 h}{\partial R^3} = 12(2R - 1) = 12 \times (2 \times 1 - 1) = 12$

Table 3.2 Summary of pure bending analysis parameters for SS line continuum

Function	Point A (R = 0)	Point C (R = 0.5)	Point B (R = 1)
h	0.0	0.3125	0.0
$\frac{\partial^2 h}{\partial R^2}$	0.0	-3.0	0.0
$\frac{\partial^3 h}{\partial R^3}$	-12	0.0	12

Maximum Deflection, w of SS line continuum

Maximum deflection will occur at the midspan of the CC line continuum were R is 0.5. At this point, shape function, h from Table 3.2 is 0.3125.

For Euler-Bernoulli method, deflection coefficient, A from Equation (3.250) is as given below.

$$A = \frac{qa^4}{24EI}$$

$$\text{Deflection}, w_{\max} = Ah = \frac{qa^4}{24EI} \times 0.3125 = \frac{5qa^4}{384EI}$$

At points, A and B, the shape functions, h is zero, hence, deflection at these points is zero.

Similarly, for Ritz method, deflection coefficient, A from Equation (3.251) is as given below.

$$A = \frac{qa^4}{24EI}$$

$$\text{Deflection}, w_{\max} = Ah = \frac{qa^4}{24EI} \times 0.3125 = \frac{5qa^4}{384EI}$$

At points, A and B, the shape functions, h is zero, hence, deflection at these points is zero.

Moment of SS line continuum, At the supports, R = 0 and 1.

General moment equation of a beam is as given below;

$$M = -\frac{EI}{qa^2}\left(\frac{\partial^2 w}{\partial R^2}\right) = -\frac{EI}{qa^2}A.\left(\frac{\partial^2 h}{\partial R^2}\right) \tag{3.256}$$

Substituting the deflection and bending moment functions for Euler Bernoulli and Ritz into Equation (3.256) will yield the value of moment at the points considered.

Euler Bernoulli method

$$A = \frac{qa^4}{24EI} \text{ and } \frac{\partial^2 h}{\partial R^2} = 0$$

$$M = -\frac{EI}{qa^2} \times \frac{qa^4}{24EI} \times 0 = 0$$

Ritz Method (weighted residual method)

$$A = \frac{qa^4}{24EI} \text{ and } \frac{\partial^2 h}{\partial R^2} = 0$$

$$M = -\frac{EI}{qa^2}\left(\frac{\partial^2 w}{\partial R^2}\right) = -\frac{EI}{qa^2} A.\left(\frac{\partial^2 h}{\partial R^2}\right)\frac{EI}{qa^2} \times \frac{qa^4}{24EI} \times 0 = 0$$

Moment of SS line continuum, At the center, R = 0.5

Euler Bernoulli method

$$A = \frac{qa^4}{24EI} \text{ and } \frac{\partial^2 h}{\partial R^2} = -3$$

$$M = -\frac{EI}{qa^2}\left(\frac{\partial^2 w}{\partial R^2}\right) = -\frac{EI}{qa^2} A.\left(\frac{\partial^2 h}{\partial R^2}\right) = -\frac{EI}{qa^2} \times \frac{0.67qa^4}{16.08EI} \times -3 = \frac{qa^2}{8EI}$$

Ritz Method (weighted residual method)

$$A = \frac{qa^4}{24EI} \text{ and } \frac{\partial^2 h}{\partial R^2} = -3$$

$$M = -\frac{EI}{qa^2}\left(\frac{\partial^2 w}{\partial R^2}\right) = -\frac{EI}{qa^2} A.\left(\frac{\partial^2 h}{\partial R^2}\right)$$

$$M = -\frac{EI}{qa^2} \times \frac{qa^4}{24EI} \times -3 = \frac{qa^2}{8EI}$$

Shear force of SS line continuum, At point A, R = 0

General shear force equation of a beam is as given below;

$$V = -\frac{EI}{qa^3}\left(\frac{\partial^3 w}{\partial R^3}\right) = -\frac{EI}{qa^3} A.\left(\frac{\partial^3 h}{\partial R^3}\right) \tag{3.257}$$

Substituting the deflection and shear force functions for Euler Bernoulli and Ritz into Equation (3.257) will yield the value of shear force at the points considered.

Euler Bernoulli method

$$\mathrm{A} = \frac{\mathrm{qa}^4}{24\mathrm{EI}} \text{ and } \frac{\partial^3 \mathrm{h}}{\partial \mathrm{R}^3} = -12$$

$$\mathrm{V} = -\frac{\mathrm{EI}}{\mathrm{qa}^3}\left(\frac{\partial^3 \mathrm{w}}{\partial \mathrm{R}^3}\right) = -\frac{\mathrm{EI}}{\mathrm{qa}^3} \times \frac{\mathrm{qa}^4}{24\mathrm{EI}} \times -12 = \frac{\mathrm{qa}}{2\mathrm{EI}}$$

Ritz Method (weighted residual method)

$$\mathrm{A} = \frac{\mathrm{qa}^4}{24\mathrm{EI}} \text{ and } \frac{\partial^3 \mathrm{h}}{\partial \mathrm{R}^3} = -12$$

General shear force equation of a beam is as given below;

$$\mathrm{V} = -\frac{\mathrm{EI}}{\mathrm{qa}^3}\left(\frac{\partial^3 \mathrm{w}}{\partial \mathrm{R}^3}\right) = -\frac{\mathrm{EI}}{\mathrm{qa}^3}\mathrm{A}.\left(\frac{\partial^3 \mathrm{h}}{\partial \mathrm{R}^3}\right)$$

$$\mathrm{V} = -\frac{\mathrm{EI}}{\mathrm{qa}^3}\left(\frac{\partial^3 \mathrm{w}}{\partial \mathrm{R}^3}\right) = -\frac{\mathrm{EI}}{\mathrm{qa}^3} \times \frac{\mathrm{qa}^4}{24\mathrm{EI}} \times -12 = \frac{\mathrm{qa}}{2\mathrm{EI}}$$

Shear force of SS line continuum, At point B, R = 1

Euler Bernoulli method

$$\mathrm{A} = \frac{\mathrm{qa}^4}{24\mathrm{EI}} \text{ and } \frac{\partial^3 \mathrm{h}}{\partial \mathrm{R}^3} = 12$$

General shear force equation of a beam is as given below;

$$\mathrm{V} = -\frac{\mathrm{EI}}{\mathrm{qa}^3}\left(\frac{\partial^3 \mathrm{w}}{\partial \mathrm{R}^3}\right) = -\frac{\mathrm{EI}}{\mathrm{qa}^3}\mathrm{A}.\left(\frac{\partial^3 \mathrm{h}}{\partial \mathrm{R}^3}\right)$$

$$\mathrm{V} = -\frac{\mathrm{EI}}{\mathrm{qa}^3}\left(\frac{\partial^3 \mathrm{w}}{\partial \mathrm{R}^3}\right) = -\frac{\mathrm{EI}}{\mathrm{qa}^3} \times \frac{\mathrm{qa}^4}{24\mathrm{EI}} \times 12 = -\frac{\mathrm{qa}}{2\mathrm{EI}}$$

Ritz Method (weighted residual method)

$$\mathrm{A} = \frac{\mathrm{qa}^4}{24\mathrm{EI}} \text{ and } \frac{\partial^3 \mathrm{h}}{\partial \mathrm{R}^3} = 12$$

General shear force equation of a beam is as given below;

$$\mathrm{V} = -\frac{\mathrm{EI}}{\mathrm{qa}^3}\left(\frac{\partial^3 \mathrm{w}}{\partial \mathrm{R}^3}\right) = -\frac{\mathrm{EI}}{\mathrm{qa}^3}\mathrm{A}.\left(\frac{\partial^3 \mathrm{h}}{\partial \mathrm{R}^3}\right)$$

$$\mathrm{V} = -\frac{\mathrm{EI}}{\mathrm{qa}^3}\left(\frac{\partial^3 \mathrm{w}}{\partial \mathrm{R}^3}\right) = -\frac{\mathrm{EI}}{\mathrm{qa}^3} \times \frac{\mathrm{qa}^4}{24\mathrm{EI}} \times 12 = -\frac{\mathrm{qa}}{2\mathrm{EI}}$$

Shear force of SS line continuum, At point C, R = 0.5

Euler-Bernoulli method

$$\mathrm{A}=\frac{\mathrm{qa}^4}{24\mathrm{EI}}\ \text{and}\ \frac{\partial^3\mathrm{h}}{\partial\mathrm{R}^3}=0.0$$

General shear force equation of a beam is as given below;

$$\mathrm{V}=-\frac{\mathrm{EI}}{\mathrm{qa}^3}\left(\frac{\partial^3\mathrm{w}}{\partial\mathrm{R}^3}\right)=-\frac{\mathrm{EI}}{\mathrm{qa}^3}\mathrm{A}.\left(\frac{\partial^3\mathrm{h}}{\partial\mathrm{R}^3}\right)$$

$$\mathrm{V}=-\frac{\mathrm{EI}}{\mathrm{qa}^3}\left(\frac{\partial^3\mathrm{w}}{\partial\mathrm{R}^3}\right)=-\frac{\mathrm{EI}}{\mathrm{qa}^3}\times\frac{\mathrm{qa}^4}{24\mathrm{EI}}\times 0.0=0$$

Ritz Method (weighted residual method)

$$\mathrm{A}=\frac{\mathrm{qa}^4}{24\mathrm{EI}}\ \text{and}\ \frac{\partial^3\mathrm{h}}{\partial\mathrm{R}^3}=0.0$$

General shear force equation of a beam is as given below;

$$\mathrm{V}=-\frac{\mathrm{EI}}{\mathrm{qa}^3}\left(\frac{\partial^3\mathrm{w}}{\partial\mathrm{R}^3}\right)=-\frac{\mathrm{EI}}{\mathrm{qa}^3}\mathrm{A}.\left(\frac{\partial^3\mathrm{h}}{\partial\mathrm{R}^3}\right)$$

$$\mathrm{V}=-\frac{\mathrm{EI}}{\mathrm{qa}^3}\left(\frac{\partial^3\mathrm{w}}{\partial\mathrm{R}^3}\right)=-\frac{\mathrm{EI}}{\mathrm{qa}^3}\times\frac{\mathrm{qa}^4}{24\mathrm{EI}}\times 0.0=0$$

Table 3.3 Summary of SS beam analysis results for Euler-Bernoulli (exact) method

Function	Point A (R = 0)	Point C (R = 0.5)	Point B (R = 1)
Deflection	0.0	$\frac{5\mathrm{qa}^4}{3848\mathrm{EI}}$	0.0
Moment	0	$\frac{\mathrm{qa}^2}{8\mathrm{EI}}$	0
Shear Force	$\frac{\mathrm{qa}}{2\mathrm{EI}}$	0	$-\frac{\mathrm{qa}}{2\mathrm{EI}}$

Table 3.4 Summary of SS beam analysis results for Ritz (weighted residual) method

Function	Point A (R = 0)	Point C (R = 0.5)	Point B (R = 1)
Deflection	0.0	$\frac{5\mathrm{qa}^4}{3848\mathrm{EI}}$	0.0
Moment	0	$-\frac{\mathrm{qa}^2}{8\mathrm{EI}}$	0
Shear Force	$-\frac{\mathrm{qa}}{2\mathrm{EI}}$	0	$\frac{\mathrm{qa}}{2\mathrm{EI}}$

CC line continuum

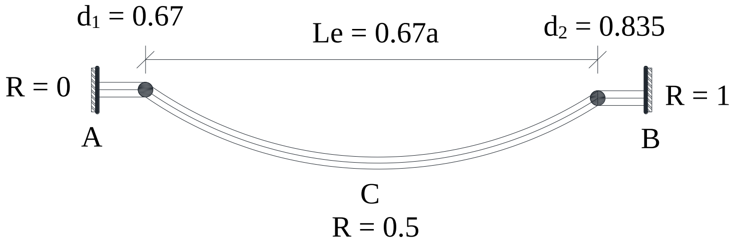

Figure 3.17 Two edges clamped, CC line continuum

Recall,

Deflection function, $h = (R^2 - 2R^3 + R^4)$

Moment function, $\frac{d^2h}{dR^2} = \ 2 - 12R + 12R^2$

Shear force function, $\frac{d^3h}{dR^3} = \ 12(2R - 1)$

At point A, R = 0,

$$h = (0 - 2(0)^3 + 0^4) = 0$$
$$\frac{\partial^2 h}{\partial R^2} = 2 - 12R + 12R^2 = 2 - 12 \times 0 \ + 12 \times 0^2 = 2$$
$$\frac{\partial^3 h}{\partial R^3} = 12(2R - 1) = 12 \times (2 \times 0 - 1) = -12$$

At point C, R = 0.5,

$$h = (R^2 - 2R^3 + R^4) = (0.5^2 - 2(0.5)^3 + 0.5^4) = 0.0625$$
$$\frac{\partial^2 h}{\partial R^2} = 2 - 12R + 12R^2 = 2 - 12 \times 0.5 \ + 12 \times 0.5^2 = -1.0$$
$$\frac{\partial^3 h}{\partial R^3} = 12(2R - 1) = 12 \times (2 \times 0.5 - 1) = 0$$

At point B, R = 1,

$$h = (1^2 - 2(1)^3 + 1^4) = 0$$
$$\frac{\partial^2 h}{\partial R^2} = 2 - 12R + 12R^2 = 2 - 12 \times 1 \ + 12 \times 1^2 = 2$$
$$\frac{\partial^3 h}{\partial R^3} = 12(2R - 1) = 12 \times (2 \times 1 - 1) = 12$$

Table 3.5 Summary of pure bending analysis parameters of CC line continuum

Function	Point A (R = 0)	Point C (R = 0.5)	Point B (R = 1)
h	0.0	0.0625	0.0
$\frac{\partial^2 h}{\partial R^2}$	2.0	-1.0	2.0
$\frac{\partial^3 h}{\partial R^3}$	-12	0.0	12

Maximum Deflection, w of CC line continuum

Maximum deflection will occur at the midspan of the CC line continuum were R is 0.5. At this point, shape function, h from Table 3.5 is 0.0625.

For Euler-Bernoulli method, deflection coefficient, A from Equation (3.252) is as given below.

$$A = \frac{qa^4}{24EI}$$

$$\text{Deflection}, w_{\max} = Ah = \frac{qa^4}{24EI} \times 0.0625 = \frac{qa^4}{384EI}$$

At points, A and B, the shape functions, h is zero, hence, deflection at these points is zero.

Similarly, for Ritz (weighted residual) method, deflection coefficient, A from Equation (3.253) is as given below.

$$A = \frac{qa^4}{24EI}$$

$$\text{Deflection}, w_{\max} = Ah = \frac{qa^4}{24EI} \times 0.0625 = \frac{qa^4}{384EI}$$

At points, A and B, the shape functions, h is zero, hence, deflection at these points is zero.

Moment of CC line continuum, At the supports, R = 0 and 1.

Euler Bernoulli method

$$A = \frac{qa^4}{24EI} \text{ and } \frac{\partial^2 h}{\partial R^2} = 2$$

General moment equation of a beam is as given below;

$$M = -\frac{EI}{qa^2}\left(\frac{\partial^2 w}{\partial R^2}\right) = -\frac{EI}{qa^2}A.\left(\frac{\partial^2 h}{\partial R^2}\right)$$

$$M = -\frac{EI}{qa^2} \times \frac{qa^4}{24EI} \times 2 = -\frac{qa^2}{12EI}$$

Ritz (weighted residual) method

$$A = \frac{qa^4}{24EI} \text{ and } \frac{\partial^2 h}{\partial R^2} = 2$$

General moment equation of a beam is as given below;

$$M = -\frac{EI}{qa^2}\left(\frac{\partial^2 w}{\partial R^2}\right) = -\frac{EI}{qL^2}A.\left(\frac{\partial^2 h}{\partial R^2}\right)$$

$$M = -\frac{EI}{qa^2} \times \frac{qa^4}{24EI} \times 2 = -\frac{qa^2}{12EI}$$

Moment of CC line continuum, At the center, R = 0.5

Euler Bernoulli (exact) method

$$A = \frac{qa^4}{24EI} \text{ and } \frac{\partial^2 h}{\partial R^2} = -1$$

$$M = -\frac{EI}{qa^2}\left(\frac{\partial^2 w}{\partial R^2}\right) = -\frac{EI}{qa^2}A.\left(\frac{\partial^2 h}{\partial R^2}\right)$$

$$M = -\frac{EI}{qa^2} \times \frac{qa^4}{24EI} \times -1 = \frac{qa^2}{24EI}$$

Ritz (weighted residual) method

$$A = \frac{qa^4}{24EI} \text{ and } \frac{\partial^2 h}{\partial R^2} = -1$$

$$M = -\frac{EI}{qa^2}\left(\frac{\partial^2 w}{\partial R^2}\right) = -\frac{EI}{qa^2}A.\left(\frac{\partial^2 h}{\partial R^2}\right)$$

$$M = -\frac{EI}{qa^2} \times \frac{qa^4}{24EI} \times -1 = \frac{qa^2}{24EI}$$

Shear force of CC line continuum, At point A, R = 0

Euler Bernoulli (exact) method

$$A = \frac{qa^4}{24EI} \text{ and } \frac{\partial^3 h}{\partial R^3} = -12$$

General shear force equation of a beam is as given below;

$$V = -\frac{EI}{qa^3}\left(\frac{\partial^3 w}{\partial R^3}\right) = -\frac{EI}{qa^3}A.\left(\frac{\partial^3 h}{\partial R^3}\right)$$

$$V = -\frac{EI}{qa^3}\left(\frac{\partial^3 w}{\partial R^3}\right) = -\frac{EI}{qa^3} \times \frac{qa^4}{24EI} \times -12 = \frac{qa}{2EI}$$

Ritz (weighted residual) method

$$A = \frac{qa^4}{24EI} \text{ and } \frac{\partial^3 h}{\partial R^3} = -12$$

General shear force equation of a beam is as given below;

$$V = -\frac{EI}{qa^3}\left(\frac{\partial^3 w}{\partial R^3}\right) = -\frac{EI}{qa^3}A.\left(\frac{\partial^3 h}{\partial R^3}\right)$$

$$V = -\frac{EI}{qa^3}\left(\frac{\partial^3 w}{\partial R^3}\right) = -\frac{EI}{qa^3} \times \frac{qa^4}{24EI} \times -12 = \frac{qa}{2EI}$$

Shear force of CC line continuum, At point B, R = 1

Euler Bernoulli (exact) method

$$A = \frac{qa^4}{24EI} \text{ and } \frac{\partial^3 h}{\partial R^3} = 12$$

General shear force equation of a beam is as given below;

$$V = -\frac{EI}{qa^3}\left(\frac{\partial^3 w}{\partial R^3}\right) = -\frac{EI}{qa^3}A.\left(\frac{\partial^3 h}{\partial R^3}\right)$$

$$V = -\frac{EI}{qa^3}\left(\frac{\partial^3 w}{\partial R^3}\right) = -\frac{EI}{qa^3} \times \frac{qa^4}{24EI} \times 12 = -\frac{qa}{2EI}$$

Ritz (weighted residual) method

$$A = \frac{qa^4}{24EI} \text{ and } \frac{\partial^3 h}{\partial R^3} = 12$$

General shear force equation of a beam is as given below;

$$V = -\frac{EI}{qa^3}\left(\frac{\partial^3 w}{\partial R^3}\right) = -\frac{EI}{qa^3}A.\left(\frac{\partial^3 h}{\partial R^3}\right)$$

$$V = -\frac{EI}{qa^3}\left(\frac{\partial^3 w}{\partial R^3}\right) = -\frac{EI}{qa^3} \times \frac{qa^4}{24EI} \times 12 = -\frac{qa}{2EI}$$

Shear force of CC line continuum, At point B, R = 1

Euler Bernoulli (exact) method

$$A = \frac{qa^4}{24EI} \text{ and } \frac{\partial^3 h}{\partial R^3} = 12$$

General shear force equation of a beam is as given below;

$$V = -\frac{EI}{qa^3}\left(\frac{\partial^3 w}{\partial R^3}\right) = -\frac{EI}{qa^3}A.\left(\frac{\partial^3 h}{\partial R^3}\right)$$

$$V = -\frac{EI}{qa^3}\left(\frac{\partial^3 w}{\partial R^3}\right) = -\frac{EI}{qa^3} \times \frac{qa^4}{24EI} \times 0 = 0$$

Ritz (weighted residual) method

$A = \frac{qa^4}{24EI}$ and $\frac{\partial^3 h}{\partial R^3} = 12$

General shear force equation of a beam is as given below;

$$V = -\frac{EI}{qa^3}\left(\frac{\partial^3 w}{\partial R^3}\right) = -\frac{EI}{qa^3} A.\left(\frac{\partial^3 h}{\partial R^3}\right)$$

$$V = -\frac{EI}{qa^3}\left(\frac{\partial^3 w}{\partial R^3}\right) = -\frac{EI}{qa^3} \times \frac{qa^4}{24EI} \times 0 = 0$$

Table 3.6 Summary of CC beam analysis results for Euler-Bernoulli (exact) method

Function	Point A (R = 0)	Point C (R = 0.5)	Point B (R = 1)
Deflection	0.0	$\frac{qa^4}{384EI}$	0.0
Moment	$-\frac{qa^2}{12EI}$	$\frac{qa^2}{24EI}$	$-\frac{qa^2}{12EI}$
Shear Force	$\frac{qa}{2EI}$	0.0	$-\frac{qa}{2EI}$

Table 3.7 Summary of CC beam analysis results for Ritz (weighted residual) method

Function	Point A (R = 0)	Point C (R = 0.5)	Point B (R = 1)
Deflection	0.0	$\frac{qa^4}{384EI}$	0.0
Moment	$-\frac{qa^2}{12EI}$	$\frac{qa^2}{24EI}$	$-\frac{qa^2}{12EI}$
Shear Force	$\frac{qa}{2EI}$	0.0	$-\frac{qa}{2EI}$

CS line continuum

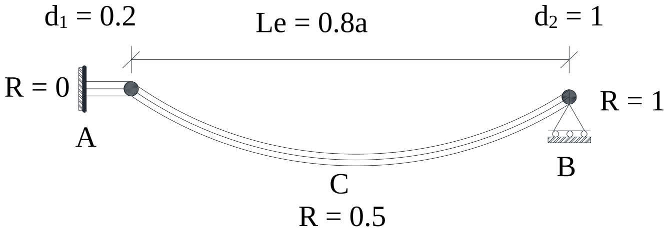

Figure 3.18 One edge clamped, opposite edge simply supported, CS line continuum

Recall,

Deflection finction, $h = (1.5R^2 - 2.5R^3 + R^4)$

Bending moment function, $\frac{\partial^2 h}{\partial R^2} = 3(1 - 5R + 4R^2)$

Shear force function, $\frac{\partial^3 h}{\partial R^3} = 3(8R - 5)$

At point A, R = 0,

$$h = 1.5R^2 - 2.5R^3 + R^4 = (1.5(0) - 2.5(0)^3 + 0^4) = 0$$

$$\frac{\partial^2 h}{\partial R^2} = 3(1 - 5R + 4R^2) = 3 \times (1 - 5(0) + 4(0)^2) = 3$$

$$\frac{\partial^3 h}{\partial R^3} = 3(8R - 5) = 3 \times (8 \times 0 - 5) = -15$$

At point C, R = 0.5,

$$h = 1.5R^2 - 2.5R^3 + R^4 = (1.5(0.5) - 2.5(0.5)^3 + 0.5^4) = 0.5$$

$$\frac{\partial^2 h}{\partial R^2} = 3(1 - 5R + 4R^2) = 3 \times (1 - 5(0.5) + 4(0.5)^2) = -1.5$$

$$\frac{\partial^3 h}{\partial R^3} = 3(8R - 5) = 3 \times (8 \times 0.5 - 5) = -3$$

At point B, R = 1,

$$h = 1.5R^2 - 2.5R^3 + R^4 = (1.5(1) - 2.5(1)^3 + 1^4) = 0$$

$$\frac{\partial^2 h}{\partial R^2} = 3(1 - 5R + 4R^2) = 3 \times (1 - 5(1) + 4(1)^2) = 0$$

$$\frac{\partial^3 h}{\partial R^3} = 3(8R - 5) = 3 \times (8 \times 1 - 5) = 9$$

Table 3.8 Summary of pure bending analysis parameters for CS line continuum

Function	Point A (R = 0)	Point C (R = 0.5)	Point B (R = 1)
h	0.0	0.5	0.0
$\frac{\partial^2 h}{\partial R^2}$	3	-1.5	0
$\frac{\partial^3 h}{\partial R^3}$	-15	-3.0	9.0

Maximum Deflection, w of CS line continuum

Maximum deflection will occur at the midspan of the CS line continuum were R is 0.5. At this point, shape function, h from Table 3.8 is 0.0625.

For Euler-Bernoulli method, deflection coefficient, A from Equation (3.254) is as given below.

$$A = \frac{qa^4}{24EI}$$

$$\text{Deflection}, w_{max} = Ah = \frac{qa^4}{24EI} \times 0.5 = \frac{qa^4}{48EI}$$

At points, A and B, the shape functions, h is zero, hence, deflection at these points is zero.

Similarly, for Ritz (weighted residual) method, deflection coefficient, A from Equation (3.255) is as given below.

$$A = \frac{qa^4}{24EI}$$

$$\text{Deflection}, w_{max} = Ah = \frac{qa^4}{24EI} \times 0.5 = \frac{qa^4}{48EI}$$

At points, A and B, the shape functions, h is zero, hence, deflection at these points is zero.

Moment of CS line continuum, At point A, R = 0

Euler-Bernoulli (exact) method

$$A = \frac{qa^4}{24EI} \text{ and } \frac{\partial^2 h}{\partial R^2} = 3$$

General moment equation of a beam is as given below;

$$M = -\frac{EI}{qa^2}\left(\frac{\partial^2 w}{\partial R^2}\right) = -\frac{EI}{qa^2}A.\left(\frac{\partial^2 h}{\partial R^2}\right)$$

$$M = -\frac{EI}{qa^2} \times \frac{qa^4}{24EI} \times 3 = -\frac{qa^2}{8EI}$$

Ritz (weighted residual) method

$$\mathrm{A} = \frac{\mathrm{qa}^4}{24\mathrm{EI}} \text{ and } \frac{\partial^2 \mathrm{h}}{\partial \mathrm{R}^2} = 3$$

General moment equation of a beam is as given below;

$$\mathrm{M} = -\frac{\mathrm{EI}}{\mathrm{qa}^2}\left(\frac{\partial^2 \mathrm{w}}{\partial \mathrm{R}^2}\right) = -\frac{\mathrm{EI}}{\mathrm{qa}^2}\mathrm{A.}\left(\frac{\partial^2 \mathrm{h}}{\partial \mathrm{R}^2}\right)$$

$$\mathrm{M} = -\frac{\mathrm{EI}}{\mathrm{qa}^2} \times \frac{\mathrm{qa}^4}{24\mathrm{EI}} \times 3 = -\frac{\mathrm{qa}^2}{8\mathrm{EI}}$$

Moment of CS line continuum, At the center, C, R = 0.5

Euler-Bernoulli (exact) method

$$\mathrm{A} = \frac{\mathrm{qa}^4}{24\mathrm{EI}} \text{ and } \frac{\partial^2 \mathrm{h}}{\partial \mathrm{R}^2} = -1.5$$

$$\mathrm{M} = -\frac{\mathrm{EI}}{\mathrm{qa}^2}\left(\frac{\partial^2 \mathrm{w}}{\partial \mathrm{R}^2}\right) = -\frac{\mathrm{EI}}{\mathrm{qa}^2}\mathrm{A.}\left(\frac{\partial^2 \mathrm{h}}{\partial \mathrm{R}^2}\right)$$

$$\mathrm{M} = -\frac{\mathrm{EI}}{\mathrm{qa}^2} \times \frac{\mathrm{qa}^4}{24\mathrm{EI}} \times = -1.5 = \frac{\mathrm{qa}^2}{16\mathrm{EI}}$$

Ritz (weighted residual) method

$$\mathrm{A} = \frac{\mathrm{qa}^4}{24\mathrm{EI}} \text{ and } \frac{\partial^2 \mathrm{h}}{\partial \mathrm{R}^2} = -1.5$$

$$\mathrm{M} = -\frac{\mathrm{EI}}{\mathrm{qa}^2}\left(\frac{\partial^2 \mathrm{w}}{\partial \mathrm{R}^2}\right) = -\frac{\mathrm{EI}}{\mathrm{qa}^2}\mathrm{A.}\left(\frac{\partial^2 \mathrm{h}}{\partial \mathrm{R}^2}\right)$$

$$\mathrm{M} = -\frac{\mathrm{EI}}{\mathrm{qa}^2} \times \frac{\mathrm{qa}^4}{24\mathrm{EI}} \times = -1.5 = \frac{\mathrm{qa}^2}{16\mathrm{EI}}$$

Moment of CS line continuum, At point B, R = 1

Euler-Bernoulli (exact) method

$$\mathrm{A} = \frac{\mathrm{qa}^4}{24\mathrm{EI}} \text{ and } \frac{\partial^2 \mathrm{h}}{\partial \mathrm{R}^2} = 0$$

$$\mathrm{M} = -\frac{\mathrm{EI}}{\mathrm{qa}^2}\left(\frac{\partial^2 \mathrm{w}}{\partial \mathrm{R}^2}\right) = -\frac{\mathrm{EI}}{\mathrm{qa}^2}\mathrm{A.}\left(\frac{\partial^2 \mathrm{h}}{\partial \mathrm{R}^2}\right)$$

$$\mathrm{M} = -\frac{\mathrm{EI}}{\mathrm{qa}^2} \times \frac{\mathrm{qa}^4}{24\mathrm{EI}} \times 0 = 0$$

Ritz (weighted residual) method

$$A = \frac{qa^4}{24EI} \text{ and } \frac{\partial^2 h}{\partial R^2} = 0$$

$$M = -\frac{EI}{qa^2}\left(\frac{\partial^2 w}{\partial R^2}\right) = -\frac{EI}{qa^2} A.\left(\frac{\partial^2 h}{\partial R^2}\right)$$

$$M = -\frac{EI}{qa^2} \times \frac{qa^4}{24EI} \times 0 = 0$$

Shear force of CS line continuum, At point A, R = 0

Euler-Bernoulli (exact) method

$$A = \frac{qa^4}{24EI} \text{ and } \frac{\partial^3 h}{\partial R^3} = -15$$

General shear force equation of a beam is as given below;

$$V = -\frac{EI}{qa^3}\left(\frac{\partial^3 w}{\partial R^3}\right) = -\frac{EI}{qa^3} A.\left(\frac{\partial^3 h}{\partial R^3}\right)$$

$$V = -\frac{EI}{qa^3}\left(\frac{\partial^3 w}{\partial R^3}\right) = -\frac{EI}{qa^3} \times \frac{qa^4}{24EI} \times -15 = \frac{5qa}{8EI}$$

Ritz (weighted residual) method

$$A = \frac{qa^4}{24EI} \text{ and } \frac{\partial^3 h}{\partial R^3} = -15$$

General shear force equation of a beam is as given below;

$$V = -\frac{1}{qa^3}\left(\frac{\partial^3 w}{\partial R^3}\right) = -\frac{1}{qa^3} A.\left(\frac{\partial^3 h}{\partial R^3}\right)$$

$$V = -\frac{EI}{qa^3}\left(\frac{\partial^3 w}{\partial R^3}\right) = -\frac{1}{qa^3} \times \frac{qa^4}{24EI} \times -15 = \frac{5qa}{8EI}$$

Shear force of CS line continuum, At point B, R = 1

Euler-Bernoulli (exact) method

$$A = \frac{qa^4}{24EI} \text{ and } \frac{\partial^3 h}{\partial R^3} = 9$$

General shear force equation of a beam is as given below;

$$V = -\frac{1}{qa^3}\left(\frac{\partial^3 w}{\partial R^3}\right) = -\frac{1}{qa^3} A.\left(\frac{\partial^3 h}{\partial R^3}\right)$$

$$V = -\frac{1}{qa^3}\left(\frac{\partial^3 w}{\partial R^3}\right) = -\frac{1}{qa^3} \times \frac{qa^4}{24EI} \times 9 = -\frac{3qa}{8EI}$$

Ritz (weighted residual) method

$$A = \frac{qa^4}{24EI} \text{ and } \frac{\partial^3 h}{\partial R^3} = 9$$

General shear force equation of a beam is as given below;

$$V = -\frac{1}{qa^3}\left(\frac{\partial^3 w}{\partial R^3}\right) = -\frac{1}{qa^3} A.\left(\frac{\partial^3 h}{\partial R^3}\right)$$

$$V = -\frac{1}{qa^3}\left(\frac{\partial^3 w}{\partial R^3}\right) = -\frac{1}{qa^3} \times \frac{qa^4}{24EI} \times 9 = -\frac{3qa}{8EI}$$

Shear force of CS line continuum, At point C, R = 0.5

Euler-Bernoulli (exact) method

$$A = \frac{qa^4}{24EI} \text{ and } \frac{\partial^3 h}{\partial R^3} = -3.0$$

General shear force equation of a beam is as given below;

$$V = -\frac{1}{qa^3}\left(\frac{\partial^3 w}{\partial R^3}\right) = -\frac{1}{qa^3} A.\left(\frac{\partial^3 h}{\partial R^3}\right)$$

$$V = -\frac{1}{qa^3}\left(\frac{\partial^3 w}{\partial R^3}\right) = -\frac{EI}{qa^3} \times \frac{qa^4}{24EI} \times -3.0 = \frac{qa}{8EI}$$

Ritz (weighted residual) method

$$A = \frac{qa^4}{24EI} \text{ and } \frac{\partial^3 h}{\partial R^3} = -3.0$$

General shear force equation of a beam is as given below;

$$V = -\frac{EI}{qa^3}\left(\frac{\partial^3 w}{\partial R^3}\right) = -\frac{1}{qa^3} A.\left(\frac{\partial^3 h}{\partial R^3}\right)$$

$$V = -\frac{1}{qa^3}\left(\frac{\partial^3 w}{\partial R^3}\right) = -\frac{1}{qa^3} \times \frac{qa^4}{24EI} \times -3.0 = \frac{qa}{8EI}$$

Table 3.9 Summary of CS beam analysis results for Euler Bernoulli (exact) method

Function	Point A (R = 0)	Point C (R = 0.5)	Point B (R = 1)
Deflection	0.0	$\frac{qa^4}{48EI}$	0.0
Moment	$-\frac{qa^2}{8EI}$	$\frac{qa^2}{16EI}$	0
Shear Force	$\frac{5qa}{8EI}$	$-\frac{3qa}{8EI}$	$\frac{qa}{8EI}$

Table 3.10 Summary of CS beam analysis results for Ritz (weighted residual) method

Function	Point A (R = 0)	Point C (R = 0.5)	Point B (R = 1)
Deflection	0.0	$\frac{qa^4}{48EI}$	0.0
Moment	$-\frac{qa^2}{8EI}$	$\frac{qa^2}{16EI}$	0
Shear Force	$\frac{5qa}{8EI}$	$-\frac{3qa}{8EI}$	$\frac{qa}{8EI}$

3.4.3 Pure bending analysis of thin plates

3.4.3.1 The derivatives of the shape functions

The shape functions of the various thin plates considered in this study were successfully determined in section 3.3. The derivatives of the shape functions will be the focus of this section.

