

**BUCKLING ANALYSIS OF THIN RECTANGULAR PLATES UNDER
VIBRATION USING SPLIT-DEFLECTION METHOD**

BY

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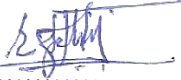
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IN CIVIL ENGINEERING (STRUCTURES).**

NOVEMBER, 2019

CERTIFICATION


This is to certify that work **“Buckling Analysis of Thin Rectangular Plates under Vibration using Split-Deflection Method”**, was carried out by Uchechukwu Christopher Nwachukwu (20154940788) in partial fulfillment of the award of the Degree of Master of Engineering in Civil Engineering (Structures) in the Department of Civil Engineering of the Federal University of Technology, Owerri.



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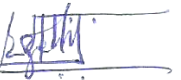
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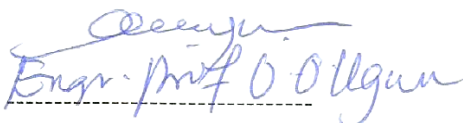
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DEDICATION

To God Almighty for his grace and mercy upon the lives of his children.

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NOTATIONS

SYMBOLS	MEANING
a	Length of the plate
b	Width of the plate
S	Simply support
C	Clamped support
X	Primary axis of the plate
Y	Secondary axis of the plate
W	Deflection equation of the plate
A	Coefficient of deflection of the plate
h	Shape function of the plate
D	Flexural rigidity of the plate
u	Displacement component along X - axis of the plate
v	Displacement component along Y - axis of the plate
ϵ	Normal strain of plate
γ	Shear strain of plate
δ	Normal stress of plate
τ	Shear stress of plate
U	Strain energy
π	Total potential energy functional

R	Non- dimensional parameter, equal to x/a
Q	Non- dimensional parameter, equal to y/b
K	Stiffness factor
n	Resonating frequency ratio, equal to $\frac{\theta}{\lambda}$
θ	Frequency below critical level
λ	Resonating frequency
P	Aspect ratio, equal to b/a
N_x	Critical buckling load coefficient, equal to $\frac{D}{a^2} \cdot \bar{N}_x$
\bar{N}_x	Non – dimensional critical buckling load parameter
%	Percentage
t	Thickness of the plate
Z	Tertiary axis of the plate

ABSTRACT

This study presents buckling analysis of thin rectangular plates under vibration using split-deflection method. In this method, the deflection functions are split into x and y components. Applying the split – deflection equation into principles of theory of elasticity, total potential energy functional was derived. By minimization of the potential energy functional, the governing equation for critical buckling loads for rectangular plates under vibration was obtained. Shape functions containing orthogonal trigonometry-trigonometry, orthogonal polynomial-polynomial and orthogonal polynomial-trigonometry are obtained for the six plates studied in this work. The boundary conditions considered were simple support and clamp support. The non – dimensional critical buckling loads for the various rectangular plates were obtained at various aspect ratios (ranging from 1 to 2) and resonating frequency ratios (ranging from 0 to 1). The non – dimensional critical buckling loads obtained using polynomial shape functions were compared with those given by Ibearugbulem in 2014. For rectangular plates simply supported at all edges, the percentage differences at the various aspect ratios (1 to 2) and various resonating frequency ratios (0 to 1) ranges 0.000% to 0.006%. It is clear from this study that the percentage differences recorded for rectangular plates of other boundary conditions are insignificant. This showed that the past and present results are in good agreement. Hence, it is recommended that rectangular plates with boundary conditions different from the ones studied here should be considered by further research works.

Keywords: Split-deflection, total potential energy functional, trigonometric functions, polynomial functions, rectangular plate boundary conditions, non – dimensional critical buckling loads.

CHAPTER ONE

INTRODUCTION

1.1 Background of study

Plates, as structural elements are widely used in many fields of engineering such as aeronautic, civil, structural and mechanical to transmit in-plane and lateral loads. Ibearugbulem et al. (2014) defined a plate as a solid that consist of two parallel plane surfaces separated by a small dimension (its thickness).

Generally, plates maybe classified based on their length - thickness ratio (that is; thin plates, thick plates, membranes), elastic properties (that is; isotropic, orthotropic, anisotropic) or shape (that is; rectangular, circular, elliptical). Thin plates represent the most extensive group of plates (Ventsel and Krauthammer, 2001). According to them, “the first impetus to a mathematical statement of plate problems was probably done by Euler, who in 1776 performed a free - vibration analysis of plate problems. Many years later, Chladni, a German physicist discovered the various modes of free - vibrations. In Chladni’s experiment on horizontal plates, evenly distributed powder was used, which formed regular patterns after induction of vibration. The powder accumulated along the nodal lines where no vertical displacement occurred”. Ibearugbulem et al. (2016) derived an equation for critical buckling loads for rectangular plates using Work – error and Split – deflection methods given in equation (1.1) as:

$$N_x = \frac{\frac{D}{a^2} [K_x + \frac{2}{P^2} K_{xy} + \frac{K_y}{P^4}]}{K_{N_x}} \quad (1.1)$$

The results of the critical buckling loads obtained for SSSS and CCCC rectangular plates were compared with those obtained by Ibearugbulem et al. (2014) using Work – error method at various aspect ratios, P (1.0 to 2.0) agreed very closely. Ibearugbulem (2012) obtained the general polynomial deflection equation for rectangular plates given in Equation (1.2) as;

$$W = (a_0 + a_1R + a_2R^2 + a_3R^3 + a_4R^4) (b_0 + b_1Q + b_2Q^2 + b_3Q^3 + b_4Q^4) \quad (1.2)$$

Ibearugbulem et al. (2016) assumed a trigonometric deflection equation in y-component for SSSS rectangular plate given in Equation (1.3) as;

$$W_y = \sqrt{A} \sin \pi Q \quad (1.3)$$

which they used polynomial deflection equation given by Ibearugbulem in 2012 and they assumed trigonometric deflection equation to analyze classical SSSS rectangular plates. The results they obtained were compared with those given by Ibearugbulem et al. (2014) showed good agreements. According to Ventsel and Krauthammer (2001), Cauchy in 1828 and Poisson in 1829 were first to formulate the problem of plate bending on the basis of general equations of theory of elasticity. Expanding all the characteristic quantities into series of powers of distance from a middle surface, they retained only terms of the first order of smallness. The governing equation of a thin rectangular plate subjected to direct compressive forces, N_x was first derived by Navier (Ventsel and

Krauthammer, 2001). The beginning of the twentieth century marked a prolific development of plate theories (Szilard, 2004). In recent times, several researchers have worked on free - vibration, buckling and pure bending of plates. For instance, free-vibration analysis of thin rectangular plate that are clamped on all edges using numerical approaches (Lee, 2004, Werfalli and Karaid, 2005 and Misra, 2012) and energy variational methods (Lal et al., 2009, Shu et al., 2007). It was seen that their approaches were very rigorous, hence Njoku et al. (2013) analyzed the same work using Taylor series formulated shape function in Galerkin's method. According to Ibearugbulem et al. (2013), Timoshenko and Woinowsky - Krieger in 1959 assumed shape function in bending analysis of plates of particular boundary conditions. They concluded that the approach is approximate, and exact solution is achieved for plates of various boundary conditions using direct integration and work principle.

Among the methods mentioned earlier in this literature on analysis of plates, the common energy methods in classical plate theory analysis include Galerkin, Raleigh, Raleigh-Ritz, Ritz methods e.t.c (Njoku et al., 2013, Ibearugbulem et al., 2014, Ibeabuchi, 2014). These methods use orthogonal polynomial functions (deflection equation). According to Ibearugbulem et al. (2016), these methods use orthogonal function in form double furrier series (that product of two mutually perpendicular trigonometric functions) or orthogonal polynomial.

They also said that Hutchinson (1992), Jianqiao (1994), Ugural (1999), Ventsel and Krauthammer (2001), Wang et al. (2002), Taylor and Goindjee (2004), Szilard (2004), Jiu et al. (2007), Erdem et al. (2007), Ezeh et al. (2013), Ibearugbulem (2014) stated that, energy equations of the methods used earlier than now are based on the orthogonal deflection functions. They also started that due to the difficulty in obtaining orthogonal function for a plate of a particular boundary condition lead to the modification of a method termed split-deflection method. Furthermore, they succeeded in using deflection function that was typically separated into two independent distinct functions (that is, $w = w_x \times w_y$), where w_x and w_y maybe both polynomial or trigonometric functions or w_x maybe polynomial function and w_y maybe trigonometric function in pure bending, buckling and free-vibration analysis of rectangular plate. It is interesting to note that though research works has been done on pure bending, buckling and free-vibration using split - deflection method but none has been recorded on determining the effect of vibration on buckling of rectangular plate using split - deflection method. Hence, this research work presents the solution to the buckling analysis of rectangular plates under vibration using split - deflection method.

A thin rectangular plate has four edges denoted as shown in figure 1.1

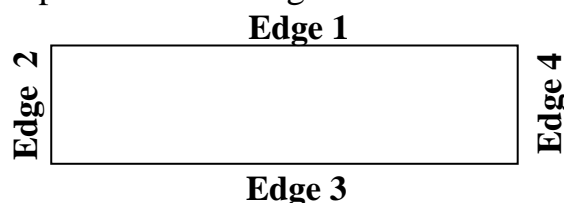


Figure 1.1: Rectangular plate showing the edge numbers.

Each of the edges could be simply supported (designated as “S”), clamped (designated as “C”) or free (designated as “F”).

Diagrammatically, they are represented as shown in Figures (1.2) to (1.4)

Figure 1.2: Simply supported edge



Figure 1.3: Clamped edge

Figure 1.4: Free edge - - - - -

Thus, a rectangular plate can be identified by its four edge conditions. For instance, SSSS represents a rectangular plate whose four edges are simply supported, CSCS represents a rectangular plate with edges 1 and 3 clamped and 2 and 4 simply supported e.t.c.

Sketches of plates with various boundary conditions considered in this work are presented in Figures 1.5 [(a) to (f)]

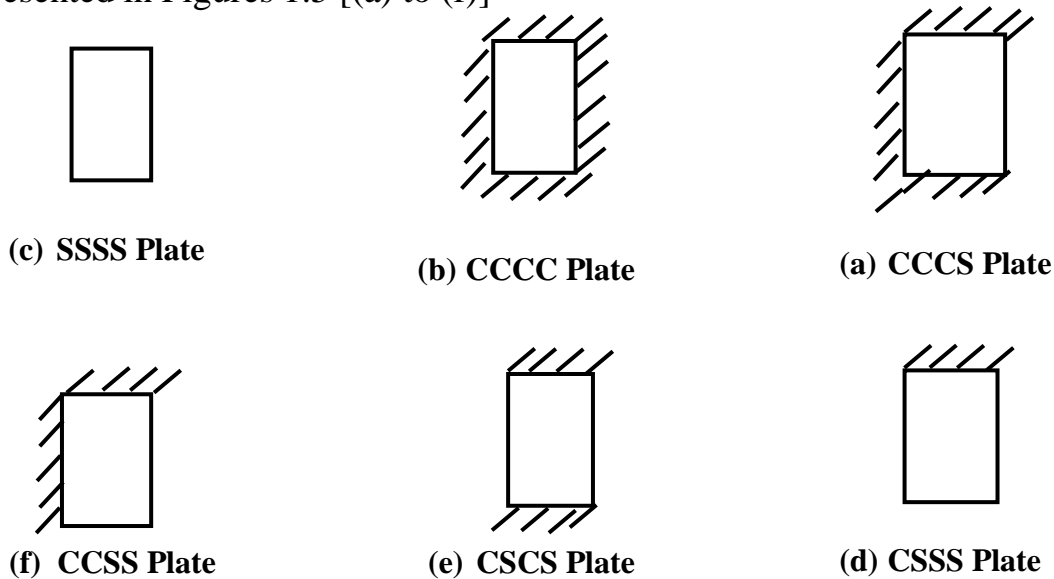


Figure 1.5: Rectangular plates with various boundary conditions

A thin rectangular plate is dimensioned as shown in Figure 1.6

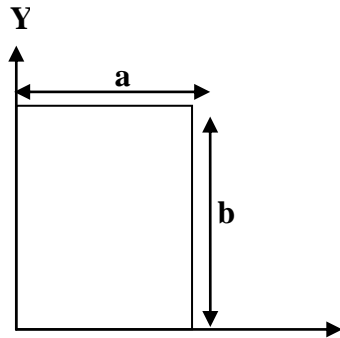


Figure 1.6: Rectangular plates showing denotations of dimensions.

The letters “ a” and “b” shown in Figure 1.6 represent the plate’s primary and secondary dimensions in the plate’s X and Y -axes respectively. The tertiary dimension of the plate is its thickness, “t “which is on the Z - axis.

1.2 Problem Statement

Buckling and free -vibration analysis of rectangular plates have been carried out by many researchers over the years. However, most of them used trigonometric functions in the development of shape functions of rectangular plates, which are limited to just few simple plate boundary conditions. Apart from the said limitation, their usage in the buckling and free -vibration analysis usually leads to complex equations which are extremely difficult and time- consuming to analyze. In recent years, some researchers have carried out buckling and free - vibration analysis of rectangular plates using polynomial functions and also combined trigonometric and polynomial functions but only for few plates (such as SSSS and CCCC rectangular plates). Based on the literature available to me,

solution to the buckling analysis of rectangular plates under vibration has not been provided. This unavailability of required results has made the design of plates for Civil, Structural, Mechanical or Aeronautic Engineering incomplete by ignoring the impact of vibration on buckling of plates. Thus, this research work is concern on the buckling of rectangular plates under vibration (such as SSSS, CCCC, CCSS, CSCS, CSSS and CCCS rectangular plates) using split - deflection method.

1.3 Objectives of Study

The main objective of this research work is to perform buckling analysis of thin rectangular plates under vibration using Split - deflection method. The specific objectives to be achieved through the above are:

- (i) To derive the governing equation for determining the critical buckling load for thin rectangular plates under vibration.
- (ii) To develop various shape functions such as trigonometric, polynomial, trigonometric-polynomial and polynomial-trigonometric for the rectangular plates such as SSSS, CCCC, CCSS, CSCS, CSSS and CCCS
- (iii) To determine the non – dimensional critical buckling loads for the rectangular plates of various boundary conditions.
- (iv) To compare the results of this study with those from previous studies.

1.4 Justification of Study

This research work will be beneficial in the following ways;

- (i) Before now, past scholars worked on few rectangular plates such as SSSS, CCCC and CSCS using trigonometric shape functions but this work added three plates such as CCSS, CSSS and CCCS in its analysis using the same functions.
- (ii) Before now, past scholars worked on buckling and free-vibration gave complex equations which are rigorous, difficult and time – consuming to analyze, but this work simplified the approach and made them easy, straight – forward and faster.
- (iii) Before now, past scholars applied split-deflection method on buckling and free-vibration of few rectangular plates such as SSSS and CCCC using polynomial, combination of polynomial and trigonometric functions, but this work considered additional four plates such as CCSS, CSCS, CSSS and CCCS in its analysis using the same functions.
- (iv) Before now past scholars worked on buckling and free-vibration of rectangular plates such as SSSS, CCCC, CCSS, CSCS, CSSS and CCCS, none based on literature available to me has work on the buckling analysis of these plates under vibration which was considered in this work.

1.5 Scope of Study

This research work focused on the study of the buckling analysis of thin rectangular plates under vibration using split-deflection method. Plates of other length - thickness ratios, elastic properties or shapes are not considered. Other methods of plate analysis such as Ritz, Galerkin's methods e.t.c are not considered. The results from such methods are only used where necessary for the purpose of comparison. The shape functions which are used in the split - deflection method are trigonometric functions, polynomial functions and their combinations.