The derivatives of SSSS plate type

Equation (3.258) through Equation (3.269) are the derivatives of the SSSS plate type polynomial deflection equation, w. Recall Equation (3.197).

$$w = A(R - 2R^3 + R^4)(Q - 2Q^3 + Q^4)$$

$$\frac{\partial w}{\partial R} = A(1 - 6R^2 + 4R^3)(Q - 2Q^3 + Q^4) \qquad (3.258)$$

$$\frac{\partial^2 w}{\partial R^2} = 12A(R^2 - R)(Q - 2Q^3 + Q^4) \qquad (3.259)$$

$$\frac{\partial^3 w}{\partial R^3} = 12A(2R - 1)(Q - 2Q^3 + Q^4) \qquad (3.260)$$

$$\frac{\partial^4 w}{\partial R^4} = 24A(Q - 2Q^3 + Q^4) \quad (3.261)$$

$$\frac{\partial w}{\partial Q} = A(1 - 6Q^2 + 4Q^3)(R - 2R^3 + R^4) \quad (3.262)$$

$$\frac{\partial^2 w}{\partial Q^2} = 12A(Q^2 - Q)(R - 2R^3 + R^4) \quad (3.263)$$

$$\frac{\partial^3 w}{\partial Q^3} = 12A(2Q - 1)(R - 2R^3 + R^4) \quad (3.264)$$

$$\frac{\partial^4 w}{\partial Q^4} = 24A(R - 2R^3 + R^4) \quad (3.265)$$

$$\frac{\partial^2 w}{\partial R\,\partial Q} = A(1 - 6R^2 + 4R^3)(1 - 6Q^2 + 4Q^3) \quad (3.266)$$

$$\frac{\partial^3 w}{\partial R^2\,\partial Q} = 12A(R^2 - R)(1 - 6Q^2 + 4Q^3) \quad (3.267)$$

$$\frac{\partial^3 w}{\partial R\,\partial Q^2} = 12A(1 - 6R^2 + 4R^3)(Q^2 - Q) \quad (3.268)$$

$$\frac{\partial^4 w}{\partial R^2\,\partial Q^2} = 144A(R^2 - R)(Q^2 - Q) \quad (3.269)$$

The derivatives of the CCCC deflection equation

Equation (3.270) through Equation (3.281) are the derivatives of the CCCC plate type polynomial deflection equation, w. Recall Equation (3.198).

$$w = A(R^2 - 2R^3 + R^4)(Q^2 - 2Q^3 + Q^4)$$

$$\frac{\partial w}{\partial R} = A(2R - 6R^2 + 4R^3)(Q^2 - 2Q^3 + Q^4) \quad (3.270)$$

$$\frac{\partial^2 w}{\partial R^2} = A(2 - 12R + 12R^2)(Q^2 - 2Q^3 + Q^4) \quad (3.271)$$

$$\frac{\partial^3 w}{\partial R^3} = 12A(2R - 1)(Q^2 - 2Q^3 + Q^4) \quad (3.272)$$

$$\frac{\partial^4 w}{\partial R^4} = 24A(Q^2 - 2Q^3 + Q^4) \quad (3.273)$$

$$\frac{\partial w}{\partial Q} = A(R^2 - 2R^3 + R^4)(2Q - 6Q^2 + 4Q^3) \quad (3.274)$$

$$\frac{\partial^2 w}{\partial Q^2} = A(R^2 - 2R^3 + R^4)(2 - 12Q + 12Q^2) \quad (3.275)$$

$$\frac{\partial^3 w}{\partial Q^3} = 12A(R^2 - 2R^3 + R^4)(2Q - 1) \quad (3.276)$$

$$\frac{\partial^4 w}{\partial Q^4} = 24A(R^2 - 2R^3 + R^4) \tag{3.277}$$

$$\frac{\partial^2 w}{\partial R\,\partial Q} = A(2R - 6R^2 + 4R^3)(2Q - 6Q^2 + 4Q^3) \tag{3.278}$$

$$\frac{\partial^3 w}{\partial R^2\,\partial Q} = 2A(1 - 6R + 6R^2)(Q - 3Q^2 + 2Q^3) \tag{3.279}$$

$$\frac{\partial^3 w}{\partial R\,\partial Q^2} = 2A(R - 3R^2 + 2R^3)(1 - 6Q + 6Q^2) \tag{3.280}$$

$$\frac{\partial^4 w}{\partial R^2\,\partial Q^2} = A(2 - 12R + 12R^2)(2 - 12Q + 12Q^2) \tag{3.281}$$

The derivatives of the CSCS deflection equation

Equation (3.282) through Equation (3.293) are the derivatives of the CSCS plate type polynomial deflection equation, w. Recall Equation (3.199).

$$w = A(R - 2R^3 + R^4)(Q^2 - 2Q^3 + Q^4)$$

$$\frac{\partial w}{\partial R} = A(1 - 6R^2 + 4R^3)(Q^2 - 2Q^3 + Q^4) \tag{3.282}$$

$$\frac{\partial^2 w}{\partial R^2} = A(12R^2 - 12R)(Q^2 - 2Q^3 + Q^4) \tag{3.283}$$

$$\frac{\partial^3 w}{\partial R^3} = A(24R - 12)(Q^2 - 2Q^3 + Q^4) \tag{3.284}$$

$$\frac{\partial^4 w}{\partial R^4} = 24A(Q^2 - 2Q^3 + Q^4) \tag{3.285}$$

$$\frac{\partial w}{\partial Q} = A(R^2 - 2R^3 + R^4)(2Q - 6Q^2 + 4Q^3) \tag{3.286}$$

$$\frac{\partial^2 w}{\partial Q^2} = A(R - 2R^3 + R^4)(2 - 12Q + 12Q^2) \tag{3.287}$$

$$\frac{\partial^3 w}{\partial Q^3} = 12A(R - 2R^3 + R^4)(2Q - 1) \tag{3.288}$$

$$\frac{\partial^4 w}{\partial Q^4} = 24A(R - 2R^3 + R^4) \tag{3.289}$$

$$\frac{\partial^2 w}{\partial R\,\partial Q} = A(1 - 6R^2 + 4R^3)(2Q - 6Q^2 + 4Q^3) \tag{3.290}$$

$$\frac{\partial^3 w}{\partial R^2\,\partial Q} = 24A(R^2 - R)(Q - 3Q^2 + 2Q^3) \tag{3.291}$$

$$\frac{\partial^3 w}{\partial R\, \partial Q^2} = A(1 - 6R^2 + 4R^3)(2 - 12Q + 12Q^2) \tag{3.292}$$

$$\frac{\partial^4 w}{\partial R^2\, \partial Q^2} = 12A(R^2 - R)(2 - 12Q + 12Q^2) \tag{3.293}$$

The derivatives of the CCSS deflection equation

Equation (3.294) through Equation (3.306) are the derivatives of the CCSS plate type polynomial deflection equation, w. Recall Equation (3.200).

$$w = A(1.5R^2 - 2.5R^3 + R^4)(1.5Q^2 - 2.5Q^3 + Q^4)$$

$$\frac{\partial w}{\partial R} = A(3R - 7.5R^2 + 4R^3)(1.5Q^2 - 2.5Q^3 + Q^4) \tag{3.294}$$

$$\frac{\partial^2 w}{\partial R^2} = A(3 - 15R + 12R^2)(1.5Q^2 - 2.5Q^3 + Q^4) \tag{3.295}$$

$$\frac{\partial^3 w}{\partial R^3} = A(-15 + 24R)(1.5Q^2 - 2.5Q^3 + Q^4) \tag{3.297}$$

$$\frac{\partial^4 w}{\partial R^4} = 24A(1.5Q^2 - 2.5Q^3 + Q^4) \tag{3.298}$$

$$\frac{\partial w}{\partial Q} = A(1.5R^2 - 2.5R^3 + R^4)(3Q - 7.5Q^2 + 4Q^3) \tag{3.299}$$

$$\frac{\partial^2 w}{\partial Q^2} = A(1.5R^2 - 2.5R^3 + R^4)(3 - 15Q + 12Q^2) \tag{3.300}$$

$$\frac{\partial^3 w}{\partial Q^3} = A(1.5R^2 - 2.5R^3 + R^4)(-15 + 24Q) \tag{3.301}$$

$$\frac{\partial^4 w}{\partial Q^4} = 24A(1.5R^2 - 2.5R^3 + R^4) \tag{3.302}$$

$$\frac{\partial^2 w}{\partial R\, \partial Q} = A(3R - 7.5R^2 + 4R^3)(3Q - 7.5Q^2 + 4Q^3) \tag{3.303}$$

$$\frac{\partial^3 w}{\partial R^2\, \partial Q} = A(3 - 15R + 12R^2)(3Q - 7.5Q^2 + 4Q^3) \tag{3.304}$$

$$\frac{\partial^3 w}{\partial R\, \partial Q^2} = 3A(3R - 7.5R^2 + 4R^3)(1 - 5Q + 4Q^2) \tag{3.305}$$

$$\frac{\partial^4 w}{\partial R^2\, \partial Q^2} = 3A(1 - 5R + 4R^2)(1 - 5Q + 4Q^2) \tag{3.306}$$

The derivatives of the CSSS deflection equation

Equation (3.307) through Equation (3.318) are the derivatives of the CSSS plate type polynomial deflection equation, w. Recall Equation (3.201).

$$w = A(R - 2R^3 + R^4)(1.5Q^2 - 2.5Q^3 + Q^4)$$

$$\frac{\partial w}{\partial R} = A(1 - 6R^2 + 4R^3)(1.5Q^2 - 2.5Q^3 + Q^4) \quad (3.307)$$

$$\frac{\partial^2 w}{\partial R^2} = 12A(R^2 - R)(1.5Q^2 - 2.5Q^3 + Q^4) \quad (3.308)$$

$$\frac{\partial^3 w}{\partial R^3} = 12A(2R - 1)(1.5Q^2 - 2.5Q^3 + Q^4) \quad (3.309)$$

$$\frac{\partial^4 w}{\partial R^4} = 24A(1.5Q^2 - 2.5Q^3 + Q^4) \quad (3.310)$$

$$\frac{\partial w}{\partial Q} = A(3Q - 7.5Q^2 + 4Q^3)(R - 2R^3 + R^4) \quad (3.311)$$

$$\frac{\partial^2 w}{\partial Q^2} = 3A(1 - 5Q + 4Q^2)(R - 2R^3 + R^4) \quad (3.312)$$

$$\frac{\partial^3 w}{\partial Q^3} = 3A(-5 + 8Q)(R - 2R^3 + R^4) \quad (3.313)$$

$$\frac{\partial^4 w}{\partial Q^4} = 24A(R - 2R^3 + R^4) \quad (3.314)$$

$$\frac{\partial^2 w}{\partial R\,\partial Q} = A(1 - 6R^2 + 4R^3)(3Q - 7.5Q^2 + 4Q^3) \quad (3.315)$$

$$\frac{\partial^3 w}{\partial R^2\,\partial Q} = 12A(R^2 - R)(3Q - 7.5Q^2 + 4Q^3) \quad (3.316)$$

$$\frac{\partial^3 w}{\partial R\,\partial Q^2} = 3A(1 - 6R^2 + 4R^3)(1 - 5Q + 4Q^2) \quad (3.317)$$

$$\frac{\partial^4 w}{\partial R^2\,\partial Q^2} = 36A(R^2 - R)(1 - 5Q + 4Q^2) \quad (3.318)$$

The derivatives of the CCCS deflection equation

Equation (3.319) through Equation (3.330) are the derivatives of the CCCS plate type polynomial deflection equation, Equation (3.202).

$$w = A(1.5R^2 - 2.5R^3 + R^4)(Q^2 - 2Q^3 + Q^4)$$

$$\frac{\partial w}{\partial R} = A(3R - 7.5R^2 + 4R^3)(Q^2 - 2Q^3 + Q^4) \qquad (3.319)$$

$$\frac{\partial^2 w}{\partial R^2} = 3A(1 - 5R + 4R^2)(Q^2 - 2Q^3 + Q^4) \qquad (3.320)$$

$$\frac{\partial^3 w}{\partial R^3} = 3A(8R - 5)(Q^2 - 2Q^3 + Q^4) \qquad (3.321)$$

$$\frac{\partial^4 w}{\partial R^4} = 24A(Q^2 - 2Q^3 + Q^4) \qquad (3.322)$$

$$\frac{\partial w}{\partial Q} = A(1.5R^2 - 2.5R^3 + R^4)(2Q - 6Q^2 + 4Q^3) \qquad (3.323)$$

$$\frac{\partial^2 w}{\partial Q^2} = A(1.5R^2 - 2.5R^3 + R^4)(2 - 12Q + 12Q^2) \qquad (3.324)$$

$$\frac{\partial^3 w}{\partial Q^3} = 12A(1.5R^2 - 2.5R^3 + R^4)(2Q - 1) \qquad (3.325)$$

$$\frac{\partial^4 w}{\partial Q^4} = 24A(1.5R^2 - 2.5R^3 + R^4) \qquad (3.326)$$

$$\frac{\partial^2 w}{\partial R\,\partial Q} = A(3R - 7.5R^2 + 4R^3)(2Q - 6Q^2 + 4Q^3) \qquad (3.327)$$

$$\frac{\partial^3 w}{\partial R^2\,\partial Q} = 6A(1 - 5R + 4R^2)(Q - 3Q^2 + 2Q^3) \qquad (3.328)$$

$$\frac{\partial^3 w}{\partial R\,\partial Q^2} = 2A(3R - 7.5R^2 + 4R^3)(1 - 6Q + 6Q^2) \qquad (3.329)$$

$$\frac{\partial^4 w}{\partial R^2\,\partial Q^2} = 6A(1 - 5R + 4R^2)(1 - 6Q + 6Q^2) \qquad (3.330)$$

3.4.3.2 The integrands of the shape functions

The definite integrals of the shape functions are presented in this section. This is by integrating the derivatives of the shape functions derived in section 3.4.3.2. The integrands of the plate types considered in this section are those that appears in the plate force equilibrium equation (Equation (3.151)).

The integrands of the SSSS deflection equation

Table 3.11 shows the numerical values of the integrands of SSSS plate type calculated as shown below. The plate is simply supported (S) at both ends in the x direction, i.e. (SS) and in the y direction, it is also simply supported (S) at both ends, i.e. (SS). The effective length of an SS line continuum is 1.0. Therefore, an SSSS plate will have an effective length of 1.0 in both directions.

$$k_x \int_0^1 \int_0^1 \frac{d^4w}{dR^4} dRdQ = 24 \int_0^1 \int_0^1 (Q - 2Q^3 + Q^4) dRdQ$$

$$= 24\left(\frac{Q^2}{2} - 2\frac{Q^4}{4} + \frac{Q^5}{5}\right]_0^1\Bigg) = 24\left(\frac{1^2}{2} - 2 \times \left(\frac{1^4}{4}\right) + \frac{1^5}{5}\right]_0^1\Bigg) = \frac{24}{5}$$

$$k_{xy} \int_0^1 \int_0^1 \frac{d^4w}{dR^2dQ^2} dRdQ = 144 \int_0^1 \int_0^1 (R^2 - R)(Q^2 - Q) dRdQ$$

$$= 144\left(\frac{R^3}{3} - \frac{R^2}{2}\right]_0^1 \times \frac{Q^3}{3} - \frac{Q^2}{2}\Bigg]_0^1\Bigg) = 144\left(\frac{1^3}{3} - \frac{1^2}{2}\right]_0^1 \times \frac{1^3}{3} - \frac{1^2}{2}\Bigg]_0^1\Bigg) = 4$$

$$k_y \int_0^1 \int_0^1 \frac{d^4w}{dQ^4} dRdQ = 24 \int_0^1 \int_0^1 (R - 2R^3 + R^4) dRdQ$$

$$= 24\left(\frac{R^2}{2} - 2\frac{R^4}{4} + \frac{R}{5}\right]_0^1\Bigg) = 24\left(\frac{1^2}{2} - 2 \times \left(\frac{1^4}{4}\right) + \frac{1^5}{5}\right]_0^1\Bigg) = \frac{24}{5}$$

$$k_q \int_0^1 \int_0^1 dRdQ = \int_0^1 \int_0^1 dRdQ = \left(\frac{1}{1}\right)\left(\frac{1}{1}\right) = 1$$

The integrals of the CCCC deflection equation

Table 3.11 shows the numerical values of the integrands of CCCC plate type calculated as shown below. The plate is clamped (C) at both ends in the x direction and in the y direction, it is also clamped (C) at both ends. The effective length of the plate is 0.835 in both directions.

$$k_x \int_{0.165}^{0.835} \int_{0.165}^{0.835} \frac{d^4w}{dR^4} dRdQ = 24 \int_{0.165}^{0.835} \int_{0.165}^{0.835} (Q^2 - 2Q^3 + Q^4) dRdQ$$

$$= 24\left[\left(\frac{Q^3}{3} - 2\frac{Q^4}{4} + \frac{Q^5}{5}\right]_{0.165}^{0.835}\Bigg) \times \left(\frac{R}{1}\right]_{0.165}^{0.835}\Bigg)\right]$$

$$= 24\left[\left(\left(\frac{0.835^3}{3} - 2\frac{0.835^4}{4} + \frac{0.835^5}{5}\right) - \left(\frac{0.165^3}{3} - 2\frac{0.165^4}{4} + \frac{0.165^5}{5}\right)\right) \times \left(\frac{0.835}{1} - \frac{0.165}{1}\right)\right] = 0.499$$

$$k_{xy}\int_{0.165}^{0.835}\int_{0.165}^{0.835}\frac{d^4w}{dR^2dQ^2}dRdQ$$

$$= 4\int_{0.165}^{0.835}\int_{0.165}^{0.835}(1-6R+6R^2)(1-6Q+6Q^2)dRdQ$$

$$= 4\left(R-\frac{6R^2}{2}+\frac{6R^3}{3}\right]_{0.165}^{0.835}\times Q-\frac{6Q^2}{2}+\frac{6Q^3}{3}\Bigg]_{0.165}^{0.835}\Bigg) = 0.847$$

$$= 4\left(\begin{array}{c}\left[\left(0.835-\frac{6(0.835)^2}{2}+\frac{6(0.835)^3}{3}\right)-\left(0.165-\frac{6(0.165)^2}{2}+\frac{6(0.165)^3}{3}\right)\right]\\ \times\left[\left(0.835-\frac{6(0.835)^2}{2}+\frac{6(0.835)^2}{3}\right)-\left(0.165-\frac{6(0.165)^2}{2}+\frac{6(0.165)^3}{3}\right)\right]\end{array}\right)$$

$$= 0.136$$

$$k_y\int_{0.165}^{0.835}\int_{0.165}^{0.835}\frac{d^4w}{dQ^4}dRdQ = 24\int_{0.165}^{0.835}\int_{0.165}^{0.835}(R^2-2R^3+R^4)dRdQ$$

$$= 24\left[\left(\frac{R^3}{3}-2\frac{R^4}{4}+\frac{R^5}{5}\right]_{0.165}^{0.835}\Bigg)\times\left(\frac{Q}{1}\right]_{0.165}^{0.835}\Bigg)\right]$$

$$= 24\left[\left(\left(\frac{0.835^3}{3}-2\frac{0.835^4}{4}+\frac{0.835^5}{5}\right)-\left(\frac{0.165^3}{3}-2\frac{0.165^4}{4}+\frac{0.165^5}{5}\right)\right)\right.$$

$$\left.\times\left(\frac{0.835}{1}-\frac{0.165}{1}\right)\right] = 0.499$$

$$k_q\int_{0.165}^{0.835}\int_{0.165}^{0.835}dRdQ = \left(\frac{R}{1}\times\frac{Q}{1}\right)_{0.165}^{0.835} = \left(\frac{0.835}{1}-\frac{0.165}{1}\right)\times\left(\frac{0.835}{1}-\frac{0.165}{1}\right) = 0.449$$

The integrands of the CSCS deflection equation

Table 3.11 shows the numerical values of the integrands of CSCS plate type calculated as shown below. The plate is simply supported (S) at both ends in the x direction while in the y direction, it is clamped (C) at both ends. The effective length in x and y – directions is 1 and 0.835 respectively.

$$k_x\int_{0}^{1}\int_{0.165}^{0.835}\frac{d^4w}{dR^4}dRdQ = 24\int_{0}^{1}\int_{0.165}^{0.835}(Q^2-2Q^3+Q^4)dRdQ$$

$$= 24\left[\left(\frac{R}{1}\right]_0^1\right)\times\left(\frac{Q^3}{3}-2\frac{Q^4}{4}+\frac{Q^5}{5}\right]_{0.165}^{0.835}\right)\right]$$

$$= 24\left[\left(\left(\frac{0.835^3}{3}-2\frac{0.835^4}{4}+\frac{0.835^5}{5}\right)-\left(\frac{0.165^3}{3}-2\frac{0.165^4}{4}+\frac{0.165^5}{5}\right)\right)\right] = 0.745$$

$$k_{xy}\int_{0}^{1}\int_{0.165}^{0.835}\frac{d^4w}{dR^2dQ^2}dRdQ = 12\int_{0}^{1}\int_{0.165}^{0.835}(R^2-R)(2-12Q+12Q^2)dRdQ$$

$$= 12\left(\frac{R^3}{3}-\frac{R^2}{2}\right]_0^1\times 2Q-\frac{12Q^2}{2}+\frac{12Q^3}{3}\right]_{0.165}^{0.835}\right)$$

$$= 12\left(\left[\left(\frac{1^3}{3}-\frac{1^2}{2}\right)\right]\times\left[\begin{array}{l}\left(0.835-\frac{12(0.835)^2}{2}+\frac{12(0.835)^3}{3}\right)\\-\left(0.165-\frac{12(0.165)^3}{2}+\frac{12(0.165)^3}{3}\right)\end{array}\right]\right) = 0.738$$

$$k_y\int_{0}^{1}\int_{0.165}^{0.835}\frac{d^4w}{dQ^4}dRdQ = 24\int_{0}^{1}\int_{0}^{0.835}(R-2R^3+R^4)dRdQ$$

$$= 24\left(\frac{R^2}{2}-2\frac{R^4}{4}+\frac{R^5}{5}\right]_0^1\times\left(\frac{Q}{1}\right]_{0.165}^{0.835}\right)\right) = 24\times\left(\frac{1^2}{2}-2\frac{1^4}{4}+\frac{1^5}{5}\right) = 3.216$$

$$k_q\int_{0}^{1}\int_{0.165}^{0.835}dRdQ = \int_{0}^{1}\int_{0.165}^{0.835}dRdQ = \frac{1}{1}\times\left[\left(\frac{0.835}{1}\right)-\left(\frac{0.165}{1}\right)\right] = 0.67$$

The integrands of the CCSS deflection equation

Table 3.11 shows the numerical values of the integrands of CCSS plate type calculated as shown below. The plate is simply supported (S) at one end and clamped (C) in the opposite end in both x and y directions. The effective length in x and y – directions is 0.8.

$$k_x\int_{0.2}^{1}\int_{0.2}^{1}\frac{d^4w}{dR^4}dRdQ = 24\int_{0.2}^{1}\int_{0.2}^{1}(1.5Q^2-2.5Q^3+Q^4)dRdQ$$

$$= 24\left[\left(\frac{R}{1}\Big]_{0.2}^{1}\right)\times\left(\frac{1.5Q^3}{3}-\frac{2.5Q^4}{4}+\frac{Q^5}{5}\Big]_{0.2}^{1}\right)\right]$$

$$= 24\left[\left(\frac{1}{1}-\frac{0.2}{1}\right)\times\left(\left(1.5\frac{1^3}{3}-2.5\frac{1^4}{4}+\frac{1^5}{5}\right)-\left(1.5\frac{0.2^3}{3}-2.5\frac{0.2^4}{4}+\frac{0.2^5}{5}\right)\right)\right] = 1.381$$

$$-k_{xy}\int_{0.2}^{1}\int_{0.2}^{1}\frac{d^4w}{dR^2dQ^2}dRdQ = 9\int_{0.2}^{1}\int_{0.2}^{1}(1-5R+4R^2)(1-5Q+4Q^2)dRdQ$$

$$= 9\left(R-\frac{5R^2}{2}+\frac{4R^3}{3}\Big]_{0.2}^{1}\times Q-\frac{5Q^2}{2}+\frac{4Q^3}{3}\Big]_{0.2}^{1}\right)$$

$$= 9\left(\begin{array}{c}\left[\left(1-\frac{5}{2}-\frac{4}{3}\right)-\left(0.2-\frac{5(0.2^2)}{2}-\frac{4(0.2^3)}{3}\right)\right]\\ \times\left[\left(1-\frac{5}{2}-\frac{4}{3}\right)-\left(0.2-\frac{5(0.2^2)}{2}-\frac{4(0.2^3)}{3}\right)\right]\end{array}\right) = 0.692$$

$$k_y\int_{0.2}^{1}\int_{0.2}^{1}\frac{d^4w}{dQ^4}dRdQ = 24\int_{0.2}^{1}\int_{0.2}^{1}(1.5R^2-2.5R^3+R^4)dRdQ$$

$$= 24\left[\left(\frac{1.5R^3}{3}-\frac{2.5R^4}{4}+\frac{R^5}{5}\Big]_{0.2}^{1}\right)\times\left(\frac{Q}{1}\Big]_{0.2}^{1}\right)\right]$$

$$= 24\left[\left(1.5\frac{1^2}{3}-2.5\frac{1^4}{4}+\frac{1^5}{5}\Big]_{0}^{1}\right)\times\left(\frac{1}{1}-\frac{0.2}{1}\right)\right] = 1.381$$

$$k_q\int_{0.2}^{1}\int_{0.2}^{1}dRdQ = \int_{0.2}^{1}\int_{0.2}^{1}dRdQ = \left[\left(\frac{1}{1}\right)-\left(\frac{0.2}{1}\right)\right]\times\left[\left(\frac{1}{1}\right)-\left(\frac{0.2}{1}\right)\right] = 0.64$$

The integrands of the CSSS deflection equation

Table 3.11 shows the numerical values of the integrands of CSSS plate type calculated as shown below. The plate is simply supported (S) at both ends in the x direction while in the y direction, one end is simply supported (S) and the other end is clamped (C). The effective length in x and y – directions is 1 and 0.8 respectively.

$$k_x\int_{0}^{1}\int_{0.2}^{1}\frac{d^4w}{dR^4}dRdQ = 24\int_{0}^{1}\int_{0.2}^{1}(1.5Q^2 - 2.5Q^3 + Q^4)dRdQ$$

$$= 24\Bigg[\Bigg(\frac{R}{1}\Bigg]_{0}^{1}\Bigg)\times\Bigg(\frac{1.5Q^3}{3} - \frac{2.5Q^4}{4} + \frac{Q^5}{5}\Bigg]_{0.2}^{1}\Bigg)\Bigg]$$

$$= 24\Bigg[\Bigg(\frac{1}{1}\Bigg]_{0}^{1}\Bigg)\times\Bigg(\Bigg(1.5\frac{1^3}{3} - 2.5\frac{1^4}{4} + \frac{1^5}{5}\Bigg) - \Bigg(1.5\frac{0.2^3}{3} - 2.5\frac{0.2^4}{4} + \frac{0.2^5}{5}\Bigg)\Bigg)\Bigg] = 1.726$$

$$k_{xy}\int_{0}^{1}\int_{0.2}^{1}\frac{d^4w}{dR^2dQ^2}dRdQ = 36\int_{0}^{1}\int_{0.2}^{1}(R^2 - R)(1 - 5Q + 4Q^2)dRdQ$$

$$= 36\Bigg(\frac{R^3}{3} - \frac{R^2}{2}\Bigg]_{0}^{1}\times Q - \frac{5Q^2}{2} + \frac{4Q^3}{3}\Bigg]_{0.2}^{1}\Bigg)$$

$$= 36\Bigg[\Bigg(\frac{1^3}{3} - \frac{1^2}{2}\Bigg)\times\Bigg(\Bigg(1 - \frac{5}{2} + \frac{4}{3}\Bigg) - \Bigg(0.2 - \frac{5(0.2^4)}{2} + \frac{4(0.2^4)}{3}\Bigg)\Bigg)\Bigg] = 1.664$$

$$k_y\int_{0}^{1}\int_{0.2}^{1}\frac{d^4w}{dQ^4}dRdQ = 24\int_{0}^{1}\int_{0.2}^{1}(R - 2R^3 + R^4)dRdQ$$

$$= 24\Bigg(\frac{R^2}{2} - 2\frac{R^4}{4} + \frac{R^5}{5}\Bigg]_{0}^{1}\times\Bigg(\frac{Q}{1}\Bigg]_{0.2}^{1}\Bigg)\Bigg) = 24\times\Bigg(\frac{1^2}{2} - 2\frac{1^4}{4} + \frac{1^5}{5}\Bigg]_{0}^{1}\Bigg)\times\Bigg[\Bigg(\frac{1}{1}\Bigg) - \Bigg(\frac{0.2}{1}\Bigg)\Bigg] = 3.84$$

$$k_q\int_{0}^{1}\int_{0.2}^{1}dRdQ = \int_{0}^{1}\int_{0.2}^{1}dRdQ = \frac{1}{1}\times\Bigg[\Bigg(\frac{1}{1}\Bigg) - \Bigg(\frac{0.2}{1}\Bigg)\Bigg] = 0.8$$

The integrands of the CCCS deflection equation

Table 3.11 shows the numerical values of the integrands of CCCS plate type calculated as shown below. The plate is simply supported (S) at one end and clamped (C) at opposite end in the x direction while in the y direction, the opposite ends are clamped (C). The effective length in x and y – directions is 0.8 and 0.835 respectively.

$$k_x\int_{0.2}^{1}\int_{0.165}^{0.835}\frac{d^4w}{dR^4}dRdQ = 24\int_{0.2}^{1}\int_{0.165}^{0.835}(Q^2 - 2Q^3 + Q^4)dRdQ$$

$$= 24\left[\left(\frac{R}{1}\right]_{0.2}^{1}\right)\times\left(\frac{Q^3}{3}-\frac{2Q^4}{4}+\frac{Q^5}{5}\right]_{0.165}^{0.835}\left.\right)\Bigg]$$

$$= 24\left[\left(\frac{1}{1}-\frac{0.2}{1}\right)\times\left(\left(\frac{0.835^3}{3}-2\frac{0.835^4}{4}+\frac{0.835^5}{5}\right)-\left(\frac{0.165^3}{3}-2\frac{0.165^4}{4}+\frac{0.165^5}{5}\right)\right)\right]$$

$$= 0.745$$

$$k_{xy}\int_{0.2}^{1}\int_{0.165}^{0.835}\frac{d^4w}{dR^2dQ^2}dRdQ = 6\int_{0.2}^{1}\int_{0.165}^{0.835}(1-5R+4R^2)(1-6Q+6Q^2)dRdQ$$

$$= 6\left(R-\frac{5R^2}{2}+\frac{4R^3}{3}\right]_{0.2}^{1}\times Q-\frac{6Q^2}{2}+\frac{6Q^3}{3}\Bigg]_{0.165}^{0.835}\Bigg)$$

$$= 6\left[\begin{array}{c}\left(\left(1-5\times\frac{1^2}{2}+4\times\frac{1^3}{3}\right)-\left(0.2-5\times\frac{0.2^2}{2}+4\times\frac{0.2^3}{3}\right)\right)\\ \times\left(\left(0.835-6\frac{0.835^2}{2}+6\frac{0.835^3}{3}\right)-\left(0.165-6\frac{0.165^2}{2}+6\frac{0.165^3}{3}\right)\right)\end{array}\right] = 0.307$$

$$k_y\int_{0.2}^{1}\int_{0.165}^{0.835}\frac{d^4w}{dQ^4}dRdQ = 24\int_{0.2}^{1}\int_{0.165}^{0.835}(1.5R^2-2.5R^3+R^4)dRdQ$$

$$= 24\left[\left(\frac{1.5R^3}{3}-\frac{2.5R^4}{4}+\frac{R^5}{5}\right]_{0.2}^{1}\right)\times\left(\frac{Q}{1}\right]_{0.165}^{0.835}\left.\right)\Bigg]$$

$$= 24\left[\left(\left(1.5\frac{1^3}{3}-2.5\frac{1^4}{4}+\frac{1^5}{5}\right)-\left(1.5\frac{0.2^3}{3}-2.5\frac{0.2^4}{4}+\frac{0.2^5}{5}\right)\right)\times\left(\frac{0.835}{1}-\frac{0.165}{1}\right)\right]$$

$$= 1.726$$

$$k_q\int_{0.2}^{1}\int_{0.165}^{0.835}dRdQ$$

$$= \int_{0.2}^{1}\int_{0.165}^{0.835}dRdQ = \left[\left(\frac{1}{1}\right)-\left(\frac{0.2}{1}\right)\right]\times\left[\left(\frac{0.835}{1}\right)-\left(\frac{0.165}{1}\right)\right] = 0.536$$

Table 3.11 Euler Bernoulli polynomial shape function stiffness coefficients for various plates

Plate type	Kx	Kxy	Ky	Kq
SSSS	4.800	4.000	4.800	1.000
CCCC	0.499	0.136	0.499	0.449
CSCS	0.745	0.738	3.216	0.670
CSSS	1.726	1.664	3.840	0.800
CCSS	1.381	0.692	1.381	0.640
CCCS	0.596	0.307	1.157	0.536

3.4.3.3 Determination of the exact coefficient of deflection of selected plates

Recall Equation (3.153), the term 'A' is the coefficient of the deflection function. here, 'A' is made the subject of the relationship and the resulting equation used to determine the exact coefficient of deflection based on the polynomial shape function for the plate types under consideration. 'A' from Equation (3.153).

$$A\int_0^1\int_0^1\left(\frac{\partial^4 h}{\partial R^4}+\frac{2}{p^2}\frac{\partial^4 h}{\partial R^2\,\partial Q^2}+\frac{1}{p^4}\frac{\partial^4 h}{\partial Q^4}\right)dRdQ=\int_0^1\int_0^1\frac{qa^4}{D}dRdQ$$

Rearranging this equation gives the value of A as shown in Equation (3.316).

$$A=\frac{\frac{qa^4}{D}\int_0^1\int_0^1 dRdQ}{\int_0^1\int_0^1\left(\frac{\partial^4 h}{\partial R^4}+\frac{2}{p^2}\frac{\partial^4 h}{\partial R^2\,\partial Q^2}+\frac{1}{p^4}\frac{\partial^4 h}{\partial Q^4}\right)dRdQ} \qquad (3.331)$$

Equation (3.331) shall be used to obtain the exact coefficient of deflection for the plate types within the scope of this work.

Exact deflection coefficient for SSSS rectangular plate

Substituting the integrands of the shape function for SSSS in Table 3.11 into Equation (3.331) and solving for 'A' yields Equation (3.332).

$$A=\frac{qa^4}{D}\frac{p^4}{(4.8p^4+8p^2+4.8)} \qquad (3.332)$$

Exact deflection coefficient for CCCC rectangular plate

Substituting the integrands of the shape function for CCCC in Table 3.11 into Equation (3.331) and solving for 'A' yields Equation (3.333).

$$A=\frac{qa^4}{D}\frac{0.449p^4}{(0.499p^4+0.272p^2+0.499)} \qquad (3.333)$$

Exact deflection coefficient for CSCS rectangular plate

Substituting the integrands of the shape function for CSCS in Table 3.11 into Equation (3.331) and solving for ‘A’ yields Equation (3.334).

$$\mathrm{A} = \frac{\mathrm{qa}^4}{\mathrm{D}} \frac{0.67\mathrm{p}^4}{(0.745\mathrm{p}^4 + 1.477\mathrm{p}^2 + 3.216)} \tag{3.334}$$

Exact deflection coefficient for CSSS rectangular plate

Substituting the integrands of the shape function for CSSS in Table 3.11 into Equation (3.331) and solving for ‘A’ yields Equation (3.335).

$$\mathrm{A} = \frac{\mathrm{qa}^4}{\mathrm{D}} \frac{0.8\mathrm{p}^4}{(1.726\mathrm{p}^4 + 3.328\mathrm{p}^2 + 3.84)} \tag{3.335}$$

Exact deflection coefficient for CCSS rectangular plate

Substituting the integrands of the shape function for CCSS in Table 3.11 into Equation (3.331) and solving for ‘A’ yields Equation (3.336).

$$\mathrm{A} = \frac{\mathrm{qa}^4}{\mathrm{D}} \frac{0.64\mathrm{p}^4}{(1.381\mathrm{p}^4 + 1.384\mathrm{p}^2 + 1.381)} \tag{3.336}$$

Exact deflection coefficient for CCCS rectangular plate

Substituting the integrands of the shape function for CCCS in Table 3.11 into Equation (3.331) and solving for ‘A’ yields Equation (3.337).

$$\mathrm{A} = \frac{\mathrm{qa}^4}{\mathrm{D}} \frac{0.536\mathrm{p}^4}{(0.596\mathrm{p}^4 + 0.614\mathrm{p}^4 + 1.157)} \tag{3.337}$$

3.4.3.4 Determination of the exact bending moments and shear forces of selected plates

Having determined the exact coefficient and exact shape functions for the plate types under consideration, next is to determine the design factors for maximum mid-span deflection, the mid-span moment coefficients and the shear force coefficients at the edges.

FACTORS FOR PLATE TYPE SSSS

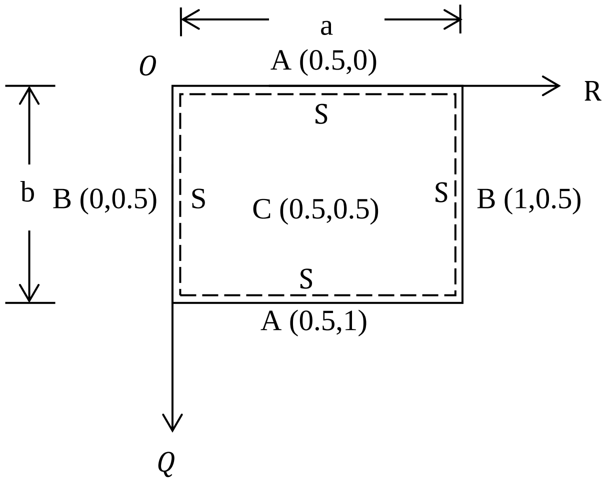

Figure 3.19: SSSS Plate Showing Points of Maximum Values

Figure 3.19 shows a thin rectangular plate under uniformly distributed lateral load, q. The plate has all edges simply supported.