The plate boundary conditions considered in the analysis are those with either simply supported edges, clamped edges or a combination of simply supported and clamped edges (such as SSSS, CCCC, CCSS, CSCS, CSSS and CCCS rectangular plates). Boundary conditions of plates with free edges (F) are not considered. The aspect ratios ($P = b/a$) ranging from 1.0 to 2.0 and resonating frequency ratios (n) ranging from 0 to 1.0 are considered. Other aspect ratios such as $P = a/b$ are not also considered.

CHAPTER TWO

LITERATURE REVIEW

2.1 Definition of Plate

According to Steele and Chad (2009), a plate is a structural element whose transverse dimension or thickness is small compared to the length and width dimensions. Chenamsetti (2009) defined a plate as a body that is bounded by surfaces which are flat in geometry and whose lateral dimensions are large compared to the separation between the surfaces. Ibearugbulem et al. (2014) also defined a plate as a solid that consist of two parallel plane surfaces separated by a small dimension (its thickness).

2.2 Classification of Plates

A plate resist transverse loads by means of bending .The flexural properties of plate highly depend upon its thickness when compared to the other dimension. According to Ventsel and Krauthammer (2001), plates maybe classified into three groups according to the ratios a/h , where “a” is a typical dimension of a plate in a plane and “h” is a plate thickness as explained in Sub - sections 2.2.1, 2.2.2 and 2.2.3.

2.2.1: Thick plates: These plates have ratios $a/h \leq (8\dots 10)$. The analysis of such bodies includes all the components of stresses, strains and displacements as for solid bodies using the general equations of three-dimensional elasticity.

2.2.2: Membranes: These plates have ratios $a/h \geq (80\dots 100)$ and are devoid of flexural rigidity. They carry the lateral loads by axial tensile forces and shear forces acting in the plate middle surface.

2.2.3: Thin plates: These are most extensive group of plates. They are intermediate type of plates with ratios ranging $(8\dots 10) \leq a/h \leq (80\dots 100)$. Depending on the ratio of the maximum deflection of the plate to its thickness, w/h , thin plates maybe also be subdivided into two different classes, namely:

2.2.3.1: Stiff plates: These plates have ratio $w/h \leq 0.2$. They are flexurally rigid thin plates. They carry loads two dimensionally, mostly by internal bending and twisting moments and by transverse shear forces. In the middle, plane deformations and membrane forces are negligible.

2.2.3.2: Flexible plates: These plates have ratio $w/h \geq 0.3$, then the lateral deflection will be accompanied by stretching the middle surface. These plates represent a combination of stiff plates and membranes and carry external loads

by the combined action of internal moments, shear forces, and membrane (axial) forces.

2.3 Buckling of Rectangular Plate

Buckling of plate occur when the plate is subjected to only in-plane load which caused it to buckling. According to Onwuka et al. (2016), when a thin rectangular panel is subjected to in - plane compressive loads and loads are gradually increased, the panel becomes prone to buckling even though lateral loads may not present. Buckling of plates maybe classified as elastic buckling or inelastic/plastic buckling depending on the stress - strain relationship as explained in Sub - sections 2.3.1 and 2.3.2

2.3.1: Elastic Buckling of Rectangular Plate

This type of buckling of plate may occur as a result of various in - plane loads such as uniaxial compression, biaxial compression, in - plane shear forces, combined loads, body forces and other forms of loads.

Several researchers have different approaches to elastic buckling of rectangular plates. According to Chen Yu (2003), Navier in 1822 derived the basic stability equation for rectangular plates under lateral load by including the twisting action. The ability of plate to resist twisting can considerably reduce deflections under lateral load. He further said that Saint Venant in 1883 modified the Navier's equation by including axial edge forces and shear forces. The modified

equation formed the basis for much of the works on plate stability of plates with various loads and boundary conditions. J. H. Kang (2005) worked on the exact solutions for the buckling of rectangular plates having linearly varying in – plane loading on two opposite simply supported edges. He achieved this by assuming the transverse displacement (w) to vary as $\sin(m\pi/a)$, the governing partial equation of motion is reduced to an ordinary differential equation in y with varying coefficients for which an exact solution is obtained as a power series (that is, method of Frobenius). The elastic buckling of thin plate could be studied using the equilibrium approach, the energy approach, and the numerical approach (Szilard, 2004). The equilibrium approach sums all the forces acting on the continuum to zero. The energy approach methods include Rayleigh – Ritz, Galerkin’s methods e.t.c. Szilard (2004) stated that Rayleigh-Ritz method uses the combination of the principle of conservation of energy and the minimum potential energy principle while Galerkin’s method uses the governing differential equation of plate buckling to express the total potential of the system. Iwuoha (2016) worked on the buckling analysis of plates subjected to biaxial forces using Galerkin’s method. In his work, he determined the critical buckling load for a biaxially loaded square all – round simply supported (SSSS) plate under uniform stress (that is, at constant relating N_y and N_x "k" = aspect ratio " α " = 1) which was compared with the result obtained by Ventsel and Krauthammer (2001), Chajes (1974) and Iyengar

(1988) on the same research work gave percentage difference of 0.047%. Hence, concluded that the polynomial shape function he used excellently agreed with the trigonometric function (exact solution) used by previous researchers. Ramu and Mohanty (2014) worked on buckling analysis of rectangular functionally graded material plates under uniaxial and biaxial compression load. They investigated using classical plate theory and finite element methods. They concluded that critical buckling load of a rectangular plate under uniaxial compression is greater than the biaxial compression. Secondly, as aspect ratio increases, buckling load reduces. Finally, the critical buckling load decreases as the volume fraction index increase. The results obtained are compared with the results of Bouazza, et al. (2012), shown excellent agreement. Onwuka and Iwuoha (2017) worked on elastic instability analysis of biaxially compressed flat rectangular isotropic all-round clamped plates. This was achieved using polynomial shape function in Galerkin's functional. The results of the critical buckling loads obtained for the various aspect ratios (1.0 to 2.0) and relationship constant between forces on the Y- axis and forces on the X- axis "K" values (0.1 to 1.0) for biaxially compressed CCCC plate were compared with the results of Ibearugbulem et al. (2014) for the same various aspect ratios at $K=0$ for uniaxial buckling of CCCC plates, agreed very closely. Ezeh et al. (2014) worked on elastic buckling analysis of thin rectangular plates with all edges clamped using indirect variation principle (Galerkin's method). The results obtained were compared with the results of Iyengar (1988) and Ibearugbulem &

Ezeh (2013) showed excellent agreement as they gave average percentage differences of 4.702% and 0.735% respectively. Hence, concluded that the assumed deflection function is very close to the exact shape function. Oguaghamba et al. (2015) worked on the buckling and post buckling loads of all edges clamped (CCCC) thin rectangular plate. Compared with the previous related studies carried out on plates of all edges simply supported (SSSS) that assumed displacement and stress profile in the form of double trigonometric function, they applied direct integration of the governing differential equations on CCCC plate and implored the work principle technique. The results showed that CCCC plate accommodates additional load beyond critical buckling load, prior to actual material failure in its post buckling regime.

Ibearugbulem et al. (2016) derived an equation for critical buckling loads for rectangular plates using Work – error and Split – deflection methods given in Equation (2.1) as:

$$N_x = \frac{\frac{D}{a^2} [K_x + \frac{2}{P^2} K_{xy} + \frac{K_y}{P^4}]}{K_{Nx}} \quad (2.1)$$

Where

$$K_x = \int_0^1 \frac{d^4 h_x}{dR^4} h_x dR \int_0^1 h_y^2 dQ \quad (2.2)$$

$$K_{xy} = \int_0^1 \frac{d^2 h_x}{dR^2} h_x dR \int_0^1 \frac{d^2 h_y}{dQ^2} h_y dQ \quad (2.3)$$

$$K_y = \int_0^1 h_x^2 dR \int_0^1 \frac{d^4 h_y}{dQ^4} h_y dQ \quad (2.4)$$

$$K_{Nx} = \int_0^1 \frac{d^2 h_x}{dR^2} h_x dR \int_0^1 h_y^2 dQ \quad (2.5)$$

$\frac{D}{a^2}$ is a constant

P is aspect ratio ($1 \ll P \ll 2$)

The results of the critical buckling loads obtained for SSSS and CCCC rectangular plates were compared with those obtained by Ibearugbulem et al. (2014) using Work – error method at various aspect ratios (1.0 to 2.0) agreed very closely.

Also, Ibearugbulem et al. (2016) worked on the buckling analysis of rectangular plate using Split – deflection method. They used polynomial – trigonometric shape functions and complete polynomial shape functions to determine the non – dimensional critical buckling loads for SSSS and CCCC rectangular plates respectively. A comparison with the previous results carried out by Ibearugbulem et al. (2014) gave very close percentage differences.

2.3.1.1: Assumptions of elastic buckling theory of thin plate

Ventsel and Krauthammer (2001) and Szilard (2004) summarized the simplest plate theory proposed by Kirchhoff in 1850 as follows:

- (i) The plate is initially flat.

- (ii) The deflection of the mid-plane is small when compared with the thickness of the plate.
- (iii) The material of the plate is assumed to be linearly elastic, homogenous and isotropic.
- (iv) The normal stress, σ_z to the middle plane is neglected in the stress - strain relations
- (v) The effect of the vertical shear strains, τ_{xz} and τ_{yz} and the normal strain, ϵ_z are negligible.
- (vi) Slopes of deflected middle surface are small compared to unity.
- (vii) The middle surface remains unstrained after bending.
- (viii) The constant thickness of the plate, t is small compared with other dimensions. That is, the small lateral dimension of the plate is at least 10 times larger than its thickness.

2.3.2: Plastic Buckling of Rectangular Plate

This is concerned with the development of plate plasticity stability theories. The most popularly used plasticity theories for plastic buckling analysis of plates are the deformation theory of plasticity (Aung, 2006). These two theories are briefly described in Sub - section 2.3.2.1 and 2.3.2.2

2.3.2.1: Deformation Theory of Plasticity

The earliest development of the deformation theory was due to Engesser in 1859 and Von Karman in 1910 (Chen, 2003). They developed a theory based on the fact that for a fibre which is compressed beyond the elastic limit, the tangent modulus assumes different values depending on whether the variation of stress constitutes an increase or a relief of the existing compressive stress. According to Aung (2006), Kaufmann in 1936 and Ilyushin in 1944 developed the basis of deformation theory of plasticity by presenting another route for application of Engesser -Von Karman theory. Thereafter, Stowell in 1948 modified Ilyushin's approach and assumed that the plate remained in the purely plastic state during buckling, hence derived the differential equation of equilibrium of plates under combined loads. He further stated that Bijlaard in 1949 used the assumption of plastic deformation and derived the stress-strain relations by writing the infinitely small excess strains as total differentials and computing the partial derivatives of the strains with respect to the stresses. According to Eziefula (2014), El - Ghazaly and Sherbourne in 1986 employed the deformation theory for the elasto - plastic buckling of plates under non-proportional external loading and non - proportional stresses. Loading, unloading and reloading situations were considered. Maarefdoust and Kadkhodayam (2013) studied the elasto - plastic buckling of rectangular plates by the deformation theory of plasticity. They studied plates of various boundary conditions involving one or

more free edges subjected to uniaxial, biaxial and shear loading. For the uniaxial load conditions, the non - loaded edges were simply supported and parallel to longitudinal direction (x - direction), with the one of the loaded edges being free, and the other loaded edge being free, clamped or simply supported. The employed material was aluminum alloy, AL 7075T6, which is usually in the aerospace industry. The generalized differential quadrature method (GDQ) was used as the method of their analysis. The results obtained using deformation theory of plasticity was in good agreement with the experimental data. An extensive parametric study for the effects of different aspect ratios, transverse shear deformations, loading ratios, thickness ratios and various boundary conditions on the buckling coefficient were presented. Eziefula et al. (2015) worked on the plastic buckling of thin flat rectangular isotropic plates under uniaxial in-plane loads. They derived the plastic buckling equation using a deformation theory of plastic (Stowell's approach) and a work principle. The plate analysis was carried out through a theoretical formulation based on Taylor-Maclaurin's series and application of energy method. Under different moduli ratio (E_t/E_s) of aspect ratios ranging from 0.1 to 1, plates with various boundary conditions; four simply supported edges (SSSS), four clamped edges (CCCC), and two clamped edges along the x - axis and two simply supported edges along the y - axis (CSCS) were considered. It was observed from the results obtained at $E_t/E_s = 0.9$ for SSSS plate were exactly the same with the

results of Onwuka et al. (2013) and they showed average percentage difference of 0.091% with the work presented by Iyengar (1988).

2.3.2.2: Incremental or Flow Theory of Plasticity

According to Wang (2006), the incremental theory was first developed by G.H. Handelman and W. Prager in 1948. Pearson in 1950 modified the calculations made by Handelman and Prager using the assumption of continuous loading. In the incremental theory of plasticity, the stresses and strains are related by a complicated non-linear function that depends on the loading path (Singh, 2008). Becque (2010) worked on inelastic buckling of plates and Becque et al. (2011) provided solution of the inelastic buckling paradox by proposing modifications to the flow theory.

2.3.2.3: Comparison of the Plasticity Theories

The deformation and the incremental theories seem to be the two competing plasticity theories for plate buckling analysis. In the deformation theory, the strain that corresponds to a certain state of stress is entirely independent of the manner in which this stress state has been reached while in the incremental theory, the strains and stresses are related by a function that depend on the loading path (Wang, 2006; Jones, 2009). Although the deformation theory contains fundamental mathematical inconsistencies which are not present in the incremental theory, results from the deformation theory tend to be in better

agreement with experimental evidence (Chen, 2003; Aung, 2006). Generally, the deformation theory gives better prediction of buckling loads for rectangular plates (Wang, 2006).

It is interesting to note that from available literatures in plastic buckling analysis of plates, all works were done by the energy and equilibrium approaches formulated deflection function using trigonometric series. According to Ibearugbulem et al. (2013), the limitation of trigonometric series functions in buckling analysis of rectangular plates is that it makes use of assumed shape functions, and it is difficult to assume satisfying shape functions for plates of various boundary condition. Polynomial deflection function have advantage of being applicable to rectangular plates with mixed boundary conditions such as rectangular plates whose opposite edges have different boundary conditions. Eziefula (2014) worked on the plastic buckling of thin rectangular isotropic plates and formulated their polynomial deflection functions using Taylor - Maclaurin's series. He provided solutions to flat, homogenous and isotropic rectangular thin plates such as simply supported edges (SSSS), four clamped edges (CCCC) and two clamped edges along the x - axis and two simply supported edges along the y - axis (CSCS).

2.4 Pure Bending of Rectangular Plate

Pure Bending is a situation where a plate is caused to bend under load that is normal to its middle plane such that the natural axis coincide with the middle

surface of the section. The bending properties of a plate depend greatly on its thickness as compared with its dimension. Previous studies have also made distinction between bending in thin plates with small deflection. Existing approaches made by some researchers such as Navier in 1823, Levy in 1899 and Timoshenko & Woinowsky - Krieger in 1959 in bending analysis of isotropic rectangular plates of particular boundary conditions assumed shape function in form of trigonometric or hyperbolic – trigonometric series (Ibearugbulem et al., 2013). Probably, the greatest problem with this approach is that, it is very difficult to assume satisfying shape functions for plates of various boundary conditions (Wang et al., 2002; Taylor & Govindjee, 2004; Wang & El-Sheikh, 2005; Imrak and Gerdemeli, 2007; Liu et al., 2007). This limitation in achieving exact solution led to a new approach. Ibearugbulem et al. (2013) worked on a new approach in bending analysis of isotropic rectangular plates using direct integration and work principle. Unlike the previous approaches that assumed shape function, they obtained the shape function by direct integration of governing differential equation for isotropic rectangular plates. They further stated that the maximum deflection and the maximum bending moments due to both uniform load throughout the plate and the centre point load obtained have excellent agreement compared with those obtained by Timoshenko & Woinowsky - Krieger (1959) for aspect ratios ranging 1.0 to 2.0 at 0.1 increments. They therefore recommended that direct integration and work principle approaches could be more easily and better used for the bending

analysis of isotropic rectangular plates. Lim et al. (2007) worked on new symplectic elasticity approach for exact bending solutions for rectangular thin plates with two opposite sides simply supported. They solved the bending problems using eigenvalue equation unlike classical approaches where eigenvalue analysis was only required in vibration and buckling problems. Compared the results obtained with those of classical solutions such as works of Levy (1899) and Timoshenko & Woinowsky - Krieger (1987) showed excellent agreement. An and Gu (2011) obtained an exact solution for the bending problem of fully clamped orthotropic rectangular plates using integral transform technique (GITT). The results obtained showed that GITT approach provided an exact numerical - analytical solution for transverse deflections, bending moments and transverse shear forces. The numerical results obtained are in good agreement compared with the exact solutions given by Timoshenko & Woinowsky - Krieger (1959) and Wojtaszak (1937). Ezeh et al. (2013) worked on a pure bending analysis of thin SSSS and CCCC rectangular flat plates using ordinary finite difference method. They transformed the differential equation of plates into finite difference form and then applied to the boundary conditions of the SSSS and CCCC plates respectively, their deflection equations are obtained. The results of the maximum deflection for SSSS plates of grid sizes $n = 7$ and $n = 9$ obtained were compared with those given by Timoshenko & Woinowsky - Krieger (1987) at aspect ratios ranging from 1 to 2 gave average percentage difference of -0.03731% and 0.05337% respectively. Also, the results of the

maximum deflection for CCCC plates of grid sizes $n = 7$, $n = 9$ and $n = 13$ obtained were compared with those given by Timoshenko & Woinowsky - Krieger (1987) at aspect ratios ranging from 1 to 2 gave average percentage difference of 7.94588%, 5.68351% and 4.14286% respectively. Hence, they concluded that the approximate solution closely agreed with the exact solution. Ibearugbulem et al. (2013) worked on pure bending analysis of thin rectangular SSSS plates through a theoretical formulation of shape function based on Taylor - Maclaurin's series and Ritz energy approaches. The results of the maximum deflection obtained were compared with those given by Ventsel and Krauthammer (2001) showed an average percentage difference of 3.63%. Hence, they concluded that Taylor - Maclaurin's shape function closely agreed with the exact displacement shape function of SSSS plates.

2.5 Free - Vibration of Rectangular Plate

Free-vibration is a situation where by a plate is vibrating freely without any external load at its own natural frequency. Vibration of rectangular plates is an interesting subject because of its wide applications in Structural Engineering and Transport Engineering. Structures like railway bridges, highway bridges, cranes, road pavements etc can actually be modeled as rectangular plates. The applicability of this subject in the study of these structures can hardly do without. This has accelerated research on the study and understanding of

rectangular plates. As a way of review, we quickly discuss some early works on the vibration of rectangular plates and some interesting results obtained. Njoku et al. (2013) worked on free-vibration analysis of thin rectangular isotropic CCCC plates using Taylor series formulated shape function in Galerkin's method. The results of the non - dimensional frequency parameter, "K" obtained were compared with those given by Chakraverty (2009) at aspect ratios (0.5 and 1.0) gave percentage differences of 0.2378% and -0.0475% respectively. Also, the same study results of Njoku et al. (2013) obtained were compared with the results of Liew et al. (1995) at aspect ratios (0.4 and 0.6) gave percentage differences of 0.3315% and 0.1516% respectively. Hence, they were convinced that the assumed shape function excellently agreed with the exact shape function for CCCC plates. However, the solution given by Sakata et al. (1996) were also compared with that given by Njoku et al. (2013) at aspect ratio of 1.0 gave the highest percentage difference of 6.055%. Hence, they concluded that Galerkin's method is a good approximate method for analyzing plates in vibration. Ezeh et al. (2013) worked on free vibration analysis of thin rectangular flat plates using ordinary finite difference method. The results of the fundamental natural frequency " λ " obtained were compared with that given by Ventsel and Krauthammer (2001) and Chakraverty (2009) that used trigonometric series for SSSS plate at aspect ratio of 1.0 gave percentage difference of -5.275% . Hence, they conclude that the approximate solution

closely agreed with the exact solution though shown a lower bound solution. For CCC plates for aspect ratio of 1.0, the results of “ λ ” given by Ezeh et al. (2013) and Szilard (2004) excellently agreed as both solutions are based on the ordinary finite difference method. The results obtained using ordinary finite difference method were compared with that given by Chakraverty (2009) that used trigonometric series gave percentage difference of -24.96% . The large difference was because ordinary finite difference method does not give better result beyond the grid size of 3. Thus, it approximated slowly to the exact solution. However, it was a lower bound solution. For CSCS plates for aspect ratio of 1.0, the results of “ λ ” given by Chakraverty (2009) and Gorman (1982) are exact solutions but were compared with the result given by Ezeh et al. (2013) gave percentage difference of -19.18% . The large difference was because ordinary finite difference method does not give better result beyond the grid size of 3. Yang et al. (2014) worked on the analytical solutions for the natural frequencies and mode shapes of rectangular plate on foundation with four edges free (FFFF) using finite cosine integral transform method. They considered classical Kirchhoff rectangular plate and the foundation were modeled as the Winkler elastic foundation. The results obtained were compared with those given by LI et al. (2009) agreed very closely.

Korhan and Ayse (2012) worked on a free-vibration analysis of thick plates on elastic foundation using modified Vlasov model. They studied the effects of the

subsoil depth, aspect ratio, the ratio of subsoil depth of span of the plate and the value of the vertical deformation parameter within the subsoil on the frequency parameter of the plate. Using an eight - noded 24 degree freedom equilateral finite element based on the Mindlin plate theory, they obtained results for the frequency parameter compared with those given by Ayvaz and Oguzan (2008). It was observed that for low frequency parameters, their results were in excellent agreement while for high frequency parameters, their results started to diverge slightly which showed that the method used by Njoku et al. (2013) are efficient for lower value of the frequency parameters. Ibearugbulem et al. (2014) worked on free - vibration analysis of orthotropic thin rectangular simply supported on all edges (SSSS) plate using Taylor's series function in Ritz method. The results of the frequency parameter " λ " obtained were compared with those given by Pilkey (2005) at various flexural rigidities for the three sets; [(1.0, 1.0, 1.0), (1.0, 0.5,1.0) and (1.0, 0.5, 0.5)]and aspect ratios (0.1 to 1.0) gave average percentage difference of 0.0458%, 0.0564% and 0.0528% respectively. Hence, they concluded that the solutions of the SSSS thin rectangular orthotropic plate using polynomial deflection functions closely agreed with those given to the same plate using other deflection functions. Ebirim et al. (2014) worked on free-vibration analysis of isotropic rectangular plates with one edge free of support (CSCF and SCFC plates) using theoretical formulation based on Ibearugbulem's shape function and application of Ritz method. The results of the fundamental natural frequency " λ " obtained were

compared with those given by Leissa (1973) at aspect ratios (0.4 to 1.5) gave average percentage difference of 1.60%. Also the same study results of Ebirim et al. (2014) obtained were compared with those given by Chakraverty (2009) at aspect ratio of 1.0 gave percentage difference of 1.92%. Hence, they concluded that the approximate solution using Ibearugbulem's shape function in free – vibration agreed excellently with the exact solutions. Onwuka et al. (2017) worked on free-vibration of a rectangular thin orthotropic plate bounded by three simply supported edges and one clamped edges (SSSC) using Taylor - Maclaurin's series truncated at the fourth term in Ritz potential energy functional. The results of the fundamental frequency “ λ ” obtained were compared with those given by Pilkey (2005) at various flexural rigidities for the three sets; [(1.0, 1.0, 1.0), (1.0, 0.5,1.0) and (1.0, 0.5, 0.5)]and aspect ratios (0.1 to 2.0) gave average percentage difference of -3.404% , -2.029% and -2.456% respectively. Hence, they concluded that the solutions of the SSSC thin rectangular orthotropic plate using Taylor - Maclaurin's series formulated shape function closely agreed with those given to the same plate using other deflection functions.

Summarily, it was realized from the literatures given in Sections 2.3 to 2.5 of this thesis that methods of structural analysis of plates such as classical plate energy approaches (Ritz, Galerkin's methods e.t.c) use either polynomial or trigonometric shape functions in their pure bending, buckling and free-vibration

analysis. This led to the literatures in section 2.6 where some of the methods of analysis of plates were explained. The limitations of these methods were erased using split – deflection method explained in sub-section 2.6.3.

2.6 Methods of Analysis of Rectangular Plate

There are many methods used in analysis of rectangular plates. They include:

- (i) Energy approaches such as Ritz, Galerkin's methods e.t.c.
- (ii) Equilibrium approach (Euler method).
- (iii) Numerical approaches such as finite element, finite difference methods.
- (iv) Taylor - Maclaurin's series.
- (v) Split - deflection method.

2.6.1: Energy Methods of Rectangular Plate Analysis

The common energy methods in classical plate theory analysis include Galerkin's, Raleigh, Raleigh-Ritz, Ritz methods e.t.c (Njoku et al., 2013; Ibearugbulem et al., 2014; Ibeabuchi, 2014). According to Ibearugbulem et al. (2016), these methods use orthogonal function in form double furrier series (that product of two mutually perpendicular trigonometric functions) or orthogonal polynomial. They also referred that Hutchinson in 1992, Jianqiao in

1994, Ugural in 1999, Ventsel and Krauthammer in 2001, Wang et al. in 2002, Taylor and Govindjee in 2004, Szilard in 2004, Jiu et al. in 2007, Erdem et al. in 2007, Ezeh et al. in 2013, Ibearugbulem in 2014 stated that energy equations of the methods used earlier than now are based on the orthogonal deflection functions. According to Ezeh et al. (2014), Iyengar in 1988 stated that energy approach sums all the work (strain energy and potential energy or external work) on the continuum to be equal to total potential energy. This approach is quite different from numerical and Euler approaches. The energy methods, also known as the direct method of calculus of variation are amongst the most important approximate method of mathematical physics used for the solution of difficult value problems. The objective of variational methods is to find from a group of admissible functions those that represent the deflection of the elastic body, pertinent to its stable equilibrium condition (Szilard, 2004). Reddy (2004) explained that it should be understood that the equations governing a physical problem are themselves approximate. He further stated that the approximations are introduced by several sources including the geometry, representation of specified loads and boundary conditions.

Few of these common energy methods are discussed in sub - sections 2.6.1.1 and 2.6.1.2.

2.6.1.1: Ritz Method

According to Ventsel and Krauthammer (2001), the energy method developed by Ritz in 1909 applies the principle of minimum potential energy,

$$(\delta\pi = 0) \tag{2.6}$$

Where $\delta\pi$ is the variation in the total potential energy.

$$\delta\pi = \delta(U - W) \tag{2.7}$$

Where δU is the strain energy,

And δW is the external work

He further stated that according to Ritz method, the deflection surface of the plate is approximated by series of the form;

$$w(x,y) = \sum_{i=1}^{\infty} C_i f_i(x,y) \tag{2.8}$$

Where $f_i(x,y)$ are coordinate functions that satisfy kinematic boundary conditions.

C_i are unknown constants to be determined from the minimum potential energy principle.

2.6.1.1.1: Advantages of Ritz Method

According to Ventsel and Krauthammer (2001),

- (i) The coordinate functions $f_i(x,y)$ must satisfy the kinematic (or geometric) boundary conditions only.
- (ii) Due to no difference between the expressions of total potential energy (π) for plates of constant and variable thickness, the method can be applied successfully to rectangular plates of variable thickness.
- (iii) The matrix of the linear algebraic equations for a system of linear non - homogenous algebraic equations for unknown constants to be determined from the minimum potential energy principle is always symmetrical, resulting in stable and powerful logarithms for their numerical solution.

Many researchers have been used Ritz method in the analysis of plates. Ibearugbulem et al. (2011) analyzed buckling of axially compressed SSSS thin rectangular plate using Taylor - Maclaurin's shape function in Ritz total potential energy functional. The results of the critical load "H" obtained were compared with those given by Iyengar (1988) at aspect ratios (0.1 to 1.0) gave average percentage difference of 0.069%. Hence, they concluded that the approximate solution closely agreed with the exact solution though was upper bound solution. Osadebe and Aginan (2011) worked on bending analysis of isotropic rectangular plate with all edges clamped (CCCC) using Ritz variational approach. The results of the deflection and bending obtained were

compared with those given by Timoshenko & Woinowsky - Krieger (1959) at aspect ratio of 0.3 and span ratios (1.0 to 2.0) gave an insignificant average percentage difference. Hence, concluded that the approximate solution closely agreed with the exact solution. Ezeh et al. (2014) worked on vibration of plate with three edges clamped and one edge free (CCFC) using Taylor series shape function in Ritz's total potential energy functional. The results of the natural frequency " λ " obtained were compared with those given by Leissa (1973) at aspect ratios (0.4 to 1.5) gave average percentage difference of 2.79%. Also, the results of the natural frequency " λ " obtained by Ezeh et al. (2014) were compared with those given by Chakraverty (2009) at aspect ratio of 1.0 gave percentage difference of 2.87%. Hence, they concluded that the approximate solution using Taylor series shape function in Ritz potential energy functional in vibration analysis of plate agreed excellently with the exact solution. Ezeh et al. (2014) examined the behavior of buckled clamped-simply-free-simple (CSFS) isotropic rectangular plate using fifth term truncated polynomial series shape function in Ritz's total potential energy functional. The results of the critical buckling load " K " obtained were compared with those given by Iyengar (1988) and Ibearugbulem (2012) at aspect ratios (0.2, 0.4 and 0.8) gave percentage difference of (-0.00394% , 4.3217% and -3.9352%) and (37.2442% , 8.6043% and -14.6265%) respectively. Also the results of the critical buckling load " K " obtained by Ezeh et al. (2014) were compared with

those given by Timoshenko (1936) at aspect ratio of 1.0 gave percentage difference of -4.1765% . Hence, they concluded that the approximate solution closely agreed with the exact solution.

2.6.1.2: Galerkin's Method

According to Ventsel and Krauthammer (2001), the method formulated by Galerkin in 1915 can be applied successfully to diverse types of problems of applied elasticity including the plate bending problems. He further stated that according to Galerkin's method, the deflection surface of the plate is approximated by series of the form;

$$w_N(x,y) = \sum_{i=1}^N \alpha_i f_i(x,y) \quad (2.9)$$

Where α_i are unknown coefficients to be determined.

$f_i(x,y)$ are the linearly independent coordinate functions that satisfy all prescribed boundary conditions.

2.6.1.2.1: Assumptions of Galerkin's Method

According to Ibearugbulem et al. (2014), the following assumptions were made in derivation of Galerkin's equilibrium equations of flat rectangular plate.

- (i) The governing differential equation is equilibrium of all force acting on the plates.

- (ii) Multiplying governing differential equation by plate displacement shall result to work done on the plate.
- (iii) Since continuity existed within the domain of the plate and throughout the loading period, then both the loads and the plate made the same displacement.
- (iv) Since the load and the made the same displacement, then the total work performed by them must be in equilibrium.

Many researchers have used Galerkin's method in their analysis of plate.

Osadebe et al. (2016) worked on stability analysis of all four edges simply supported (SSSS) thin rectangular plate using multi-degree of freedom Taylor – Maclaurin's series polynomial function in Galerkin's variational method. The results of the critical buckling load coefficients "K" obtained were compared with those given by Iyengar (1988) and Ibearugbulem et al. (2011); Ezeh et al. (2014) and Iyengar (1988); and Iyengar (1988) at aspect ratios (0.1 to 1.0) gave average percentage differences of 0.066%; 0.011%; and 0.002% respectively. Hence, they concluded that the approximate solution closely agreed with the exact solution. Onwuka et al. (2016) worked on buckling analysis of biaxially compressed all-round simply supported (SSSS) thin rectangular isotropic plate using Galerkin's method. The results of the critical buckling load coefficient obtained were compared with those given by Ventsel and Krauthammer (2001),

Chajes (1974) and Iyengar (1988) at aspect ratio “ α ” and relationship constant between forces on the Y-axis and forces on the X-axis “K” equal to 1 ($\alpha = K = 1$) gave percentage difference of 0.047%. Hence, they concluded that the approximate solution closely agreed with the exact solution.