The shape function for SSSS as determined in Equation (3.197) was given as:

$$W = A(R - 2R^3 + R^4)(Q - 2Q^3 + Q^4)$$

Similarly, the exact coefficient, 'A' from Equation (3.332) is given as:

$$A = \frac{qa^4}{D}\frac{p^4}{(4.8p^4 + 8p^2 + 4.8)}$$

The critical (maximum) values of deflections and moments will occur at the center of the plate as shown in Equation (3.338).

$$x = \frac{a}{2}\,;\left(x = \frac{Pb}{2} \text{ since } P = \frac{a}{b}\right) \text{ and } y = \frac{b}{2} \tag{3.338}$$

In non – dimensional form, for critical (maximum) deflections and moments will occur at the centre of the plate where:

$$R = \frac{1}{2} \text{ and } Q = \frac{1}{2} \tag{3.339}$$

MAXIMUM DEFLECTION

Using Equation (3.197), the maximum deflection of the plate is given as shown in Equation (3.340).

$$W_{max} = A\left(\frac{1}{2} - 2\left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^4\right)\left(\frac{1}{2} - 2\left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^4\right)$$

$$W_{max} = A\left(\frac{5}{16}\right)\left(\frac{5}{16}\right) = A\left(\frac{25}{256}\right) \tag{3.340}$$

Substituting the expression of A in Equation (3.332) into Equation (3.340) yields Equation (3.341).

$$W_{max} = \left(\frac{25}{256}\right)\frac{qa^4}{D}\frac{P^4}{4.8P^4 + 8P^2 + 4.8}$$

$$W_{max} = \left(\frac{25}{256}\right)\frac{P^4}{4.8 + 4.8P^4 + 8P^2}\frac{qa^4}{D} \qquad (3.341)$$

Equation for maximum deflection is as shown in Equation (3.342).

$$W_{max} = k\frac{qa^4}{D} \qquad (3.342)$$

Equating Equation (3.341) and Equation (3.342) and solving for k_D in terms of the aspect ratio, P, yields Equation (3.343).

$$k_D = \left(\frac{25}{256}\right)\frac{P^4}{4.8 + 4.8P^4 + 8P^2} \qquad (3.343)$$

Where k_D is the factor of deflection.

MID-SPAN MOMENTS, M_{xC} and M_{yC} ($\mu = 0.3$)

Substituting Equations (3.259) and (3.263) into Equation (3.70b) and simplifying the equation yields Equation (3.344).

$$M_x = -D\left[\frac{\partial^2 w}{\partial x^2} + \mu\frac{\partial^2 w}{\partial y^2}\right]$$

$$= \frac{-AD}{a^2}\left[\frac{\partial^2 h}{\partial R^2} + \mu\frac{\partial^2 h}{P^2\,\partial Q^2}\right]$$

$$= \frac{-AD}{a^2}\left[\left(12(R^2 - R)(Q - 2Q^3 + Q^4)\right) + \mu\frac{12(Q^2 - Q)(R - 2R^3 + R^4)}{P^2}\right] \qquad (3.344)$$

For maximum moment at point C (See Figure 3.19), M_{xC}, substitute Equation (3.339) into Equation (3.344) and simplifying the equation yields Equation (3.345)

$$= \frac{-12AD}{a^2}\left[\left(\left(\left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)\right)\left(\left(\frac{1}{2}\right) - 2\left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^4\right)\right) + 0.3\frac{\left(\left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)\right)\left(\left(\frac{1}{2}\right) - 2\left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^4\right)}{P^2}\right]$$

$$M_{xC} = \frac{-12AD}{a^2}\left[-0.078125 - \frac{0.023438}{P^2}\right]$$

$$M_{xC} = \frac{AD}{a^2}\left[\frac{0.281250}{P^2} + 0.93750\right] \tag{3.345}$$

Substituting the exact coefficient equation (Equation (3.332)) yields Equation (3.346)

$$M_{xC} = \frac{D}{a^2}\left[\frac{0.281250}{P^2} + 0.93750\right]\frac{qa^4}{D}\left(\frac{P^4}{4.8 + 4.8P^4 + 8P^2}\right)$$

$$M_{xC} = \left[\frac{0.281250}{P^2} + 0.93750\right]\left(\frac{P^4}{4.8 + 4.8P^4 + 8P^2}\right)qa^2 \tag{3.346}$$

Equation for maximum bending moment in x – direction is as shown in Equation (3.347).

$$M_{xC} = \beta_{xC}qa^2 \tag{3.347}$$

Equating Equation (3.346) and Equation (3.347) and solving for β_{xC} in terms of the aspect ratio, P yields Equation (3.348).

$$\beta_x = \left[\frac{0.281250}{P^2} + 0.93750\right]\left(\frac{P^4}{4.8 + 4.8P^4 + 8P^2}\right) \tag{3.348}$$

Where, β_x is bending moment factor in the x – direction.

Similarly, substituting Equations (3.259) and (3.263) into Equation (3.71b) and simplifying the equation yields Equation (3.349).

$$\begin{aligned} M_{yC} \quad &= -D\left[\mu\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right] \\ &= \frac{-AD}{a^2}\left[\mu\frac{\partial^2 h}{\partial R^2} + \frac{\partial^2 h}{P^2\,\partial Q^2}\right] \\ &= \frac{-AD}{a^2}\left[12\mu(R^2 - R)(Q - 2Q^3 + Q^4) + \frac{\left(12(R - 2R^3 + R^4)(Q^2 - Q)\right)}{P^2}\right] \end{aligned} \tag{3.349}$$

For maximum moment at point C (See Figure 3.19), M_{yC}, substitute Equation (3.339) into Equation (3.349) and simplifying the equation yields Equation (3.350)

$$M_{yC} = \frac{-12AD}{a^2}\left[0.3\left(\left(\left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)\right)\left(\left(\frac{1}{2}\right) - 2\left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^4\right)\right) + \frac{\left(\left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)\right)\left(\left(\frac{1}{2}\right) - 2\left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^4\right)}{P^2}\right]$$

$$M_{yC} = \frac{-12AD}{a^2}\left[-0.023438 - \frac{0.078125}{P^2}\right]$$

$$M_{yC} = \frac{AD}{a^2}\left[\frac{0.93750}{P^2} + 0.281250\right] \qquad (3.350)$$

Substituting the exact coefficient equation (Equation (3.332)) yields Equation (3.351)

$$M_{yC} = \frac{D}{a^2}\left[\frac{0.93750}{P^2} + 0.281250\right]\frac{qa^4}{D}\frac{P^4}{4.8 + 4.8P^4 + 8P^2}$$

$$M_{yC} = \left[\frac{0.93750}{P^2} + 0.281250\right]\left(\frac{P^4}{4.8 + 4.8P^4 + 8P^2}\right)qa^2 \qquad (3.351)$$

Equation for maximum bending moment in y – direction is as shown in Equation (3.352).

$$M_{yC} = \beta_{yC}qa^2 \qquad (3.352)$$

Equating Equation (3.351) and Equation (3.352) and solving for β_{yC} in terms of the aspect ratio, P yields Equation (3.353).

$$\beta_{yC} = \left[\frac{0.93750}{P^2} + 0.281250\right]\left(\frac{P^4}{4.8 + 4.8P^4 + 8P^2}\right) \qquad (3.353)$$

Where β_y is bending moment factor in the y– directions

MAXIMUM SHEAR FORCE, V_{xB} and V_{yA} ($\mu = 0.3$)

Substituting Equations (3.260), (3.268) into Equation (3.82b) and simplifying the equation yields Equation (3.354).

Shear force is maximum at R = 0 and 1 and Q = 0.5 (See Figure 3.19)

$$V_x = -D\left[\frac{\partial^3 w}{\partial x^3} + (2-\mu)\frac{\partial^3 w}{\partial x\,\partial y^2}\right]$$

$$V_x = \frac{-AD}{a^3}\left[\frac{\partial^3 h}{\partial R^3} + (2-\mu)\frac{\partial^3 h}{P^2\,\partial R\,\partial Q^2}\right]$$

$$= \frac{-AD}{a^3}\left[12(2R-1)(Q - 2Q^3 + Q^4) + (2-\mu)\frac{\left(12(1 - 6R^2 + 4R^3)(Q^2 - Q)\right)}{P^2}\right] \qquad (3.354)$$

For maximum shear at point B (See Figure 3.19), V_{xB}, substitute the values of R = 0 and Q = 0.5 into Equation (3.354) and simplifying the equation yields Equation (3.355).

$$V_{xB} = \frac{-AD}{a^3}\left[12\left((2(0)-1)\left(\left(\frac{1}{2}\right)-2\left(\frac{1}{2}\right)^3+\left(\frac{1}{2}\right)^4\right)\right)+(2-\right.$$

$$\left.0.3)\frac{12\left(1-6(0)^2+4(0)^3\right)\left(\left(\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)\right)}{P^2}\right]7$$

$$V_{xB} = \frac{-AD}{a^3}\left[-3.75-\frac{5.1}{P^2}\right]$$

$$V_{xB} = \frac{AD}{a^3}\left[3.75+\frac{5.1}{P^2}\right] \qquad (3.355)$$

Substituting the exact coefficient equation (Equation (3.332)) into Equation (3.355) yields Equation (3.356).

$$V_{xB} = \frac{D}{a^3}\left[\frac{5.1}{P^2}+3.75\right]\frac{qa^4}{D}\frac{P^4}{4.8+4.8P^4+8P^2}$$

$$V_{xB} = \left[\frac{5.1}{P^2}+3.75\right]\left(\frac{P^4}{4.8+4.8P^4+8P^2}\right)qa \qquad (3.356)$$

Equation for maximum shear force in x – direction is as shown in Equation (3.357).

$$V_{xB} = k_{sxB}qa \qquad (3.357)$$

Equating Equation (3.356) and Equation (3.357) and solving for k_{sxB} in terms of the aspect ratio, P yields Equation (3.358).

$$k_{sxB} = \left[\frac{5.1}{P^2}+3.75\right]\left(\frac{P^4}{4.8+4.8P^4+8P^2}\right) \qquad (3.358)$$

Where k_{sxB} is shear force factor in x – directions,

Similarly, substituting Equations (3.264), (3.267) into Equation (3.83b) and simplifying the equation yields Equation (3.359).

$$V_y = -D\left[\frac{\partial^3 w}{\partial y^3}+(2-\mu)\frac{\partial^3 w}{\partial x^2\,\partial y}\right]$$

$$= \frac{-AD}{a^3}\left[\frac{\partial^3 h}{P^3\,\partial Q^3}+(2-\mu)\frac{\partial^3 h}{P\,\partial QR^2\,\partial Q}\right]$$

$$= \frac{-AD}{a^3}\left[\frac{12(R-2R^3+R^4)(2Q-1)}{P^3}\right.$$

$$\left.+(2-\mu)\frac{\left(12(R^2-R)(1-6Q^2+4Q^3)\right)}{P}\right] \qquad (3.359)$$

For maximum shear at point A (See Figure 3.19), V_{yA}, R = 0.5 and Q = 0 and 1. Substituting the values of R and Q and simplifying the equation yields Equation (3.360).

$$V_{yA} = \frac{-AD}{a^3}\left[\frac{12\left(\left(\left(\frac{1}{2}\right) - 2\left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^4\right)(2(0) - 1)\right)}{P^3} + (2 - 0.3)\frac{12\left(\left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)\right)(1 - 6(0)^2 + 4(0)^3)}{P}\right]$$

$$V_{yA} = \frac{-AD}{a^3}\left[\frac{3.75}{P^3} + \frac{5.1}{P}\right]$$

$$V_{yA} = \frac{AD}{a^3}\left[\frac{3.75}{P^3} + \frac{5.1}{P}\right] \tag{3.360}$$

Substituting the exact coefficient equation, Equation (3.332) into Equation (3.360) yields Equation (3.361).

$$V_{yA} = \frac{D}{a^3}\left[\frac{3.75}{P^3} + \frac{5.1}{P}\right]\frac{qa^4}{D}\frac{P^4}{4.8 + 4.8P^4 + 8P^2}$$

$$V_{yA} = \left[\frac{3.75}{P^3} + \frac{5.1}{P}\right]\left(\frac{P^4}{4.8 + 4.8P^4 + 8P^2}\right)qa \tag{3.361}$$

Equation for maximum shear force in y – direction is as shown in Equation (3.347).

$$V_{yA} = k_{syA}qa \tag{3.362}$$

Equating Equation (3.346) and Equation (3.347) and solving for k_{sxB} in terms of the aspect ratio, P yields Equation (3.363).

$$k_{syA} = \left[\frac{3.75}{P^3} + \frac{5.1}{P}\right]\left(\frac{P^4}{4.8 + 4.8P^4 + 8P^2}\right) \tag{3.363}$$

Where k_{syA} is shear force factor in y – directions,

FACTORS FOR PLATE TYPE CCCC

Figure 3.20 shows a thin rectangular plate under uniformly distributed lateral load, q. The plate has all edges clamped.

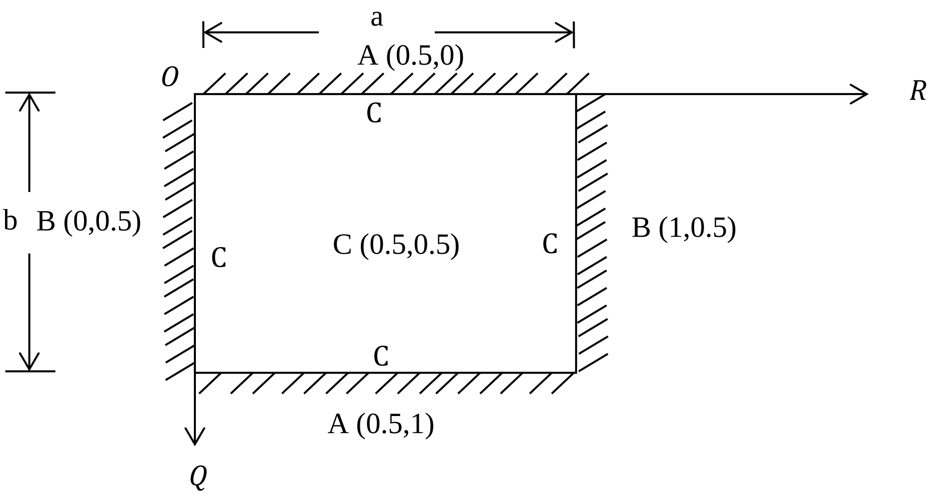

Figure 3.20: CCCC Plate Showing Points of Maximum Values

From Equation (3.198), the shape function for CCCC is given as:

$$W = A(R^2 - 2R^3 + R^4)(Q^2 - 2Q^3 + Q^4)$$

From Equation (3.333), the exact coefficient, A:

$$A = \frac{qa^4}{D}\frac{0.449P^4}{0.499P^4 + 0.272P^4 + 0.499}$$

The critical (maximum) values of deflections and moments will occur at the center of the plate where:

$$x = \frac{a}{2}\,;\left(x = \frac{Pb}{2} \text{ since } P = \frac{a}{b}\right) \text{ and } y = \frac{b}{2} \qquad (3.364)$$

In non – dimensional form, for critical (maximum) deflections and moments will occur at the center of the plate where:

$$R = \frac{1}{2} \text{ and } Q = \frac{1}{2} \qquad (3.365)$$

MAXIMUM DEFLECTIONS

Using Equation (3.198), the maximum deflection of the plate is given as shown in Equation (3.366) for Q = 0.5 and R = 0.5.

$$\mathrm{W_{max}} = \mathrm{A}\left(\left[\frac{1}{2}\right]^2 - 2\left[\frac{1}{2}\right]^3 + \left[\frac{1}{2}\right]^4\right)\left(\left[\frac{1}{2}\right]^2 - 2\left[\frac{1}{2}\right]^3 + \left[\frac{1}{2}\right]^4\right)$$

$$\mathrm{W_{max}} = \mathrm{A}\left(\frac{1}{16}\right)\left(\frac{1}{16}\right)$$

$$\mathrm{W_{max}} = \mathrm{A}\left(\frac{1}{256}\right) \tag{3.366}$$

Substituting the expression of A in Equation (3.333) into Equation (3.366) will yield Equation (3.367)

$$\mathrm{W_{max}} = \left(\frac{1}{256}\right)\frac{\mathrm{qa^4}}{\mathrm{D}}\frac{0.449\mathrm{P^4}}{0.499\mathrm{P^4} + 0.272\mathrm{P^4} + 0.499}$$

$$\mathrm{W_{max}} = \left(\left(\frac{1}{256}\right)\frac{0.449\mathrm{P^4}}{0.499\mathrm{P^4} + 0.272\mathrm{P^4} + 0.499}\right)\frac{\mathrm{qa^4}}{\mathrm{D}} \tag{3.367}$$

$$\mathrm{W_{max}} = \mathrm{k_D}\frac{\mathrm{qa^4}}{\mathrm{D}} \tag{3.368}$$

Where the factor of deflection, $\mathrm{k_D}$ is obtained by equating Equations (3.367) and (3.368) as shown in Equation (3.369).

$$\mathrm{k_D} = \left(\left(\frac{1}{256}\right)\frac{0.449\mathrm{P^4}}{0.499\mathrm{P^4} + 0.272\mathrm{P^4} + 0.499}\right) \tag{3.369}$$

Where $\mathrm{k_D}$ is the factor of deflection.

MID-SPAN MOMENTS, $\mathrm{M_{xc}}$ and $\mathrm{M_{yc}}$ ($\mu = 0.3$)

Substituting Equations (3.271) and (3.275) into Equation (3.70b) and simplifying the equation yields Equation (3.370).

$$\mathrm{M_x} = -\mathrm{D}\left[\frac{\partial^2 \mathrm{w}}{\partial \mathrm{x}^2} + \mu\frac{\partial^2 \mathrm{w}}{\partial \mathrm{y}^2}\right]$$

$$= \frac{-\mathrm{AD}}{\mathrm{a^2}}\left[\frac{\partial^2 \mathrm{h}}{\partial \mathrm{R}^2} + \mu\frac{\partial^2 \mathrm{h}}{\mathrm{P^2}\,\partial \mathrm{Q}^2}\right]$$

$$= \frac{-\mathrm{AD}}{\mathrm{a^2}}\Bigg[(2 - 12\mathrm{R} + 12\mathrm{R^2})(\mathrm{Q^2} - 2\mathrm{Q^3} + \mathrm{Q^4})$$

$$+ \frac{\mu(\mathrm{R^2} - 2\mathrm{R^3} + \mathrm{R^4})(2 - 12\mathrm{Q} + 12\mathrm{Q^2})}{\mathrm{P^2}}\Bigg] \tag{3.370}$$

For mid-span moment at point C (See Figure 3.20), Where R = 0.5 and Q = 0.5, Equation (3.355) will be as shown in Equation (3.371).

$$M_{xC} = \frac{-AD}{a^2}\left[\left(2 - 12\left(\frac{1}{2}\right) + 12\left(\frac{1}{2}\right)^2\right)\left(\left(\frac{1}{2}\right)^2 - 2\left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^4\right)\right.$$

$$\left. + \frac{0.3\left(\left(\frac{1}{2}\right)^2 - 2\left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^4\right)\left(2 - 12\left(\frac{1}{2}\right) + 12\left(\frac{1}{2}\right)^2\right)}{P^2}\right]$$

$$M_{xC} = \frac{-AD}{a^2}\left[-0.0625 - \frac{0.01875}{P^2}\right]$$

$$M_{xC} = \frac{AD}{a^2}\left[\frac{0.01875}{P^2} + 0.0625\right] \qquad (3.371)$$

Substituting the expression of A in Equations (3.333) into Equation (3.371) yields Equation (3.372).

$$M_{xC} = \frac{D}{a^2}\left[\frac{0.01875}{P^2} + 0.0625\right]\frac{qa^4}{D}\frac{0.449P^4}{0.499P^4 + 0.272P^4 + 0.499}$$

$$M_{xC} = \left[\frac{0.01875}{P^2} + 0.0625\right]\frac{0.449P^4}{0.499P^4 + 0.272P^4 + 0.499}\ qa^2 \qquad (3.372)$$

Equation for mid-span bending moment in x – direction is as shown in Equation (3.358).

$$M_{xC} = \beta_{xC}qa^2 \qquad (3.373)$$

Equating Equation (3.372) and Equation (3.3738) and solving for β_{xC} in terms of the aspect ratio, P yields Equation (3.374).

$$\beta_{xC} = \left[\frac{0.01875}{P^2} + 0.0625\right]\frac{0.449P^4}{0.499P^4 + 0.272P^4 + 0.499} \qquad (3.374)$$

Where βxc is mid-span bending moment factor in the x – directions.

Similarly, substituting Equations (3.271) and (3.275) into Equation (3.71b) and simplifying the equation yields Equation (3.375).

$$M_{yC} = -D\left[\mu\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right]$$

$$= \frac{-AD}{a^2}\left[\mu\frac{\partial^2 h}{\partial R^2} + \frac{\partial^2 h}{P^2\,\partial Q^2}\right]$$

$$M_{yC} = \frac{-AD}{a^2}\left[0.3(2-12(R)+12(R)^2)((Q)^2-2(Q)^3+(Q)^4) + \frac{((R)^2-2(R)^3+(R)^4)(2-12(Q)+12(Q)^2)}{P^2}\right] \quad (3.375)$$

For mid-span moment at point C (See Figure 3.20), Where R = 0.5 and Q = 0.5, Equation (3.375) will be as shown in Equation (3.376).

$$M_{yC} = \frac{-AD}{a^2}\left[0.3\left(2-12\left(\frac{1}{2}\right)+12\left(\frac{1}{2}\right)^2\right)\left(\left(\frac{1}{2}\right)^2-2\left(\frac{1}{2}\right)^3+\left(\frac{1}{2}\right)^4\right) + \frac{\left(\left(\frac{1}{2}\right)^2-2\left(\frac{1}{2}\right)^3+\left(\frac{1}{2}\right)^4\right)\left(2-12\left(\frac{1}{2}\right)+12\left(\frac{1}{2}\right)^2\right)}{P^2}\right]$$

$$M_{yC} = \frac{-AD}{a^2}\left[-0.01875-\frac{0.0625}{P^2}\right]$$

$$M_{yC} = \frac{AD}{a^2}\left[\frac{0.0625}{P^2}+0.01875\right] \quad (3.376)$$

Substituting the expression of A in Equations (3.333) into Equation (3.376) yields Equation (3.377).

$$M_{yC} = \frac{D}{a^2}\left[\frac{0.0625}{P^2}+0.01875\right]\frac{qa^4}{D}\frac{0.449P^4}{0.499P^4+0.272P^4+0.499}$$

$$M_{yC} = \left[\frac{0.0625}{P^2}+0.01875\right]\frac{0.449P^4}{0.499P^4+0.272P^4+0.499}\ qa^2 \quad (3.377)$$

Equation for mid-span bending moment in y – direction is as shown in Equation (3.378).

$$M_{yC} = \beta_{yC}qa^2 \quad (3.378)$$

Equating equation (3.377) and equation (3.378) and solving for β_{yC} in terms of the aspect ratio, P yields Equation (3.379).

$$\beta_{yC} = \left[\frac{0.0625}{P^2}+0.01875\right]\frac{0.449P^4}{0.499P^4+0.272P^4+0.499} \quad (3.379)$$

Where βyc is mid-span bending moment factor about y– direction.

EDGE SHEAR FORCE, V_{xB} and V_{yB} ($\mu = 0.3$)

Substituting Equations (3.272), (3.280) into Equation (3.82b) and simplifying the equation yields Equation (3.380).

$$V_x = -D\left[\frac{\partial^3 w}{\partial x^3}+(2-\mu)\frac{\partial^3 w}{\partial x\,\partial y^2}\right]$$

$$= \frac{-AD}{a^3}\left[\frac{\partial^3 h}{\partial R^3} + (2-\mu)\frac{\partial^3 h}{P^2\,\partial R\,\partial Q^2}\right]$$

$$= \frac{-AD}{a^3}\Bigg[12(2R-1)(Q^2-2Q^3+Q^4)$$

$$+ (2-\mu)\frac{\left(2(R-3R^2+2R^3)(1-6Q+6Q^2)\right)}{P^2}\Bigg] \qquad (3.380)$$

Highest shear force occurs at the edges B where R = 0, Q = 0.5 and R = 1, Q = 0.5. Substituting R = 0 and Q = 0.5 into Equation (3.380) yields Equation (3.381).

$$V_{xB} = \frac{-AD}{a^3}\left[12\left((2(0)-1)\left(\left(\frac{1}{2}\right)^2 - 2\left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^4\right)\right)\right.$$

$$\left. + (2-0.3)\frac{2\left[\left((0)-3(0)^2+2(0)^3\right)\left(1-6\left(\frac{1}{2}\right)+6\left(\frac{1}{2}\right)^2\right)\right]}{P^2}\right]$$

$$V_{xB} = \frac{-AD}{a^3}\left[0.75 + \frac{0}{P^2}\right]$$

$$V_{xB} = \frac{AD}{a^3}[0.75] \qquad (3.381)$$

Substituting the expression for A in equation (3.333) into Equation (3.381) yields Equation (3.382).

$$V_{xB} = \frac{D}{a^3}[0.75]\frac{qa^4}{D}\frac{0.449P^4}{0.499P^4 + 0.272P^4 + 0.499}$$

$$V_{xB} = 0.75\left(\frac{0.449P^4}{0.499P^4 + 0.272P^4 + 0.499}\right)qa \qquad (3.382)$$

Equation for maximum shear force in x – direction is as shown in Equation (3.383).

$$V_{xB} = k_{sxB}qa \qquad (3.383)$$

Equating Equation (3.382) and Equation (3.383) and solving for k_{sxB} in terms of the aspect ratio, P, yields Equation (3.384).

$$k_{sxB} = 0.75\left(\frac{0.449P^4}{0.499P^4 + 0.272P^4 + 0.499}\right) \qquad (3.384)$$

Where k_{sxB} is shear force factor in the x – direction.

Similarly, substituting Equations (3.276) and (3.279) into Equation (3.83b) and simplifying the equation yields Equation (3.385).

$$\mathrm{V_y} = -\mathrm{D}\left[\frac{\partial^3 \mathrm{w}}{\partial \mathrm{y}^3} + (2-\mu)\frac{\partial^3 \mathrm{w}}{\partial \mathrm{x}^2\,\partial \mathrm{y}}\right]$$

$$= \frac{-\mathrm{AD}}{\mathrm{a}^3}\left[\frac{\partial^3 \mathrm{h}}{\mathrm{P}^3\,\partial \mathrm{Q}^3} + (2-\mu)\frac{\partial^3 \mathrm{h}}{\mathrm{P}\,\partial \mathrm{QR}^2\,\partial \mathrm{Q}}\right]$$

$$= \frac{-\mathrm{AD}}{\mathrm{a}^3}\left[\frac{12(\mathrm{R}^2 - 2\mathrm{R}^3 + \mathrm{R}^4)(2\mathrm{Q}-1)}{\mathrm{P}^3} + (2-\mu)\frac{\left(2(1-6\mathrm{R}+6\mathrm{R}^2)(\mathrm{Q}-3\mathrm{Q}^2+2\mathrm{Q}^3)\right)}{\mathrm{P}}\right] \tag{3.385}$$

For maximum shear at point A (See Figure 3.20), $\mathrm{V_{yA}}$, where R = 0.5 and Q = 0 and 1. Substituting the values of R and Q and simplifying the equation yields Equation (3.386).

$$\mathrm{V_{yA}} = \frac{-\mathrm{AD}}{\mathrm{a}^3}\left[\frac{12\left(\left(\left(\frac{1}{2}\right)^2 - 2\left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^4\right)(2(0)-1)\right)}{\mathrm{P}^3} + (2-0.3)\frac{2\left(1-6\left(\frac{1}{2}\right)+6\left(\frac{1}{2}\right)^2\right)\left((0)-3(0)^2+2(0)^3\right)}{\mathrm{P}}\right]$$

$$\mathrm{V_{yA}} = \frac{-\mathrm{AD}}{\mathrm{a}^3}\left[\frac{0.75}{\mathrm{P}^3} + \frac{0}{\mathrm{P}}\right]$$

$$\mathrm{V_{yA}} = \frac{\mathrm{AD}}{\mathrm{a}^3}\left[\frac{0.75}{\mathrm{P}^3}\right] \tag{3.386}$$

Substituting the exact coefficient equation, Equation (3.333) into Equation (3.386) yields Equation (3.387).

$$\mathrm{V_{yA}} = \frac{\mathrm{D}}{\mathrm{a}^3}\left[\frac{0.75}{\mathrm{P}^3}\right]\frac{\mathrm{qa}^4}{\mathrm{D}}\,\frac{0.449\mathrm{P}^4}{0.499\mathrm{P}^4 + 0.272\mathrm{P}^4 + 0.499}$$

$$\mathrm{V_{yA}} = \left[\frac{0.75}{\mathrm{P}^3}\right]\left(\frac{0.449\mathrm{P}^4}{0.499\mathrm{P}^4 + 0.272\mathrm{P}^4 + 0.499}\right)\mathrm{qa} \tag{3.387}$$

Equation for maximum shear force in y – direction is as shown in Equation (3.388).

$$\mathrm{V_{yA}} = \mathrm{k_{syA}}\mathrm{qa} \tag{3.388}$$

Equating Equation (3.387) and Equation (3.388) and solving for $\mathrm{k_{syA}}$ in terms of the aspect ratio, P yields Equation (3.389).

$$\mathrm{k_{syA}} = \left[\frac{0.75}{\mathrm{P}^3}\right]\left(\frac{0.449\mathrm{P}^4}{0.499\mathrm{P}^4 + 0.272\mathrm{P}^4 + 0.499}\right) \tag{3.389}$$

Where $\mathrm{k_{yA}}$ is shear force factor in the y– direction.

FACTORS FOR PLATE TYPE CSCS

Figure 3.21 shows a thin rectangular plate under uniformly distributed lateral load, q. The plate is clamped on two opposite short edges and supported on two opposite long edges.

From Equation (3.199), the shape function for CSCS is given as:

$$W = A(R - 2R^3 + R^4)(Q^2 - 2Q^3 + Q^4)$$

Where A, from Equation (3.334) is given as,

$$A = \frac{qa^4}{D}\frac{0.67p^4}{(0.745p^4 + 1.477p^2 + 3.216)}$$

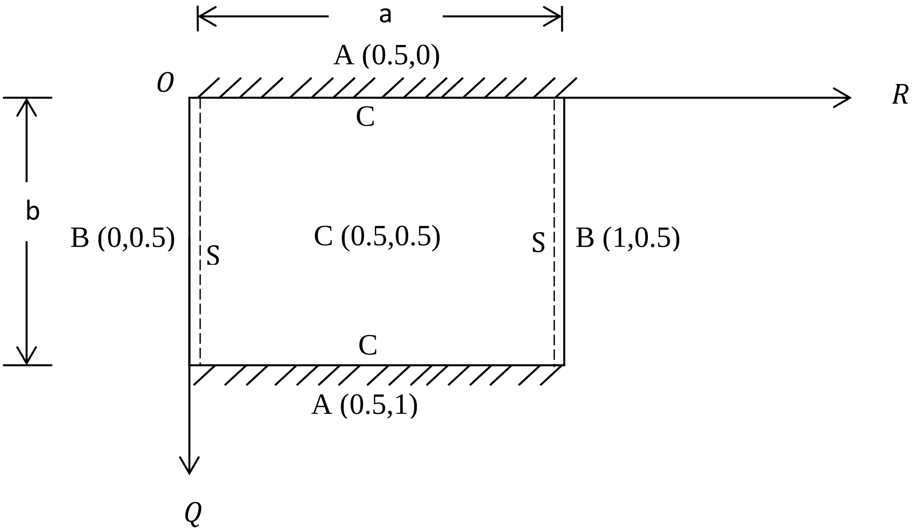

Figure 3.21: CSCS Plate Showing Points of Maximum Values

The critical (maximum) values of deflections and moments will occur at the center of the plate where:

$$x = \frac{a}{2}\ ; \left(x = \frac{Pb}{2} \text{ since } P = \frac{a}{b}\right) \text{ and } y = \frac{b}{2} \qquad (3.390)$$

In non – dimensional form, for critical (maximum) deflections and moments will occur at the center of the plate where:

$$R = \frac{1}{2} \text{ and } Q = \frac{1}{2} \qquad (3.391)$$

MAXIMUM DEFLECTIONS

Substituting Equation (3.391) into Equation (3.199) yields Equation (3.392)

$$W_{max} = A\left(\frac{1}{2} - 2\left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^4\right)\left(\left(\frac{1}{2}\right)^2 - 2\left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^4\right)$$

$$\mathrm{W_{max}} = \mathrm{A}\left(\frac{5}{16}\right)\left(\frac{1}{16}\right)$$

$$\mathrm{W_{max}} = \mathrm{A}\left(\frac{5}{256}\right) \qquad (3.392)$$

Substituting the expression of A in Equation (3.334) into Equation (3.392) yields Equation (3.393).

$$\mathrm{W_{max}} = \left(\frac{5}{256}\right)\frac{\mathrm{qa^4}}{\mathrm{D}}\frac{0.67\mathrm{p}^4}{(0.745\mathrm{p}^4 + 1.477\mathrm{p}^2 + 3.216)}$$

$$\mathrm{W_{max}} = \left(\left(\frac{5}{256}\right)\frac{0.67\mathrm{p}^4}{(0.745\mathrm{p}^4 + 1.477\mathrm{p}^2 + 3.216)}\right)\frac{\mathrm{qa^4}}{\mathrm{D}} \qquad (3.393)$$

Equation for maximum deflection is as shown in Equation (3.394).

$$\mathrm{W_{max}} = \mathrm{k_D}\frac{\mathrm{qa^4}}{\mathrm{D}} \qquad (3.394)$$

Equating Equation (3.393) and Equation (3.394) and solving for $\mathrm{k_D}$ in terms of the aspect ratio, P yields Equation (3.395).

$$\mathrm{k_D} = \left(\frac{5}{256}\right)\frac{0.67\mathrm{p}^4}{(0.745\mathrm{p}^4 + 1.477\mathrm{p}^2 + 3.216)} \qquad (3.395)$$

Where $\mathrm{k_D}$ is the factor of deflection.

MAXIMUM MOMENTS, $\mathrm{M_{xC}}$ and $\mathrm{M_{yC}}$ ($\mu = 0.3$)

Substituting Equations (3.283) and (3.287) into Equation (3.70b) and simplifying the equation yields Equation (3.396).

$$\mathrm{M_x} = -\mathrm{D}\left[\frac{\partial^2 \mathrm{w}}{\partial \mathrm{x}^2} + \mu\frac{\partial^2 \mathrm{w}}{\partial \mathrm{y}^2}\right]$$

$$= \frac{-\mathrm{AD}}{\mathrm{a}^2}\left[\frac{\partial^2 \mathrm{w}}{\partial \mathrm{R}^2} + \mu\frac{\partial^2 \mathrm{w}}{\mathrm{P}^2\,\partial \mathrm{Q}^2}\right]$$

$$= \frac{-\mathrm{AD}}{\mathrm{a}^2}\left[(12\mathrm{R}^2 - 12\mathrm{R})(\mathrm{Q}^2 - 2\mathrm{Q}^3 + \mathrm{Q}^4) + \frac{\mu(\mathrm{R} - 2\mathrm{R}^3 + \mathrm{R}^4)(2 - 12\mathrm{Q} + 12\mathrm{Q}^2)}{\mathrm{P}^2}\right] \qquad (3.396)$$

For mid-span moment, $\mathrm{M_{xC}}$, substitute the values of R and Q (See Figure 3.21), into Equation (3.396) to obtain Equation (3.397).

$$M_{xC} = \frac{-AD}{a^2}\left[\left(12\left(\frac{1}{2}\right)^2 - 12\left(\frac{1}{2}\right)\right)\left(\left(\frac{1}{2}\right)^2 - 2\left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^4\right) + \frac{0.3\left(\left(\frac{1}{2}\right) - 2\left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^4\right)\left(2 - 12\left(\frac{1}{2}\right) + 12\left(\frac{1}{2}\right)^2\right)}{P^2}\right]$$

$$M_{xC} = \frac{-AD}{a^2}\left[-0.1875 - \frac{0.09375}{P^2}\right]$$

$$M_{xC} = \frac{AD}{a^2}\left[\frac{0.09375}{P^2} + 0.1875\right] \qquad (3.397)$$

Substituting the expression of A in Equation (3.334) into Equation (3.397) yields Equation (3.398).

$$M_{xC} = \frac{D}{a^2}\left[\frac{0.09375}{P^2} + 0.1875\right]\frac{qa^4}{D}\frac{0.67p^4}{(0.745p^4 + 1.477p^2 + 3.216)}$$

$$M_{xC} = \left[\frac{0.09375}{P^2} + 0.1875\right] \times \frac{0.67p^4}{(0.745p^4 + 1.477p^2 + 3.216)} \times qa^2 \qquad (3.398)$$

Equation for maximum bending moment in x – direction is as shown in Equation (3.385).

$$M_{xC} = \beta_{xC}qa^2 \qquad (3.399)$$

Equating Equation (3.398) and Equation (3.399) and solving for β_{xC} in terms of the aspect ratio, P yields Equation (3.400).

$$\beta_{xC} = \left[\frac{0.09375}{P^2} + 0.1875\right]\frac{0.67p^4}{(0.745p^4 + 1.477p^2 + 3.216)} \qquad (3.400)$$

Where β_{xC} is bending moment factor about x – directions.