2.6.2: Taylor - Maclaurin’s Series

The Taylor’s series refers to any representation of a function as an infinite sum of terms calculated from the values of the derivatives of the function at single point. Glyn (2011) expressed this series mathematically as:

$$f_x = \sum_{n=0}^{\infty} \frac{(x-x_0)^n}{n!} f^{(n)}(x_0) \quad (2.10)$$

where $f^{(n)}(x_0)$ is the n^{th} derivative of f_x determined at point x_0 , and $n!$ is the factorial of n .

Expanding equation (2.5) gives equation (2.6)

$$f(x) = \frac{(x-x_0)^0}{0!} f^{(0)}(x_0) + \frac{(x-x_0)^1}{1!} f'(x_0) + \frac{(x-x_0)^2}{2!} f''(x_0) + \dots \quad (2.11)$$

Where $f^{(0)}(x_0) = f(x_0)$

Differentiating $f'(x_0)$ gives the first and second derivatives as $f'(x_0)$ and $f''(x_0)$ respectively. Mathematically, it can be proved that

$$0! = 1! = 1 \quad (2.12)$$

$$(x - x_0)^0 = 1 \tag{2.13}$$

$$(x - x_0)^1 = x - x_0 \tag{2.14}$$

Therefore, the Taylor series can be re-written by substituting Equations (2.12) to (2.14) into equation (2.11). This gives equation (2.15)

$$f(x) = f(x_0) + (x - x_0)f'(x_0) + \frac{(x - x_0)^2}{2!}f''(x_0) + \dots \tag{2.15}$$

According to Stroud and Booth (2003), if x_0 is zero, then the series expansion about the origin becomes Maclaurin series expansion and it is expressed as given in equation (2.16).

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \dots + \frac{x^n}{n!}f^{(n)}(0) \tag{2.16}$$

Bird (2006) stated that Maclaurin series maybe used to represent any function, say $f(x)$ as a power series if:

$$(i) \quad f(0) \neq \infty \tag{2.17}$$

$$(ii) \quad f(0), f'(0), f''(0), \dots \neq \infty \tag{2.18}$$

(iii) The resultant Maclaurin series is convergent.

From the mathematical expressions above, the Maclaurin series can be said to be special type of Taylor's series when $(x_0) = 0$. This special type of Taylor's

series is called Taylor - Maclaurin's series. The Taylor - Maclaurin's series is a power series of x representing functions $f(x)$.

Taking $\frac{f^n(x_0)}{n!}$ as a_n and x_0 as zero, the Taylor - Maclaurin series in polynomial series maybe re-written as given in Equation (2.19).

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \dots \dots a_n x^n \quad (2.19)$$

Generally, deformed shapes of continua under the action of loads take the form of sinusoidal shapes of sine or cosine curves with the angle measured in radians. Taylor - Maclaurin's series can be used to approximate the deflection function of structural elements such as plates and beams. In estimating deflection function, Taylor - Maclaurin's series have the property to converge into many functions including trigonometric functions.

2.6.3: Split - Deflection Method

According to Ibearugbulem et al. (2016), the common energy methods in classical plate theory analysis include Galerkin's, Raleigh, Raleigh-Ritz, Ritz methods etc. (Njoku et al., 2013, Ibearugbulem et al., 2014, Ibeabuchi, 2014). They also stated that energy methods use orthogonal functions in form double furrier series (that product of two mutually perpendicular trigonometric functions) or orthogonal polynomial. Split-deflection is an exact method used to eradicate the difficulties facing energy method in analysis of plates. In

comparison with energy methods that used deflection function, W in a complex orthogonal trigonometric or orthogonal polynomial, Split - deflection method separated deflection functions into two independent distinct functions, ($W = W_x \times W_y$). This created a possible route for a particular deflection function to have more than one function, that is polynomial and trigonometric functions.

Comparing the work of Ibearugbulem et al. (2014) that used complete orthogonal polynomial deflection function with that of Ibearugbulem et al. (2016) that used partly polynomial and partly trigonometric deflection functions, the percentage differences between the results of both works with various aspect ratios, P and different boundary conditions under pure bending, buckling or free - vibration are given in Table 2.1, Table 2.2, Table 2.3, Table 2.4 and Table 2.5. The details of Split-deflection method is discussed in chapter three of this work

Table 2.1: Centre deflection (pure bending) of SSSS isotropic thin plate under uniform distributed load.

Aspect ratio, P	Deflection at centre (qa^4/D) meters		Percentage difference
	Present (Ibearugbulem et. al. 2016)	Past (Ibearugbulem et. al. 2014)	
1	0.00414	0.00414	0.000
1.1	0.00497	0.00496	0.202
1.2	0.00577	0.00576	0.174
1.3	0.00654	0.00653	0.153
1.4	0.00727	0.00725	0.276
1.5	0.00794	0.00793	0.126
1.6	0.00857	0.00856	0.117
1.7	0.00915	0.00913	0.219
1.8	0.00968	0.00966	0.207
1.9	0.01016	0.01015	0.099
2	0.01061	0.01059	0.189

Source: Ibearugbulem et al. (2016)

Table 2.2: Non - dimensional form of critical buckling load of SSSS Isotropic thin plate

Aspect ratio, P	Deflection at centre $N_x(\frac{D}{a^2})$ meters		Percentage difference
	Present (Ibearugbulem et. al. 2016)	Past (Ibearugbulem et. al. 2014)	
1	39.49	39.51	0.04
1.1	39.85	39.87	0.04
1.2	40.82	40.84	0.03
1.3	42.28	42.29	0.03
1.4	44.14	44.16	0.03
1.5	46.36	46.37	0.02
1.6	48.89	48.90	0.02
1.7	51.71	51.72	0.02
1.8	54.80	54.81	0.02
1.9	58.15	58.16	0.02
2	61.74	61.75	0.01

Source: Ibearugbulem et al. (2016)

Table 2.3: Non - dimensional form of critical buckling load of CCCC**Isotropic thin plate**

Aspect ratio, P	Deflection at centre $N_x(\frac{D}{a^2})$ meters		Percentage difference
	Present (Ibearugbulem et. al. 2016)	Past (Ibearugbulem et. al. 2014)	
1	108.00	108.67	0.61
1.1	109.53	110.21	0.62
1.2	113.65	114.36	0.62
1.3	119.83	120.59	0.63
1.4	127.75	128.57	0.64
1.5	137.17	138.06	0.65
1.6	147.93	148.91	0.66
1.7	159.91	160.99	0.67
1.8	173.04	174.23	0.68
1.9	187.25	188.55	0.69
2	202.50	203.92	0.69

Source: Ibearugbulem et al. (2016)

Table 2.4: Non - dimensional form of resonating frequency of SSSS**Isotropic thin plate**

Aspect ratio, P	Resonating frequency, $\lambda(\frac{1}{a^2} \sqrt{\frac{D}{m}})$ hertz		Percentage difference
	Present (Ibearugbulem et. al. 2016)	Past (Ibearugbulem et. al. 2014)	
1	19.74	19.75	-0.05
1.1	18.03	18.04	-0.06
1.2	16.73	16.73	0.00
1.3	15.71	15.72	-0.06
1.4	14.91	14.91	0.00
1.5	14.26	14.26	0.00
1.6	13.73	13.73	0.00
1.7	13.29	13.29	0.00
1.8	12.92	12.92	0.00
1.9	12.61	12.61	0.00
2	12.34	12.34	0.00

Source: Ibearugbulem et al. (2016)

Table 2.5: Non - dimensional form of resonating frequency of CCCC**Isotropic thin plate**

Aspect ratio, P	Resonating frequency, $\lambda(\frac{1}{a^2} \sqrt{\frac{D}{m}})$ hertz		Percentage difference
	Present (Ibearugbulem et. al. 2016)	Past (Ibearugbulem et. al. 2014)	
1	36	35.97	0.08
1.1	32.96	32.93	0.09
1.2	30.77	30.75	0.07
1.3	29.17	29.15	0.07
1.4	27.97	27.95	0.07
1.5	27.05	27.03	0.07
1.6	26.33	26.31	0.08
1.7	25.77	25.75	0.08
1.8	25.32	25.30	0.08
1.9	24.95	24.94	0.04
2	24.65	24.64	0.04

Source: Ibearugbulem et al. (2016)

CHAPTER THREE

MATERIALS AND METHODS

3.1 Materials

The materials used in this research work are already existing equations that are of common knowledge, given by Ibearugbulem et al. (2016) as seen in Equations (3.1, 3.2, 3.3, 3.4 and 3.5) and Ibearugbulem (2014) as seen in Equations (3.6, 3.7, 3.8, 3.9, 3.10, 3.11, 3.12, 3.13, 3.14, 3.15 and 3.16).

3.1.1: Split - Deflection

The split – deflection equation is given in equation (3.1) as:

$$w = w_x \cdot w_y \quad (3.1)$$

Where the x and y components of the deflection are given in Equations (3.2) and (3.3) as:

$$w_x = A_x \cdot h_x \quad (3.2)$$

$$w_y = A_y \cdot h_y \quad (3.3)$$

The deflection coefficient of the x and y components are given in Equation (3.4) as:

$$A = A_x \cdot A_y \quad (3.4)$$

Substituting Equations (3.2), (3.3) and (3.4) into Equation (3.1) gives Equation (3.5)

$$w = A \cdot h_x \cdot h_y \quad (3.5)$$

3.1.2: In - Plane Displacements, u and v

The in-plane displacement for thin plates of small deflection is given in Equations (3.6) and (3.7) as:

$$u = -Z \frac{dw}{dx} + u_0 \quad (3.6)$$

$$v = -Z \frac{dw}{dy} + v_0 \quad (3.7)$$

Where u and v are the displacement components along x and y coordinate directions respectively.

3.1.3: In – Plane Strains

The in – plane strains for thin plates of small deflection are given as:

$$\epsilon_{xx} = \frac{du}{dx} = -Z \frac{d^2w}{dx^2} \quad (3.8)$$

$$\epsilon_{yy} = \frac{dv}{dy} = -Z \frac{d^2w}{dy^2} \quad (3.9)$$

$$\gamma_{xy} = \frac{du}{dy} + \frac{dv}{dx} = -2Z \frac{d^2w}{dxdy} \quad (3.10)$$

Where $(\epsilon_{xx}, \epsilon_{yy})$ are the normal strains and γ_{xy} is the shear strain.

3.1.4: Constitutive Relations

The in - plane constitutive equations are given in Equations (3.11), (3.12) and (3.13) as:

$$\sigma_{xx} = \frac{E}{1-\mu^2}[\varepsilon_{xx} + \mu\varepsilon_{yy}] \quad (3.11)$$

$$\sigma_{yy} = \frac{E}{1-\mu^2}[\mu\varepsilon_{xx} + \varepsilon_{yy}] \quad (3.12)$$

$$\tau_{xy} = \frac{E(1-\mu)\gamma_{xy}}{2(1-\mu^2)} \quad (3.13)$$

Where $(\delta_{xx}, \delta_{yy})$ are the normal stresses and τ_{xy} is the shear stress.

3.1.5: Total Potential Energy

The total potential energy functional, π of thin plate subjected to both inplane load and vibration is given in Equation (3.14) as:

$$\pi = U - V \quad (3.14)$$

Where U is the strain energy given in Equation (3.15) as:

$$U = \frac{1}{2} \int_x \int_y \left[\int_{-t/2}^{t/2} (\sigma_{xx}\varepsilon_{xx} + \sigma_{yy}\varepsilon_{yy} + \tau_{xy}\gamma_{xy}) dz \right] dx dy \quad (3.15)$$

And

V is the external work under buckling and free – vibration given in Equation (3.16) as:

$$V = \frac{1}{2} \int_x \int_y \left[N_x \left(\frac{dw}{dx} \right)^2 + m\lambda^2 w^2 \right] dx dy \quad (3.16)$$

3.2 Methods

Four procedures were presented here to justify the five specific objectives.

3.2.1: Derivation of the Governing Equation for determining the Critical Buckling Loads for Rectangular Plate under Vibration using Split – deflection

3.2.1.1: Kinematics

Substituting Equation (3.5) into Equations (3.8), (3.9) and (3.10) give Equations (3.17), (3.18) and (3.19)

$$\varepsilon_{xx} = -Z \frac{d^2w}{dx^2} = -Z \frac{Ad^2h_x \cdot h_y}{dx^2} \quad (3.17)$$

$$\varepsilon_{yy} = -Z \frac{d^2w}{dy^2} = -Z \frac{Ah_x \cdot d^2h_y}{dy^2} \quad (3.18)$$

$$\gamma_{xy} = -2Z \frac{d^2w}{dxdy} = -2ZA \frac{dh_x}{dx} \cdot \frac{dh_y}{dy} \quad (3.19)$$

Multiplying Equation (3.11) with ε_{xx} gives Equation (3.20)

$$\sigma_{xx}\varepsilon_{xx} = \frac{E}{1-\mu^2} [\varepsilon_{xx}^2 + \mu\varepsilon_{xx}\varepsilon_{yy}] \quad (3.20)$$

Multiplying Equation (3.12) with ε_{yy} gives Equation (3.21)

$$\sigma_{yy}\varepsilon_{yy} = \frac{E}{1-\mu^2} [\mu\varepsilon_{xx}\varepsilon_{yy} + \varepsilon_{yy}^2] \quad (3.21)$$

Multiplying Equation (3.13) with γ_{xy} gives Equation (3.22)

$$\tau_{xy}\gamma_{xy} = \frac{E(1-\mu)\gamma_{xy}^2}{2(1-\mu^2)} \quad (3.22)$$

Substituting Equations (3.17) and (3.18) into Equations (3.20) and (3.21) respectively give Equations (3.23) and (3.24)

$$\sigma_{xx}\varepsilon_{xx} = \frac{EA^2Z^2}{1-\mu^2} \left[\left(\frac{d^2h_x}{dx^2} \right)^2 \cdot h_y^2 + \mu \left(\frac{dh_x}{dx} \cdot \frac{dh_y}{dy} \right)^2 \right] \quad (3.23)$$

$$\sigma_{yy}\varepsilon_{yy} = \frac{EA^2Z^2}{1-\mu^2} \left[\mu \left(\frac{dh_x}{dx} \cdot \frac{dh_y}{dy} \right)^2 + \left(\frac{d^2h_y}{dy^2} \right)^2 \cdot h_x^2 \right] \quad (3.24)$$

Substituting Equation (3.19) into Equation (3.22) gives Equation (3.25)

$$\tau_{xy}\gamma_{xy} = \frac{2EA^2(1-\mu)Z^2}{(1-\mu^2)} \left(\frac{dh_x}{dx} \cdot \frac{dh_y}{dy} \right)^2 \quad (3.25)$$

3.2.1.2: Strain Energy (U)

Recall, Strain energy equation was given in Equation (3.15) as:

$$U = \frac{1}{2} \int_x \int_y \left[\int_{-t/2}^{t/2} (\sigma_{xx}\varepsilon_{xx} + \sigma_{yy}\varepsilon_{yy} + \tau_{xy}\gamma_{xy}) dz \right] dx dy$$

Substituting Equations (3.23), (3.24), and (3.25) into Equation (3.15) gives

Equation (3.26)

$$U = \frac{EA^2}{2(1-\mu^2)} \int_x \int_y \left[\left(\frac{d^2h_x}{dx^2} \right)^2 \cdot h_y^2 + 2\mu \left(\frac{dh_x}{dx} \cdot \frac{dh_y}{dy} \right)^2 + \left(\frac{d^2h_y}{dy^2} \right)^2 \cdot h_x^2 + \right. \\ \left. 2 \left(\frac{dh_x}{dx} \cdot \frac{dh_y}{dy} \right)^2 - 2\mu \left(\frac{dh_x}{dx} \cdot \frac{dh_y}{dy} \right)^2 \right] dx dy \cdot \int_{-t/2}^{t/2} Z^2 dz$$

That is,

$$U = \frac{Et^3}{24(1-\mu^2)} A^2 \int_x \int_y \left[\left(\frac{d^2h_x}{dx^2} \right)^2 h_y^2 + 2 \left(\frac{dh_x}{dx} \cdot \frac{dh_y}{dy} \right)^2 + \left(\frac{d^2h_y}{dy^2} \right)^2 h_x^2 \right] dx dy \quad (3.26)$$

Where

$$\int_{-t/2}^{t/2} Z^2 dz = \frac{t^3}{12} \quad (3.27)$$

But

$$D = \frac{Et^3}{12(1-\mu^2)} \quad (3.28)$$

E, μ and t are Young's modulus, Poison's ratio, and the thickness of the plate.