Similarly, for mid-span moment in y – direction, substituting Equations (3.283) and (3.287) into Equation (3.71b) for R = 0.5 and Q = 0.5 and simplifying the equation yields Equation (3.401).

$$M_y = -D\left[\mu\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right]$$

$$= \frac{-AD}{a^2}\left[\mu\frac{\partial^2 h}{\partial R^2} + \frac{\partial^2 h}{P^2\,\partial Q^2}\right]$$

$$M_{yC} = \frac{-AD}{a^2}\left[0.3\left(12\left(\frac{1}{2}\right)^2 - 12\left(\frac{1}{2}\right)\right)\left(\left(\frac{1}{2}\right)^2 - 2\left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^4\right) + \frac{\left(\left(\frac{1}{2}\right) - 2\left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^4\right)\left(2 - 12\left(\frac{1}{2}\right) + 12\left(\frac{1}{2}\right)^2\right)}{P^2}\right]$$

$$M_{yC} = \frac{-AD}{a^2}\left[-0.05625 - \frac{0.3125}{P^2}\right]$$

$$M_{yC} = \frac{AD}{a^2}\left[\frac{0.3125}{P^2} + 0.05625\right] \qquad (3.401)$$

Substituting the expression of A in Equation (3.334) into Equation (3.401) yields Equation (3.402).

$$M_{yC} = \frac{D}{a^2}\left[\frac{0.3125}{P^2} + 0.05625\right]\frac{qa^4}{D}\frac{0.67p^4}{(0.745p^4 + 1.477p^2 + 3.216)}$$

$$M_{yC} = \left[\frac{0.3125}{P^2} + 0.05625\right]\frac{0.67p^4}{(0.745p^4 + 1.477p^2 + 3.216)}qa^2 \qquad (3.402)$$

Equation for maximum bending moment in y – direction is as shown in Equation (3.389).

$$M_{yC} = \beta_{yC}qa^2 \qquad (3.403)$$

Equating Equation (3.402) and Equation (3.403) and solving for β_{xC} in terms of the aspect ratio, P yields Equation (3.404).

$$\beta_{yC} = \left[\frac{0.3125}{P^2} + 0.05625\right]\frac{0.67p^4}{(0.745p^4 + 1.477p^2 + 3.216)} \qquad (3.404)$$

Where β_{yC} is bending moment factor in the y– direction.

EDGE SHEAR FORCE, V_{xB} and V_{yA} ($\mu = 0.3$)

Substituting Equations (3.284) and (3.292) into Equation (3.82b) for R = 1 and Q = 0.5 and simplifying the equation yields Equation (3.405).

$$V_x = -D\left[\frac{\partial^3 w}{\partial x^3} + (2-\mu)\frac{\partial^3 w}{\partial x\,\partial y^2}\right]$$

$$= \frac{-AD}{a^3}\left[\frac{\partial^3 h}{\partial R^3} + (2-\mu)\frac{\partial^3 h}{P^2\,\partial R\,\partial Q^2}\right]$$

$$= \frac{-AD}{a^3}\left[(24R - 12)(Q^2 - 2Q^3 + Q^4) + (2-\mu)\frac{\left((1 - 6R^2 + 4R^3)(2 - 12Q + 12Q^2)\right)}{P^2}\right]$$

$$V_{xB} = \frac{-AD}{a^3}\left[\left((24(1) - 12)\left(\left(\frac{1}{2}\right)^2 - 2\left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^4\right)\right) + (2 - 0.3)\frac{\left[(1 - 6(1)^2 + 4(1)^3)\left(2 - 12\left(\frac{1}{2}\right) + 12\left(\frac{1}{2}\right)^2\right)\right]}{P^2}\right]$$

$$V_{xB} = \frac{-AD}{a^3}\left[0.75 + \frac{1.7}{P^2}\right]$$

$$V_{xB} = \frac{AD}{a^3}\left[\frac{1.7}{P^2} + 0.75\right] \qquad (3.405)$$

Substituting the expression of A in Equation (3.334) into Equation (3.405) yields Equation (3.406).

$$V_{xB} = \frac{D}{a^3}\left[\frac{1.7}{P^2} + 0.75\right]\frac{qa^4}{D}\frac{0.67p^4}{(0.745p^4 + 1.477p^2 + 3.216)}$$

$$V_{xB} = \left[\frac{1.7}{P^2} + 0.75\right]\left(\frac{0.67p^4}{0.745p^4 + 1.477p^2 + 3.216}\right)qa \qquad (3.406)$$

Equation for maximum edge shear force in x – direction is as shown in Equation (3.393).

$$V_{xB} = k_{sxB}qa \qquad (3.407)$$

Equating Equation (3.392) and Equation (3.393) and solving for k_{sxB} in terms of the aspect ratio, P yields Equation (3.394).

$$k_{sxB} = \left[\frac{1.7}{P^2} + 0.75\right]\left(\frac{0.67p^4}{0.745p^4 + 1.477p^2 + 3.216}\right) \qquad (3.408)$$

Where k_{sxB} is shear force factor in the x – directions.

Similarly, substituting Equations (3.288) and (3.291) into Equation (3.83b) and simplifying the equation for R = 1 and Q = 0.5, yields Equation (3.409).

$$V_{yA} = -D\left[\frac{\partial^3 w}{\partial y^3} + (2 - \mu)\frac{\partial^3 w}{\partial x^2\,\partial y}\right]$$

$$= \frac{-AD}{a^3}\left[\frac{\partial^3 h}{P^3\,\partial Q^3} + (2 - \mu)\frac{\partial^3 h}{P\,\partial Q\,\partial R^2}\right]$$

$$= \frac{-AD}{a^3}\left[\frac{12(2Q - 1)(R - 2R^3 + R^4)}{P^3} + (2 - \mu)\frac{\left(24(R^2 - R)(Q - 3Q^2 + 2Q^3)\right)}{P}\right]$$

$$V_{yA} = \frac{-AD}{a^3}\left[\frac{12(2(1)-1)\left(\left(\frac{1}{2}\right)-2\left(\frac{1}{2}\right)^3+\left(\frac{1}{2}\right)^4\right)}{P^3} + (2-0.3)\frac{24\left(\left(\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)\right)(1-3(1)^2+2(1)^3)}{P}\right]$$

$$V_{yA} = \frac{-AD}{a^3}\left[\frac{3.75}{P^3}+\frac{0}{P}\right]$$

$$V_{yA} = \frac{AD}{a^3}\left[\frac{3.75}{P^3}\right] \tag{3.409}$$

Substituting the expression of A in Equation (3.334) into Equation (3.409) yields Equation (3.410).

$$V_{yA} = \frac{D}{a^3}\left[\frac{3.75}{P^3}\right]\frac{qa^4}{D}\frac{0.67p^4}{(0.745p^4+1.477p^2+3.216)}$$

$$V_{yA} = \left[\frac{3.75}{P^3}\right]\left(\frac{0.67p^4}{(0.745p^4+1.477p^2+3.216)}\right)qa \tag{3.410}$$

Equation for maximum edge shear force in y – direction is as shown in Equation (3.376).

$$V_{yA} = k_{syA}qa \tag{3.411}$$

Equating Equation (3.410) and Equation (3.411) and solving for k_{syA} in terms of the aspect ratio, P yields Equation (3.412).

$$k_{syA} = \left[\frac{3.75}{P^3}\right]\left(\frac{0.67p^4}{0.745p^4+1.477p^2+3.216}\right) \tag{3.412}$$

Where k_{syA} is shear force factor in the y– direction.

FACTORS FOR PLATE TYPE CCSS

Figure 3.22 shows a thin rectangular plate under uniformly distributed lateral load, q. The plate is clamped on one short and one long edge and simply supported on one short and one long edges.

From Equation (3.200), the shape function for CCSS is given as:

$$W = A(1.5R^2-2.5R^3+R^4)(1.5Q^2-2.5Q^3+Q^4)$$

From Equation (3.335), the exact coefficient, A:

$$A = \frac{qa^4}{D}\frac{0.64p^4}{(1.381p^4+1.384p^2+1.381)}$$

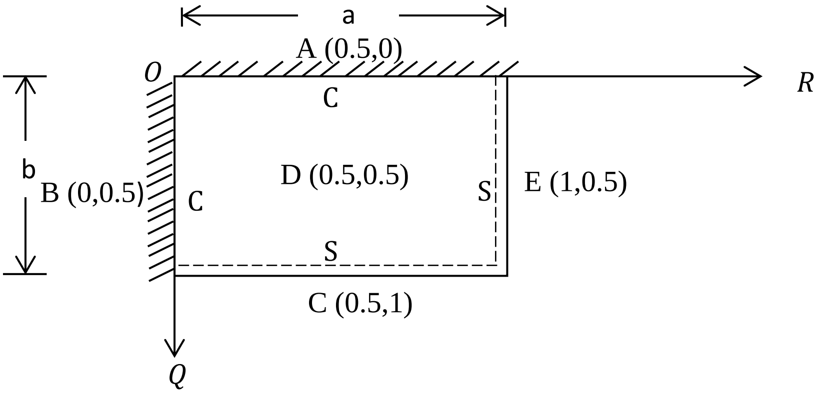

Figure 3.22: CCSS Plate Showing Points of Maximum Values

Considering the value of deflection at the center, D of the plate where:

$$x = \frac{a}{2}\ ;\left(x = \frac{Pb}{2} \text{ since } P = \frac{a}{b}\right) \text{ and } y = \frac{b}{2} \tag{3.413}$$

In non – dimensional form, the center deflection will occur on the plate where at points,

$$R = \frac{1}{2} \text{ and } Q = \frac{1}{2} \tag{3.414}$$

DEFLECTION AT THE CENTRE

Substituting Equation (3.414) into Equation (3.200) yields Equation (3.415).

$$W_D = A\left(1.5\left(\frac{1}{2}\right)^2 - 2.5\left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^4\right)\left(1.5\left(\frac{1}{2}\right)^2 - 2.5\left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^4\right)$$

$$W_D = A\left(\frac{1}{8}\right)\left(\frac{1}{8}\right)$$

$$W_D = A\left(\frac{1}{64}\right) \tag{3.415}$$

Substituting the expression of A in Equation (3.335) into Equation (3.415) yields Equation (3.416).

$$W_D = \left(\frac{1}{64}\right)\frac{qa^4}{D}\frac{0.64p^4}{(1.381p^4 + 1.384p^2 + 1.381)}$$

$$W_D = \left(\frac{1}{64}.\frac{0.64p^4}{(1.381p^4 + 1.384p^2 + 1.381)}\right)\frac{qa^4}{D} \tag{3.416}$$

Equation for central deflection is as shown in Equation (3.417).

$$W_D = k_D \frac{qa^4}{D} \tag{3.417}$$

Equating Equation (3.416) and Equation (3.417) and solving for k_D in terms of the aspect ratio, P yields Equation (3.418).

$$k_D = \left(\frac{1}{64}\right)\frac{0.64p^4}{(1.381p^4 + 1.384p^2 + 1.381)} \tag{3.418}$$

Where k_D is the factor of deflection.

MID-SPAN MOMENTS, M_{xD} and M_{yD} ($\mu = 0.3$)

Substituting Equations (3.295) and (3.300) into Equation (3.70b) and simplifying the equation for R = 0.5 and Q = 0.5 at point D, yields Equation (3.419).

$$\begin{aligned} M_x &= -D\left[\frac{\partial^2 w}{\partial x^2} + \mu\frac{\partial^2 w}{\partial y^2}\right] \\ &= \frac{-AD}{a^2}\left[\frac{\partial^2 h}{\partial R^2} + \mu\frac{\partial^2 h}{P^2\,\partial Q^2}\right] \\ &= \frac{-AD}{a^2}\left[(3 - 15R + 12R^2)(1.5Q^2 - 2.5Q^3 + Q^4)\right. \\ &\qquad\left. + \frac{\mu(1.5R^2 - 2.5R^3 + R^4)(3 - 15Q + 12Q^2)}{P^2}\right] \end{aligned} \tag{3.419}$$

For mid-span moment at point D, M_{xD}, substituting the values of R and Q into the Equation (3.419) yields Equation (3.420).

$$\begin{aligned} M_{xD} = \frac{-AD}{a^2}&\left[\left(3 - 15\left(\frac{1}{2}\right) + 12\left(\frac{1}{2}\right)^2\right)\left(1.5\left(\frac{1}{2}\right)^2 - 2.5\left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^4\right)\right. \\ &\left. + \frac{0.3\left(1.5\left(\frac{1}{2}\right)^2 - 2.5\left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^4\right)\left(3 - 15\left(\frac{1}{2}\right) + 12\left(\frac{1}{2}\right)^2\right)}{P^2}\right] \end{aligned}$$

$$M_{xD} = \frac{-AD}{a^2}\left[-0.1875 - \frac{0.05625}{P^2}\right]$$

$$M_{xD} = \frac{AD}{a^2}\left[\frac{0.05625}{P^2} + 0.1875\right] \tag{3.420}$$

Substituting the expression of A in Equation (3.335) into Equation (3.420) yields Equation (3.421).

$$M_{xD} = \frac{D}{a^2}\left[\frac{0.05625}{P^2} + 0.1875\right]\frac{qa^4}{D}\frac{0.64p^4}{(1.381p^4 + 1.384p^2 + 1.381)}$$

$$M_{xD} = \left[\frac{0.05625}{P^2} + 0.1875\right]\frac{0.64p^4}{(1.381p^4 + 1.384p^2 + 1.381)}qa^2 \tag{3.421}$$

Equation for mid-span bending moment at point D in x - direction is as shown in Equation (3.422).

$$M_{xD} = \beta_{xD}qa^2 \tag{3.422}$$

Equating Equation (3.421) and Equation (3.422) and solving for β_{xD} in terms of the aspect ratio, P yields Equation (3.423).

$$\beta_{xD} = \left[\frac{0.05625}{P^2} + 0.1875\right]\frac{0.64p^4}{(1.381p^4 + 1.384p^2 + 1.381)} \tag{3.423}$$

Where β_{xD} is bending moment factor in the x – directions.

Similarly, for mid-span moment in y – direction, substituting Equations (3.295) and (3.300) into Equation (3.71b) for R = 0.5 and Q = 0.5 and simplifying the equation yields Equation (3.424).

$$M_{yD} = -D\left[\mu\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right]$$

$$= \frac{-AD}{a^2}\left[\mu\frac{\partial^2 w}{\partial R^2} + \frac{\partial^2 w}{P^2\,\partial Q^2}\right]$$

$$M_{yD} = \frac{-AD}{a^2}\left[0.3\left(3 - 15\left(\frac{1}{2}\right) + 12\left(\frac{1}{2}\right)^2\right)\left(1.5\left(\frac{1}{2}\right)^2 - 2.5\left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^4\right) + \frac{\left(1.5\left(\frac{1}{2}\right)^2 - 2.5\left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^4\right)\left(3 - 15\left(\frac{1}{2}\right) + 12\left(\frac{1}{2}\right)^2\right)}{P^2}\right]$$

$$M_{yD} = \frac{-AD}{a^2}\left[-0.05625 - \frac{0.1875}{P^2}\right]$$

$$M_{yD} = \frac{AD}{a^2}\left[\frac{0.1875}{P^2} + 0.05625\right] \tag{3.424}$$

Substituting the expression of A in Equation (3.335) into Equation (3.424) yields Equation (3.425).

$$M_{yD} = \frac{D}{a^2}\left[\frac{0.1875}{P^2} + 0.05625\right]\frac{qa^4}{D}\frac{0.64p^4}{(1.381p^4 + 1.384p^2 + 1.381)}$$

$$M_{yD} = \left[\frac{0.1875}{P^2} + 0.05625\right]\frac{0.64p^4}{(1.381p^4 + 1.384p^2 + 1.381)}qa^2 \tag{3.425}$$

Equation for mid-span bending moment at point D in y - direction is as shown in Equation (3.426).

$$M_{yD} = \beta_{yD} qa^2 \tag{3.426}$$

Equating Equation (3.425) and Equation (3.426) and solving for β_{yD} in terms of the aspect ratio, P yields Equation (3.427).

$$\beta_{yD} = \left[\frac{0.1875}{P^2} + 0.05625\right]\frac{0.64p^4}{(1.381p^4 + 1.384p^2 + 1.381)} \tag{3.427}$$

Where β_{yD} is bending moment factor in the y – direction.

EDGE SHEAR FORCE, V_{xB} and V_{yA} ($\mu = 0.3$)

Substituting Equations (3.297) and (3.305) into Equation (3.82b) for R = 0 and Q = 0.5 and simplifying the equation yields Equation (3.428).

$$V_x = -D\left[\frac{\partial^3 w}{\partial x^3} + (2-\mu)\frac{\partial^3 w}{\partial x\,\partial y^2}\right]$$

$$= \frac{-AD}{a^3}\left[\frac{\partial^3 h}{\partial R^3} + (2-\mu)\frac{\partial^3 h}{P^2\,\partial R\,\partial Q^2}\right]$$

$$= \frac{-AD}{a^3}\left[(-15+24R)(1.5Q^2 - 2.5Q^3 + Q^4) + (2-\mu)\frac{(3R - 7.5R^2 + 4R^3)(3 - 15Q + 12Q^2)}{P^2}\right]$$

$$V_{xB} = \frac{-AD}{a^3}\left[\left((24(0)-15)\left(1.5\left(\frac{1}{2}\right)^2 - 2.5\left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^4\right)\right) + (2-0.3)\frac{\left[(3(0) - 7.5(0)^2 + 4(0)^3)\left(3 - 15\left(\frac{1}{2}\right) + 12\left(\frac{1}{2}\right)^2\right)\right]}{P^2}\right]$$

$$V_{xB} = \frac{-AD}{a^3}\left[-1.875 + \frac{0}{P^2}\right]$$

$$V_{xB} = \frac{AD}{a^3}[1.875] \tag{3.428}$$

Substituting the expression of A in Equation (3.335) into Equation (3.428) yields Equation (3.429).

$$V_{xB} = \frac{D}{a^3}[1.875]\frac{qa^4}{D}\frac{0.64p^4}{(1.381p^4 + 1.384p^2 + 1.381)}$$

$$V_{xB} = [1.875]\left(\frac{0.64p^4}{1.381p^4 + 1.384p^2 + 1.381}\right)qa \tag{3.429}$$

Equation for edge shear force at point B in x - direction is as shown in Equation (3.440).

$$V_{xB} = k_{sxB}qa \tag{3.430}$$

Equating Equation (3.429) and Equation (3.430) and solving for k_{sxB} in terms of the aspect ratio, P yields Equation (3.431).

$$k_{sxB} = 1.875\frac{0.64p^4}{(1.381p^4 + 1.384p^2 + 1.381)} \tag{3.431}$$

Where k_{sxB} is shear force factor in the x – direction.

Similarly, substituting Equations (3.301) and (3.304) into Equation (3.83b) and simplifying the equation for R = 0.5 and Q = 1 at point A, yields Equation (3.432).

$$V_y = -D\left[\frac{\partial^3 w}{\partial y^3} + (2-\mu)\frac{\partial^3 w}{\partial x^2\,\partial y}\right]$$

$$= \frac{-AD}{a^3}\left[\frac{\partial^3 h}{P^3\,\partial Q^3} + (2-\mu)\frac{\partial^3 h}{P\,\partial QR^2\,\partial Q}\right]$$

$$= \frac{-AD}{a^3}\left[\frac{(24Q-15)(1.5R^2-2.5R^3+R^4)}{P^3} + (2-\mu)\frac{\left((3Q-7.5Q^2+4Q^3)(3-15R+12R^2)\right)}{P}\right]$$

$$V_{yC} = \frac{-AD}{a^3}\left[\frac{(24(1)-15)\left(1.5\left(\frac{1}{2}\right)^2 - 2.5\left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^4\right)}{P^3} + (2-0.3)\frac{\left((3(1)-7.5(1)^2+4(1)^3)\right)\left(3-15\left(\frac{1}{2}\right)+12\left(\frac{1}{2}\right)^2\right)}{P}\right]$$

$$V_{yC} = \frac{-AD}{a^3}\left[\frac{1.125}{P^3} - \frac{1.275}{P}\right]$$

$$V_{yC} = \frac{AD}{a^3}\left[\frac{1.125}{P^3} + \frac{1.275}{P}\right] \tag{3.432}$$

Substituting the expression of A in Equation (3.335) into Equation (3.432) yields Equation (3.433).

$$V_{yC} = \frac{D}{a^3}\left[\frac{1.125}{P^3} + \frac{0.75}{P}\right]\frac{qa^4}{D}\frac{0.64p^4}{(1.381p^4 + 1.384p^2 + 1.381)}$$

$$V_{yC} = \left[\frac{1.125}{P^3} + \frac{0.75}{P}\right]\left(\frac{0.64p^4}{1.381p^4 + 1.384p^2 + 1.381}\right)qa \tag{3.433}$$

Equation for edge shear force at point C in y - direction is as shown in Equation (3.444).

$$V_{yC} = k_{syC}qa \tag{3.434}$$

Equating Equation (3.433) and Equation (3.434) and solving for k_{syC} in terms of the aspect ratio, P yields Equation (3.445).

$$k_{syC} = \left[\frac{1.125}{P^3} + \frac{0.75}{P}\right]\left(\frac{0.64p^4}{1.381p^4 + 1.384p^2 + 1.381}\right) \tag{3.435}$$

Where k_{syC} is shear force factor in the y – direction.

FACTORS FOR PLATE TYPE CSSS

Figure 3.23 shows a thin rectangular plate under uniformly distributed lateral load, q. The plate is clamped on one short edge, simply supported one the opposite edge, and simply supported on two opposite long edges.

From Equation (3.201), the shape function for CSSS is given as:

$$W = A(R - 2R^3 + R^4)(1.5Q^2 - 2.5Q^3 + Q^4)$$

Where,

$$A = \frac{qa^4}{D}\frac{0.8p^4}{(1.726p^4 + 1.664p^2 + 3.84)}$$

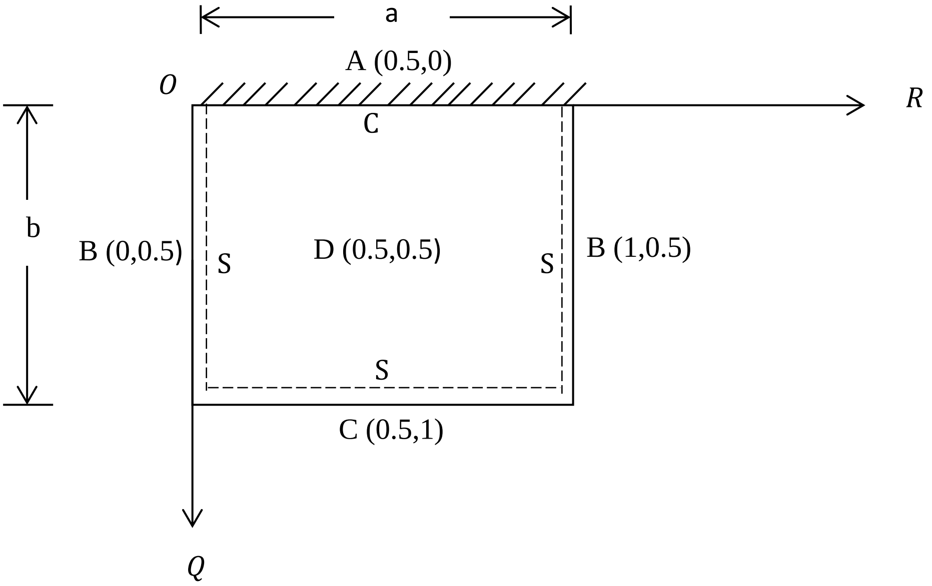

Figure 3.23: CSSS Plate Showing Points of Maximum Values

The critical value of deflection will occur at the center, D of the plate where:

$$x = \frac{a}{2}\ ; \left(x = \frac{Pb}{2} \text{ since } P = \frac{a}{b}\right) \text{ and } y = \frac{b}{2} \qquad (3.436)$$

In non – dimensional form, the center deflection will occur on the plate where at points,

$$R = \frac{1}{2} \text{ and } Q = \frac{1}{2} \qquad (3.437)$$

DEFLECTION AT THE CENTRE

Substituting Equation (3.437) into Equation (3.201) yields Equation (3.438).

$$W_D = A\left(\frac{1}{2} - 2\left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^4\right)\left(1.5\left(\frac{1}{2}\right)^2 - 2.5\left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^4\right)$$

$$W_D = A\left(\frac{5}{16}\right)\left(\frac{1}{8}\right)$$

$$W_D = A\left(\frac{5}{128}\right) \qquad (3.438)$$

Substituting the expression of A in Equation (3.336) into Equation (3.438) yields Equation (3.439).

$$W_D = \left(\frac{5}{128}\right)\frac{qa^4}{D}\frac{0.8p^4}{(1.726p^4 + 3.328p^2 + 3.84)}$$

$$W_D = \left(\left(\frac{5}{128}\right)\frac{0.8p^4}{(1.726p^4 + 3.328p^2 + 3.84)}\right)\frac{qa^4}{D} \qquad (3.439)$$

Equation for central deflection is as shown in Equation (3.440).

$$W_D = k_D\frac{qa^4}{D} \qquad (3.440)$$

Equating Equation (3.439) and Equation (3.440) and solving for k_D in terms of the aspect ratio, P yields Equation (3.441).

$$k_D = \left(\left(\frac{5}{128}\right)\frac{0.8p^4}{(1.726p^4 + 3.328p^2 + 3.84)}\right) \qquad (3.441)$$

Where k_D is the factor of deflection.

MID-SPAN MOMENTS, M_{xD} and M_{yD} ($\mu = 0.3$)

Substituting Equations (3. 308) and (3.312) into Equation (3.70b) and simplifying the equation yields Equation (3.405).

$$M_x = -D\left[\frac{\partial^2 w}{\partial x^2} + \mu\frac{\partial^2 w}{\partial y^2}\right]$$

$$= \frac{-\mathrm{AD}}{\mathrm{a}^2}\left[\frac{\partial^2 \mathrm{h}}{\partial \mathrm{R}^2} + \mu \frac{\partial^2 \mathrm{h}}{\mathrm{P}^2\,\partial \mathrm{Q}^2}\right]$$

$$= \frac{-\mathrm{AD}}{\mathrm{a}^2}\Bigg[(12\mathrm{R}^2 - 12\mathrm{R})(1.5\mathrm{Q}^2 - 2.5\mathrm{Q}^3 + \mathrm{Q}^4) + \frac{3\mu(\mathrm{R} - 2\mathrm{R}^3 + \mathrm{R}^4)(1 - 5\mathrm{Q} + 4\mathrm{Q}^2)}{\mathrm{P}^2}\Bigg] \tag{3.442}$$

For central moment, MxD, at point D, substitute the values of R and Q (See Figure 3.23) into Equation (3.442).

$$\mathrm{M}_{\mathrm{xD}} = \frac{-\mathrm{AD}}{\mathrm{a}^2}\left[\left(12\left(\frac{1}{2}\right)^2 - 12\left(\frac{1}{2}\right)\right)\left(1.5\left(\frac{1}{2}\right)^2 - 2.5\left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^4\right) + \frac{0.9\left(\left(\frac{1}{2}\right) - 2\left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^4\right)\left(1 - 5\left(\frac{1}{2}\right) + 4\left(\frac{1}{2}\right)^2\right)}{\mathrm{P}^2}\right]$$

$$\mathrm{M}_{\mathrm{xD}} = \frac{-\mathrm{AD}}{\mathrm{a}^2}\left[-0.375 - \frac{0.1406}{\mathrm{P}^2}\right]$$

$$\mathrm{M}_{\mathrm{xD}} = \frac{\mathrm{AD}}{\mathrm{a}^2}\left[\frac{0.1406}{\mathrm{P}^2} + 0.375\right] \tag{3.443}$$

Substituting the expression of A in Equation (3.336) into Equation (3.443) yields Equation (3.444).

$$\mathrm{M}_{\mathrm{xD}} = \frac{\mathrm{D}}{\mathrm{a}^2}\left[\frac{0.1406}{\mathrm{P}^2} + 0.375\right]\frac{\mathrm{qa}^4}{\mathrm{D}}\frac{0.8\mathrm{p}^4}{(1.726\mathrm{p}^4 + 3.328\mathrm{p}^2 + 3.84)}$$

$$\mathrm{M}_{\mathrm{xD}} = \left[\frac{0.1406}{\mathrm{P}^2} + 0.375\right]\frac{0.8\mathrm{p}^4}{(1.726\mathrm{p}^4 + 3.328\mathrm{p}^2 + 3.84)}\mathrm{qa}^2 \tag{3.444}$$

Equation for central bending moment in the x-direction is as shown in Equation (3.408).

$$\mathrm{M}_{\mathrm{xD}} = \beta_{\mathrm{xD}}\mathrm{qa}^2 \tag{3.445}$$

Equating Equation (3.444) and Equation (3.445) and solving for β_{xD} in terms of the aspect ratio, P yields Equation (3.446).

$$\beta_{\mathrm{xD}} = \left[\frac{0.1406}{\mathrm{P}^2} + 0.375\right]\frac{0.8\mathrm{p}^4}{(1.726\mathrm{p}^4 + 3.328\mathrm{p}^2 + 3.84)}\mathrm{qa}^2 \tag{3.446}$$

Where β_{xD} is bending moment factor in the x – direction.

Similarly, for mid-span moment in y – direction, substituting Equations (3.308) and (3.312) into Equation (3.71b) for R = 0.5 and Q = 0.5 and simplifying the equation yields Equation (3.447).

$$\mathrm{M_y} = -\mathrm{D}\left[\mu\frac{\partial^2 \mathrm{w}}{\partial \mathrm{x}^2} + \frac{\partial^2 \mathrm{w}}{\partial \mathrm{y}^2}\right]$$

$$= \frac{-\mathrm{AD}}{\mathrm{a}^2}\left[\mu\frac{\partial^2 \mathrm{h}}{\partial \mathrm{R}^2} + \frac{\partial^2 \mathrm{h}}{\mathrm{P}^2\,\partial \mathrm{Q}^2}\right]$$

$$\mathrm{M_{yD}} = \frac{-\mathrm{AD}}{\mathrm{a}^2}\left[0.3\left(12\left(\frac{1}{2}\right)^2 - 12\left(\frac{1}{2}\right)\right)\left(1.5\left(\frac{1}{2}\right)^2 - 2.5\left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^4\right) + \frac{3\left(\left(\frac{1}{2}\right) - 2\left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^4\right)\left(1 - 5\left(\frac{1}{2}\right) + 4\left(\frac{1}{2}\right)^2\right)}{\mathrm{P}^2}\right]$$

$$\mathrm{M_{yD}} = \frac{-\mathrm{AD}}{\mathrm{a}^2}\left[-0.1125 - \frac{0.4687}{\mathrm{P}^2}\right]$$

$$\mathrm{M_{yD}} = \frac{\mathrm{AD}}{\mathrm{a}^2}\left[\frac{0.4687}{\mathrm{P}^2} + 0.1125\right] \qquad (3.447)$$

Substituting the expression of A in Equation (3.336) into Equation (3.447) yields Equation (3.448).

$$\mathrm{M_{yD}} = \frac{\mathrm{D}}{\mathrm{a}^2}\left[\frac{0.4687}{\mathrm{P}^2} + 0.1125\right]\frac{\mathrm{qa}^4}{\mathrm{D}}\frac{0.8\mathrm{p}^4}{(1.726\mathrm{p}^4 + 3.328\mathrm{p}^2 + 3.84)}$$

$$\mathrm{M_{yD}} = \left[\frac{0.4687}{\mathrm{P}^2} + 0.1125\right]\frac{0.8\mathrm{p}^4}{(1.726\mathrm{p}^4 + 3.328\mathrm{p}^2 + 3.84)}\mathrm{qb}^2 \qquad (3.448)$$

Equation for bending moment at the centre in y - direction is as shown in Equation (3.449).

$$\mathrm{M_{yD}} = \beta_{\mathrm{yD}}\mathrm{qa}^2 \qquad (3.449)$$

Equating Equation (3.448) and Equation (3.449) and solving for β_{xD} in terms of the aspect ratio, P yields Equation (3.450).

$$\beta_{\mathrm{yD}} = \left[\frac{0.4687}{\mathrm{P}^2} + 0.1125\right]\frac{0.8\mathrm{p}^4}{(1.726\mathrm{p}^4 + 3.328\mathrm{p}^2 + 3.84)} \qquad (3.450)$$

Where β_{yD} is center bending moment factor about y– directions.

MAXIMUM SHEAR FORCE, V_{xB} and V_{yA} ($\mu = 0.3$)

Substituting Equations (3.309) and (3.317) into Equation (3.82b) for R = 1 and Q = 0.5 and simplifying the equation yields Equation (3.451).

$$V_{xB} = -D\left[\frac{\partial^3 w}{\partial x^3} + (2-\mu)\frac{\partial^3 w}{\partial x\,\partial y^2}\right]$$

$$= \frac{-AD}{a^3}\left[\frac{\partial^3 h}{\partial R^3} + (2-\mu)\frac{\partial^3 h}{P^2\,\partial R\,\partial Q^2}\right]$$

$$= \frac{-AD}{a^3}\left[12(2R-1)(1.5Q^2 - 2.5Q^3 + Q^4) + (2-\mu)\frac{\left(3(1-6R^2+4R^3)(1-5Q+4Q^2)\right)}{P^2}\right]$$

$$= \frac{-AD}{a^3}\left[12\left((2(1)-1)\left(1.5\left(\frac{1}{2}\right)^2 - 2.5\left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^4\right)\right) + (2-0.3)\frac{3\left[(1-6(1)^2+4(1)^3)\left(1-5\left(\frac{1}{2}\right)+4\left(\frac{1}{2}\right)^2\right)\right]}{P^2}\right]$$

$$V_{xB} = \frac{-AD}{a^3}\left[1.5 + \frac{2.55}{P^2}\right]$$

$$V_{xB} = \frac{-AD}{a^3}\left[\frac{2.55}{P^2} + 1.5\right] \tag{3.451}$$

Substituting the expression of A in Equation (3.336) into Equation (3.451) yields Equation (3.452).

$$V_{xB} = \frac{D}{a^3}\left[\frac{2.55}{P^2} + 1.5\right]\frac{qa^4}{D}\frac{0.8p^4}{(1.726p^4 + 3.328p^2 + 3.84)}$$

$$V_{xB} = \left[\frac{2.55}{P^2} + 1.5\right]\left(\frac{0.8p^4}{1.726p^4 + 3.328p^2 + 3.84}\right)qa \tag{3.452}$$

Equation for edge shear force at point B in x - direction is as shown in Equation (3.453).

$$V_{xB} = k_{sxB}qa \tag{3.453}$$

Equating Equation (3.452) and Equation (3.453) and solving for k_{sxB} in terms of the aspect ratio, P yields Equation (3.454).

$$k_{sxB} = \left[\frac{2.55}{P^2} + 1.5\right]\frac{0.8p^4}{(1.726p^4 + 3.328p^2 + 3.84)} \tag{3.454}$$

Where k_{sxB} is shear force factor in the x – direction.

Similarly, substituting Equations (3.313) and (3.316) into Equation (3.83b) and simplifying the equation for R = 0.5 and Q = 0 at point A, yields Equation (3.455).

$$V_y = -D\left[\frac{\partial^3 w}{\partial y^3} + (2-\mu)\frac{\partial^3 w}{\partial x^2\,\partial y}\right]$$

$$= \frac{AD}{a^3}\left[\frac{\partial^3 h}{P^3\,\partial Q^3} + (2-\mu)\frac{\partial^3 h}{P\,\partial QR^2\,\partial Q}\right]$$

$$= \frac{-AD}{a^3}\left[\frac{3(8Q-5)(R-2R^3+R^4)}{P^3} + (2-\mu)\frac{\left(12(R^2-R)(3Q-7.5Q^2+4Q^3)\right)}{P}\right]$$

$$V_{yC} = \frac{-AD}{a^3}\left[\frac{3(8(1)-5)\left(\left(\frac{1}{2}\right)-2\left(\frac{1}{2}\right)^3+\left(\frac{1}{2}\right)^4\right)}{P^3} + (2-0.3)\frac{12\left(\left(\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)\right)(3(1)-7.5(1)^2+4(1)^3)}{P}\right]$$

$$V_{yC} = \frac{-AD}{a^3}\left[\frac{2.8125}{P^3} + \frac{2.55}{P}\right]$$

$$V_{yC} = \frac{AD}{a^3}\left[\frac{2.8125}{P^3} + \frac{2.55}{P}\right] \qquad (3.455)$$

Substituting the expression of A in Equation (3.336) into Equation (3.455) yields Equation (3.456).

$$V_{yC} = \frac{D}{a^3}\left[\frac{2.8125}{P^3} + \frac{2.55}{P}\right]\frac{qa^4}{D}\frac{0.8p^4}{(1.726p^4+3.328p^2+3.84)}$$

$$V_{yC} = \left[\frac{2.8125}{P^3} + \frac{2.55}{P}\right]\left(\frac{0.8p^4}{(1.726p^4+3.328p^2+3.84)}\right)qa \qquad (3.456)$$

Equation for edge shear force at point C in y - direction is as shown in Equation (3.457).

$$V_{yC} = k_{syC}qa \qquad (3.457)$$

Equating Equation (3.456) and Equation (3.457) and solving for k_{syC} in terms of the aspect ratio, P yields Equation (3.458).

$$k_{syC} = \left[\frac{2.8125}{P^3} + \frac{2.55}{P}\right]\left(\frac{0.8p^4}{(1.726p^4+3.328p^2+3.84)}\right) \qquad (3.458)$$

Where k_{syC} is shear force factor in the y – direction.