Substituting Equation (3.28) into Equation (3.26) gives Equation (3.29)

$$U = \frac{A^2 D}{2} \int_x \int_y \left[\left(\frac{d^2 h_x}{dx^2} \right)^2 h_y^2 + 2 \left(\frac{dh_x}{dx} \cdot \frac{dh_y}{dy} \right)^2 + \left(\frac{d^2 h_y}{dy^2} \right)^2 h_x^2 \right] dx dy \quad (3.29)$$

3.2.1.3: Buckling Load and Vibration External Work (V)

Substituting Equation (3.5) into Equation (3.16) gives Equation (3.30)

$$V = \frac{A^2}{2} \int_x \int_y \left[N_x \left(\frac{dh_x}{dx} \right)^2 h_y^2 + m \lambda^2 h_x^2 h_y^2 \right] dx dy \quad (3.30)$$

3.2.1.4: Total Potential Energy (π)

Substituting Equation (3.29) and (3.30) into Equation (3.14) gives Equation

(3.31)

$$\begin{aligned} \pi = & \frac{A^2 D}{2} \int_x \int_y \left[\left(\frac{d^2 h_x}{dx^2} \right)^2 h_y^2 + 2 \left(\frac{dh_x}{dx} \cdot \frac{dh_y}{dy} \right)^2 + \left(\frac{d^2 h_y}{dy^2} \right)^2 h_x^2 \right] dx dy - \\ & \frac{A^2}{2} \int_x \int_y \left[N_x \left(\frac{dh_x}{dx} \right)^2 h_y^2 + m \lambda^2 h_x^2 h_y^2 \right] dx dy \end{aligned} \quad (3.31)$$

Let the non – dimensional cordinates, R and Q be defined as given in Equations

(3.32) and (3.33)

$$R = \frac{x}{a} \quad (3.32)$$

$$Q = \frac{y}{b} \quad (3.33)$$

Where a and b are the plate lengths along x and y axes respectively.

Let the aspect ratio be defined as given in Equation (3.34)

$$P = b/a \quad (3.34)$$

Substituting Equations (3.32), (3.33) and (3.34) into Equation (3.31) gives Equation (3.35)

$$\pi = \frac{A^2 Dab}{2a^4} \int_0^1 \int_0^1 \left[\left(\frac{d^2 h_x}{dR^2} \right)^2 h_y^2 + \frac{2}{P^2} \left(\frac{dh_x}{dR} \cdot \frac{dh_y}{dQ} \right)^2 + \frac{1}{P^4} \left(\frac{d^2 h_y}{dQ^2} \right)^2 h_x^2 \right] dRdQ - \frac{A^2 ab}{2a^2} \int_0^1 \int_0^1 \left[N_x \left(\frac{dh_x}{dR} \right)^2 h_y^2 + m\lambda^2 h_x^2 h_y^2 \right] dRdQ \quad (3.35)$$

3.2.1.5: Minimization of Total Potential Energy

Differentiating equation (3.35) with respect to deflection coefficient, A and equating the resulting equation to zero gives Equation (3.36)

$$\frac{d\pi}{dA} = \frac{AD}{a^4} \left[K_x + \frac{2}{P^2} K_{xy} + \frac{1}{P^4} K_y \right] - \frac{A}{a^2} [N_x K_N + m\lambda^2 K_\lambda] = 0$$

That is,

$$\frac{D}{a^2} \left[K_x + \frac{2}{P^2} K_{xy} + \frac{1}{P^4} K_y \right] = [N_x K_N + m\lambda^2 K_\lambda] \quad (3.36)$$

Where

$$K_x = \int_0^1 \int_0^1 \left(\frac{d^2 h_x}{dR^2} \right)^2 h_y^2 dRdQ \quad (3.37)$$

$$K_{xy} = \int_0^1 \int_0^1 \left(\frac{dh_x}{dR} \cdot \frac{dh_y}{dQ} \right)^2 dRdQ \quad (3.38)$$

$$K_y = \int_0^1 \int_0^1 \left(\frac{d^2 h_y}{dQ^2} \right)^2 h_x^2 dRdQ \quad (3.39)$$

$$K_{N_x} = \int_0^1 \int_0^1 \left(\frac{dh_x}{dR} \right)^2 h_y^2 dRdQ \quad (3.40)$$

$$K_\lambda = \int_0^1 \int_0^1 h_x^2 h_y^2 dRdQ \quad (3.41)$$

Under free – vibration only, Equation (3.36) becomes Equation (3.42)

$$\frac{D}{a^2} \left[K_x + \frac{2}{P^2} K_{xy} + \frac{1}{P^4} K_y \right] = m\lambda^2 K_\lambda \quad (3.42)$$

Making the buckling load, N_x the subject of Equation (3.36) gives Equation (3.43)

$$N_x = \frac{\frac{D}{a^2} \left[K_x + \frac{2}{P^2} K_{xy} + \frac{K_y}{P^4} \right] - m\theta^2 K_\lambda}{K_{N_x}} \quad (3.43)$$

Substituting Equation (3.42) into Equation (3.43) gives Equation (3.44)

$$N_x = \frac{m\lambda^2 K_\lambda - m\theta^2 K_\lambda}{K_{N_x}} \quad (3.44)$$

Re – arranging Equation (3.44) gives Equation (3.45)

$$N_x = \frac{m\lambda^2 K_\lambda}{K_{N_x}} \left(1 - \frac{\theta^2}{\lambda^2} \right) \quad (3.45)$$

Substituting Equation (3.42) into Equation (3.45) gives Equation (3.46)

$$N_x = \frac{\frac{D}{a^2} \left[K_x + \frac{2}{P^2} K_{xy} + \frac{K_y}{P^4} \right]}{K_{N_x}} \left(1 - \frac{\theta^2}{\lambda^2} \right) \quad (3.46)$$

Where;

θ is the forcing frequency ($0 \leq \theta \leq 1$)

λ is the resonating frequency

Let the resonating frequency ratio (n) = $\frac{\theta}{\lambda}$; where ($0 \leq n \leq 1$),

Aspect ratio (P) = b/a ; where ($1 \leq P \leq 2$), $\frac{D}{a^2}$ is a constant and the non –

dimensional critical buckling load, (\bar{N}_x); where

$$(\bar{N}_x) = \frac{\left[K_x + \frac{2}{P^2} K_{xy} + \frac{K_y}{P^4} \right]}{K_{N_x}} (1 - n^2)$$

Hence, equation (3.46) becomes equation (3.47)

$$N_x = \frac{\frac{D}{a^2} \left[K_x + \frac{2}{P^2} K_{xy} + \frac{K_y}{P^4} \right]}{K_{N_x}} (1 - n^2) = \frac{D}{a^2} \cdot \bar{N}_x \quad (3.47)$$

This equation (3.47) is the governing equation for determining the critical buckling load for a rectangular plate under vibration.

3.2.2 Derivation of various Shape Functions for the Rectangular Plates

Using Split – deflection method

The various shape functions for the rectangular plates are derived using split – deflection method as shown in Sub – sections 3.2.2.1 and 3.2.2.2.

3.2.2.1: Derivation of the General Trigonometric Shape Functions for a Rectangular Plate

Recall that strain energy in Equation (3.29) was given as seen in Equation (3.48)

$$U = \frac{A^2 D}{2} \int_x \int_y \left[\left(\frac{d^2 h_x}{dx^2} \right)^2 h_y^2 + 2 \left(\frac{dh_x}{dx} \cdot \frac{dh_y}{dy} \right)^2 + \left(\frac{d^2 h_y}{dy^2} \right)^2 h_x^2 \right] dx dy \quad (3.48)$$

Also recall that deflection components were given in Equations (3.2), (3.3) and (3.4) as:

$$w_x = A_x \cdot h_x \quad (3.49)$$

$$w_y = A_y \cdot h_y \quad (3.50)$$

$$A = A_x \cdot A_y \quad (3.51)$$

Substituting Equations (3.49), (3.50) and (3.51) into Equation (3.48) gives Equation (3.52)

$$U = \frac{D}{2} \int_x \int_y \left[\left(\frac{d^2 w_x}{dx^2} \right)^2 w_y^2 + 2 \left(\frac{dw_x}{dx} \right)^2 \left(\frac{dw_y}{dy} \right)^2 + \left(\frac{d^2 w_y}{dy^2} \right)^2 w_x^2 \right] dx dy \quad (3.52)$$

From Equation (3.30), external work for buckling is given in Equation (3.53) as:

$$V = \frac{A^2}{2} \int_x \int_y \left[N_x \left(\frac{dw_x}{dx} \right)^2 w_y^2 \right] dx dy \quad (3.53)$$

Substituting Equations (3.49), (3.50) and (3.51) into Equation (3.53) gives Equation (3.54)

$$V = \int_x \int_y \left[\frac{N_x}{2} \left(\frac{dw_x}{dx} \right)^2 w_y^2 \right] dx dy \quad (3.54)$$

Subtracting Equation (3.54) from Equation (3.52) gives the total potential energy functional as given in Equation (3.55)

$$\pi = \frac{D}{2} \int_x \int_y \left[\left(\frac{d^2 w_x}{dx^2} \right)^2 w_y^2 + 2 \left(\frac{dw_x}{dx} \right)^2 \left(\frac{dw_y}{dy} \right)^2 + \left(\frac{d^2 w_y}{dy^2} \right)^2 w_x^2 - \frac{N_x}{2} \left(\frac{dw_x}{dx} \right)^2 w_y^2 \right] dx dy \quad (3.55)$$

Expanding Equation (3.55) gives Equation (3.56)

$$\pi = \frac{D}{2} \int_x \int_y \left[\frac{d^4 w_x^2}{dx^4} w_y^2 + 2 \frac{d^2 w_x^2}{dx^2} \cdot \frac{d^2 w_y^2}{dy^2} + \frac{d^4 w_y^2}{dy^4} w_x^2 - \frac{N_x}{D} \frac{d^2 w_x^2}{dx^2} w_y^2 \right] dx dy \quad (3.56)$$

Differentiating the total potential energy functional (π) with respect to deflection (w) and equating the resulting equation to zero gives Equation (3.57)

$$\frac{d\pi}{dw} = D \int_x \int_y \left[\frac{d^4 w_x}{dx^4} w_y + 2 \frac{d^2 w_x}{dx^2} \cdot \frac{d^2 w_y}{dy^2} + \frac{d^4 w_y}{dy^4} w_x + \frac{N_x}{D} \frac{d^2 w_x}{dx^2} w_y \right] dx dy = 0 \quad (3.57)$$

This Equation (3.57) is the strong form of equilibrium of forces acting on rectangular plates.

For Equation (3.57) to be true, Equation (3.58) exists.

$$W_y \frac{d^4 w_x}{dx^4} + 2 \frac{d^2 w_x}{dx^2} \cdot \frac{d^2 w_y}{dy^2} + W_x \frac{d^4 w_y}{dy^4} + W_y \frac{N_x}{D} \frac{d^2 w_x}{dx^2} = 0 \quad (3.58)$$

Re - arranging Equation (3.58) into like terms gives Equation (3.59)

$$W_y \left[\frac{d^4 w_x}{dx^4} + \frac{N_x}{D} \frac{d^2 w_x}{dx^2} \right] + W_x \left[\frac{d^4 w_y}{dy^4} + \frac{2}{W_x} \frac{d^2 w_x}{dx^2} \cdot \frac{d^2 w_y}{dy^2} \right] = 0 \quad (3.59)$$

For Equation (3.59) to be true, Equations (3.60), (3.61), (3.62) and (3.63) exist.

$$W_y \left[\frac{d^4 w_x}{dx^4} + \frac{N_x}{D} \frac{d^2 w_x}{dx^2} \right] = 0 \quad (3.60)$$

And

$$W_x \left[\frac{d^4 w_y}{dy^4} + \left(\frac{2}{W_x} \cdot \frac{d^2 w_x}{dx^2} \right) \cdot \frac{d^2 w_y}{dy^2} \right] = 0 \quad (3.61)$$

That is,

$$\frac{d^4 w_x}{dx^4} + \frac{N_x}{D} \frac{d^2 w_x}{dx^2} = 0 \quad (3.62)$$

And

$$\frac{d^4 w_y}{dy^4} + \left(\frac{2}{W_x} \cdot \frac{d^2 w_x}{dx^2} \right) \cdot \frac{d^2 w_y}{dy^2} = 0 \quad (3.63)$$

Substituting $k^2 = \frac{N_x}{D}$ and $C^2 = \frac{2}{w_x} \cdot \frac{d^2 w_x}{dx^2}$ into Equations (3.62) and (3.63) give Equations (3.64) and (3.65)

$$\frac{d^4 w_x}{dx^4} + k^2 \frac{d^2 w_x}{dx^2} = 0 \quad (3.64)$$

$$\frac{d^4 w_y}{dy^4} + C^2 \frac{d^2 w_y}{dy^2} = 0 \quad (3.65)$$

Where k^2 and C^2 are constants

Let differential equation given in Equation (3.66) as;

$$w = A + Bx + Ce^{hx} \quad (3.66)$$

Differentiating Equation (3.66) once with respect to x gives Equation (3.67)

$$\frac{dw}{dx} = B + Che^{hx} \quad (3.67)$$

Differentiating Equation (3.67) once with respect to x gives Equation (3.68)

$$\frac{d^2 w}{dx^2} = Ch^2 e^{hx} \quad (3.68)$$

Differentiating Equation (3.68) twice with respect to x gives Equation (3.69)

$$\frac{d^4 w}{dx^4} = Ch^4 e^{hx} \quad (3.69)$$

Substituting Equations (3.68) and (3.69) into Equation (3.64) gives Equation (3.70)

$$Ch^4 e^{hx} + k^2 Ch^2 e^{hx} = 0 \quad (3.70)$$

That is,

$$Ce^{hx} [h^4 + k^2 h^2] = 0 \quad (3.71)$$

$$\text{Let } h^4 + k^2 h^2 = h^2 (h^2 + k^2) = 0 \quad (3.72)$$

This implies that,

$$h^2 + k^2 = 0 \quad (3.73)$$

$$h^2 = -k^2 \quad (3.74)$$

$$h = \pm \sqrt{-k^2} = \pm \sqrt{-k} \cdot \sqrt{k^2} \quad (3.75)$$

Therefore,

$$h = \pm i k \quad (3.76)$$

That is,

$$h = i k \text{ or } -i k \quad (3.77)$$

Substituting Equation (3.77) into Equation (3.66) gives Equation (3.78)

$$w = A + Bx + C_1 e^{ikx} + C_2 e^{-ikx} \quad (3.78)$$

Converting the exponential functions into trigonometric functions give

Equations (3.79) and (3.80)

$$e^{ikx} = \cos kx + i \sin kx \quad (3.79)$$

$$e^{-ikx} = \cos kx - i \sin kx \quad (3.80)$$

Substituting Equations (3.79) and (3.80) into Equation (3.78) gives Equation

(3.81)

$$w = A + Bx + C_1 \cos kx + C_1 i \sin kx + C_2 \cos kx - C_2 i \sin kx \quad (3.81)$$

Re –arranging Equation (3.81) gives Equation (3.82)

$$w = A + Bx + [C_1 + C_2] \cos kx + [iC_1 - iC_2] \sin kx \quad (3.82)$$

Let $D = C_1 + C_2$ and $E = iC_1 - iC_2$

Hence, Equation (3.82) becomes Equation (3.83)

$$w = A + Bx + D \cos kx + E \sin kx \quad (3.83)$$

Substituting $a_0 = A, a_1 = B, a_2 = D$ and $a_3 = E$ into Equation (3.83) gives Equation (3.84)

$$w = a_0 + a_1x + a_2 \cos kx + a_3 \sin kx \quad (3.84)$$

For a rectangular plate whose lengths in x and y directions are “a” and “b”, non - dimensional parameter in x and y directions are defined as:

$$R = \frac{x}{a}, \text{ that is } x = aR \quad (3.85)$$

$$Q = \frac{y}{b}, \text{ that is } y = bQ \quad (3.86)$$

Substituting Equation (3.85) and (3.86) into Equation (3.84) gives Equation (3.87) and (3.88)

$$w_x = a_0 + a_1R + a_2 \cos kR + a_3 \sin kR \quad (3.87)$$

And

$$w_y = b_0 + b_1Q + b_2 \cos kQ + b_3 \sin kQ \quad (3.88)$$

This Equation (3.87) is the general trigonometric shape function for a rectangular plate in x - direction. Similarly, that of y - direction is given in Equation (3.88)

3.2.2.1.1: Derivation of the Trigonometric Shape Function for SSSS

Rectangular Plate

Considering SSSS rectangular plate. The boundary conditions are shown in figure 3.1



Figure 3.1: SSSS Rectangular Plate's Boundary Conditions.

Recall that the general trigonometric shape function for a rectangular plate given in Equation (3.87) as;

$$w = a_0 + a_1 R + a_2 \cos kR + a_3 \sin kR \quad (3.89)$$

Differentiating Equation (3.89) twice with respect to R gives Equation (3.90)

$$w'' = -a_2 k^2 \cos kR - a_3 k^2 \sin kR \quad (3.90)$$

Substituting the boundary conditions into Equations (3.89) and (3.90) each at $R = 0$ and $R = 1$ respectively in matrix form gives Equation (3.91)

$$\begin{bmatrix} W_{(0)} \\ W''_{(0)} \\ W_{(1)} \\ W''_{(1)} \end{bmatrix} = 0 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & -k^2 & 0 \\ 1 & 1 & \text{Cos}k & \text{Sink} \\ 0 & 0 & -k^2\text{Cos}k & -k^2\text{Sink} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = 0 \quad (3.91)$$

The determinant of the matrix in equation (3.91) gives Equation (3.92)

$$1 \begin{vmatrix} 0 & -k & 0 \\ 1 & \text{Cos}k & \text{Sink} \\ 0 & -k^2\text{Cos}k & -k^2\text{Sink} \end{vmatrix} + 1 \begin{vmatrix} 0 & 0 & 0 \\ 1 & 1 & \text{Sink} \\ 0 & 0 & -k^2\text{sink} \end{vmatrix} = 0$$

That is,

$$-(k^2) \begin{vmatrix} 1 & \text{Sink} \\ 0 & -k^2\text{sink} \end{vmatrix} = k^2 (-k^2 \text{Sink} - 0) = 0 \quad (3.92)$$

Re - arranging Equation (3.92) give Equations (3.93)

$$-k^4 \sin k = 0 \quad (3.93)$$

For Equation (3.93) to be true;

$$\sin k = 0 \quad (3.94)$$

Therefore,

$$k = \pi \quad (3.95)$$

Substituting Equation (3.95) into Equations (3.89) and (3.90) give Equations (3.96) and (3.97) respectively.

$$w = a_0 + a_1 R + a_2 \cos \pi R + a_3 \sin \pi R \quad (3.96)$$

$$w'' = -a_2\pi^2 \cos kR - a_3\pi^2 \sin kR \quad (3.97)$$

Substituting the boundary conditions into Equations (3.96) and (3.97) each at $R = 0$ and $R = 1$ respectively give equation (3.98) to (3.101)

$$w_{(0)} = 0 = a_0 + a_2 \Rightarrow a_0 = -a_2 \quad (3.98)$$

$$w''_{(0)} = 0 = -a_2\pi^2 \Rightarrow a_2 = 0 \quad (3.99)$$

$$w_{(1)} = 0 = a_1 \Rightarrow a_1 = 0 \quad (3.100)$$

$$w''_{(1)} = 0 = -(-a_2\pi^2) \Rightarrow a_2 = 0 \quad (3.101)$$

Substituting Equations (3.98) to (3.101) into Equation (3.96) gives Equation (3.102)

$$w = a_3 \sin \pi R \quad (3.102)$$

Substituting $a_3 = A$ into Equation (3.102) gives Equation (3.103)

$$w_x = A \sin \pi R \quad (3.103)$$

This Equation (3.103) is the trigonometric shape function for a rectangular plate with four edges simply supported (SSSS) in x - direction. Similarly, that of y - direction is given in Equation (3.104)

$$w_y = A \sin \pi Q \quad (3.104)$$

3.2.2.1.2: Derivation of the Trigonometric Shape Function for CCCC

Rectangular Plate

Considering CCCC rectangular plate. The boundary conditions are shown in

Figure 3.2

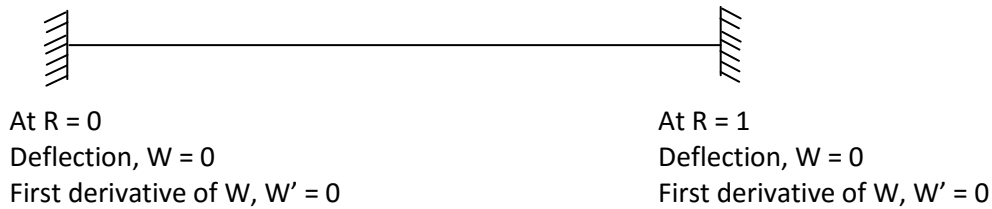


Figure 3.2: CCCC Rectangular Plate's Boundary Conditions.

The general trigonometric shape function for a rectangular plate given in Equation (3.87) as;

$$w = a_0 + a_1 R + a_2 \cos kR + a_3 \sin kR \quad (3.105)$$

Differentiating Equation (3.105) once with respect to R gives Equation (3.106)

$$w' = a_1 - a_2 k \sin kR + a_3 k \cos kR \quad (3.106)$$

Substituting the boundary conditions into Equations (3.105) and (3.106) each at $R = 0$ and $R = 1$ respectively in matrix form gives Equation (3.107)

$$\begin{bmatrix} W_{(0)} \\ W'_{(0)} \\ W_{(1)} \\ W'_{(1)} \end{bmatrix} = 0 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & k \\ 1 & 1 & \text{Cos}k & \text{Sink} \\ 0 & 1 & -k\text{Sink} & k\text{Cos}k \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = 0 \quad (3.107)$$

The determinant of the matrix in Equation (3.107) gives Equation (3.108)

$$1 \begin{vmatrix} 1 & 0 & k \\ 1 & \text{Cos}k & \text{Sink} \\ 1 & -k\text{Sink} & k\text{Cos}k \end{vmatrix} + 1 \begin{vmatrix} 0 & 1 & k \\ 1 & 1 & \text{Sink} \\ 0 & 1 & k\text{Cos}k \end{vmatrix} = 0$$

That is,

$$1 \begin{vmatrix} \text{Cos}k & \text{Sink} \\ -k\text{Sink} & k\text{Cos}k \end{vmatrix} + k \begin{vmatrix} 1 & \text{Cos}k \\ 1 & -k\text{Sink} \end{vmatrix} - 1 \begin{vmatrix} 1 & k \\ 1 & k\text{Cos}k \end{vmatrix} = 0$$

$$1(k \cos^2 k + k \sin^2 k) + k(-k \sin k - \cos k) - 1(k \cos k - k) = 0 \quad (3.108)$$

Re - arranging Equation (3.108) gives Equation (3.109)

$$k \cos^2 k + k \sin^2 k - k^2 \sin k - k \cos k - k \cos k + k = 0 \quad (3.109)$$

Re - arranging Equation (3.109) gives Equation (3.110)

$$\cos^2 k + \sin^2 k - k \sin k - 2 \cos k + 1 = 0 \quad (3.110)$$

$$\text{Let } k = 2\pi \quad (3.111)$$

Substituting Equations (3.111) into Equations (3.105) and (3.106) give Equations (3.112) and (3.113)

$$w = a_0 + a_1 R + a_2 \cos 2\pi R + a_3 \sin 2\pi R \quad (3.112)$$

$$w' = a_1 - 2a_2\pi \sin 2\pi R + 2a_3\pi \cos 2\pi R \quad (3.113)$$

Substituting the boundary conditions into Equations (3.112) and (3.113) each at $R=0$ and $R=1$ respectively give Equations (3.114) to (3.117)

$$w_{(0)} = 0 = a_0 + a_2; \Rightarrow a_0 = -a_2 \quad (3.114)$$

$$w'_{(0)} = 0 = -a_0 + 2a_3\pi; \Rightarrow a_1 = -2a_3\pi \quad (3.115)$$

$$w_{(1)} = 0 = a_0 + a_1 + a_2 = -a_2 + a_1 + a_2; \Rightarrow a_1 = 0 \quad (3.116)$$

$$w'_{(1)} = 0 = a_1 + 2a_3\pi = 0 \quad (3.117)$$

Substituting Equation (3.116) into Equation (3.115) gives Equation (3.118)

$$a_3 = 0 \quad (3.118)$$

Substituting Equations (3.116) and (3.118) into Equation (3.112) gives Equation (3.119)

$$w = -a_2 + a_2 \cos 2\pi R \quad (3.119)$$

Re - arranging Equation (3.119) gives Equation (3.120)

$$w = -a_2(1 - \cos 2\pi R) \quad (3.120)$$

Substituting $-a_2 = A$ into Equation (3.120) gives Equation (3.121)

$$w_x = A(1 - \cos 2\pi R) \quad (3.121)$$

This Equation (3.121) is the trigonometric shape function for a rectangular plate with four edges clamped (CCCC) in x - direction. Similarly, that of y - direction is given in Equation (3.122)

$$w_y = A(1 - \cos 2\pi Q) \quad (3.122)$$

3.2.2.1.3: Derivation of the Trigonometric Shape Function for CCSS

Rectangular Plate.

Considering CCSS rectangular plate. The boundary conditions are shown in Figure 3.3

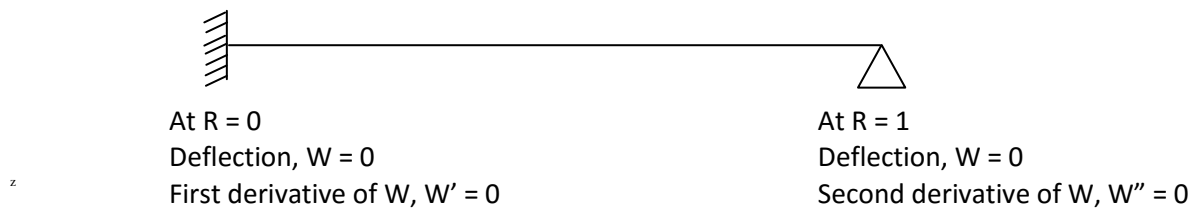


Figure 3.3: CCSS Rectangular Plate's Boundary Conditions.

The general trigonometric shape function for a rectangular plate given in Equation (3.87) as;

$$w = a_0 + a_1 R + a_2 \cos kR + a_3 \sin kR \quad (3.123)$$

Differentiating equation (3.123) once with respect to R gives Equation (3.124)

$$w' = a_1 - a_2 k \sin kR + a_3 k \cos kR \quad (3.124)$$

Differentiating Equation (3.124) once with respect to R gives Equation (3.125)

$$w'' = -a_2 k^2 \cos kR - a_3 k^2 \sin kR \quad (3.125)$$

Substituting the boundary conditions into Equations (3.123) and (3.124) at $R = 0$ and into Equations (3.123) and (3.125) at $R = 1$ respectively in matrix form gives Equation (3.126)

$$\begin{bmatrix} W_{(0)} \\ W'_{(0)} \\ W_{(1)} \\ W''_{(1)} \end{bmatrix} = 0 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & k \\ 1 & 1 & \text{Cos}k & \text{Sink} \\ 0 & 0 & -k^2\text{Cos}k & -k^2\text{Sink} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = 0 \quad (3.126)$$

The determinant of the matrix in Equation (3.126) gives Equation (3.127)

$$1 \begin{vmatrix} 1 & 0 & k \\ 1 & \text{Cos}k & \text{Sink} \\ 1 & -k\text{Sink} & k\text{Cos}k \end{vmatrix} + 1 \begin{vmatrix} 0 & 1 & k \\ 1 & 1 & \text{Sink} \\ 0 & 0 & -k^2\text{Sink} \end{vmatrix} = 0$$

That is,

$$1 \begin{vmatrix} \text{Cos}k & \text{Sink} \\ -k^2\text{Cos}k & -k^2\text{Sink} \end{vmatrix} - 1 \begin{vmatrix} 0 & k \\ -k^2\text{Cos}k & -k^2\text{Sink} \end{vmatrix} - 1 \begin{vmatrix} 1 & k \\ 0 & -k^2\text{Sink} \end{vmatrix} = 0$$

$$1(-k^2 \cos k \sin k + k^2 \cos k \sin k) - 1(0 + k^3 \cos k) - 1(-k^2 \sin k - 0) = 0 \quad (3.127)$$

Re - arranging equation (3.127) gives Equation (3.128)

$$-k^2 \cos k \sin k + k^2 \cos k \sin k - k^3 \cos k + k^2 \sin k = 0 \quad (3.128)$$

Re - arranging Equation (3.128) gives Equation (3.129)

$$-k^3 \cos k + k^2 \sin k = 0 \quad (3.129)$$

Simplifying Equation (3.129) gives Equation (3.130)

$$k \cos k = \sin k \quad (3.130)$$

Therefore, Equation (3.130) can be written as Equation (3.131)

$$k = \frac{\sin k}{\cos k} = \tan k \quad (3.131)$$

$$\text{Take } k = 4.49340946 \quad (3.132)$$

Substituting the boundary conditions into Equations (3.123) and (3.124) at $R=0$ and into Equation (3.125) at $R=1$ respectively give Equations (3.133) to (3.137)

$$w_{(0)} = 0 = a_0 + a_2 = 0; \Rightarrow a_0 = -a_2 \quad (3.133)$$

$$w'_{(0)} = 0 = a_1 + ka_3 = 0 \Rightarrow a_1 = -ka_3 \quad (3.134)$$

$$w_{(1)} = 0 = a_0 + a_1 + a_2 \cos k + a_3 \sin k = 0 \quad (3.135)$$

That is,

$$w_{(1)} = 0 = -a_2 - ka_3 + a_2 \cos k + a_3 \sin k = 0 \quad (3.136)$$

$$w''_{(1)} = 0 = -a_2 k^2 \cos k - a_3 k^2 \sin k = 0 \quad (3.137)$$

Re - arranging Equation (3.137) gives Equation (3.138)

$$a_2 \cos k = -a_3 \sin k \quad (3.138)$$

Substituting Equation (3.138) into Equation (3.136) gives Equation (3.139)

$$-a_2 - ka_3 - a_3 \sin k + a_3 \sin k = 0 \quad (3.139)$$

Re - arranging Equation (3.139) gives Equation (3.140)

$$a_2 = -ka_3 \quad (3.140)$$

Substituting Equation (3.133) into Equation (3.140) gives Equation (3.141)

$$a_0 = ka_3 \quad (3.141)$$

Substituting Equations (3.134),(3.140) and (3.141) into Equation (3.123) gives Equation (3.142)

$$w = ka_3 - ka_3 R - ka_3 \cos kR + a_3 \sin kR$$

That is,

$$w = a_3 [k - kR - k \cos kR + \sin kR] \quad (3.142)$$

Substituting Equations (3.134) and (3.140) into Equation (3.124) gives Equation (3.143)

$$w' = -ka_3 + k^2 a_3 \sin kR + a_3 k \cos k$$

That is,

$$w' = a_3 [-k + k^2 \sin kR + k \cos k] \quad (3.143)$$

Substituting Equation (3.140) into Equation (3.125) gives Equation (3.144)

$$w'' = a_3 k^3 \cos kR - a_3 k^2 \sin kR$$

That is,

$$w'' = a_3 [k^3 \cos kR - k^2 \sin kR] \quad (3.144)$$

Substituting Equations (3.130) and (3.132) into Equations (3.142) and (3.143) at $R=0$ and into Equation (3.142) and (3.144) at $R=1$ respectively give Equations (3.145) to (3.148).

$$w_{(0)} = a_3 [k - k] = 0 \quad (3.145)$$

$$w'_{(0)} = a_3 [-k + k] = 0 \quad (3.146)$$

$$w_{(1)} = a_3 [k - k - 4.479598301 + 4.479598301] = 0 \quad (3.147)$$

$$w''_{(1)} = a_3 [90.44635342 - 90.44635342] = 0 \quad (3.148)$$

Substituting $a_3 = A$ into Equation (3.142) gives Equation (3.149)

$$w_x = A_x [k - kR - k \cos kR + \sin kR] \quad (3.149)$$

This Equation (3.149) is the trigonometric shape function for a rectangular CCSS plate in x - direction. Similarly, that of y - direction is given in Equation (3.150)

$$w_y = A_y [k - kQ - k \cos kQ + \sin kQ] \quad (3.150)$$

3.2.2.2: Expressions of the General Polynomial Shape Function for a Rectangular Plate.