FACTORS FOR PLATE TYPE CCCS

Figure 3.24 shows a thin rectangular plate under uniformly distributed lateral load, q. The plate is clamped on three edges and simply supported on one long edge.

From Equation (3.202), the shape function for CCCS is given as:

$$W = A(1.5R^2 - 2.5R^3 + R^4)(Q^2 - 2Q^3 + Q^4)$$

Where A as given in Equation (3.337) is as shown below.

$$\mathrm{A} = \frac{\mathrm{qa}^4}{\mathrm{D}}\frac{0.536\mathrm{p}^4}{(0.596\mathrm{p}^4 + 0.614\mathrm{p}^2 + 1.157)}$$

Considering the value of deflection at the center, D of the plate where:

$$\mathrm{x} = \frac{\mathrm{a}}{2}\,;\left(\mathrm{x} = \frac{\mathrm{Pb}}{2}\ \text{since}\ \mathrm{P} = \frac{\mathrm{a}}{\mathrm{b}}\right)\ \text{and}\ \mathrm{y} = \frac{\mathrm{b}}{2} \qquad (3.459)$$

In non – dimensional form, the center deflection will occur on the plate where at points,

$$\mathrm{R} = \frac{1}{2}\ \text{and}\ \mathrm{Q} = \frac{1}{2} \qquad (3.460)$$

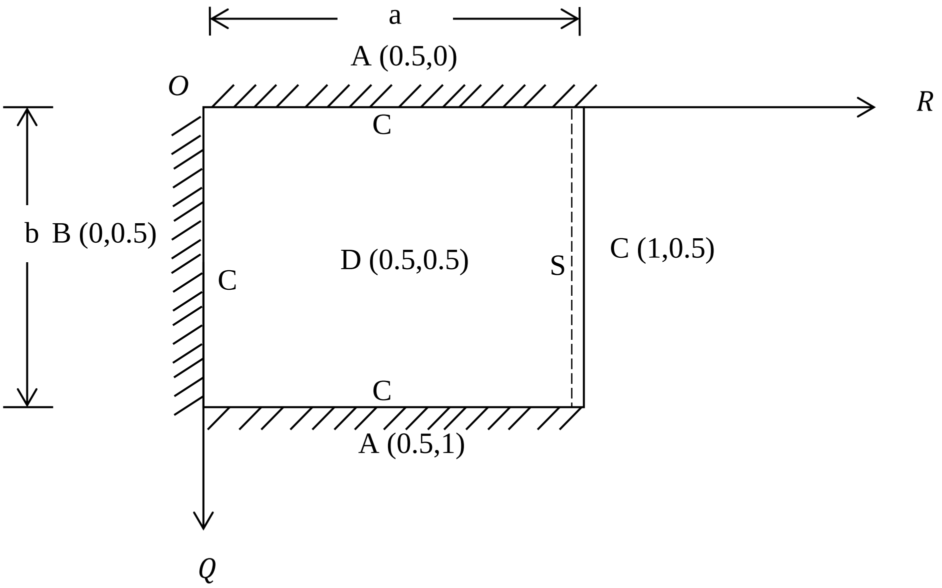

Figure 3.24: CCCS Plate Showing Points of Maximum Values

DEFLECTIONS AT THE CENTRE

Substituting Equation (3.460) into Equation (3.202) yields Equation (3.461).

$$\mathrm{W_D} = \mathrm{A}\left(1.5\left(\frac{1}{2}\right)^2 - 2.5\left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^4\right)\left(\left(\frac{1}{2}\right)^2 - 2\left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^4\right)$$

$$\mathrm{W_D} = \mathrm{A}\left(\frac{1}{8}\right)\left(\frac{1}{16}\right)$$

$$\mathrm{W_D} = \mathrm{A}\left(\frac{1}{128}\right) \qquad (3.461)$$

Substituting the expression of A in Equation (3.337) into Equation (3.461) yields Equation (3.462).

$$\mathrm{W_D} = \left(\frac{1}{128}\right) \times \frac{\mathrm{qa}^4}{\mathrm{D}} \times \frac{0.536\mathrm{p}^4}{(0.596\mathrm{p}^4 + 0.614\mathrm{p}^2 + 1.157)}$$

$$W_D = \left(\left(\frac{1}{128}\right) \times \frac{0.536p^4}{(0.596p^4 + 0.614p^2 + 1.157)}\right)\frac{qa^4}{D} \qquad (3.462)$$

Equation for center deflection at point D is as shown in Equation (3.463).

$$W_D = k_D\frac{qa^4}{D} \qquad (3.463)$$

Equating Equation (3.462) and Equation (3.463) and solving for k_D in terms of the aspect ratio, P yields Equation (3.464).

$$k_D = \left(\frac{1}{128}\right) \times \frac{0.536p^4}{(0.596p^4 + 0.614p^2 + 1.157)} \qquad (3.464)$$

Where k_D is the factor of deflection.

MID-SPAN MOMENTS, M_{xD} and M_{yD} ($\mu = 0.3$)

Substituting Equations (3.320) and (3.324) into Equation (3.70b) and simplifying the equation for R = 0.5 and Q = 0.5 at point D, yields Equation (3.465).

$$\begin{aligned} M_x &= -D\left[\frac{\partial^2 w}{\partial x^2} + \mu\frac{\partial^2 w}{\partial y^2}\right] \\ &= \frac{-AD}{a^2}\left[\frac{\partial^2 h}{\partial R^2} + \mu\frac{\partial^2 h}{P^2\,\partial Q^2}\right] \\ &= \frac{-AD}{a^2}\Bigg[(3 - 15R + 12R^2)(Q^2 - 2Q^3 + Q^4) \\ &\qquad + \frac{\mu(1.5R^2 - 2.5R^3 + R^4)(2 - 12Q + 12Q^2)}{P^2}\Bigg] \qquad (3.465) \end{aligned}$$

For central moment, MxD, substituting the values of R and Q (See Figure 3.24) into Equation (3.465) yields Equation (3.466).

$$\begin{aligned} M_{xD} = \frac{-AD}{a^2}\Bigg[&\left(3 - 15\left(\frac{1}{2}\right) + 12\left(\frac{1}{2}\right)^2\right)\left(\left(\frac{1}{2}\right)^2 - 2\left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^4\right) \\ &+ \frac{0.3\left(1.5\left(\frac{1}{2}\right)^2 - 2.5\left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^4\right)\left(2 - 12\left(\frac{1}{2}\right) + 12\left(\frac{1}{2}\right)^2\right)}{P^2}\Bigg] \end{aligned}$$

$$M_{xD} = \frac{-AD}{a^2}\left[-0.09375 - \frac{0.0375}{P^2}\right]$$

$$M_{xD} = \frac{AD}{a^2}\left[\frac{0.0375}{P^2} + 0.09375\right] \qquad (3.466)$$

Substituting the expression of A in Equation (3.337) into Equation (3.466) yields Equation (3.467).

$$M_{xD} = \frac{D}{a^2}\left[\frac{0.0375}{P^2} + 0.09375\right]\frac{qa^4}{D}\frac{0.536p^4}{(0.596p^4 + 0.614p^2 + 1.157)}$$

$$M_{xD} = \left[\frac{0.0375}{P^2} + 0.09375\right]\frac{0.536p^4}{(0.596p^4 + 0.614p^2 + 1.157)}qa^2 \qquad (3.467)$$

Equation for bending moment at point D in x – direction, is as shown in Equation (3.468).

$$M_{xD} = \beta_{xD}qa^2 \qquad (3.468)$$

Equating Equation (3.467) and Equation (3.468) and solving for β_{xD} in terms of the aspect ratio, P yields Equation (3.469).

$$\beta_{xD} = \left[\frac{0.0375}{P^2} + 0.09375\right]\frac{0.536p^4}{(0.596p^4 + 0.614p^2 + 1.157)} \qquad (3.469)$$

Where β_{xD} is bending moment factor in the x – direction.

Similarly, for mid-span moment in y – direction, substituting Equations (3.320) and (3.324) into Equation (3.71b) for R = 0.5 and Q = 0.5 and simplifying the equation yields Equation (3.470).

$$M_{yD} = -D\left[\mu\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right]$$

$$= \frac{-AD}{a^2}\left[\mu\frac{\partial^2 w}{\partial R^2} + \frac{\partial^2 w}{P^2\,\partial Q^2}\right]$$

$$M_{yD} = \frac{-AD}{a^2}\left[0.3\left(3 - 15\left(\frac{1}{2}\right) + 12\left(\frac{1}{2}\right)^2\right)\left(\left(\frac{1}{2}\right)^2 - 2\left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^4\right) + \frac{\left(1.5\left(\frac{1}{2}\right)^2 - 2.5\left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^4\right)\left(2 - 12\left(\frac{1}{2}\right) + 12\left(\frac{1}{2}\right)^2\right)}{P^2}\right]$$

$$M_{yD} = \frac{-AD}{a^2}\left[-0.028125 - \frac{0.125}{P^2}\right]$$

$$M_{yD} = \frac{AD}{a^2}\left[\frac{0.125}{P^2} + 0.028125\right] \qquad (3.470)$$

Substituting the expression of A in Equation (3.337) into Equation (3.470) yields Equation (3.471).

$$M_{yD} = \frac{D}{a^2}\left[\frac{0.125}{P^2} + 0.028125\right]\frac{qa^4}{D}\frac{0.536p^4}{(0.596p^4 + 0.614p^2 + 1.157)}$$

$$M_{yD} = \left[\frac{0.125}{P^2} + 0.028125\right]\frac{0.536p^4}{(0.596p^4 + 0.614p^2 + 1.157)}qa^2 \tag{3.471}$$

Equation for bending moment at point D in y – direction, is as shown in Equation (3.472).

$$M_{yD} = \beta_{yD}qa^2 \tag{3.472}$$

Equating Equation (3.458) and Equation (3.459) and solving for β_{yD} in terms of the aspect ratio, P yields Equation (3.473).

$$\beta_{yD} = \left[\frac{0.125}{P^2} + 0.028125\right]\frac{0.536p^4}{(0.596p^4 + 0.614p^2 + 1.157)} \tag{3.473}$$

Where β_{yD} is bending moment factor in the y– direction.

EDGE SHEAR FORCE, V_{xB} and V_{yA} ($\mu = 0.3$)

Substituting Equations (3.321) and (3.329) into Equation (3.82b) for R = 0 and Q = 0.5 and simplifying the equation yields Equation (3.474).

$$V_x = -D\left[\frac{\partial^3 w}{\partial x^3} + (2-\mu)\frac{\partial^3 w}{\partial x\,\partial y^2}\right]$$

$$= \frac{-AD}{a^3}\left[\frac{\partial^3 h}{\partial R^3} + (2-\mu)\frac{\partial^3 h}{P^2\,\partial R\,\partial Q^2}\right]$$

$$= \frac{-AD}{a^3}\left[(24R-15)(Q^2-2Q^3+Q^4) + (2-\mu)\frac{\left(2(3R-7.5R^2+4R^3)(1-6Q+6Q^2)\right)}{P^2}\right]$$

$$V_{xB} = \frac{-AD}{a^3}\left[\left((24(0)-15)\left(\left(\frac{1}{2}\right)^2 - 2\left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^4\right)\right) + (2-0.3)\frac{2\left[(3(0)-7.5(0)^2+4(0)^3)\left(1-6\left(\frac{1}{2}\right)+6\left(\frac{1}{2}\right)^2\right)\right]}{P^2}\right]$$

$$V_{xB} = \frac{-AD}{a^3}\left[-0.9375 + \frac{0}{P^2}\right]$$

$$V_{xB} = \frac{AD}{a^3}\left[\frac{0.9375}{P^2}\right] \tag{3.474}$$

Substituting the expression of A in Equation (3.337) into Equation (3.474) yields Equation (3.475).

$$V_{xB} = \frac{D}{a^3}\left[\frac{0.9375}{P^2}\right]\frac{qa^4}{D}\frac{0.536p^4}{(0.596p^4 + 0.614p^2 + 1.157)}$$

$$V_{xB} = \left[\frac{0.9375}{P^2}\right] \frac{0.536p^4}{(0.596p^4 + 0.614p^2 + 1.157)} qa \tag{3.475}$$

Equation for shear force at point B in x – direction, is as shown in Equation (3.476).

$$V_{xB} = k_{sxB} qa \tag{3.476}$$

Equating Equation (3.475) and Equation (3.476) and solving for β_{yD} in terms of the aspect ratio, P yields Equation (3.477).

$$k_{sxB} = \left[\frac{0.9375}{P^2}\right] \frac{0.536p^4}{(0.596p^4 + 0.614p^2 + 1.157)} \tag{3.477}$$

Where k_{sxB} shear force factor in the x– direction.

Similarly, substituting Equations (3.325) and (3.328) into Equation (3.83b) and simplifying the equation for R = 0.5 and Q = 1 at point A, yields Equation (3.478).

$$V_y = -D\left[\frac{\partial^3 w}{\partial y^3} + (2 - \mu)\frac{\partial^3 w}{\partial x^2\, \partial y}\right]$$

$$= \frac{-AD}{a^3}\left[\frac{\partial^3 h}{P^3\, \partial Q^3} + (2 - \mu)\frac{\partial^3 h}{P\, \partial QR^2\, \partial Q}\right]$$

$$= \frac{-AD}{a^3}\left[\frac{12(2Q - 1)(1.5R^2 - 2.5R^3 + R^4)}{P^3} + (2 - \mu)\frac{6\big((Q - 3Q^2 + 2Q^3)(1 - 5R + 4R^2)\big)}{P}\right]$$

$$V_{yA} = \frac{-AD}{a^3}\left[\frac{(24(1) - 12)\left(1.5\left(\frac{1}{2}\right)^2 - 2.5\left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^4\right)}{P^3} + (2 - 0.3)\frac{\left(3 - 15\left(\frac{1}{2}\right) + 12\left(\frac{1}{2}\right)^2\right)(2(1) - 6(1)^2 + 4(1)^3)}{P}\right]$$

$$V_{yA} = \frac{-AD}{a^3}\left[\frac{1.5}{P^3} + \frac{0}{P}\right]$$

$$V_{yA} = \frac{AD}{a^3}\left[\frac{1.5}{P^3}\right] \tag{3.478}$$

Substituting the expression of A in Equation (3.337) into Equation (3.478) yields Equation (3.479).

$$V_{yA} = \frac{D}{a^3}\left[\frac{1.5}{P^3}\right]\frac{qa^4}{D}\frac{0.536p^4}{(0.596p^4 + 0.614p^2 + 1.157)}$$

$$V_{yA} = \left[\frac{1.5}{P^3}\right]\left(\frac{0.536p^4}{0.596p^4 + 0.614p^2 + 1.157}qa\quad\right)qa \tag{3.479}$$

Equation for shear force at point A in y – direction, is as shown in Equation (3.480).

$$V_{yA} = k_{syA}qa \tag{3.480}$$

Equating Equation (3.479) and Equation (3.480) and solving for k_{syA} in terms of the aspect ratio, P yields Equation (3.481).

$$k_{syA} = \left[\frac{1.5}{P^3}\right]\left(\frac{0.536p^4}{0.596p^4 + 0.614p^2 + 1.157}qa\right) \tag{3.481}$$

Where k_{syA} is shear force factor in the y– direction.

3.5 Euler-Bernoulli residual forces for the selected plates.

In this section, values of Euler-Bernoulli residual forces of the plates considered were obtained. The results of the present study were similarly compared with the result from a weighted residual approach from a previous study.

3.5.1 Values of Euler-Bernoulli residual force of plates considered

Newton's law of motion, states that for a body at rest the summation of all the forces must be equal to zero. This shows that the forces must be as such that they cancel out. Hence, attempt was made in this section to verify that the factors determined in this study are in agreement with this law. Furthermore, factors from previous study were substituted into the governing Euler-Bernoulli partial differential equation for isotropic thin rectangular plate to determine the residual forces for the selected plate boundary conditions.

3.5.1.1 Euler-Bernoulli residual force for SSSS plate

Recall Equation (3.151)

$$F = \frac{d\Pi}{dw} = A\int_0^1\int_0^1\left(\frac{\partial^4 h}{\partial R^4} + \frac{2}{p^2}\frac{\partial^4 h}{\partial R^2\,\partial Q^2} + \frac{1}{p^4}\frac{\partial^4 h}{\partial Q^4} - \frac{qa^4}{D}\right)dR\,dQ = 0$$

Where;

$$k_x = \int_0^1\int_0^1\left(\frac{\partial^4 h}{\partial R^4}\right)dR\,dQ\,, k_{xy} = \int_0^1\int_0^1\left(\frac{\partial^4 h}{\partial R^2\,\partial Q^2}\right)dR\,dQ\,, k_q = \int_0^1\int_0^1\left(\frac{qa^4}{D}\right)dR\,dQ$$

Therefore, Equation (3.151) can be rewritten as;

$$F = \frac{\partial\Pi}{\partial w} = A\left[k_x + \left(\frac{2}{p^2}\right)k_{xy} + \left(\frac{1}{p^4}\right)k_y\right] - \left(\frac{qa^4}{D}\right)k_q = 0 \tag{3.481}$$

From Table 3.11,

$k_x = 4.8; \quad k_{xy} = 4; \quad k_y = 4.8; \quad k_q = 1$

From Equation (3.332), ‘A’ when p is 1 is given as;

$$A = \frac{qa^4}{D} \frac{p^4}{(4.8p^4 + 8p^2 + 4.8)} = \frac{qa^4}{17.6D}$$

Substituting the values of A, Kx, Kxy, Ky and Kq into Equation (3.481) with an aspect ratio, p of 1, gives;

$$F = \frac{\partial \Pi}{\partial w} = \frac{qa^4}{17.6D}\left[\frac{24}{5} + \left(\frac{2}{(1)^2}\right)4 + \left(\frac{1}{(1)^4}\right)\frac{24}{5}\right] - \left(\frac{qa^4}{D}\right)1 = 0$$

$$F = \frac{\partial \Pi}{\partial w} = 17.6\left(\frac{qa^4}{17.6D}\right) - \left(\frac{qa^4}{D}\right)1 = 0$$

$$F = \frac{\partial \Pi}{\partial w} = \frac{qa^4}{D} - \frac{qa^4}{D} = 0$$

K - Factors from Ibearugbulem (2014) study, an example of a weighted residual approach;

$\bar{k}_x = 0.23621; \quad \bar{k}_{xy} = 0.23591; \quad \bar{k}_y = 0.23621; \quad \bar{k}_q = 0.04$

$$A = \frac{qa^4}{17.6D}$$

Substituting the values of A, Kx, Kxy, Ky and Kq into Equation (3.481) with an aspect ratio, p of 1 and solving the resulting equation, gives;

$$F = \frac{\partial \Pi}{\partial w} = \frac{qa^4}{17.6D}\left[0.23621 + \left(\frac{2}{(1)^2}\right)0.23591 + \left(\frac{1}{(1)^4}\right)0.23621\right] - \left(\frac{qa^4}{D}\right)0.04 = 0$$

$$F = \frac{\partial \Pi}{\partial w} = 0.05365\frac{qa^4}{D} - 0.04\frac{qa^4}{D} = 0.0136$$

3.5.1.2 Euler-Bernoulli residual force for CCCC plate

From Table 3.11,

$k_x = 0.499; \quad k_{xy} = 0.136; \quad k_y = 0.499; \quad k_q = 0.449$

From Equation (3.333), ‘A’ when p is 1 is given as;

$$A = \frac{qa^4}{D} \frac{0.449p^4}{(0.499p^4 + 0.272p^2 + 0.499)} = 0.3535\frac{qa^4}{D}$$

Substituting the values of A, Kx, Kxy, Ky and Kq into Equation (3.481) with an aspect ratio, p of 1, gives;

$$F = \frac{\partial \Pi}{\partial w} = 0.3535\frac{qa^4}{D}\left[0.499 + \left(\frac{2}{(1)^2}\right)0.136 + \left(\frac{1}{(1)^4}\right)0.499\right] - \left(\frac{qa^4}{D}\right)0.449 = 0$$

$$F = \frac{\partial \Pi}{\partial w} = 0.3535\left(1.27\frac{qa^4}{D}\right) - \left(\frac{qa^4}{D}\right)0.449 = 0.449\left(\frac{qa^4}{D}\right) - \left(\frac{qa^4}{D}\right)0.449 = 0$$

$$F = \frac{\partial \Pi}{\partial w} = \frac{qa^4}{D} - \frac{qa^4}{D} = 0$$

K - Factors from Ibearugbulem (2014) study, an example of a weighted residual approach;

$\bar{k}_x = 0.00127;\ \ \bar{k}_{xy} = 0.00036;\ \ \ \bar{k}_y = 0.00127;\ \ \ \ \bar{k}_q = 0.0011$

$$A = 0.3535\frac{qa^4}{D}$$

Substituting the values of A, Kx, Kxy, Ky and Kq into Equation (3.481) with an aspect ratio, p of 1 and solving the resulting equation, gives;

$$F = \frac{\partial \Pi}{\partial w} = 0.3535\frac{qa^4}{D}\left[0.00127 + \left(\frac{2}{(1)^2}\right)0.00036 + \left(\frac{1}{(1)^4}\right)0.00127\right] - \left(\frac{qa^4}{D}\right)0.00111111$$

$$F = \frac{\partial \Pi}{\partial w} = 0.001152\frac{qa^4}{D} - 0.0011\frac{qa^4}{D} = 0.000041$$

3.5.1.3 Euler-Bernoulli residual force for CSCS plate

From Table 3.11,

$k_x = 0.745;\ \ k_{xy} = 0.738;\ \ \ k_y = 3.216;\ \ \ \ k_q = 0.670$

From Equation (3.334), ‘A’ when p is 1 is given as;

$$A = \frac{qa^4}{D}\frac{0.67p^4}{(0.745p^4 + 2 \times 0.738p^2 + 3.216)} = 0.123214\frac{qa^4}{D}$$

Substituting the values of A, Kx, Kxy, Ky and Kq into Equation (3.481) with an aspect ratio, p of 1, gives;

$$F = \frac{\partial \Pi}{\partial w} = 0.123214\frac{qa^4}{D}\left[0.745 + \left(\frac{2}{(1)^2}\right) \times 0.738 + \left(\frac{1}{(1)^4}\right)3.216\right] - \left(\frac{qa^4}{D}\right)0.67 = 0$$

$$F = \frac{\partial \Pi}{\partial w} = 0.67\frac{qa^4}{D} - \frac{qa^4}{D}0.67 = 0$$

$$F = \frac{\partial \Pi}{\partial w} = \frac{qa^4}{D} - \frac{qa^4}{D} = 0$$

K - Factors from Ibearugbulem (2014) study, an example of a weighted residual approach;

$\bar{k}_x = 0.00763;\ \ \bar{k}_{xy} = 0.00925;\ \ \ \bar{k}_y = 0.03937;\ \ \ \ \bar{k}_q = 0.007$

$$A = 0.123214\frac{qa^4}{D}$$

Substituting the values of A, Kx, Kxy, Ky and Kq into Equation (3.481) with an aspect ratio, p of 1 and solving the resulting equation, gives;

$$F = \frac{\partial \Pi}{\partial w} = 0.123214\frac{qa^4}{D}\left[0.00763 + \left(\frac{2}{(1)^2}\right)0.00925 + \left(\frac{1}{(1)^4}\right)0.03937\right] - \left(\frac{qa^4}{D}\right)0.007$$

$$F = \frac{\partial \Pi}{\partial w} = 0.008071\frac{qa^4}{D} - 0.0655\frac{qa^4}{D} = 0.0014$$

3.5.1.4 Euler-Bernoulli residual force for CSSS plate

From Table 3.11,

$k_x = 1.726;\quad k_{xy} = 1.664;\quad k_y = 3.840;\quad k_q = 0.80$

From Equation (3.320), ‘A’ when p is 1 is given as;

$$A = \frac{qa^4}{D}\frac{0.8p^4}{(1.726p^4 + 2 \times 1.664p^2 + 3.840)} = 0.089944\frac{qa^4}{D}$$

Substituting the values of A, Kx, Kxy, Ky and Kq into Equation (3.445) with an aspect ratio, p of 1, gives;

$$F = \frac{\partial \Pi}{\partial w} = 0.089944\frac{qa^4}{D}\left[1.726 + \left(\frac{2}{(1)^2}\right) \times 1.664 + \left(\frac{1}{(1)^4}\right) \times 3.840\right] - \left(\frac{qa^4}{D}\right) \times 0.8$$

$$= 0$$

$$F = \frac{\partial \Pi}{\partial w} = 0.8\frac{qa^4}{D} - \frac{qa^4}{D}0.8 = 0$$

$$F = \frac{\partial \Pi}{\partial w} = \frac{qa^4}{D} - \frac{qa^4}{D} = 0$$

K - Factors from Ibearugbulem (2014) study, an example of a weighted residual approach;

$\bar{k}_x = 0.036192;\quad \bar{k}_{xy} = 0.0416321;\quad \bar{k}_y = 0.088571;\quad \bar{k}_q = 0.015$

$$A = 0.089944\frac{qa^4}{D}$$

Substituting the values of A, Kx, Kxy, Ky and Kq into Equation (3.445) with an aspect ratio, p of 1 and solving the resulting equation, gives;

$$F = \frac{\partial \Pi}{\partial w} = 0.089944\frac{qa^4}{D}\left[0.036192 + \left(\frac{2}{(1)^2}\right)0.0416321 + \left(\frac{1}{(1)^4}\right)0.088571\right] - \left(\frac{qa^4}{D}\right)0.015$$

$$\mathrm{F} = \frac{\partial \Pi}{\partial \mathrm{w}} = 0.018711\frac{\mathrm{qa}^4}{\mathrm{D}} - 0.015\frac{\mathrm{qa}^4}{\mathrm{D}} = 0.003711$$

3.5.1.5 Euler-Bernoulli residual force for CCSS plate

From Table 3.11,

$\mathrm{k_x} = 1.381; \quad \mathrm{k_{xy}} = 0.692; \quad \mathrm{k_y} = 1.381; \quad \mathrm{k_q} = 0.536$

From Equation (3.321), 'A' when p is 1 is given as;

$$\mathrm{A} = \frac{\mathrm{qa}^4}{\mathrm{D}}\frac{0.64\mathrm{p}^4}{(1.381\mathrm{p}^4 + 2 \times 0.692\mathrm{p}^2 + 1.381)} = 0.154336\frac{\mathrm{qa}^4}{\mathrm{D}}$$

Substituting the values of A, Kx, Kxy, Ky and Kq into Equation (3.445) with an aspect ratio, p of 1, gives;

$$\mathrm{F} = \frac{\partial \Pi}{\partial \mathrm{w}} = 0.154336\frac{\mathrm{qa}^4}{\mathrm{D}}\left[1.381 + \left(\frac{2}{(1)^2}\right) \times 0.692 + \left(\frac{1}{(1)^4}\right) \times 1.381\right] - \left(\frac{\mathrm{qa}^4}{\mathrm{D}}\right)0.64 = 0$$

$$\mathrm{F} = \frac{\partial \Pi}{\partial \mathrm{w}} = 0.64\frac{\mathrm{qa}^4}{\mathrm{D}} - \frac{\mathrm{qa}^4}{\mathrm{D}}0.64 = 0$$

$$\mathrm{F} = \frac{\partial \Pi}{\partial \mathrm{w}} = \frac{\mathrm{qa}^4}{\mathrm{D}} - \frac{\mathrm{qa}^4}{\mathrm{D}} = 0$$

K - Factors from Ibearugbulem (2014) study, an example of a weighted residual approach;

$\bar{\mathrm{k}}_\mathrm{x} = 0.013572; \quad \bar{\mathrm{k}}_\mathrm{xy} = 0.0073469; \quad \bar{\mathrm{k}}_\mathrm{y} = 0.013572; \quad \bar{\mathrm{k}}_\mathrm{q} = 0.006$

$$\mathrm{A} = 0.154336\frac{\mathrm{qa}^4}{\mathrm{D}}$$

Substituting the values of A, Kx, Kxy, Ky and Kq into Equation (3.481) with an aspect ratio, p of 1 and solving the resulting equation, gives;

$$\mathrm{F} = \frac{\partial \Pi}{\partial \mathrm{w}} = 0.154336\frac{\mathrm{qa}^4}{\mathrm{D}}\left[0.013572 + \left(\frac{2}{(1)^2}\right)0.0073469 + \left(\frac{1}{(1)^4}\right)0.013572\right] - \left(\frac{\mathrm{qa}^4}{\mathrm{D}}\right)0.006$$

$$\mathrm{F} = \frac{\partial \Pi}{\partial \mathrm{w}} = 0.006457\frac{\mathrm{qa}^4}{\mathrm{D}} - 0.005625\frac{\mathrm{qa}^4}{\mathrm{D}}$$

$$\mathrm{F} = \frac{\partial \Pi}{\partial \mathrm{w}} = \frac{\mathrm{qa}^4}{\mathrm{D}} - \frac{\mathrm{qa}^4}{\mathrm{D}} = 0.000832$$

3.5.1.6 Euler-Bernoulli residual force for CCCS plate

From Table 3.11,

$\mathrm{k_x = 0.596; \quad k_{xy} = 0.307; \quad k_y = 1.157; \quad k_q = 0.536}$

From Equation (3.322), ‘A’ when p is 1 is given as;

$$\mathrm{A = \frac{qa^4}{D}\frac{0.536p^4}{(0.596p^4 + 2 \times 0.307p^2 + 1.157)} = 0.22645\frac{qa^4}{D}}$$

Substituting the values of A, Kx, Kxy, Ky and Kq into Equation (3.445) with an aspect ratio, p of 1, gives;

$$\mathrm{F = \frac{\partial \Pi}{\partial w} = 0.22645\frac{qa^4}{D}\left[0.596 + \left(\frac{2}{(1)^2}\right) \times 0.307 + \left(\frac{1}{(1)^4}\right) \times 1.157\right] - \left(\frac{qa^4}{D}\right)0.536 = 0}$$

$$\mathrm{F = \frac{\partial \Pi}{\partial w} = 0.536\frac{qa^4}{D} - \frac{qa^4}{D}0.536 = 0}$$

$$\mathrm{F = \frac{\partial \Pi}{\partial w} = \frac{qa^4}{D} - \frac{qa^4}{D} = 0}$$

K - Factors from Ibearugbulem (2014) study, an example of a weighted residual approach;

$\mathrm{\bar{k}_x = 0.002857; \quad \bar{k}_{xy} = 0.00163268; \quad \bar{k}_y = 0.0060318; \quad \bar{k}_q = 0.0025}$

$$\mathrm{A = 0.22645\frac{qa^4}{D}}$$

Substituting the values of A, Kx, Kxy, Ky and Kq into Equation (3.445) with an aspect ratio, p of 1 and solving the resulting equation, gives;

$$\mathrm{F = \frac{\partial \Pi}{\partial w} = 0.22645\frac{qa^4}{D}\left[0.002857 + \left(\frac{2}{(1)^2}\right)0.00163268 + \left(\frac{1}{(1)^4}\right)0.0060318\right] - \left(\frac{qa^4}{D}\right)0.0025}$$

$$\mathrm{F = \frac{\partial \Pi}{\partial w} = 0.003546\frac{qa^4}{D} - 0.0025\frac{qa^4}{D} = 0.001046}$$

3.5.2 Comparison with results of past researchers.

The results of the factors for deflection, shear force and bending moments obtained from this study were compared with the result of Ibearugbulem (2014), a weighted residual approach based on Ritz method. The result comparison was done using the percentage comparison tool. The tool is mathematically expressed as in Equation (3.482).

$$\%\text{diff.} = \left|\frac{\text{Result of Present Study} - \text{Result of Ibearugbulem}(2014)}{\text{Result of Presnt Study}}\right| \times 100 \qquad (3.482)$$

The results of the percentage differences between the present study and Ibearugbulem (2014), in this case, Ibearugbulem (2014), are presented on Tables A1 through F3 for deflection, bending moment and shear force factors of the plates considered this study in Appendices A to F.

CHAPTER FOUR

RESULTS AND DISCUSSIONS

4.1 Results

The results emanating from this research are presented in this section. They include the following, result of the general expressions of total force equilibrium (F = 0), result of the exact shape functions for rectangular Kirchhoff's plates of selected boundary conditions, the results of the exact coefficient of deflection, and the results of the exact bending moments and shear forces of the selected plates and boundary conditions.

4.1.1 Total energy functional of the plate

Euler-Bernoulli Form of Total Equilibrium of Forces

$$\mathrm{F} = \frac{\mathrm{d\Pi}}{\mathrm{dw}}\int_0^1\int_0^1 \mathrm{g\,dR\,dQ} = \int_0^1\int_0^1\left(\frac{\partial^4\mathrm{w}}{\partial\mathrm{R}^4} + \frac{2}{\mathrm{p}^2}\frac{\partial^4\mathrm{w}}{\partial\mathrm{R}^2\,\partial\mathrm{Q}^2} + \frac{1}{\mathrm{p}^4}\frac{\partial^4\mathrm{w}}{\partial\mathrm{Q}^4} - \frac{\mathrm{qa}^4}{\mathrm{D}}\right)\mathrm{dR\,dQ} = 0 \qquad (4.1)$$

$$\mathrm{g} = \frac{\partial^4\mathrm{w}}{\partial\mathrm{R}^4} + \frac{2}{\mathrm{p}^2}\frac{\partial^4\mathrm{w}}{\partial\mathrm{R}^2\,\partial\mathrm{Q}^2} + \frac{1}{\mathrm{p}^4}\frac{\partial^4\mathrm{w}}{\partial\mathrm{Q}^4} - \frac{\mathrm{qa}^4}{\mathrm{D}} = 0 \qquad (4.2)$$

Weighted Residual Form of Total Equilibrium of Forces

$$\bar{\mathrm{F}} = \frac{\partial\Pi}{\partial\mathrm{A}} = \frac{\mathrm{abD}}{\mathrm{a}^4}\int_0^{\mathrm{a}}\int_0^{\mathrm{b}}\left(\mathrm{A}\left[\frac{\partial^2\mathrm{h}}{\partial\mathrm{R}^2}\right]^2 + \frac{2}{\mathrm{p}^2}\left[\frac{\partial^2\mathrm{h}}{\partial R\partial Q}\right]^2 + \left[\frac{\partial^2\mathrm{h}}{\partial\mathrm{Q}^2}\right]^2\right)\mathrm{dR\,dQ} - \mathrm{abq}\int_0^{\mathrm{a}}\int_0^{\mathrm{b}}\mathrm{h\,dR\,dQ} = 0 \qquad (4.3)$$

Where F = Total force equilibrium of plate

g = Force equilibrium of a plate at an arbitrary point

$\bar{\mathrm{F}}$ = Weighted residual force equilibrium of plate

4.1.2 Exact shape functions and deflection coefficients of selected plates

Equations (4.4) to (4.9) are the exact polynomial shape functions for deflection of Kirchhoff's plates for the selected boundary conditions.