Ibearugbulem (2012) obtained the general polynomial shape function for a rectangular plate which are expressed in Equations (3.151), (3.156) and (3.157)

$$w = \sum_{m=0}^4 \sum_{n=0}^n J_m K_n R^m \cdot Q^n \quad (3.151)$$

$$\text{Where } J_m = \frac{a^m \cdot F^m(0)}{m!} = \text{constant} \quad (3.152)$$

$$K_n = \frac{B^n \cdot F^n(0)}{n!} = \text{constant} \quad (3.153)$$

a and b are plate dimensions in x - direction and y - direction respectively.

Non - dimensional parameter in x - direction and y - direction defined in Equation (3.154) and (3.155) respectively.

$$R = \frac{x}{a}, \text{ that is } x = aR \quad (3.154)$$

$$Q = \frac{y}{b}, \text{ that is } y = bQ \quad (3.155)$$

Equation (3.151) can be written in expanded form as given in Equation (3.156)

$$w_x = (a_0 + a_1 R + a_2 R^2 + a_3 R^3 + a_4 R^4) \quad (3.156)$$

This Equation (3.156) is the general polynomial shape function for a rectangular plate in x - direction. Similarly, that of y - direction is given in Equation (3.157)

$$w_y = (b_0 + b_1 Q + b_2 Q^2 + b_3 Q^3 + b_4 Q^4) \quad (3.157)$$

3.2.2.2.1: Derivation of the Polynomial Shape Function for SSSS

Rectangular Plate.

Considering the SSSS rectangular plate. The boundary conditions are shown in

Figure 3.4



Figure 3.4: SSSS Rectangular Plate's Boundary Conditions.

Recall that the general polynomial shape function for a rectangular plate given

in Equation (3.156) as:

$$w = (a_0 + a_1R + a_2R^2 + a_3R^3 + a_4R^4) \quad (3.158)$$

Differentiating Equation (3.158) twice with respect to R gives Equation (3.159)

$$w'' = 2a_2 + 6a_3R + 12a_4R^2 \quad (3.159)$$

Substituting the boundary conditions into Equations (3.158) and (3.159) each at

$R = 0$ and $R = 1$ respectively gives Equations (3.160) to (3.163)

$$w_{(0)} = 0 \Rightarrow a_0 = 0 \quad (3.160)$$

$$w''_{(0)} = 0 \Rightarrow a_2 = 0 \quad (3.161)$$

$$w_{(1)} = 0 \Rightarrow a_1 + a_3 + a_4 = 0 \quad (3.162)$$

$$w''_{(1)} = 0 \Rightarrow 6a_3 + 12a_4 = 0 \quad (3.163)$$

Re - arranging Equation (3.163) gives Equation (3.164)

$$a_3 = -2a_4 \quad (3.164)$$

Substituting Equation (3.164) into Equation (3.162) gives Equation (3.165)

$$a_1 - 2a_4 + a_4 = 0 \quad (3.165)$$

Simplifying Equation (3.165) gives Equation (3.166)

$$a_1 = a_4 \quad (3.166)$$

Substituting Equation (3.160), (3.161), (3.164) and (3.166) into Equation (3.158) gives Equation (3.167)

$$w = a_4 R - 2a_4 R^3 + a_4 R^4 \quad (3.167)$$

Re - arranging Equation (3.167) gives Equation (3.168)

$$w_x = a_4 [R - 2R^3 + R^4] \quad (3.168)$$

This Equation (3.168) is the polynomial shape function for SSSS rectangular plate in x - direction. Similarly, that of y - direction is given in Equation (3.169)

$$w_y = b_4 [Q - 2Q^3 + Q^4] \quad (3.169)$$

3.2.2.2.2: Derivation of the Polynomial Shape Functions for CCCC Rectangular Plate.

Considering the CCCC rectangular plate. The boundary conditions are shown in Figure 3.5

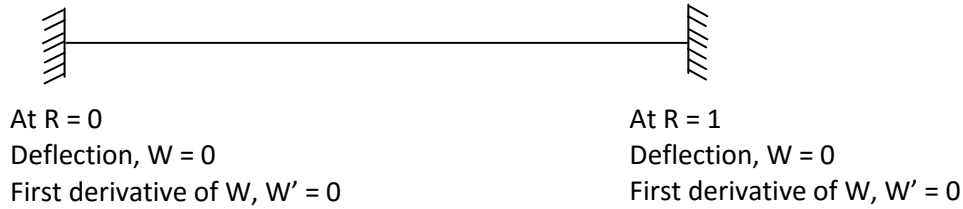


Figure 3.5: CCCC Rectangular Plate's Boundary Conditions.

Recall that the general polynomial shape function for a rectangular plate given in Equation (3.156) as:

$$w = (a_0 + a_1R + a_2R^2 + a_3R^3 + a_4R^4) \quad (3.170)$$

Differentiating Equation (3.170) once with respect to R gives Equation (3.171)

$$w' = a_1 + 2a_2R + 3a_3R^2 + 4a_4R^3 \quad (3.171)$$

Substituting the boundary conditions into Equations (3.170) and (3.171) each at $R = 0$ and $R = 1$ respectively gives Equations (3.172) to (3.175)

$$w_{(0)} = 0 \Rightarrow a_0 = 0 \quad (3.172)$$

$$w'_{(0)} = 0 \Rightarrow a_1 = 0 \quad (3.173)$$

$$w_{(1)} = 0 \Rightarrow a_2 + a_3 + a_4 = 0 \quad (3.174)$$

$$w'_{(1)} = 0 \Rightarrow 2a_2 + 3a_3 + 4a_4 = 0 \quad (3.175)$$

Solving Equations (3.174) and (3.175) simultaneously gives Equation (3.176)

$$a_3 = -2a_4 \quad (3.176)$$

Substituting Equation (3.176) into Equation (3.174) gives Equation (3.177)

$$a_2 - 2a_4 + a_4 = 0 \quad (3.177)$$

Simplifying Equation (3.177) gives Equation (3.178)

$$a_2 = a_4 \quad (3.178)$$

Substituting Equation (3.172), (3.173), (3.176) and (3.178) into Equation (3.170) gives Equation (3.179)

$$w = a_4 R^2 - 2a_4 R^3 + a_4 R^4 \quad (3.179)$$

Re - arranging Equation (3.179) gives Equation (3.180)

$$w_x = a_4 [R^2 - 2R^3 + R^4] \quad (3.180)$$

This Equation (3.180) is the polynomial shape function for CCCC rectangular plate in x - direction. Similarly, that of y - direction is given in Equation (3.181)

$$w_y = b_4 [Q^2 - 2Q^3 + Q^4] \quad (3.181)$$

3.2.2.2.3: Derivation of the Polynomial Shape Function for CCSS Rectangular Plate.

Considering the CCSS rectangular plate. The boundary conditions are shown in

Figure 3.6

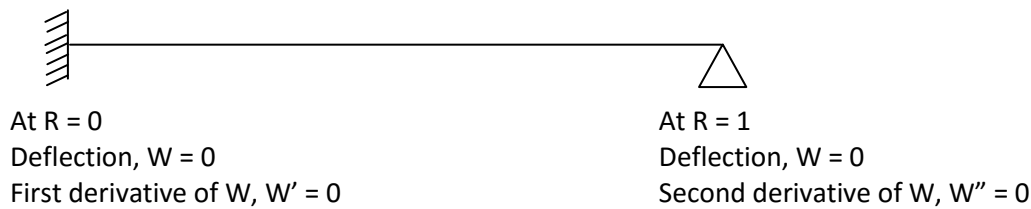


Figure 3.6: CCSS Rectangular Plate's Boundary Conditions

Recall that the general polynomial shape function for a rectangular plate given in Equation (3.156) as:

$$w = (a_0 + a_1R + a_2R^2 + a_3R^3 + a_4R^4) \quad (3.182)$$

Differentiating Equation (3.182) once with respect to R gives Equation (3.183)

$$w' = a_1 + 2a_2R + 3a_3R^2 + 4a_4R^3 \quad (3.183)$$

Differentiating Equation (3.183) once with respect to R gives Equation (3.184)

$$w'' = 2a_2 + 6a_3R + 12a_4R^2 \quad (3.184)$$

Substituting the boundary conditions into Equations (3.182) and (3.183) at $R = 0$ and into Equations (3.182) and (3.184) at $R = 1$ give Equations (3.185) and (3.186) and Equations (3.187) and (3.188) respectively.

$$w_{(0)} = 0 \Rightarrow a_0 = 0 \quad (3.185)$$

$$w'_{(0)} = 0 \Rightarrow a_1 = 0 \quad (3.186)$$

$$w_{(1)} = 0 \Rightarrow a_2 + a_3 + a_4 = 0 \quad (3.187)$$

$$w''_{(1)} = 0 \Rightarrow 2a_2 + 6a_3 + 12a_4 = 0 \quad (3.188)$$

Solving Equations (3.187) and (3.188) simultaneously gives Equation (3.189)

$$a_3 = -2.5a_4 \quad (3.189)$$

Substituting Equation (3.189) into Equation (3.187) gives Equation (3.190)

$$a_2 - 2.5a_4 + a_4 = 0 \quad (3.190)$$

Simplifying Equation (3.190) gives Equation (3.191)

$$a_2 = 1.5a_4 \quad (3.191)$$

Substituting Equation (3.185), (3.186), (3.189) and (3.191) into Equation (3.182) gives Equation (3.192)

$$w = 1.5a_4R^2 - 2.5a_4R^3 + a_4R^4 \quad (3.192)$$

Re - arranging Equation (3.192) gives Equation (3.193)

$$w_x = a_4[1.5R^2 - 2.5R^3 + R^4] \quad (3.193)$$

This Equation (3.193) is the polynomial shape function for CCSS rectangular plate in x – direction. Similarly, that of y – direction is given in Equation (3.194)

$$w_y = b_4[1.5Q^2 - 2.5Q^3 + Q^4] \quad (3.194)$$

3.2.3 Determination of the non – dimensional critical buckling loads (\bar{N}_x) of Rectangular Plates

The critical buckling load for a rectangular plate under vibration given in Equation (3.47) as:

$$N_x = \frac{\frac{D}{a^2} \left[K_x + \frac{2}{P^2} K_{xy} + \frac{K_y}{P^4} \right]}{K_{N_x}} (1 - n^2) = \frac{D}{a^2} \cdot \bar{N}_x \quad (3.195)$$

Hence, the non – dimensional critical buckling load, \bar{N}_x for a rectangular plate under vibration is given in Equation (3.196) as:

$$\bar{N}_x = \frac{\left[K_x + \frac{2}{P^2} K_{xy} + \frac{K_y}{P^4} \right]}{K_{N_x}} (1 - n^2) \quad (3.196)$$

The deflection equation for a rectangular plate given in Equation (3.5) as:

$$w = Ah_x h_y \quad (3.197)$$

Where A is the coefficient of deflection and h_x and h_y are the shape functions of the deflected plate in x and y directions respectively.

3.2.3.1: Determination of the non – dimensional critical buckling loads (\bar{N}_x) Of SSSS Rectangular Plate.

(i) The complete trigonometric shape functions for SSSS rectangular plates given in Equations (3.103) and (3.104) are expressed in Equations (3.198) and (3.199) respectively

$$h_x = \sin \pi R \quad (3.198)$$

$$h_y = \sin \pi Q \quad (3.199)$$

Squaring the shape functions, h_x and h_y in Equations (3.198) and (3.199) gives Equations (3.200) and (3.201) respectively.

$$h_x^2 = \sin^2 \pi R \quad (3.200)$$

$$h_y^2 = \sin^2 \pi Q \quad (3.201)$$

the shape function derivatives are given in Equations (3.202) to (3.205)

$$\left(\frac{dh_x}{dR}\right)^2 = \pi^2 \cos^2 \pi R \quad (3.202)$$

$$\left(\frac{dh_y}{dQ}\right)^2 = \pi^2 \cos^2 \pi Q \quad (3.203)$$

$$\left(\frac{d^2h_x}{dR^2}\right)^2 = \pi^4 \sin^2 \pi R \quad (3.204)$$

$$\left(\frac{d^2h_y}{dQ^2}\right)^2 = \pi^4 \sin^2 \pi Q \quad (3.205)$$

Integrating Equations (3.200) to (3.205) give Equations (3.206) to (3.211) respectively.

$$\int_0^1 h_x^2 dR = 0.5 \quad (3.206)$$

$$\int_0^1 h_y^2 dQ = 0.5 \quad (3.207)$$

$$\int_0^1 \left(\frac{dh_x}{dR}\right)^2 dR = 0.5\pi^2 \quad (3.208)$$

$$\int_0^1 \left(\frac{dh_y}{dQ}\right)^2 dQ = 0.5\pi^2 \quad (3.209)$$

$$\int_0^1 \left(\frac{d^2h_x}{dR}\right)^2 dR = 0.5\pi^4 \quad (3.210)$$

$$\int_0^1 \left(\frac{d^2h_y}{dQ}\right)^2 dQ = 0.5\pi^4 \quad (3.211)$$

Substituting the integrands obtained in Equations (3.206) to (3.211) into Equations (3.37) to (3.40) accordingly give Equations (3.212) to (3.215)

$$K_x = 0.5\pi^4 \times 0.5 = 24.35227276 \quad (3.212)$$

$$K_{xy} = 0.5\pi^2 \times 0.5\pi^2 = 24.35227276 \quad (3.213)$$

$$K_y = 0.5 \times 0.5\pi^4 = 24.35227276 \quad (3.214)$$

$$K_{Nx} = 0.5\pi^2 \times 0.5 = 2.4674011 \quad (3.215)$$

Substituting Equations (3.212) to (3.215) into Equation (3.196) gives the non – dimensional critical buckling load for SSSS rectangular plate under vibration using complete trigonometric shape functions expressed in Equation (3.216)

$$\bar{N}_x = \frac{\left[24.35227276 + \frac{2}{p^2}(24.35227276) + \frac{24.35227276}{p^4}\right]}{2.4674011} (1 - n^2) \quad (3.216)$$