SSSS Plate

$$\mathrm{w} = \mathrm{A}\,(\mathrm{R} - 2\mathrm{R}^3 + \mathrm{R}^4)\,.\,(\mathrm{Q} - 2\mathrm{Q}^3 + \mathrm{Q}^4) \qquad (4.4)$$

$$\mathrm{A} = \frac{\mathrm{qa}^4}{\mathrm{D}}\frac{\mathrm{p}^4}{(4.8\mathrm{p}^4 + 8\mathrm{p}^2 + 4.8)} \qquad (4.5)$$

CCCC Plate

$$\mathrm{w} = \mathrm{A}(\mathrm{R}^2 - 2\mathrm{R}^3 + \mathrm{R}^4)\,.\,(\mathrm{Q}^2 - 2\mathrm{Q}^3 + \mathrm{Q}^4) \qquad (4.6)$$

$$A = \frac{qa^4}{D}\frac{0.449p^4}{(0.499p^4 + 0.272p^2 + 0.499)} \tag{4.7}$$

CSCS Plate

$$w = A(R - 2R^3 + R^4)\,.\,(Q^2 - 2Q^3 + Q^4) \tag{4.8}$$

$$A = \frac{qa^4}{D}\frac{0.67p^4}{(0.745p^4 + 1.476p^2 + 3.216)} \tag{4.9}$$

CCSS Plate

$$w = A(1.5R^2 - 2.5R^3 + R^4)\,.\,(1.5Q^2 - 2.5Q^3 + Q^4) \tag{4.10}$$

$$A = \frac{qa^4}{D}\frac{0.64p^4}{(1.381p^4 + 1.384p^2 + 1.381)} \tag{4.11}$$

CSSS Plate

$$w = A(R - 2R^3 + R^4)\,.\,(1.5Q^2 - 2.5Q^3 + Q^4) \tag{4.12}$$

$$A = \frac{qa^4}{D}\frac{0.8p^4}{(1.726p^4 + 3.328p^2 + 3.84)} \tag{4.13}$$

CCCS Plate

$$w = A(1.5R^2 - 2R^3 + R^4)\,.\,(Q^2 - 2Q^3 + Q^4) \tag{4.14}$$

$$A = \frac{qa^4}{D}\frac{0.536p^4}{(0.596p^4 + 0.614p^2 + 1.157)} \tag{4.15}$$

4.1.3 Exact bending moments and shear forces of the selected plates.

SSSS Plate Type

Employing the coefficient of deflection for SSSS plate type in Equations (3.70b), (3.71b), (3.82b) and (3.83b) gives the mid-span moment (β_{xC} and β_{yC}) and edge shear forces (K_{sxB} and K_{syA}) factors in x and y directions respectively as presented on Table 4.1 with aspect ratios range of $1 \leq p \leq 2$. Points of maximum values, A, B, and C are shown on Figure 4.1.

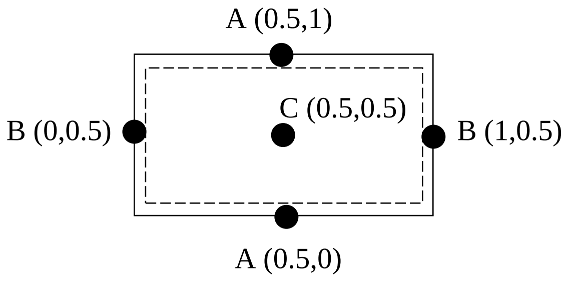

Figure 4.1: SSSS Plate showing points of maximum values

Table 4.1. Coefficients of maximum deflections, bending moment and shear forces for plate type SSSS

Aspect ratio, P	KDC	βxc	βyc	ksx	ksy
1	0.00555	0.06925	0.06925	0.50284	0.50284
1.1	0.00665	0.07964	0.07189	0.54220	0.50740
1.2	0.00771	0.08941	0.07358	0.57549	0.50670
1.3	0.00871	0.09844	0.07455	0.60349	0.50203
1.4	0.00964	0.10670	0.07497	0.62698	0.49446
1.5	0.01050	0.11420	0.07501	0.64670	0.48487
1.6	0.01128	0.12098	0.07479	0.66327	0.47393
1.7	0.01199	0.12708	0.07438	0.67724	0.46216
1.8	0.01264	0.13257	0.07385	0.68908	0.44993
1.9	0.01322	0.13751	0.07326	0.69914	0.43754
2	0.01375	0.14195	0.07262	0.70775	0.42518

CCCC Plate Type

Employing the coefficient of deflection for CCCC plate type in Equations (3.70b), (3.71b), (3.82b) and (3.83b) gives the mid-span moment (βxC and βyC) and edge shear forces (KsxB and KsyA) factors in x and y directions respectively as presented on Table 4.2 with aspect ratios range of $1 \leq p \leq 2$. Points of maximum values, A, B, and C and central point D are shown on Figure 4.2.

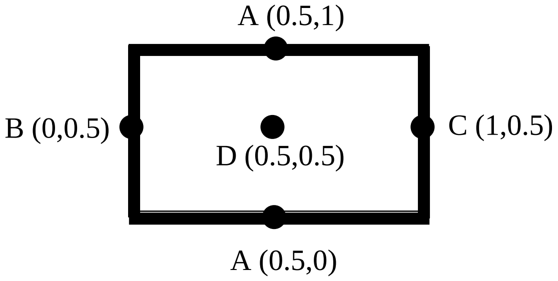

Figure 4.2: CCCC Plate showing points of maximum values

Table 4.2. Coefficients of maximum deflections, bending moment and shear forces for plate type CCCC

Aspect ratio, P	KDD	βxD	βyD	ksx	ksy
1	0.00138	0.02870	0.02870	0.26497	0.26497
1.1	0.00165	0.03287	0.02967	0.31609	0.23748
1.2	0.00189	0.03649	0.03003	0.36242	0.20973
1.3	0.00210	0.03956	0.02996	0.40319	0.18352
1.4	0.00228	0.04212	0.02960	0.43839	0.15976
1.5	0.00244	0.04424	0.02906	0.46843	0.13879
1.6	0.00257	0.04598	0.02843	0.49393	0.12059
1.7	0.00269	0.04742	0.02775	0.51553	0.10493
1.8	0.00278	0.04861	0.02708	0.53384	0.09154
1.9	0.00286	0.04959	0.02642	0.54941	0.08010
2	0.00293	0.05041	0.02579	0.56269	0.07034

CSCS Plate Type

Employing the coefficient of deflection for CSCS plate type in Equations (3.70b), (3.71b), (3.82b) and (3.83b) gives the mid-span moment (βxC and βyC) and edge shear forces (KsxB and KsyA) factors in x and y directions respectively as presented on Table 4.3 with aspect ratios range of 1 ≤ p ≤ 2. Points of maximum values, A, B, and C are shown on Figure 4.3.

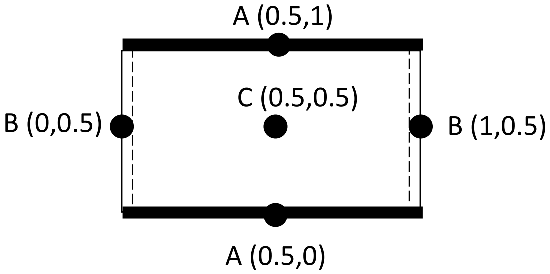

Figure 4.3: CSCS Plate showing points of maximum values

Table 4.3; Coefficients of maximum deflections, bending moment and shear forces for plate type CSCS

Aspect ratio, P	KDC	βxc	βyc	ksx	ksy
1	0.00241	0.03465	0.04544	0.30187	0.46205
1.1	0.00314	0.04266	0.05063	0.34691	0.45356
1.2	0.00394	0.05096	0.05512	0.38944	0.43777
1.3	0.00477	0.05931	0.05887	0.42863	0.41666
1.4	0.00560	0.06751	0.06188	0.46399	0.39206
1.5	0.00643	0.07540	0.06420	0.49534	0.36557
1.6	0.00722	0.08285	0.06592	0.52275	0.33845
1.7	0.00798	0.08981	0.06712	0.54644	0.31167
1.8	0.00868	0.09623	0.06789	0.56674	0.28588
1.9	0.00934	0.10211	0.06832	0.58403	0.26153
2	0.00995	0.10748	0.06847	0.59868	0.23883

CSSS Plate Type

Employing the coefficient of deflection for CSSS plate type in Equations (3.70b), (3.71b), (3.82b) and (3.83b) gives the mid-span moment (βxD and βyD) and edge shear forces (KsxB and KsyA) factors in x and y directions respectively as presented on Table 4.4 with aspect ratios range of $1 \leq p \leq 2$. Points of maximum values, A, B, and C are shown on Figure 4.4.

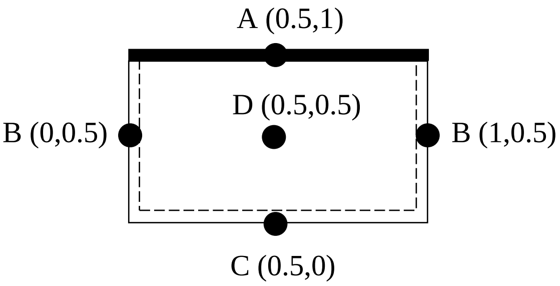

Figure 4.4: CSSS Plate showing points of maximum values

Table 4.4; Coefficients of maximum deflections, bending moment and shear forces for plate type CSSS

Aspect ratio, P	K_{DD}	β_{xD}	β_{yD}	ksx	ksy
1	0.00351	0.04637	0.05228	0.36427	0.48232
1.1	0.00440	0.05535	0.05632	0.40649	0.49932
1.2	0.00531	0.06420	0.05949	0.44430	0.50974
1.3	0.00620	0.07273	0.06188	0.47758	0.51454
1.4	0.00706	0.08078	0.06359	0.50651	0.51472
1.5	0.00788	0.08829	0.06474	0.53144	0.51126
1.6	0.00865	0.09521	0.06546	0.55278	0.50501
1.7	0.00936	0.10154	0.06584	0.57100	0.49673
1.8	0.01002	0.10730	0.06595	0.58653	0.48699
1.9	0.01062	0.11252	0.06587	0.59976	0.47629
2	0.01117	0.11725	0.06566	0.61105	0.46499

CCSS Plate Type

Employing the coefficient of deflection for CSSS plate type in Equations (3.70b), (3.71b), (3.82b) and (3.83b) gives the mid-span moment (βxD and βyD) and edge shear forces (KsxB and KsyA) factors in x and y directions respectively as presented on Table 4.5 with aspect ratios range of 1 ≤ p ≤ 2. Points of maximum values, A, B, and C are shown on Figure 4.5.

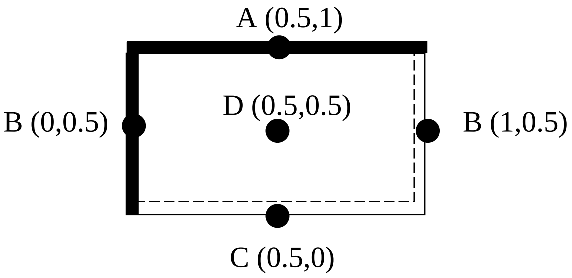

Figure 4.5: CCSS Plate showing points of maximum values

Table 4.5; Coefficients of maximum deflections, bending moment and shear forces for plate type CCSS

Aspect ratio, P	KDD	βxD	βyD	ksx	ksy
1	0.00241	0.03762	0.03762	0.28938	0.28938
1.1	0.00288	0.04317	0.03897	0.34595	0.28175
1.2	0.00332	0.04819	0.03966	0.39885	0.27144
1.3	0.00373	0.05265	0.03987	0.44710	0.25967
1.4	0.00409	0.05654	0.03973	0.49039	0.24734
1.5	0.00441	0.05993	0.03937	0.52880	0.23502
1.6	0.00469	0.06286	0.03886	0.56266	0.22309
1.7	0.00494	0.06539	0.03827	0.59242	0.21174
1.8	0.00515	0.06758	0.03765	0.61854	0.20109
1.9	0.00535	0.06948	0.03701	0.64149	0.19116
2	0.00551	0.07113	0.03639	0.66167	0.18196

CCCS Plate Type

Employing the coefficient of deflection for CCCS plate type in Equations (3.70b), (3.71b), (3.82b) and (3.83b) gives the mid-span moment (βxD and βyD) and edge shear forces (KsxB and KsyA) factors in x and y directions respectively as presented on Table 4.6 with aspect ratios range of 1 ≤ p ≤ 2. Points of maximum values, A, B, and C are shown on Figure 4.6.

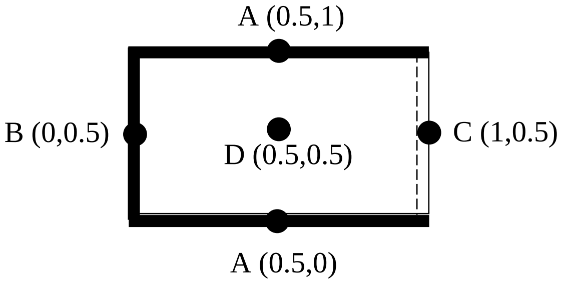

Figure 4.6: CCCS Plate showing points of maximum values

Table 4.6; Coefficients of maximum deflections, bending moment and shear forces for plate type CCCS

Aspect ratio, P	KDD	βxD	βyD	ksx	ksy
1	0.00177	0.02972	0.03468	0.21230	0.33968
1.1	0.00221	0.03531	0.03720	0.21931	0.31899
1.2	0.00265	0.04063	0.03898	0.22082	0.29442
1.3	0.00307	0.04555	0.04011	0.21793	0.26823
1.4	0.00346	0.04999	0.04070	0.21182	0.24208
1.5	0.00382	0.05393	0.04087	0.20352	0.21709
1.6	0.00414	0.05740	0.04075	0.19390	0.19390
1.7	0.00442	0.06041	0.04040	0.18363	0.17283
1.8	0.00468	0.06303	0.03992	0.17317	0.15393
1.9	0.00490	0.06530	0.03935	0.16285	0.13714
2	0.00510	0.06727	0.03873	0.15289	0.12231

4.1.4 Values of Euler-Bernoulli residual force for the selected plates.

The coefficients of the components of rectangular plate (k_x, k_{xy}, k_y, k_q) governing differential obtained by Ibearugbulem, 2014 were substituted into Euler-Bernoulli force equilibrium equation, Equation (3.445), the residual force (F) obtained for the selected plate types are as shown on Table 4.7. Similarly, the coefficients of the components of rectangular plate (k_x, k_{xy}, k_y, k_q) governing differential equation obtained from the present study and the residual forces are as shown on Table 4.8.

Table 4.7; Values of Euler-Bernoulli residual force from previous study (Ibearugbulem, 2014)

S/N	Plate type	k_x	k_{xy}	k_y	k_q	F
1	SSSS	0.236210	0.23591000	0.2362100	0.0400	0.013600
2	CCCC	0.001270	0.00036000	0.0012700	0.0011	0.000041
3	CSCS	0.007630	0.00925000	0.0393700	0.0070	0.001401
4	CSSS	0.036192	0.04163210	0.0885710	0.0150	0.003711
5	CCSS	0.013572	0.00734690	0.0135720	0.0060	0.000832
6	CCCS	0.002857	0.00163268	0.0060318	0.0025	0.001046

Table 4.8**;** Result of the values of Euler-Bernoulli residual force from present study

S/N	**Plate type**	**Kx**	**Kxy**	**Ky**	**Kq**	**F**
1	SSSS	4.800	8.000	4.800	1.000	0
2	CCCC	0.499	0.272	0.499	0.449	0
3	CSCS	0.745	1.477	3.216	0.670	0
4	CSSS	1.726	3.328	3.840	0.800	0
5	CCSS	1.381	1.384	1.381	0.640	0
6	CCCS	0.596	0.614	1.157	0.536	0

4.1.5 Comparison with results of past reserchers.

The results of pure bending analysis of SS, CC and CS line continuums for Euler-Bernoulli (exact) and Ritz (weighted residual) methods were presented in Tables 3.3 and 3.4, Tables 3.6 and 3.7, and Tables 3.9 and 3.10 respectively. The values of the polynomial shape functions and coefficients of deflection from the residual force approach by Euler-Bernoulli coincides with the corresponding values obtained from the minimization approach used by Ritz. Hence, the same exact result was obtained from the two approaches in consideration. Consequently, many researchers and scholars assume that minimization is equilibrium. The same idea was quickly used on two-dimensional analysis such as plate. However, this work has is of different view on the concept of equilibrium and minimization based on the findings of the study.

Results from the two approaches were plotted on a graph with the aspect ratio as the abscissa and the design factors as the ordinate. The percentage difference between the two approaches were stated. Tabular presentation of the comparison is attached in Appendices A to F.

SSSS Plate type

The percentage difference between the deflection factors (K_D) obtained from the present study and by Ibearugbulem, 2014 for SSSS plate is as shown on Figure 4.7. The average percentage difference from the table for aspect ratio, p of range $1 \leq p \leq 2$ is 24.42%.

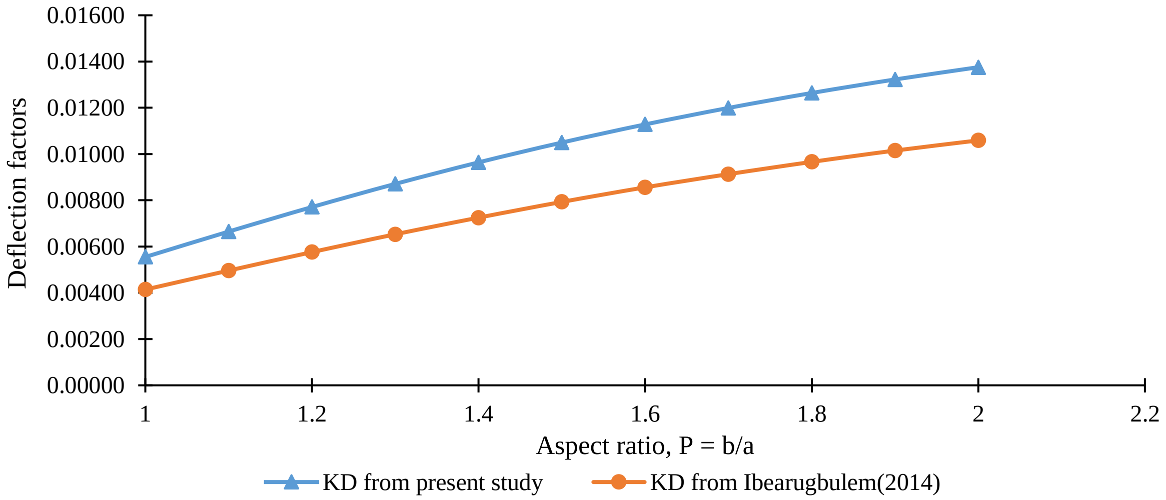

Figure 4.7; Factor for deflection, K_D for plate type SSSS for different aspect ratios

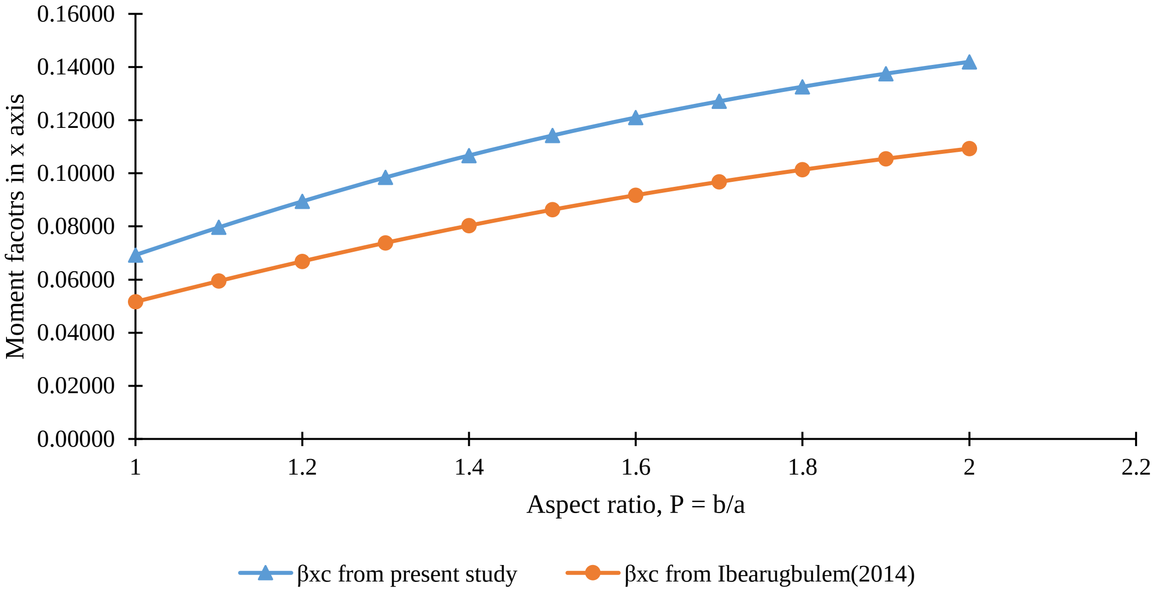

Figure 4.8; Factor for moment, in x - axis for plate type SSSS

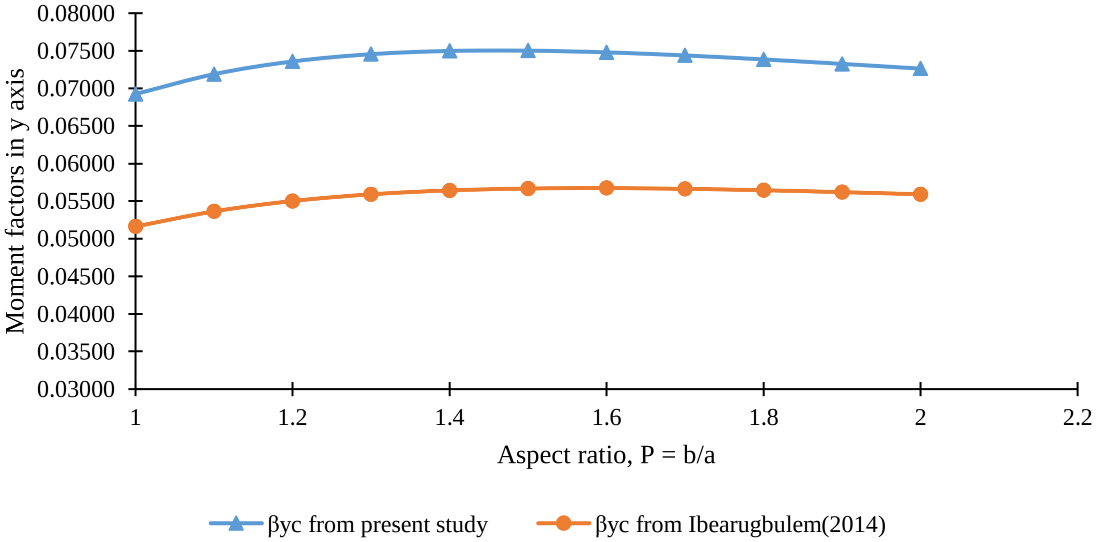

Figure 4.9; Factor for moment, in y - axis for plate type SSSS

The percentage difference between the bending moment factors (βx and βy) obtained from the present study and by Ibearugbulem, 2014 for SSSS plate are as shown on Figures 4.8 and 4.9. The average percentage difference from the table for aspect ratio, p of range $1 \leq p \leq 2$ is 23.37% for mid-span moment in the x and y directions.

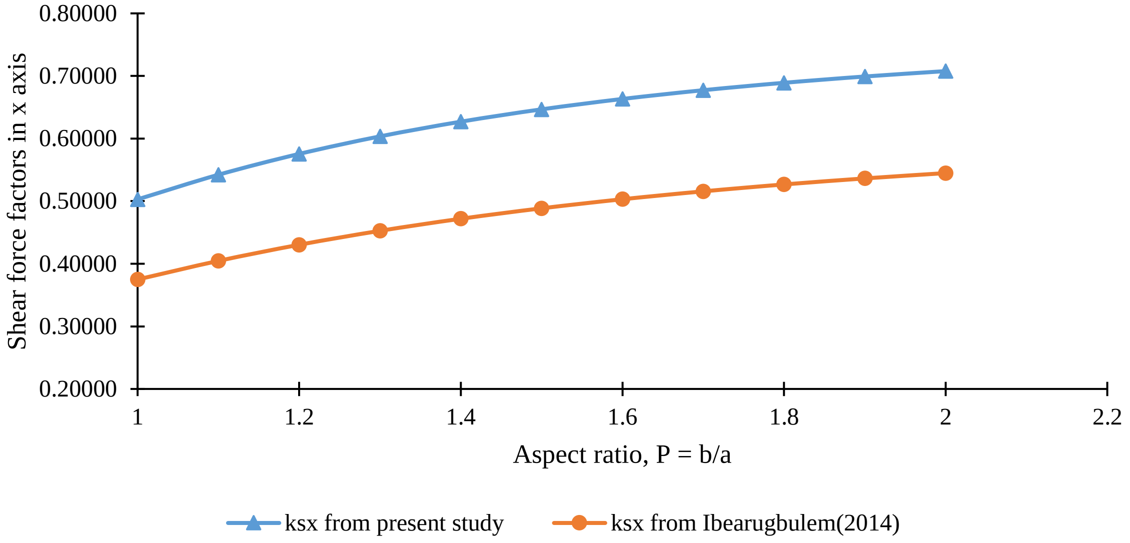

Figure 4.10; Factor for shear force, in x - axis for plate type SSSS

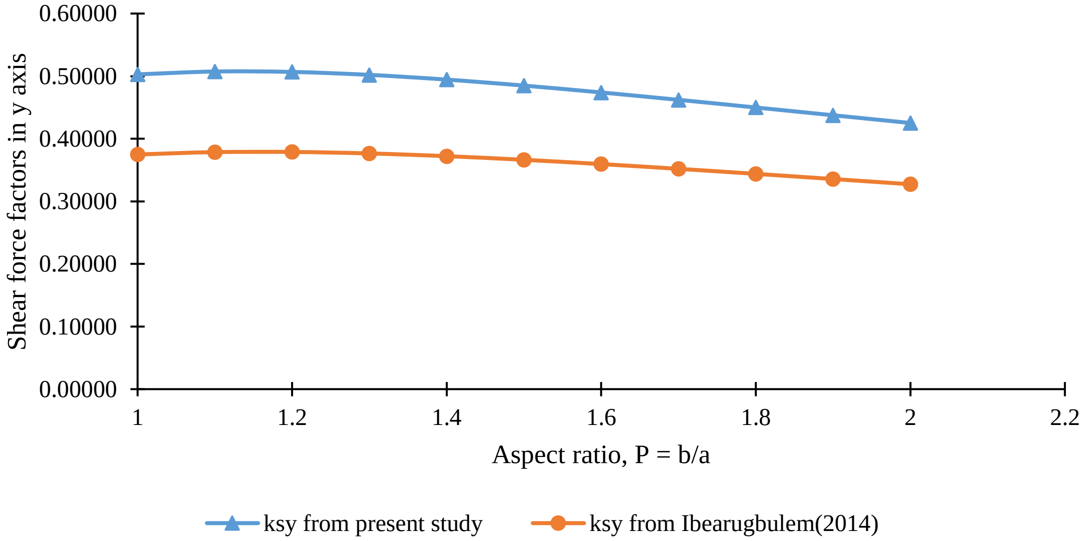

Figure 4.11; Factor for shear force, in y - axis for plate type SSSS

The percentage difference between the shear force factors (Ksx and Ksy) obtained from the present study and by Ibearugbulem, 2014 for SSSS plate are as shown on Figures 4.10 and 4.11. The average percentage difference from the table for aspect ratio, p of range $1 \leq p \leq 2$ is 23.37% for mid-span moment in the x and y directions.

CCCC Plate type

The percentage difference between the deflection factors (K_D) obtained from the present study and by Ibearugbulem, 2014 for CCCC plate is as shown on Figure 4.12. The average percentage difference from the table for aspect ratio, p of range $1 \leq p \leq 2$ is 3.35%.

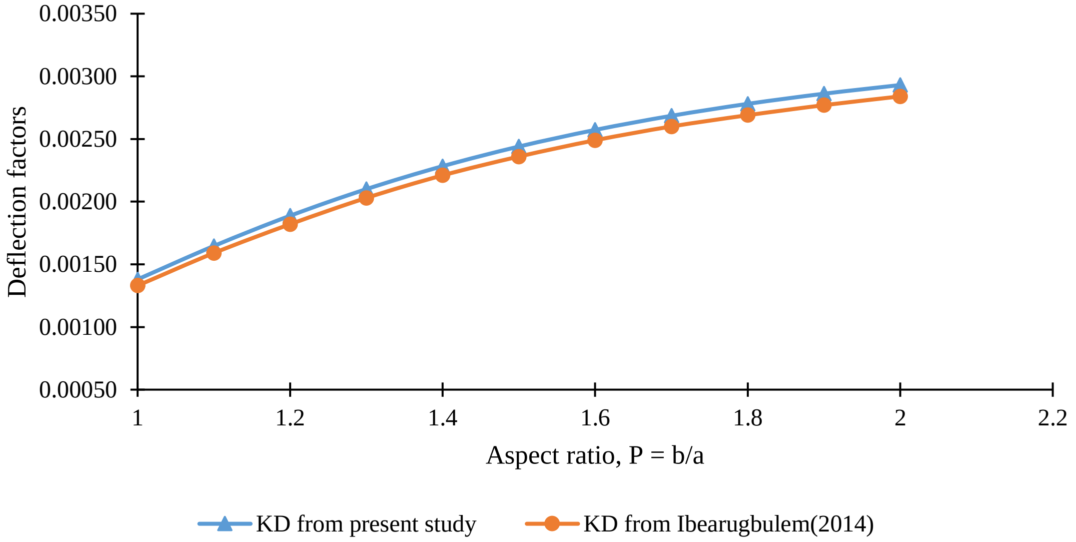

Figure 4.12; Factor for deflection, K_D for plate type CCCC for different aspect ratios

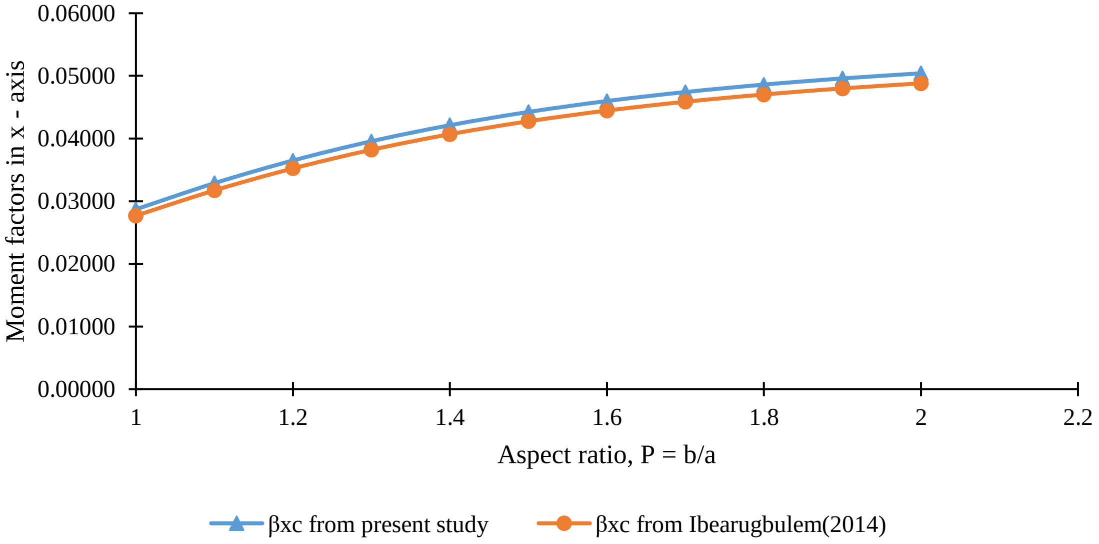

Figure 4.13; Factor for moment, in x - axis for plate type CCCC

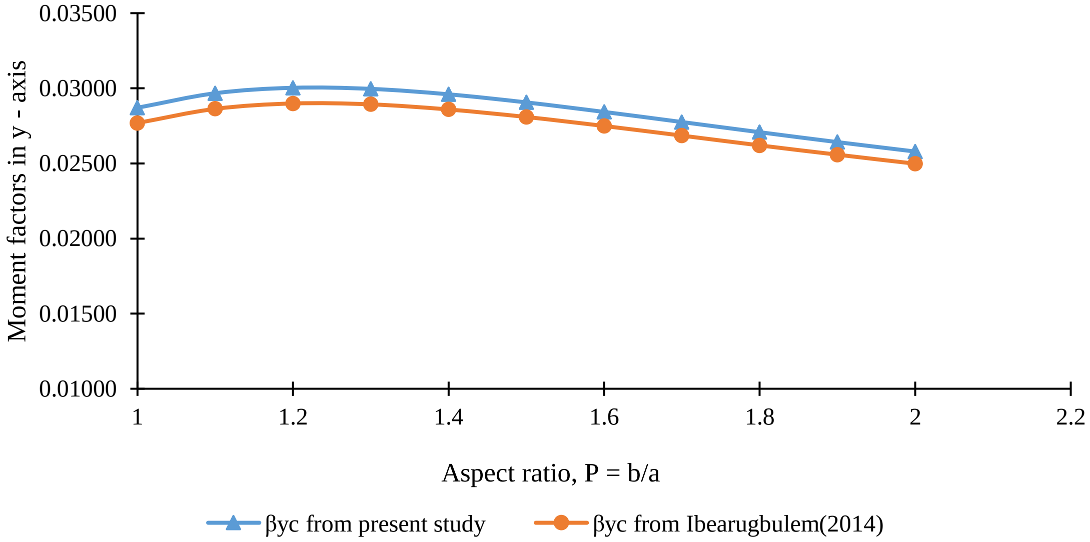

Figure 4.14; Factor for moment, in y - axis for plate type CCCC

The percentage difference between the bending moment factors (βx and βy) obtained from the present study and by Ibearugbulem, 2014 for CCCC plate are as shown on Figures 4.13 and 4.14. The average percentage difference from the table for aspect ratio, p of range $1 \leq p \leq 2$ is 3.36% for mid-span moment in the x and y directions.

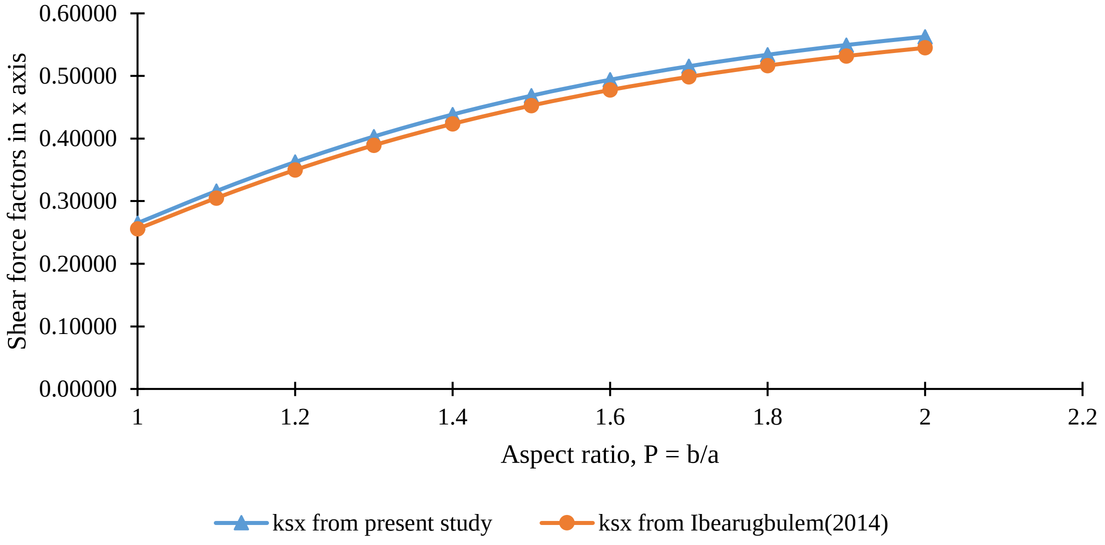

Figure 4.15; Factor for shear force, in x - axis for plate type CCCC

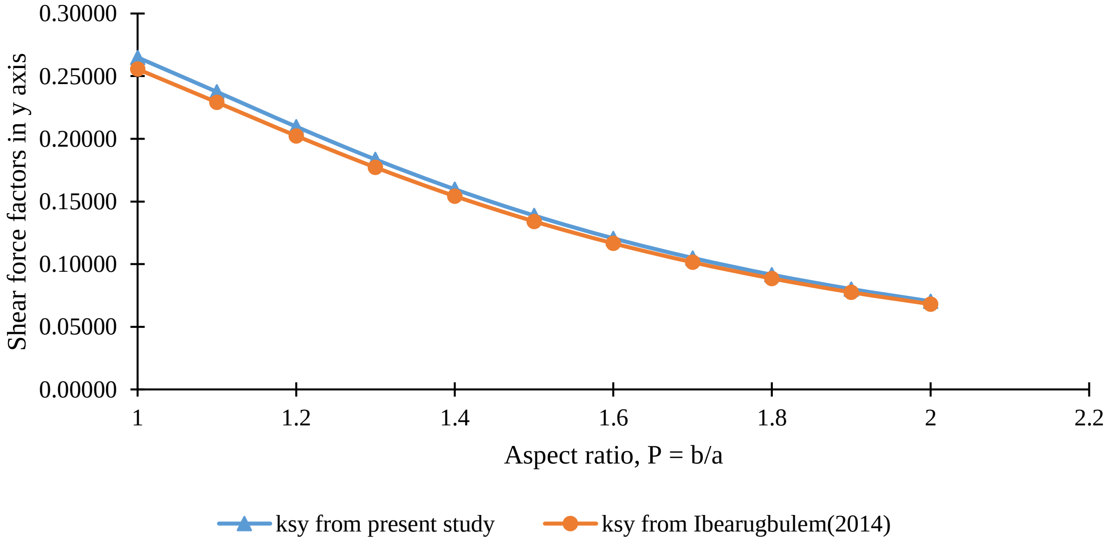

Figure 4.16; Factor for shear force, in y - axis for plate type CCCC

The percentage difference between the shear force factors (Ksx and Ksy) obtained from the present study and by Ibearugbulem, 2014 for CCCC plate are as shown on Figures 4.15 and 4.16. The average percentage difference from the table for aspect ratio, p of range $1 \leq p \leq 2$ is 3.36% for mid-span moment in the x and y directions.

CSCS Plate type

The percentage difference between the deflection factors (K_D) obtained from the present study and by Ibearugbulem, 2014 for CSCS plate is as shown on Figure 4.17. The average percentage difference from the table for aspect ratio, p of range $1 \leq p \leq 2$ is 14.2%.

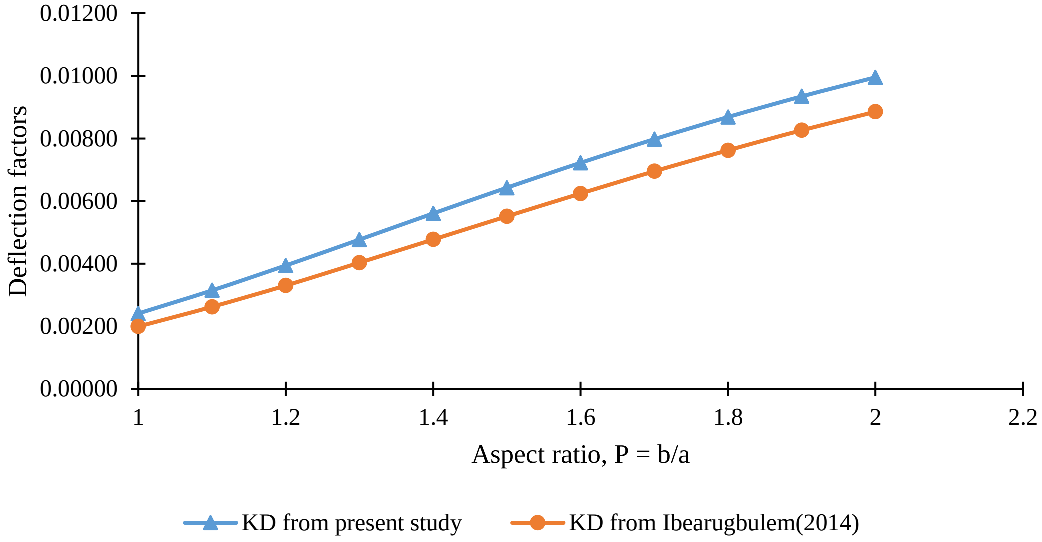

Figure 4.17; Factor for deflection, K_D for plate type CSCS for different aspect ratios

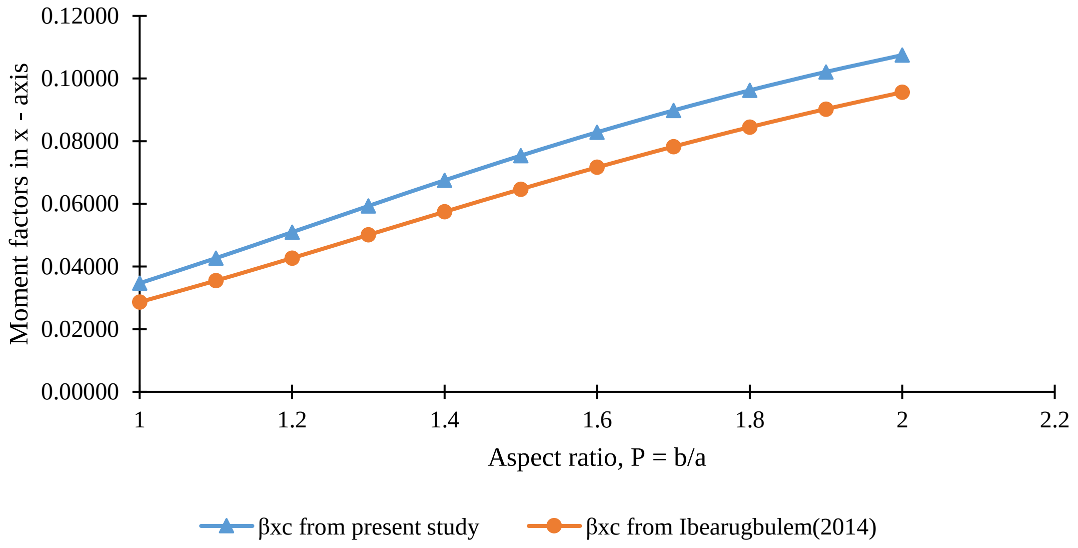

Figure 4.18; Factor for moment, in x - axis for plate type CSCS

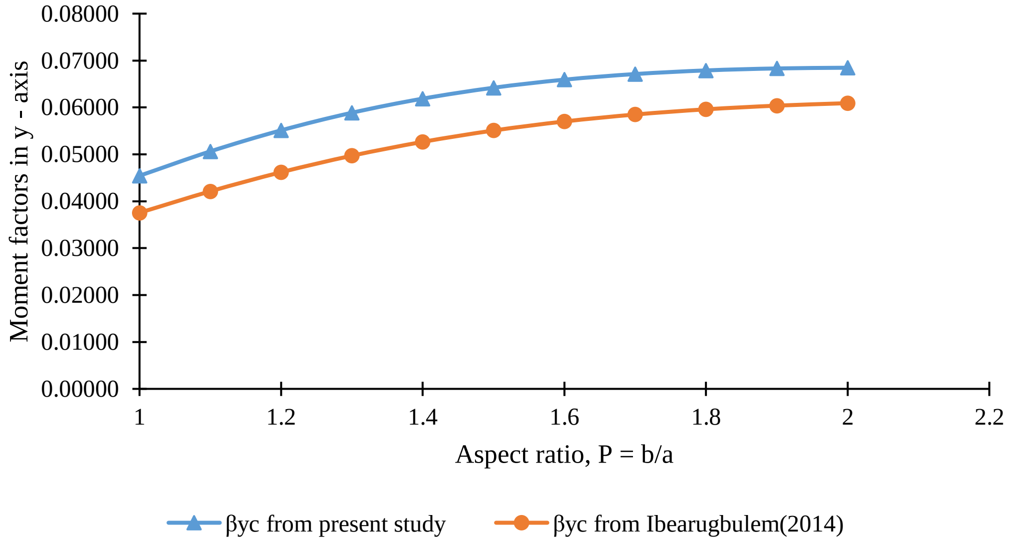

Figure 4.19; Factor for moment, in y - axis for plate type CSCS

The percentage difference between the bending moment factors (βx and βy) obtained from the present study and by Ibearugbulem, 2014 for CSCS plate are as shown on Figures 4.18 and 4.19. The average percentage difference from the table for aspect ratio, p of range $1 \leq p \leq 2$ is 14.21% for mid-span moment in the x and y directions.

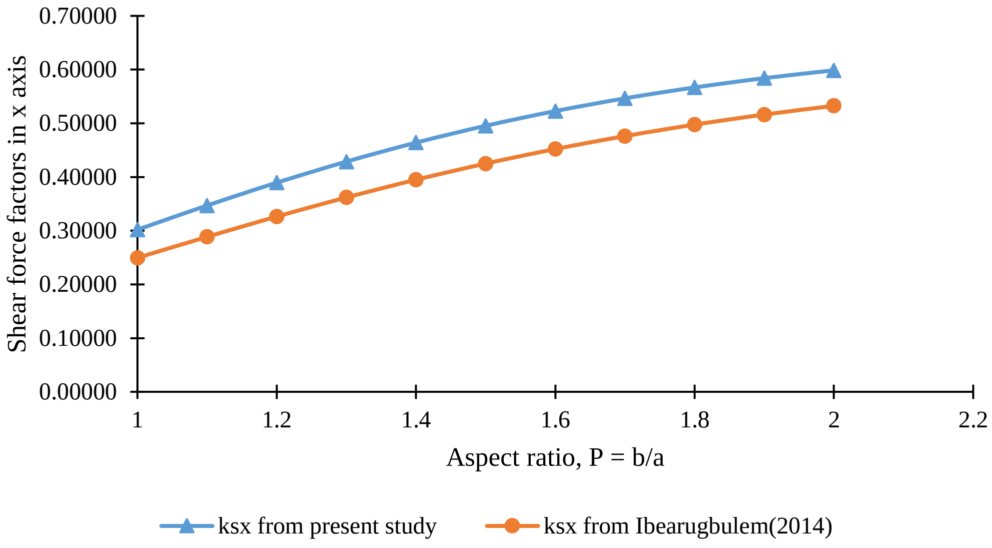

Figure 4.20; Factor for shear force, in x - axis for plate type CSCS

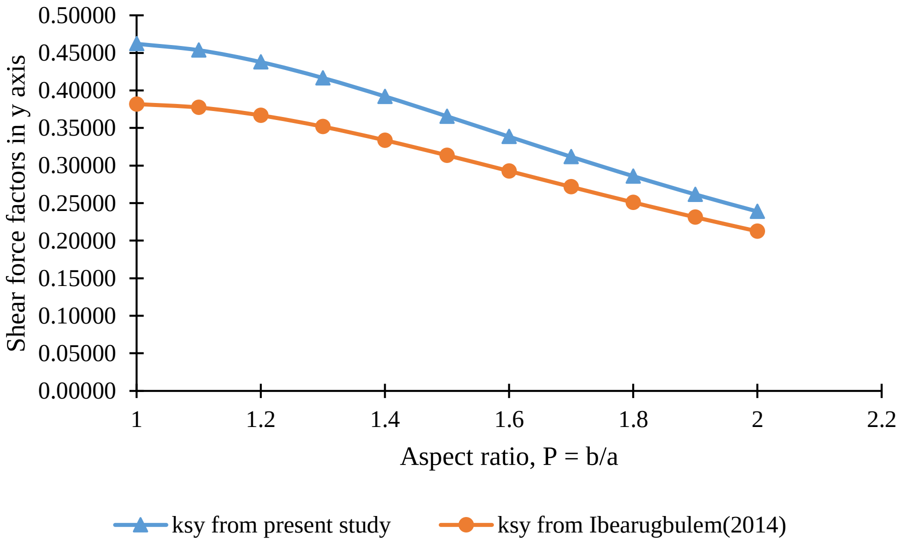

Figure 4.21; Factor for shear force, in y - axis for plate type CSCS

The percentage difference between the shear force factors (Ksx and Ksy) obtained from the present study and by Ibearugbulem, 2014 for CSCS plate are as shown on Figures 4.20 and 4.21. The average percentage difference from the table for aspect ratio, p of range $1 \leq p \leq 2$ is 14.21% for mid-span moment in the x and y directions.

CSSS Plate type

The percentage difference between the deflection factors (K_D) obtained from the present study and by Ibearugbulem, 2014 for CSSS plate is as shown on Figure 4.17. The average percentage difference from the table for aspect ratio, p of range $1 \leq p \leq 2$ is 18.0%.

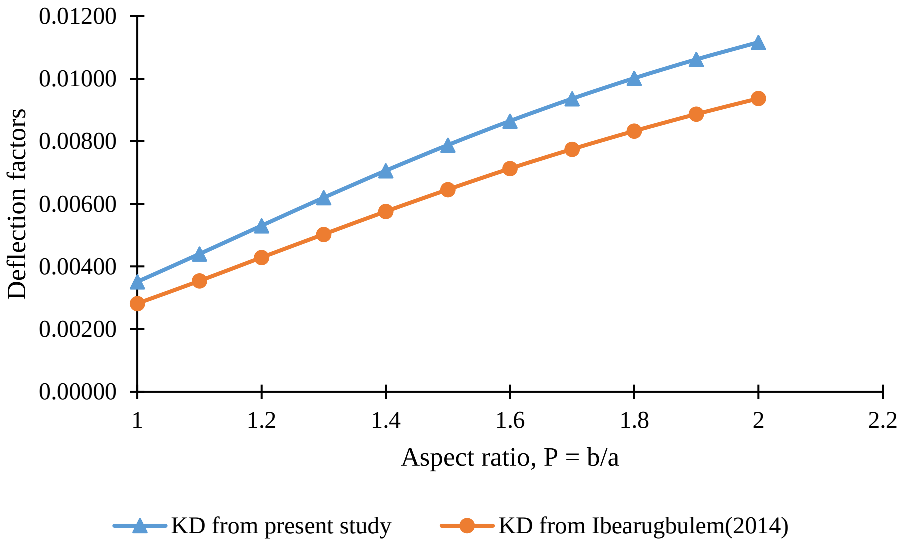

Figure 4.22; Factor for deflection, K_D for plate type CSSS for different aspect ratios

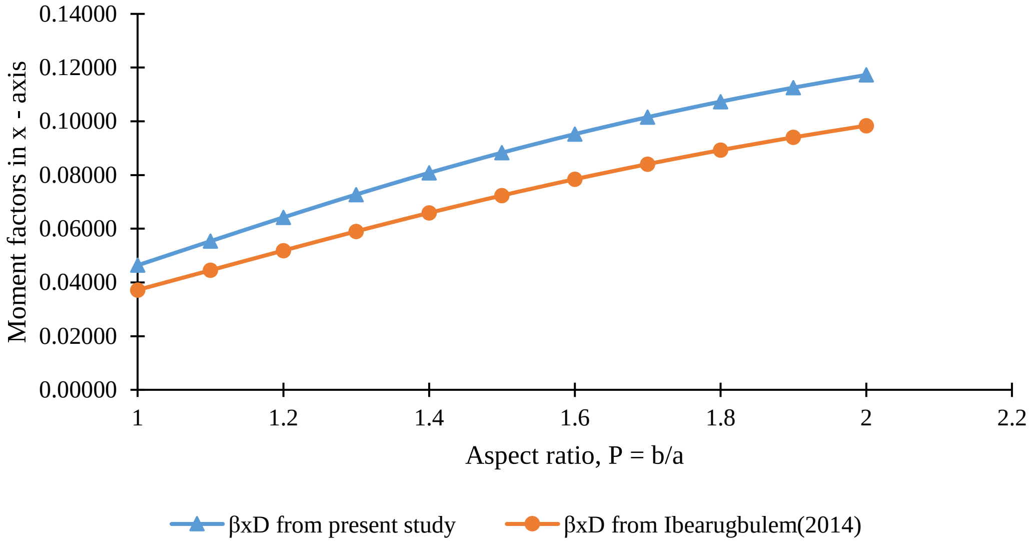

Figure 4.23; Factor for moment, in x - axis for plate type CSSS

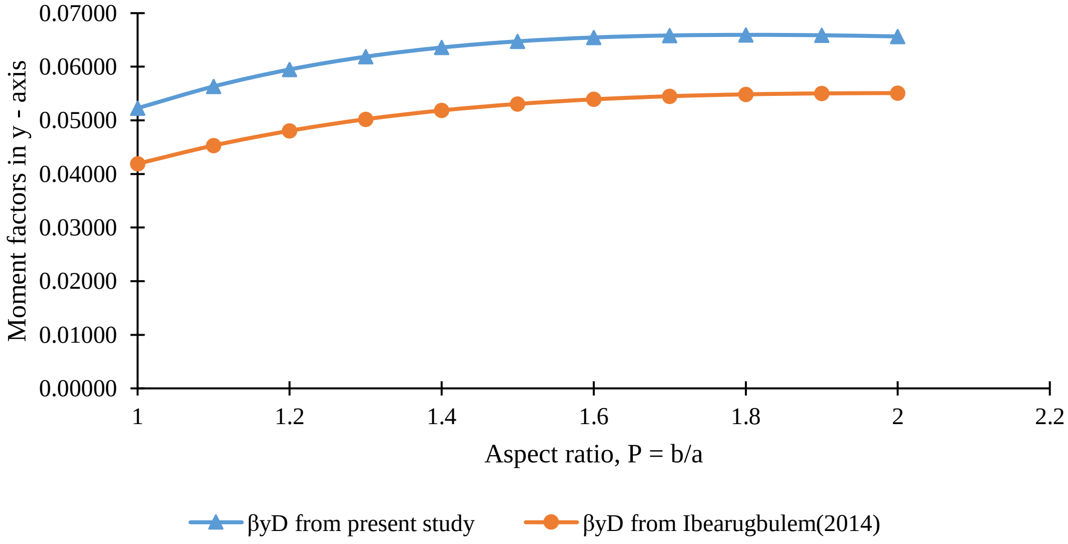

Figure 4.24; Factor for moment, in y - axis for plate type CSSS

The percentage difference between the bending moment factors (βx and βy) obtained from the present study and by Ibearugbulem, 2014 for CSCS plate are as shown on Figures 4.23 and 4.24. The average percentage difference from the table for aspect ratio, p of range $1 \leq p \leq 2$ is 18.02% for mid-span moment in the x and y directions.

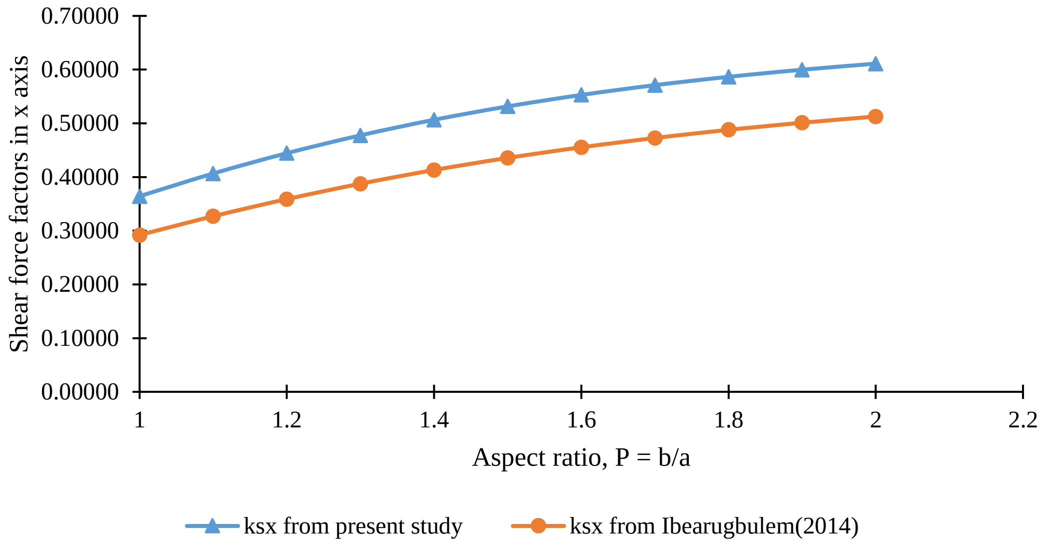

Figure 4.25; Factor for shear force, in x - axis for plate type CSSS

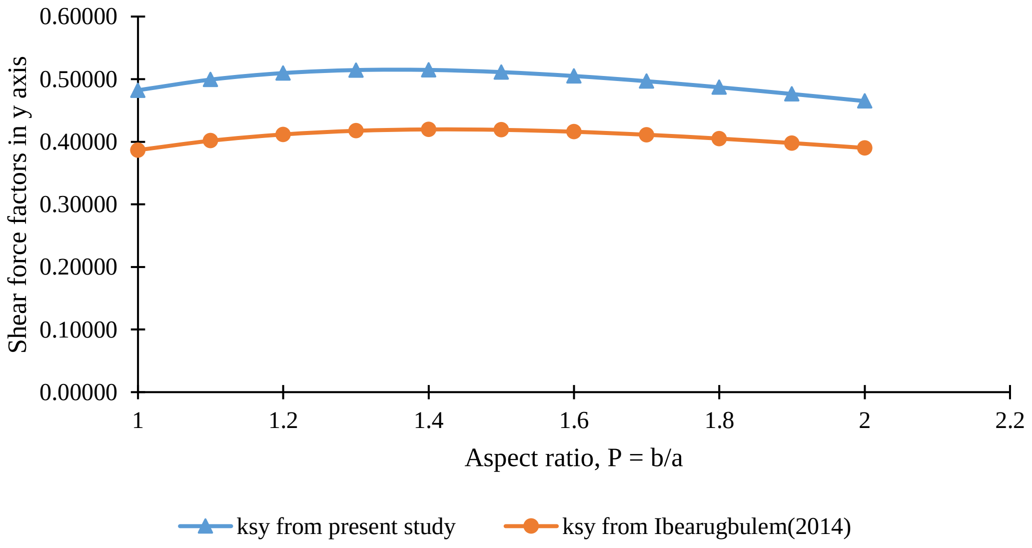

Figure 4.26; Factor for shear force, in y - axis for plate type CSSS

The percentage difference between the shear force factors (Ksx and Ksy) obtained from the present study and by Ibearugbulem, 2014 for CSSS plate are as shown on Figures 4.25 and 4.26. The average percentage difference from the table for aspect ratio, p of range $1 \leq p \leq 2$ is 18.02% for mid-span moment in the x and y directions.

CCSS Plate type

The percentage difference between the deflection factors (K_D) obtained from the present study and by Ibearugbulem, 2014 for CCSS plate is as shown on Figure 4.17. The average percentage difference from the table for aspect ratio, p of range $1 \leq p \leq 2$ is 12.46%.

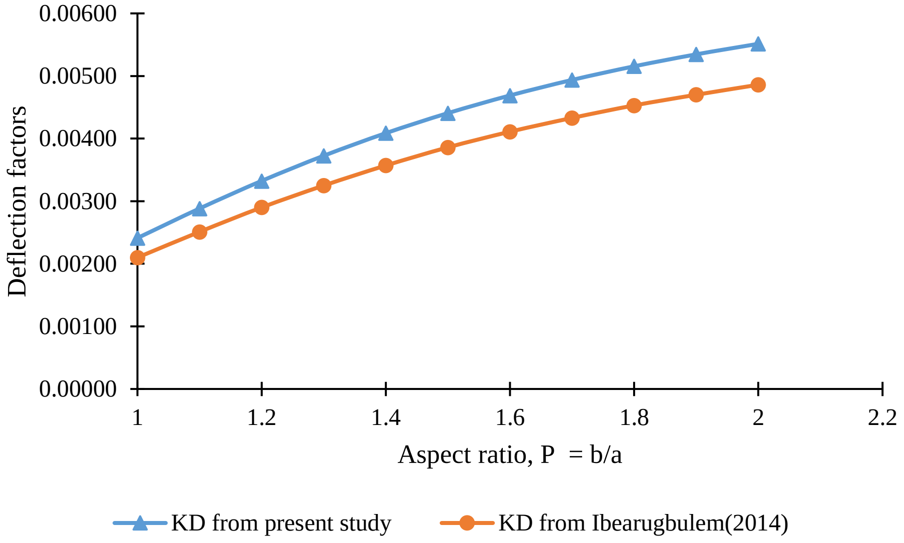

Figure 4.27; Factor for deflection, K_D for plate type CCSS for different aspect ratios

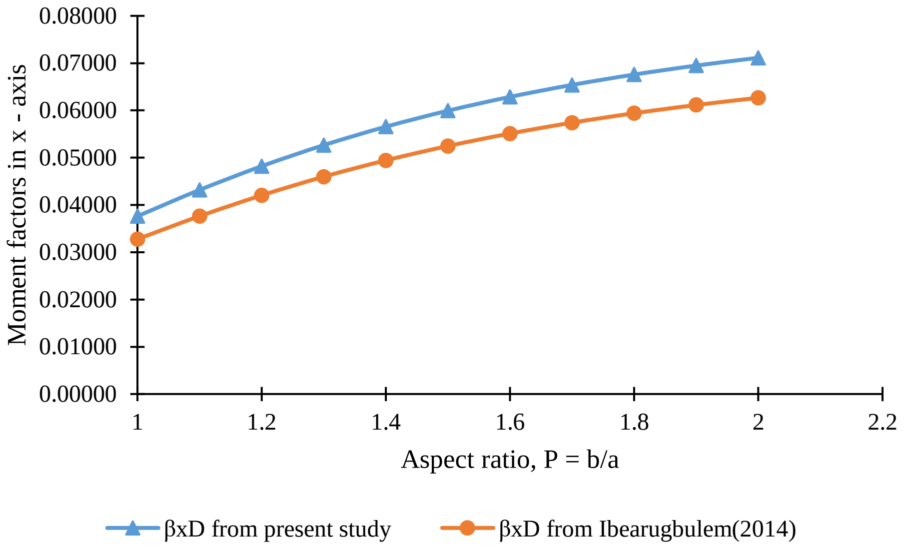

Figure 4.28; Factor for moment, in x - axis for plate type CCSS

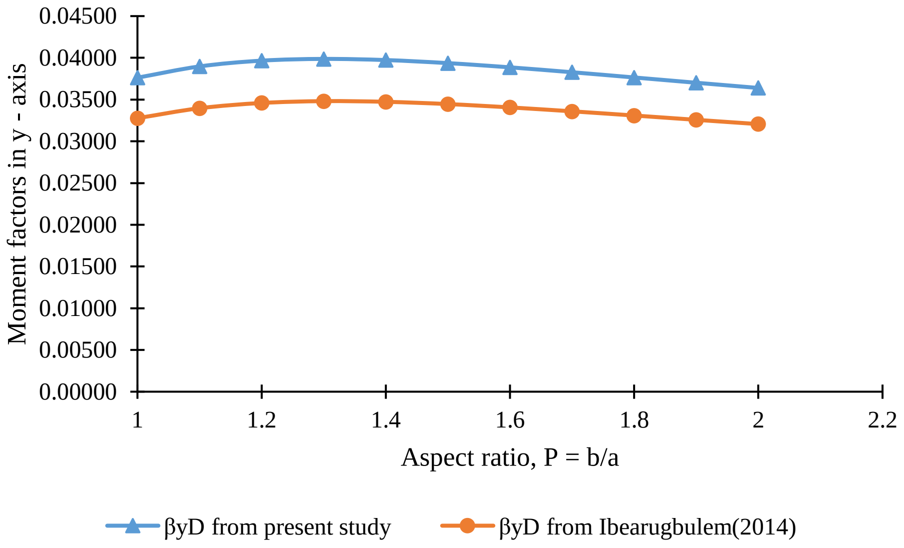

Figure 4.29; Factor for moment, in y - axis for plate type CCSS

The percentage difference between the bending moment factors (βx and βy) obtained from the present study and by Ibearugbulem, 2014 for CSCS plate are as shown on Figures 4.28 and 4.29. The average percentage difference from the table for aspect ratio, p of range $1 \leq p \leq 2$ is 12.44% for mid-span moment in the x and y directions.

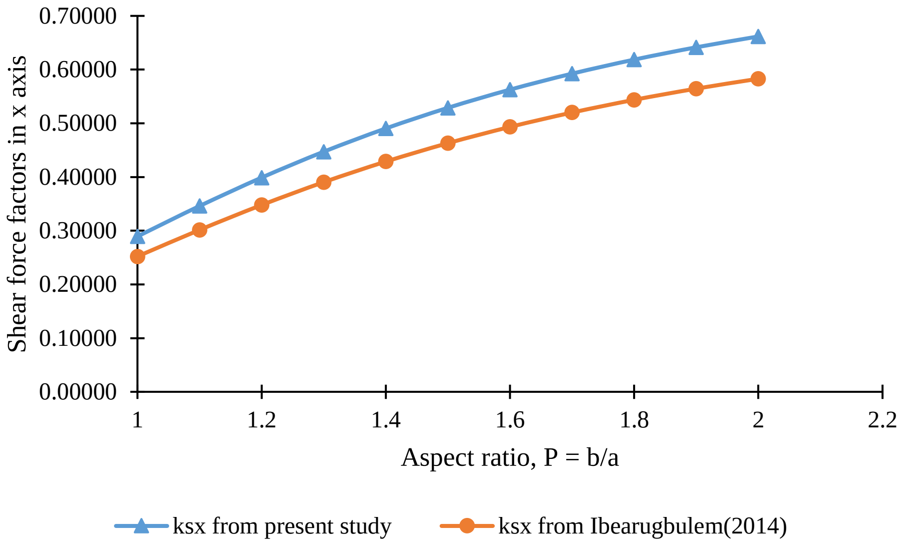

Figure 4.30; Factor for shear force, in x - axis for plate type CCSS

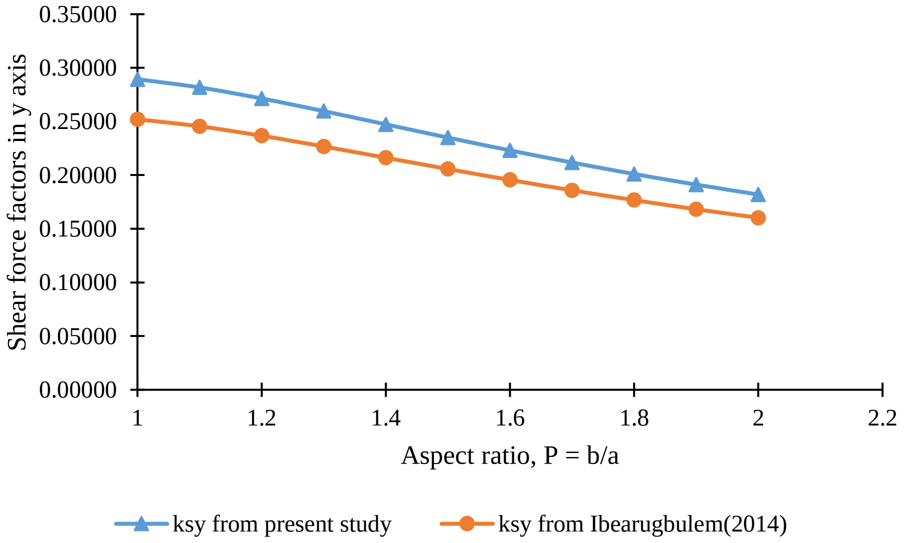

Figure 4.31; Factor for shear force, in y - axis for plate type CCSS

The percentage difference between the shear force factors (Ksx and Ksy) obtained from the present study and by Ibearugbulem, 2014 for CCSS plate are as shown on Figures 4.30 and 4.31. The average percentage difference from the table for aspect ratio, p of range $1 \leq p \leq 2$ is 12.44% for mid-span moment in the x and y directions.

CCCS Plate type

The percentage difference between the deflection factors (K_D) obtained from the present study and by Ibearugbulem, 2014 for CCCS plate is as shown on Figure 4.17. The average percentage difference from the table for aspect ratio, p of range $1 \leq p \leq 2$ is 7.93%.

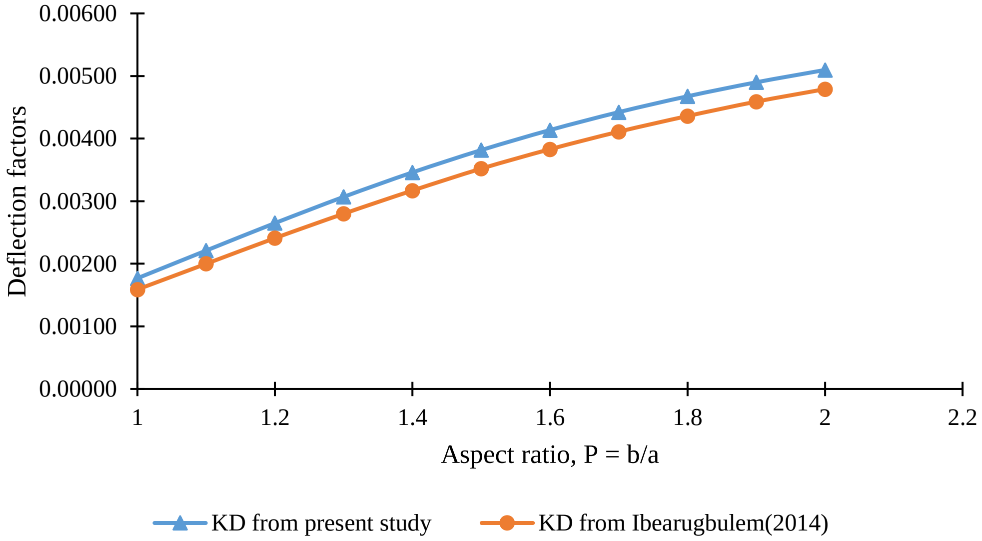

Figure 4.32; Factor for deflection, K_D for plate type CCCS for different aspect ratios

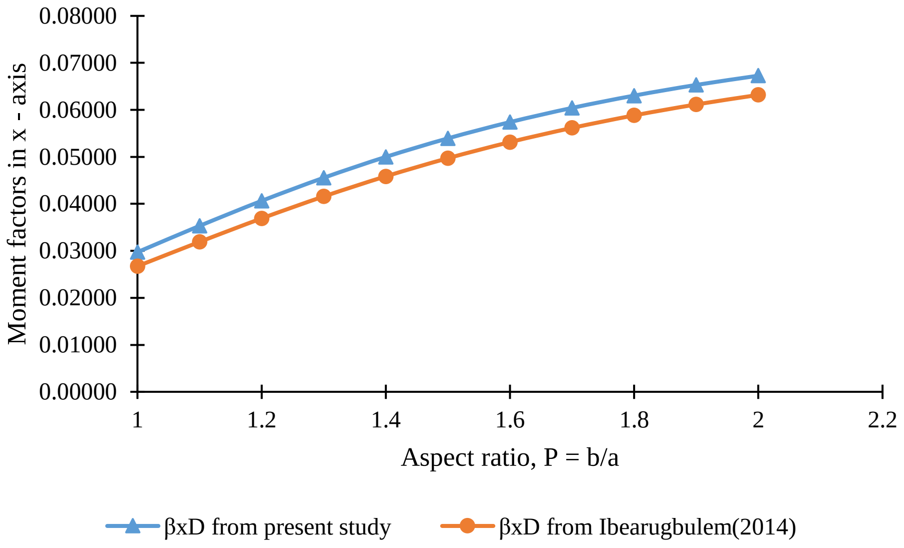

Figure 4.33; Factor for moment, in x - axis for plate type CCCS

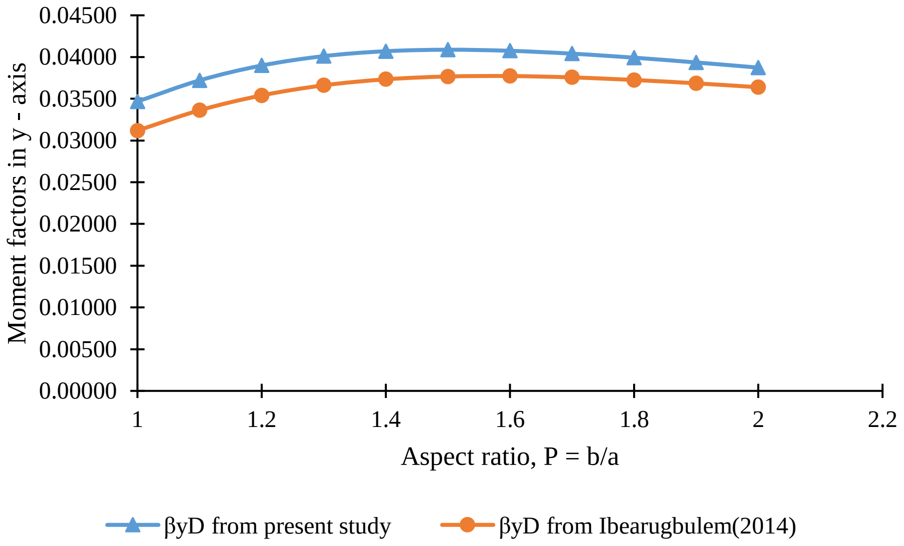

Figure 4.34; Factor for moment, in y - axis for plate type CCCS

The percentage difference between the bending moment factors (βx and βy) obtained from the present study and by Ibearugbulem, 2014 for CSCS plate are as shown on Figures 4.33 and 4.34. The average percentage difference from the table for aspect ratio, p of range $1 \leq p \leq 2$ is 7.94% for mid-span moment in the x and y directions.

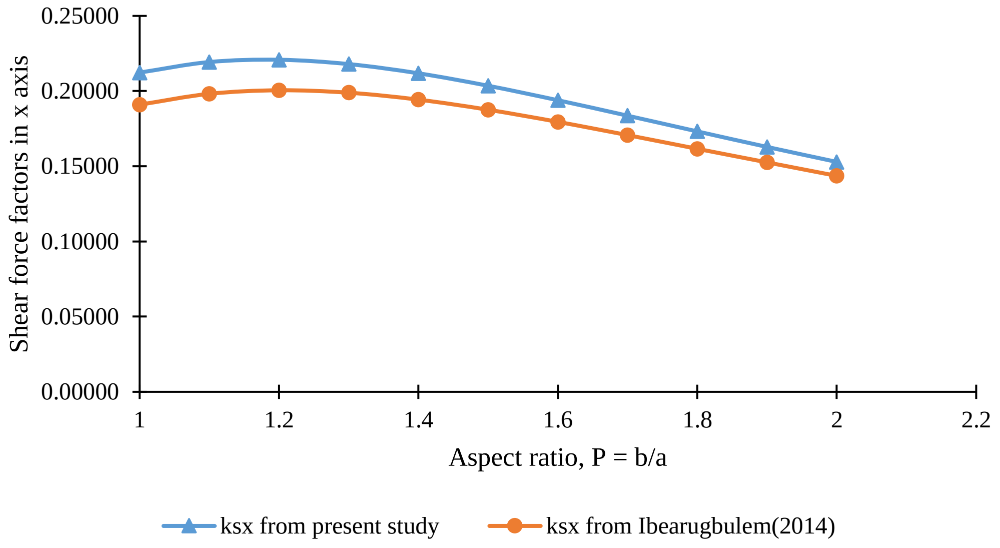

Figure 4.35; Factor for shear force, in x - axis for plate type CCCS

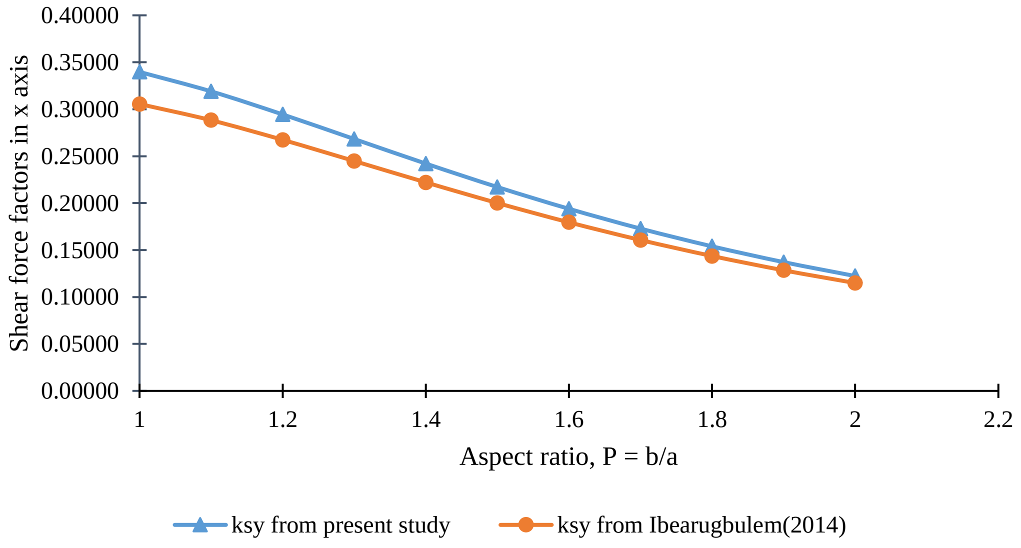

Figure 4.36; Factor for shear force, in y - axis for plate type CCCS

The percentage difference between the shear force factors (Ksx and Ksy) obtained from the present study and by Ibearugbulem, 2014 for CCCS plate are as shown on Figures 4.35 and 4.36. The average percentage difference from the table for aspect ratio, p of range $1 \leq p \leq 2$ is 7.94% for mid-span moment in the x and y directions.

4.2 Discussion of results

4.2.1 Total energy functional of the plate

Equation (4.1) is the whole plate force equilibrium equation (F = 0), Equation (4.2) is the plate force equilibrium equation at any arbitrary point on the plate (g = 0). Equation 4.3 on the other hand, is the total force equilibrium equation used in Rayleigh-Ritz method ($\bar{F} = 0$). Equation (4.3) was formulated based on Ritz energy method (See Equation (2.43)). Ibearugbulem obtained his results from direct integration of Equation (4.3). The work of other scholars such as Timoshenko, Levy and Navier were based on Equation (4.2), (See Equation (2.42)). This research work considered Equation (4.1) as superior to Equations (4.2) and (4.3) on the ground that Equation (4.3) considers whole plate equilibrium, Equation (4.2) considers plate equilibrium at an arbitrary point on the plate and Equation (4.3) is an approximate formulation.

4.2.2 Exact shape functions and deflection coefficients for the selected plates

The shape function derived in this study turns out to be same with the shape functions obtained by Ibearugbulem (2014) (see Equations (2.44) – (2.49)) using direct integration of the approximate form of total force equilibrium equation as seen in Equation (4.3). This implies that any variation in the result of the present study and that of the Ibearugbulem (2014) is not as a result of the shape function. The shape function used by Timoshenko & Woinosky Krieger (1959) were expressed in trigonometric functions, while the ones obtained in this study were expressed in polynomial functions. Hence, there was no base for comparing the result of shape function from the present study with the result from Timoshenko & Woinosky Krieger (1959). The result of the coefficient of deflection from this study are shown in Equations (4.5), (4.7), (4.9), (4.11), (4.3) and (4.15). The coefficients obtained in this work is based on exact satisfaction of Euler-Bernoulli equation (Equation (1.1)) for a plane continuum in equilibrium. The result from the work of Ibearugbulem (2014) were shown in Equations (2.50) to (2.55). Ibearugbulem obtained the coefficients of deflection using Equation (1.4) which is otherwise described in this study as quasi-force equilibrium equation. The results for deflection, bending moment, and shear force obtained in the present study therefore differ from those of Ibearugbulem (2014). The numerical values of the coefficient of deflection shows an upper bound as compared to the results of Ibearugbulem (2014).

4.2.3 Exact bending moments and shear forces of the selected plates

The section discusses the results of the bending moment and shear force factors emanating from the present study.

4.2.3.1 Factors for plate type SSSS

The factors for deflection, bending moment and shear force for SSSS plate type are presented in Table 4.1. The x and y axes were considered in the calculation. Aspect ratio from 1 to 2.0 were considered. The dots on the plate in Figure 4.1 show points of maximum values.

The results were obtained by employing the exact coefficient of deflection (Equation 4.5) obtained from this study for SSSS plate type in bending moment equations, Equation (3.70b) for x-direction and Equation (3.71b) for y-direction, and shear force equations, Equation (3.82b) for x-direction and Equation (3.83b) for y – direction. The results are mid-span moment (β_{xC} and β_{yC}) and edge shear force (K_{sxB} and K_{syA}) factors in x and y directions.

4.2.3.2 Factors for plate type CCCC

The factors for deflection, bending moment and shear force for CCCC plate type are presented in Table 4.2. The x and y axes were considered in the calculation. Aspect ratio from 1 to 2.0 were considered. The dots on the plate in Figure 4.2 show points of maximum values.

The results were obtained by employing the exact coefficient of deflection (Equation 4.7) obtained from this study for CCCC plate type in bending moment equations, Equation (3.70b) for x-direction and Equation (3.71b) for y-direction, and shear force equations, Equation (3.82b) for x-direction and Equation (3.83b) for y – direction. The results are mid-span moment (β_{xD} and β_{yD}) and edge shear force (K_{sxB} and K_{syA}) factors in x and y directions.

4.2.3.3 Factors for plate type CSCS

The factors for deflection, bending moment and shear force for CSCS plate type are presented in Table 4.3. The x and y axes were considered in the calculation. Aspect ratio from 1 to 2.0 were considered. The dots on the plate in Figure 4.3 show points of maximum values.

The results were obtained by employing the exact coefficient of deflection (Equation 4.9) obtained from this study for CSCS plate type in bending moment equations, Equation (3.70b) for x-direction and Equation (3.71b) for y-direction, and shear force equations, Equation (3.82b) for x-direction and Equation (3.83b) for y – direction. The results are mid-span moment (β_{xC} and β_{yC}) and edge shear force (K_{sxB} and K_{syA}) factors in x and y directions.

4.2.3.4 Factors for plate type CSSS

The factors for deflection, bending moment and shear force for CSSS plate type are presented in Table 4.4. The x and y axes were considered in the calculation. Aspect ratio from 1 to 2.0 were considered. The dots on the plate in Figure 4.4 show points of maximum values.

The results were obtained by employing the exact coefficient of deflection (Equation 4.11) obtained from this study for CSSS plate type in bending moment equations, Equation (3.70b) for x-direction and Equation (3.71b) for y-direction, and shear force equations, Equation (3.82b) for x-direction and Equation (3.83b) for y – direction. The results are mid-span moment (βxD and βyD) and edge shear force (KsxB and KsyA) factors in x and y directions.

4.2.3.5 Factors for plate type CCSS

The factors for deflection, bending moment and shear force for CCSS plate type are presented in Table 4.5. The x and y axes were considered in the calculation. Aspect ratio from 1 to 2.0 were considered. The dots on the plate in Figure 4.5 show points of maximum values.

The results were obtained by employing the exact coefficient of deflection (Equation 4.13) obtained from this study for CCSS plate type in bending moment equations, Equation (3.70b) for x-direction and Equation (3.71b) for y-direction, and shear force equations, Equation (3.82b) for x-direction and Equation (3.83b) for y – direction. The results are mid-span moment (βxD and βyD) and edge shear force (KsxB and KsyA) factors in x and y directions.

4.2.3.6 Factors for plate type CCCS

The factors for deflection, bending moment and shear force for CCCS plate type are presented in Table 4.6. The x and y axes were considered in the calculation. Aspect ratio from 1 to 2.0 were considered. The dots on the plate in Figure 4.6 show points of maximum values.

The results were obtained by employing the exact coefficient of deflection (Equation 4.15) obtained from this study for CCCS plate type in bending moment equations, Equation (3.70b) for x-direction and Equation (3.71b) for y-direction, and shear force equations, Equation (3.82b) for x-direction and Equation (3.83b) for y – direction. The results are mid-span moment (βxD and βyD) and edge shear force (KsxB and KsyA) factors in x and y directions.

4.2.4 Euler-Bernoulli residual forces for selected plates

Results of the Euler-Bernoulli residual force and the design factors for deflection, bending moment and shear force obtained from the present study are here compared with results from other classical methods.

4.2.4.1 Values of Euler-Bernoulli residual force

The result of the Euler-Bernoulli residual force for the selected plates boundary conditions are as tabulated in Tables 4.7 and 4.8. The results from Table 4.7 were obtained from Ibearugbulem (2014), which uses Raleigh-Ritz weighted residual method. While the values in Table 4.8 were obtained from the present study. As can be seen in the tables, residual forces from the present study are zeros while residual forces from the previous study are not zeros. The reason for this difference is that the previous study used the weighted residual method coefficient equation, Equation (1.5) to obtain the coefficient, A, of the assumed shape functions while the present study used the Euler-Bernoulli coefficient equation, Equation (1.3) to obtain the coefficient, A of the shape functions. Recall that Equation (1.5) is characterized by accelerated convergence. Hence, the sole reason why it is always employed in analysis, especially when dealing with approximate deflection functions with unknown coefficients.

4.2.4.2 Comparison with results from previous study

The results of the factors for pure bending analysis of the selected plate types for the present study are here compared with earlier results obtained by Ibearugbulem (2014) based on Ritz approach.

4.2.4.2.1 Factors for plate type SSSS

From Figures 4.7 and 4.9, it is apparent that the previous work underestimated the design factors for pure bending of a thin rectangular plate of type SSSS by an average value 23.73%. The difference is as a result of the difference in the coefficient of deflection obtained using approximate form of the equilibrium equation.

In similar way, from Figures 4.10 and 4.11, it is seen that the shear force coefficient for SSSS plate type is underestimated by an average percentage of 23.73%. The results obtained from this study are upper bound values. See also appendix A1 – A3 for Tables of factors for deflection, bending moment and shear force against the aspect ratio, p for SSSS plate.

4.2.4.2.2 Factors for plate type CCCC

From Figures 4.12 to 4.14, it is apparent that the previous work underestimated the design factors for pure bending of a thin rectangular plate of type CCCC by an average value 3.36%. The difference is as a result of the difference in the coefficient of deflection obtained using approximate form of the equilibrium equation.

In similar way, from Figures 4.15 and 4.16, it is seen that the shear force coefficient for CCCC plate type is underestimated by an average percentage of 3.36%. The results obtained from this study are upper bound values. See also appendix B1 – B3 for Tables of factors for deflection, bending moment and shear force against the aspect ratio, p for CCCC plate.

4.2.4.2.3 Factors for plate type CSCS

From Figures 4.17 to 4.19, it is apparent that the previous work underestimated the design factors for pure bending of a thin rectangular plate of type CSCS by an average value 14.21%. The difference is as a result of the difference in the coefficient of deflection obtained using approximate form of the equilibrium equation.

In similar way, from Figures 4.20 and 4.21, it is seen that the shear force coefficient for CSCS plate type is underestimated by an average percentage of 14.21%. The results obtained from this study are upper bound values. See also appendix C1 – C3 for Tables of factors for deflection, bending moment and shear force against the aspect ratio, p for CSCS plate.

4.2.4.2.4 Factors for plate type CSSS

From Figures 4.22 to 4.24, it is apparent that the previous work underestimated the design factors for pure bending of a thin rectangular plate of type CSSS by an average value 18.02%. The difference is as a result of the difference in the coefficient of deflection obtained using approximate form of the equilibrium equation.

In similar way, from Figures 4.25 and 4.26, it is seen that the shear force coefficient for CSSS plate type is underestimated by an average percentage of 18.02%. The results obtained from this study are upper bound values. See also appendix D1 – D3 for Tables of factors for deflection, bending moment and shear force against the aspect ratio, p for CSSS plate.

4.2.4.2.5 Factors for plate type CCSS

From Figures 4.27 to 4.29, it is apparent that the previous work underestimated the design factors for pure bending of a thin rectangular plate of type CCSS by an average value 12.44%.

The difference is as a result of the difference in the coefficient of deflection obtained using approximate form of the equilibrium equation.

In similar way, from Figures 4.30 and 4.31, it is seen that the shear force coefficient for CCSS plate type is underestimated by an average percentage of 12.44%. The results obtained from this study are upper bound values. See also appendix E1 – E3 for Tables of factors for deflection, bending moment and shear force against the aspect ratio, p for CCSS plate.

4.2.4.2.6 Factors for plate type CCCS

From Figures 4.32 to 4.34, it is apparent that the previous work underestimated the design factors for pure bending of a thin rectangular plate of type CCCS by an average value 7.94%. The difference is as a result of the difference in the coefficient of deflection obtained using approximate form of the equilibrium equation.

In similar way, from Figures 4.35 and 4.36, it is seen that the shear force coefficient for CCCS plate type is underestimated by an average percentage of 7.94%. The results obtained from this study are upper bound values. See also appendix F1 – F5 for Tables of factors for deflection, bending moment and shear force against the aspect ratio, p for CCCS plate.

Thus, from the on-going, it is apparent that the previous work based on the approximate form of equilibrium equation underestimated the factors by a percentage difference too obvious to ignore. The relaxation of the form of the equilibrium equation for a 2 – dimensional problem has a limitation. Though the process yielded accurate results for a one – dimensional problem.

CHAPTER FIVE

CONCLUSIONS AND RECOMMENDATIONS

5.1 Conclusions

The study successfully obtained the energy functional of a plate using Euler-Bernoulli method (exact method). The energy functional obtained was used successfully to determine the exact polynomial shape (deflection) functions for thin rectangular plates of selected boundary conditions. The study went further to determine the exact coefficient of deflection based on polynomial deflection shape functions for the selected plate types. Exact bending moments and shear forces factors of line continuums and thin rectangular plates of selected boundary conditions were equally determined. Euler-Bernoulli residual forces for the present study and for the previous studies were determined for comparison. Lastly, results from the present study were compared with results from previous study (Ibearugbulem, 2014). Hence, the objectives of this study were successfully achieved.

Consequently, since only the basic governing equations of the plates are used and there are no predetermined or assumed shape functions, the present method overcomes the deficiency of the conventional semi-inverse methods and thus serves as a completely rational model in solving plate bending problems. Similarly, the whole plate equilibrium of forces expression derived in this study is satisfactory in determining the exact deformed shape of thin rectangular plates of various boundary conditions. The coefficient of deflection derived herein can be used in confidence for isotropic thin rectangular plates analysis for the selected boundary conditions covered in this study. The whole plate equilibrium equation could as well be used to satisfactorily analyze real time isotropic thin rectangular plates of various boundary conditions under lateral loadings. Lastly, the results obtained herein are exact results as there were no presumptions of any kind made to arrive at the results.

5.2 Recommendations

The following recommendations are made:

i. Future research work should consider solving the whole plate equilibrium equation directly for plates under in-plane loads.

ii. Future research work should consider solving the whole plate equilibrium equation directly for plates under dynamic loads.

iii. Future research work should consider conducting experiments to verify the authenticity of these results as against the results obtained from previous studies.

iv. Future studies should consider the application of the direct solution of the whole plate equilibrium equation to plate with other boundary conditions such as free ends.

v. Future research can find the trigonometric equivalence of the polynomial shape functions derived in this study for comparison.

5.3 Contributions to knowledge

The following contributions to knowledge where possible through this study;

i. The exact coefficient of deflection for SSSS plate type

$$A = \frac{qa^4}{D}\frac{p^4}{(4.8p^4 + 8p^2 + 4.8)} \qquad (4.5)$$

ii. The exact coefficient of deflection for CCCC plate type

$$A = \frac{qa^4}{D}\frac{0.449p^4}{(0.499p^4 + 0.272p^2 + 0.499)} \qquad (4.7)$$

iii. The exact coefficient of deflection for CSCS plate type

$$A = \frac{qa^4}{D}\frac{0.67p^4}{(0.745p^4 + 1.476p^2 + 3.216)} \qquad (4.9)$$

iv. The exact coefficient of deflection for CCSS plate type

$$A = \frac{qa^4}{D}\frac{0.64p^4}{(1.381p^4 + 1.384p^2 + 1.381)} \qquad (4.11)$$

v. The exact coefficient of deflection for CSSS plate type

$$A = \frac{qa^4}{D}\frac{0.8p^4}{(1.726p^4 + 3.328p^2 + 3.84)} \qquad (4.13)$$

vi. The exact coefficient of deflection for CCCS plate type

$$A = \frac{qa^4}{D}\frac{0.536p^4}{(0.596p^4 + 0.614p^2 + 1.157)} \qquad (4.15)$$

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APPENDINCES

Appendix A: Table of design factors against aspect ratio from present study and Ibearugbulem, 2014 for SSSS Plate type.

Table A1. Factor for deflection for plate type SSSS.

Aspect ratio, P	K_D from present study	K_D from Ibearugbulem (2014)	Percentage difference b/w Ibearugbulem (2014) & present study (%)
1.0	0.00555	0.00414	25.39
1.1	0.00665	0.00496	25.39
1.2	0.00771	0.00576	25.27
1.3	0.00871	0.00653	25.01
1.4	0.00964	0.00725	24.79
1.5	0.01050	0.00793	24.45
1.6	0.01128	0.00856	24.11
1.7	0.01199	0.00913	23.87
1.8	0.01264	0.00966	23.57
1.9	0.01322	0.01015	23.25
2.0	0.01375	0.01059	23.01

Table A2. Factor for bending moment for plate type SSSS.

Aspect ratio, P	βxc from present study	βxc from Ibearugbulem (2014)	Percentage difference b/w Ibearugbulem (2014) & present study (%)	βyc from present study	βyc from Ibearugbulem (2014)	Percentage difference b/w Ibearugbulem (2014) & present study (%)
1.0	0.06925	0.05163	25.44	0.06925	0.05163	25.44
1.1	0.07964	0.05943	25.38	0.07189	0.05364	25.38
1.2	0.08941	0.06685	25.23	0.07358	0.05502	25.22
1.3	0.09844	0.07383	25.00	0.07455	0.05591	25.00
1.4	0.10670	0.08031	24.73	0.07497	0.05643	24.73
1.5	0.11420	0.08628	24.45	0.07501	0.05668	24.44
1.6	0.12098	0.09176	24.15	0.07479	0.05673	24.14
1.7	0.12708	0.09677	23.85	0.07438	0.05664	23.85
1.8	0.13257	0.10133	23.57	0.07385	0.05645	23.56
1.9	0.13751	0.10549	23.28	0.07326	0.05620	23.28
2.0	0.14195	0.10927	23.02	0.07262	0.05591	23.01

Table A3. Factor for shear force for plate type SSSS.

Aspect ratio, P	ksx from present study	ksx from Ibearugbulem (2014)	Percentage difference b/w Ibearugbulem (2014) & present study (%)	ksy from present study	ksy from Ibearugbulem (2014)	Percentage difference b/w Ibearugbulem (2014) & present study (%)
1.0	0.50284	0.37490	25.44	0.50284	0.37490	25.44
1.1	0.54220	0.40458	25.38	0.50740	0.37862	25.38
1.2	0.57549	0.43033	25.22	0.50670	0.37889	25.22
1.3	0.60349	0.45262	25.00	0.50203	0.37652	25.00
1.4	0.62698	0.47190	24.73	0.49446	0.37216	24.73
1.5	0.64670	0.48860	24.45	0.48487	0.36634	24.45
1.6	0.66327	0.50309	24.15	0.47393	0.35948	24.15
1.7	0.67724	0.51569	23.85	0.46216	0.35191	23.85
1.8	0.68908	0.52670	23.56	0.44993	0.34391	23.56
1.9	0.69914	0.53634	23.29	0.43754	0.33565	23.29
2.0	0.70775	0.54482	23.02	0.42518	0.32730	23.02

Appendix B: Table of design factors against aspect ratio from present study and Ibearugbulem, 2014 for CCCC Plate type.

Table B1. Factor for deflection for plate type CCCC.

Aspect ratio, P	KD from present study	KD from Ibearugbulem (2014)	Percentage difference b/w Ibearugbulem (2014) & present study (%)
1.0	0.00138	0.00133	3.63
1.1	0.00165	0.00159	3.42
1.2	0.00189	0.00182	3.58
1.3	0.00210	0.00203	3.33
1.4	0.00228	0.00221	3.21
1.5	0.00244	0.00236	3.27
1.6	0.00257	0.00249	3.21
1.7	0.00269	0.0026	3.17
1.8	0.00278	0.00269	3.25
1.9	0.00286	0.00277	3.20
2.0	0.00293	0.00284	3.09

Table B2. Factor for bending moment for plate type CCCC.

Aspect ratio, P	βxc from present study	βxc from Ibearugbulem (2014)	Percentage difference b/w Ibearugbulem (2014) & present study (%)	βyc from present study	βyc from Ibearugbulem (2014)	Percentage difference b/w Ibearugbulem (2014) & present study (%)
1.0	0.02870	0.02769	3.54	0.02870	0.02769	3.54
1.1	0.03287	0.03172	3.50	0.02967	0.02863	3.51
1.2	0.03649	0.03522	3.49	0.03003	0.02899	3.48
1.3	0.03956	0.03820	3.45	0.02996	0.02893	3.44
1.4	0.04212	0.04069	3.40	0.02960	0.02859	3.41
1.5	0.04424	0.04275	3.37	0.02906	0.02809	3.34
1.6	0.04598	0.04446	3.31	0.02843	0.02749	3.29
1.7	0.04742	0.04587	3.27	0.02775	0.02685	3.26
1.8	0.04861	0.04703	3.24	0.02708	0.02620	3.24
1.9	0.04959	0.04800	3.20	0.02642	0.02558	3.17
2.0	0.05041	0.04881	3.17	0.02579	0.02498	3.14

Table B3. Factor for shear force for plate type CCCC.

Aspect ratio, P	ksx from present study	ksx from Ibearugbulem (2014)	Percentage difference b/w Ibearugbulem (2014) & present study (%)	ksy from present study	ksy from Ibearugbulem (2014)	Percentage difference b/w Ibearugbulem (2014) & present study (%)
1.0	0.26497	0.25562	3.53	0.26497	0.25562	3.53
1.1	0.31609	0.30497	3.52	0.23748	0.22913	3.52
1.2	0.36242	0.34978	3.49	0.20973	0.20242	3.49
1.3	0.40319	0.38928	3.45	0.18352	0.17719	3.45
1.4	0.43839	0.42345	3.41	0.15976	0.15432	3.41
1.5	0.46843	0.45269	3.36	0.13879	0.13413	3.36
1.6	0.49393	0.47754	3.32	0.12059	0.11659	3.32
1.7	0.51553	0.49865	3.27	0.10493	0.10150	3.27
1.8	0.53384	0.51657	3.24	0.09154	0.08858	3.23
1.9	0.54941	0.53184	3.20	0.08010	0.07754	3.20
2.0	0.56269	0.54488	3.17	0.07034	0.06811	3.17

Appendix C: Table of design factors against aspect ratio from present study and Ibearugbulem, 2014 for CSCS Plate type.

Table C1. Factor for deflection for plate type CSCS.

Aspect ratio, P	KD from present study	KD from Ibearugbulem (2014)	Percentage difference b/w Ibearugbulem (2014) & present study (%)
1.0	0.00241	0.00199	17.31
1.1	0.00314	0.00262	16.67
1.2	0.00394	0.00330	16.24
1.3	0.00477	0.00403	15.47
1.4	0.00560	0.00477	14.87
1.5	0.00643	0.00551	14.25
1.6	0.00722	0.00624	13.58
1.7	0.00798	0.00695	12.85
1.8	0.00868	0.00762	12.25
1.9	0.00934	0.00826	11.59
2.0	0.00995	0.00885	11.07

Table C2. Factor for bending moment for plate type CSCS.

Aspect ratio, P	βxc from present study	βxc from Ibearugbulem (2014)	Percentage difference b/w Ibearugbulem (2014) & present study (%)	βyc from present study	βyc from Ibearugbulem (2014)	Percentage difference b/w Ibearugbulem (2014) & present study (%)
1.0	0.03465	0.02864	17.35	0.04544	0.03755	17.35
1.1	0.04266	0.03548	16.83	0.05063	0.04212	16.81
1.2	0.05096	0.04269	16.22	0.05512	0.04619	16.21
1.3	0.05931	0.05008	15.56	0.05887	0.04971	15.56
1.4	0.06751	0.05746	14.89	0.06188	0.05266	14.90
1.5	0.07540	0.06469	14.20	0.06420	0.05508	14.21
1.6	0.08285	0.07165	13.52	0.06592	0.05701	13.52
1.7	0.08981	0.07826	12.86	0.06712	0.05849	12.86
1.8	0.09623	0.08447	12.22	0.06789	0.05960	12.21
1.9	0.10211	0.09026	11.61	0.06832	0.06038	11.62
2.0	0.10748	0.09561	11.04	0.06847	0.06091	11.04

Table C3. Factor for shear force for plate type CSCS.

Aspect ratio, P	ksx from present study	ksx from Ibearugbulem (2014)	Percentage difference b/w Ibearugbulem (2014) & present study (%)	ksy from present study	ksy from Ibearugbulem (2014)	Percentage difference b/w Ibearugbulem (2014) & present study (%)
1.0	0.30187	0.24949	17.35	0.46205	0.38187	17.35
1.1	0.34691	0.28857	16.82	0.45356	0.37728	16.82
1.2	0.38944	0.32630	16.21	0.43777	0.36679	16.21
1.3	0.42863	0.36191	15.57	0.41666	0.35180	15.57
1.4	0.46399	0.39491	14.89	0.39206	0.33369	14.89
1.5	0.49534	0.42499	14.20	0.36557	0.31364	14.20
1.6	0.52275	0.45206	13.52	0.33845	0.29269	13.52
1.7	0.54644	0.47618	12.86	0.31167	0.27159	12.86
1.8	0.56674	0.49749	12.22	0.28588	0.25095	12.22
1.9	0.58403	0.51621	11.61	0.26153	0.23116	11.61
2.0	0.59868	0.53258	11.04	0.23883	0.21247	11.04

Appendix D: Table of design factors against aspect ratio from present study and Ibearugbulem, 2014 for CSSS Plate type.

Table D1. Factor for deflection for plate type CSSS.

Aspect ratio, P	K_D from present study	K_D from Ibearugbulem (2014)	Percentage difference b/w Ibearugbulem (2014) & present study (%)
1.0	0.00351	0.00282	19.74
1.1	0.00440	0.00354	19.58
1.2	0.00531	0.00429	19.15
1.3	0.00620	0.00503	18.87
1.4	0.00706	0.00576	18.46
1.5	0.00788	0.00646	18.05
1.6	0.00865	0.00713	17.58
1.7	0.00936	0.00775	17.22
1.8	0.01002	0.00833	16.85
1.9	0.01062	0.00887	16.47
2.0	0.01117	0.00937	16.09

Table D2. Factor for bending moment for plate type CSSS.

Aspect ratio, P	β_{xD} from present study	β_{xD} from Ibearugbulem (2014)	Percentage difference b/w Ibearugbulem (2014) & present study (%)	β_{yD} from present study	β_{yD} from Ibearugbulem (2014)	Percentage difference b/w Ibearugbulem (2014) & present study (%)
1.0	0.04637	0.03718	19.83	0.05228	0.04191	19.83
1.1	0.05535	0.04452	19.56	0.05632	0.04531	19.56
1.2	0.06420	0.05185	19.24	0.05949	0.04805	19.24
1.3	0.07273	0.05901	18.86	0.06188	0.05021	18.85
1.4	0.08078	0.06587	18.46	0.06359	0.05185	18.46
1.5	0.08829	0.07236	18.04	0.06474	0.05306	18.05
1.6	0.09521	0.07842	17.63	0.06546	0.05392	17.63
1.7	0.10154	0.08405	17.23	0.06584	0.05450	17.22
1.8	0.10730	0.08924	16.83	0.06595	0.05485	16.83
1.9	0.11252	0.09400	16.46	0.06587	0.05503	16.46
2.0	0.11725	0.09837	16.10	0.06566	0.05509	16.09

Table D3. Factor for shear force for plate type CSSS.

Aspect ratio, P	ksx from present study	ksx from Ibearugbulem (2014)	Percentage difference b/w Ibearugbulem (2014) & present study (%)	ksy from present study	ksy from Ibearugbulem (2014)	Percentage difference b/w Ibearugbulem (2014) & present study (%)
1.0	0.36427	0.29203	19.83	0.48232	0.38667	19.83
1.1	0.40649	0.32696	19.57	0.49932	0.40162	19.57
1.2	0.44430	0.35883	19.24	0.50974	0.41169	19.24
1.3	0.47758	0.38750	18.86	0.51454	0.41749	18.86
1.4	0.50651	0.41301	18.46	0.51472	0.41970	18.46
1.5	0.53144	0.43553	18.05	0.51126	0.41899	18.05
1.6	0.55278	0.45531	17.63	0.50501	0.41597	17.63
1.7	0.57100	0.47264	17.23	0.49673	0.41116	17.23
1.8	0.58653	0.48779	16.83	0.48699	0.40501	16.83
1.9	0.59976	0.50104	16.46	0.47629	0.39790	16.46
2.0	0.61105	0.51264	16.10	0.46499	0.39010	16.11

Appendix E: Table of design factors against aspect ratio from present study and Ibearugbulem, 2014 for CCSS Plate type.

Table E1. Factor for deflection for plate type CCSS.

Aspect ratio, P	K_D from present study	K_D from Ibearugbulem (2014)	Percentage difference b/w Ibearugbulem (2014) & present study (%)
1.0	0.00241	0.00210	12.92
1.1	0.00288	0.00251	12.94
1.2	0.00332	0.00290	12.75
1.3	0.00373	0.00325	12.77
1.4	0.00409	0.00357	12.64
1.5	0.00441	0.00386	12.40
1.6	0.00469	0.00411	12.35
1.7	0.00494	0.00433	12.29
1.8	0.00515	0.00453	12.12
1.9	0.00535	0.00470	12.08
2.0	0.00551	0.00486	11.86

Table E2. Factor for bending moment for plate type CCSS.

Aspect ratio, P	β_{xD} from present study	β_{xD} from Ibearugbulem (2014)	Percentage difference b/w Ibearugbulem (2014) & present study (%)	β_{yD} from present study	β_{yD} from Ibearugbulem (2014)	Percentage difference b/w Ibearugbulem (2014) & present study (%)
1.0	0.03762	0.03277	12.89	0.03762	0.03277	12.89
1.1	0.04317	0.03762	12.86	0.03897	0.03396	12.85
1.2	0.04819	0.04203	12.79	0.03966	0.03459	12.79
1.3	0.05265	0.04597	12.68	0.03987	0.03481	12.69
1.4	0.05654	0.04943	12.58	0.03973	0.03473	12.59
1.5	0.05993	0.05246	12.46	0.03937	0.03446	12.46
1.6	0.06286	0.05510	12.34	0.03886	0.03406	12.35
1.7	0.06539	0.05740	12.22	0.03827	0.03359	12.23
1.8	0.06758	0.05940	12.11	0.03765	0.03309	12.10
1.9	0.06948	0.06114	12.00	0.03701	0.03257	12.01
2.0	0.07113	0.06266	11.91	0.03639	0.03206	11.90

Table E3. Factor for shear force for plate type CCSS.

Aspect ratio, P	ksx from present study	ksx from Ibearugbulem (2014)	Percentage difference b/w Ibearugbulem (2014) & present study (%)	ksy from present study	ksy from Ibearugbulem (2014)	Percentage difference b/w Ibearugbulem (2014) & present study (%)
1.0	0.28938	0.25209	12.89	0.28938	0.25209	12.89
1.1	0.34595	0.30146	12.86	0.28175	0.24552	12.86
1.2	0.39885	0.34784	12.79	0.27144	0.23672	12.79
1.3	0.44710	0.39036	12.69	0.25967	0.22672	12.69
1.4	0.49039	0.42870	12.58	0.24734	0.21623	12.58
1.5	0.52880	0.46292	12.46	0.23502	0.20574	12.46
1.6	0.56266	0.49324	12.34	0.22309	0.19556	12.34
1.7	0.59242	0.52003	12.22	0.21174	0.18587	12.22
1.8	0.61854	0.54365	12.11	0.20109	0.17674	12.11
1.9	0.64149	0.56450	12.00	0.19116	0.16822	12.00
2.0	0.66167	0.5829	11.90	0.18196	0.16030	11.90

Appendix F: Table of design factors against aspect ratio from present study and Ibearugbulem, 2014 for CCCS Plate type.

Table F1. Factor for deflection for plate type CCCS

Aspect ratio, P	K_D from present study	K_D from Ibearugbulem (2014)	Percentage difference b/w Ibearugbulem (2014) & present study (%)
1.0	0.00177	0.00159	10.13
1.1	0.00221	0.00200	9.56
1.2	0.00265	0.00241	9.05
1.3	0.00307	0.00280	8.77
1.4	0.00346	0.00317	8.37
1.5	0.00382	0.00352	7.76
1.6	0.00414	0.00383	7.41
1.7	0.00442	0.00411	7.06
1.8	0.00468	0.00436	6.75
1.9	0.00490	0.00459	6.31
2.0	0.00510	0.00479	6.01

Table F2. Factor for bending moment for plate type CCCS.

Aspect ratio, P	β_{xD} from present study	β_{xD} from Ibearugbulem (2014)	Percentage difference b/w Ibearugbulem (2014) & present study (%)	β_{yD} from present study	β_{yD} from Ibearugbulem (2014)	Percentage difference b/w Ibearugbulem (2014) & present study (%)
1.0	0.02972	0.02673	10.07	0.03468	0.03119	10.05
1.1	0.03531	0.0319	9.65	0.03720	0.03362	9.63
1.2	0.04063	0.0369	9.18	0.03898	0.03540	9.19
1.3	0.04555	0.04157	8.73	0.04011	0.03661	8.72
1.4	0.04999	0.04585	8.28	0.04070	0.03733	8.27
1.5	0.05393	0.04971	7.83	0.04087	0.03767	7.84
1.6	0.05740	0.05314	7.41	0.04075	0.03772	7.43
1.7	0.06041	0.05616	7.04	0.04040	0.03756	7.04
1.8	0.06303	0.05882	6.68	0.03992	0.03725	6.69
1.9	0.06530	0.06115	6.36	0.03935	0.03685	6.35
2.0	0.06727	0.06318	6.08	0.03873	0.03638	6.07

Table F3. Factor for shear force for plate type CCCS.

Aspect ratio, P	ksx from present study	ksx from Ibearugbulem (2014)	Percentage difference b/w Ibearugbulem (2014) & present study (%)	ksy from present study	ksy from Ibearugbulem (2014)	Percentage difference b/w Ibearugbulem (2014) & present study (%)
1.0	0.21230	0.19095	10.06	0.33968	0.30552	10.06
1.1	0.21931	0.19816	9.64	0.31899	0.28824	9.64
1.2	0.22082	0.20052	9.19	0.29442	0.26736	9.19
1.3	0.21793	0.19891	8.73	0.26823	0.24482	8.73
1.4	0.21182	0.19430	8.27	0.24208	0.22206	8.27
1.5	0.20352	0.18758	7.83	0.21709	0.20009	7.83
1.6	0.19390	0.17952	7.42	0.19390	0.17952	7.42
1.7	0.18363	0.17071	7.04	0.17283	0.16067	7.03
1.8	0.17317	0.16160	6.68	0.15393	0.14364	6.68
1.9	0.16285	0.15259	6.30	0.13714	0.12841	6.36
2.0	0.15289	0.14360	6.07	0.12231	0.11488	6.07