(ii) The complete polynomial shape functions for SSSS rectangular plates given in Equations (3.168) and (3.169) are expressed in Equations (3.217) and (3.218) respectively.

$$h_x = (R - 2R^3 + R^4) \quad (3.217)$$

$$h_y = (Q - 2Q^3 + Q^4) \quad (3.218)$$

Squaring the shape functions, h_x and h_y in Equations (3.217) and (3.218) gives Equations (3.219) and (3.220) respectively.

$$h_x^2 = (R^2 - 4R^4 + 2R^5 + 4R^6 - 4R^7 + R^8) \quad (3.219)$$

$$h_y^2 = (Q^2 - 4Q^4 + 2Q^5 + 4Q^6 - 4Q^7 + Q^8) \quad (3.220)$$

the shape function derivatives are given in Equations (3.221) to (3.224)

$$\left(\frac{dh_x}{dR}\right)^2 = (1 - 12R^2 + 8R^3 + 36R^4 - 48R^5 + 16R^6) \quad (3.221)$$

$$\left(\frac{dh_y}{dQ}\right)^2 = (1 - 12Q^2 + 8Q^3 + 36Q^4 - 48Q^5 + 16Q^6) \quad (3.222)$$

$$\left(\frac{d^2h_x}{dR^2}\right)^2 = 144 (R^4 - 2R^3 + R^2) \quad (3.223)$$

$$\left(\frac{d^2h_y}{dQ^2}\right)^2 = 144 (Q^4 - 2Q^3 + Q^2) \quad (3.224)$$

Integrating Equations (3.219) to (3.224) give Equations (3.225) to (3.230) respectively.

$$\int_0^1 h_x^2 dR = 0.049206349 \quad (3.225)$$

$$\int_0^1 h_y^2 dQ = 0.049206349 \quad (3.226)$$

$$\int_0^1 \left(\frac{dh_x}{dR}\right)^2 dR = 0.485714285 \quad (3.227)$$

$$\int_0^1 \left(\frac{dh_y}{dQ}\right)^2 dQ = 0.485714285 \quad (3.228)$$

$$\int_0^1 \left(\frac{d^2h_x}{dR}\right)^2 dR = 4.8 \quad (3.229)$$

$$\int_0^1 \left(\frac{d^2h_y}{dQ}\right)^2 dQ = 4.8 \quad (3.230)$$

Substituting the integrands obtained in Equations (3.225) to (3.230) into Equations (3.37) to (3.40) accordingly give Equations (3.231) to (3.234)

$$K_x = 4.8 \times 0.049206349 = 0.236190475 \quad (3.231)$$

$$K_{xy} = 0.485714285 \times 0.485714285 = 0.235918366 \quad (3.232)$$

$$K_y = 0.049206349 \times 4.8 = 0.236190475 \quad (3.233)$$

$$K_{Nx} = 0.485714285 \times 0.049206349 = 0.023900226 \quad (3.234)$$

Substituting Equations (3.231) to (3.234) into Equation (3.196) gives the non – dimensional critical buckling load for SSSS rectangular plate under vibration using complete polynomial shape functions expressed in Equation (3.235)

$$\bar{N}_x = \frac{\left[0.236190475 + \frac{2}{P^2}(0.235918366) + \frac{0.236190475}{P^4}\right]}{0.023900226} (1 - n^2) \quad (3.235)$$

(iii) The trigonometric - polynomial shape functions for SSSS rectangular plates given in Equations (3.103) and (3.169) are expressed in Equations (3.236) and (3.237) respectively.

$$h_x = \sin \pi R \quad (3.236)$$

$$h_y = (Q - 2Q^3 + Q^4) \quad (3.237)$$

Squaring the shape functions, h_x and h_y in Equations (3.236) and (3.237) gives Equations (3.238) and (3.239) respectively.

$$h_x^2 = \sin^2 \pi R \quad (3.238)$$

$$h_y^2 = (Q^2 - 4Q^4 + 2Q^5 + 4Q^6 - 4Q^7 + Q^8) \quad (3.239)$$

the shape function derivatives are given in Equations (3.240) to (3.243)

$$\left(\frac{dh_x}{dR}\right)^2 = \pi^2 \cos^2 \pi R \quad (3.240)$$

$$\left(\frac{dh_y}{dQ}\right)^2 = (1 - 12Q^2 + 8Q^3 + 36Q^4 - 48Q^5 + 16Q^6) \quad (3.241)$$

$$\left(\frac{d^2h_x}{dR^2}\right)^2 = \pi^4 \sin^2 \pi R \quad (3.242)$$

$$\left(\frac{d^2h_y}{dQ^2}\right)^2 = 144 (Q^4 - 2Q^3 + Q^2) \quad (3.243)$$

Integrating Equations (3.238) to (3.243) give Equations (3.244) to (3.249) respectively.

$$\int_0^1 h_x^2 dR = 0.5 \quad (3.244)$$

$$\int_0^1 h_y^2 dQ = 0.049206349 \quad (3.245)$$

$$\int_0^1 \left(\frac{dh_x}{dR}\right)^2 dR = 0.5\pi^2 \quad (3.246)$$

$$\int_0^1 \left(\frac{dh_y}{dQ}\right)^2 dQ = 0.485714285 \quad (3.247)$$

$$\int_0^1 \left(\frac{d^2h_x}{dR}\right)^2 dR = 0.5\pi^4 \quad (3.248)$$

$$\int_0^1 \left(\frac{d^2h_y}{dQ}\right)^2 dQ = 4.8 \quad (3.249)$$

Substituting the integrands obtained in Equations (3.244) to (3.249) into Equations (3.37) to (3.40) accordingly give Equations (3.250) to (3.253)

$$K_x = 0.5\pi^4 \times 0.049206349 = 2.396572865 \quad (3.250)$$

$$K_{xy} = 0.5\pi^2 \times 0.485714285 = 2.396903922 \quad (3.251)$$

$$K_y = 0.5 \times 4.8 = 2.4 \quad (3.252)$$

$$K_{Nx} = 0.5\pi^2 \times 0.049206349 = 0.242823599 \quad (3.253)$$

Substituting Equations (3.250) to (3.253) into Equation (3.196) gives the non – dimensional critical buckling load for SSSS rectangular plate under vibration using trigonometric - polynomial shape functions expressed in Equation (3.254)

$$N_x = \frac{\left[2.396572865 + \frac{2}{P^2}(2.396903922) + \frac{2.4}{P^4}\right]}{0.242823599} (1 - n^2) \quad (3.254)$$

**3.2.3.2: Determination of the non – dimensional critical buckling loads (\bar{N}_x)
Of CCCC Rectangular Plate.**

(i) The complete trigonometric shape functions for CCCC rectangular plates given in Equations (3.121) and (3.122) are expressed in Equations (3.255) and (3.256) respectively.

$$h_x = (1 - \cos 2\pi R) \quad (3.255)$$

$$h_y = (1 - \cos 2\pi Q) \quad (3.256)$$

Squaring the shape functions, h_x and h_y in Equations (3.255) and (3.256) gives Equations (3.257) and (3.258) respectively.

$$h_x^2 = (1 - 2\cos 2\pi R + \cos^2 2\pi R) \quad (3.257)$$

$$h_y^2 = (1 - 2\cos 2\pi Q + \cos^2 2\pi Q) \quad (3.258)$$

the shape function derivatives are given in Equations (3.259) to (3.262)

$$\left(\frac{dh_x}{dR}\right)^2 = 4\pi^2 \sin^2 2\pi R \quad (3.259)$$

$$\left(\frac{dh_y}{dQ}\right)^2 = 4\pi^2 \sin^2 2\pi Q \quad (3.260)$$

$$\left(\frac{d^2h_x}{dR^2}\right)^2 = 16\pi^4 \cos^2 2\pi R \quad (3.261)$$

$$\left(\frac{d^2h_y}{dQ^2}\right)^2 = 16\pi^4 \cos^2 2\pi Q \quad (3.262)$$

Integrating Equations (3.257) to (3.262) give Equations (3.263) to (3.268) respectively.

$$\int_0^1 h_x^2 dR = 1.5 \quad (3.263)$$

$$\int_0^1 h_y^2 dQ = 1.5 \quad (3.264)$$

$$\int_0^1 \left(\frac{dh_x}{dR}\right)^2 dR = 2\pi^2 \quad (3.265)$$

$$\int_0^1 \left(\frac{dh_y}{dQ}\right)^2 dQ = 2\pi^2 \quad (3.266)$$

$$\int_0^1 \left(\frac{d^2h_x}{dR^2}\right)^2 dR = 8\pi^4 \quad (3.267)$$

$$\int_0^1 \left(\frac{d^2h_y}{dQ^2}\right)^2 dQ = 8\pi^4 \quad (3.268)$$

Substituting the integrands obtained in Equations (3.263) to (3.268) into Equations (3.37) to (3.40) accordingly give Equations (3.269) to (3.272)

$$K_x = 8\pi^4 \times 1.5 = 1168.909092 \quad (3.269)$$

$$K_{xy} = 2\pi^2 \times 2\pi^2 = 389.6363641 \quad (3.270)$$

$$K_y = 1.5 \times 8\pi^4 = 1168.909092 \quad (3.271)$$

$$K_{Nx} = 2\pi^2 \times 1.5 = 29.6088132 \quad (3.272)$$

Substituting Equations (3.269) to (3.272) into Equation (3.196) gives the non-dimensional critical buckling load for CCCC rectangular plate under vibration using complete trigonometric shape functions expressed in Equation (3.273)

$$\bar{N}_x = \frac{\left[1168.909092 + \frac{2}{P^2}(389.6363641) + \frac{1168.909092}{P^4}\right]}{29.6088132} (1 - n^2) \quad (3.273)$$

(ii) The complete polynomial shape functions for CCCC rectangular plates given in Equations (3.180) and (3.181) are expressed in Equations (3.274) and (3.275) respectively.

$$h_x = (R^2 - 2R^3 + R^4) \quad (3.274)$$

$$h_y = (Q^2 - 2RQ^3 + Q^4) \quad (3.275)$$

Squaring the shape functions, h_x and h_y in Equations (3.274) and (3.275) gives Equations (3.276) and (3.277) respectively.

$$h_x^2 = (R^4 - 4R^5 + 6R^6 + 4R^7 + R^8) \quad (3.276)$$

$$h_y^2 = (Q^4 - 4Q^5 + 6Q^6 + 4Q^7 + Q^8) \quad (3.277)$$

the shape function derivatives are given in Equations (3.278) to (3.281)

$$\left(\frac{dh_x}{dR}\right)^2 = (4R^2 - 24R^3 + 52R^4 - 48R^5 + 16R^6) \quad (3.278)$$

$$\left(\frac{dh_y}{dQ}\right)^2 = (4Q^2 - 24Q^3 + 52Q^4 - 48Q^5 + 16Q^6) \quad (3.279)$$

$$\left(\frac{d^2h_x}{dR^2}\right)^2 = (4 - 48R + 192R^2 - 288R^3 + 144R^4) \quad (3.280)$$

$$\left(\frac{d^2h_y}{dQ^2}\right)^2 = (4 - 48Q + 192Q^2 - 288Q^3 + 144Q^4) \quad (3.281)$$

Integrating Equations (3.276) to (3.281) give Equations (3.282) to (3.287) respectively.

$$\int_0^1 h_x^2 dR = 0.001587301587 \quad (3.282)$$

$$\int_0^1 h_y^2 dQ = 0.001587301587 \quad (3.283)$$

$$\int_0^1 \left(\frac{dh_x}{dR}\right)^2 dR = 0.019047619 \quad (3.284)$$

$$\int_0^1 \left(\frac{dh_y}{dQ}\right)^2 dQ = 0.019047619 \quad (3.285)$$

$$\int_0^1 \left(\frac{d^2h_x}{dR^2}\right)^2 dR = 0.8 \quad (3.286)$$

$$\int_0^1 \left(\frac{d^2h_y}{dQ^2}\right)^2 dQ = 0.8 \quad (3.287)$$

Substituting the integrands obtained in Equations (3.282) to (3.287) into Equations (3.37) to (3.40) accordingly give Equations (3.288) to (3.291)

$$K_x = 0.8 \times 0.001587301587 = 0.00126984127 \quad (3.288)$$

$$K_{xy} = 0.019047619 \times 0.019047619 = 0.00036281117896 \quad (3.289)$$

$$K_y = 0.001587301587 \times 0.8 = 0.00126984127 \quad (3.290)$$

$$K_{N_x} = 0.019047619 \times 0.001587301587 = 0.00003023431587 \quad (3.291)$$

Substituting Equations (3.288) to (3.291) into Equation (3.196) gives the non – dimensional critical buckling load for CCCC rectangular plate under vibration using complete polynomial shape functions expressed in Equation (3.292)

$$\bar{N}_x = \frac{\left[0.00126984127 + \frac{2}{P^2}(0.0003628117896) + \frac{0.00126984127}{P^4}\right]}{0.00003023431587} (1 - n^2) \quad (3.292)$$

(iii) The trigonometric - polynomial shape functions for CCCC rectangular plates given in Equations (3.121) and (3.181) are expressed in Equations (3.293) and (3.295) respectively.

$$h_x = (1 - \cos 2\pi R) \quad (3.293)$$

$$h_y = (Q^2 - 2Q^3 + Q^4) \quad (3.294)$$

Squaring the shape functions, h_x and h_y in equations (3.293) and (3.294) gives Equations (3.295) and (3.296) respectively.

$$h_x^2 = (1 - 2\cos 2\pi R + \cos^2 2\pi R) \quad (3.295)$$

$$h_y^2 = (Q^4 - 4Q^5 + 6Q^6 + 4Q^7 + Q^8) \quad (3.296)$$

the shape function derivatives are given in Equations (3.297) to (3.300)

$$\left(\frac{dh_x}{dR}\right)^2 = 4\pi^2 \sin^2 2\pi R \quad (3.297)$$

$$\left(\frac{dh_y}{dQ}\right)^2 = (4Q^2 - 24Q^3 + 52Q^4 - 48Q^5 + 16Q^6) \quad (3.298)$$

$$\left(\frac{d^2h_x}{dR^2}\right)^2 = 16\pi^4 \cos^2 2\pi R \quad (3.299)$$

$$\left(\frac{d^2h_y}{dQ^2}\right)^2 = (4 - 48Q + 192Q^2 - 288Q^3 + 144Q^4) \quad (3.300)$$

Integrating Equations (3.295) to (3.300) give Equations (3.301) to (3.306) respectively.

$$\int_0^1 h_x^2 dR = 1.5 \quad (3.301)$$

$$\int_0^1 h_y^2 dQ = 0.001587301587 \quad (3.302)$$

$$\int_0^1 \left(\frac{dh_x}{dR}\right)^2 dR = 2\pi^2 \quad (3.303)$$

$$\int_0^1 \left(\frac{dh_y}{dQ}\right)^2 dQ = 0.019047619 \quad (3.304)$$

$$\int_0^1 \left(\frac{d^2h_x}{dR^2}\right)^2 dR = 8\pi^4 \quad (3.305)$$

$$\int_0^1 \left(\frac{d^2h_y}{dQ^2}\right)^2 dQ = 0.8 \quad (3.306)$$

Substituting the integrands obtained in Equations (3.301) to (3.306) into Equations (3.37) to (3.40) accordingly give Equations (3.307) to (3.310)

$$K_x = 8\pi^4 \times 0.001587301587 = 1.236940838 \quad (3.307)$$

$$K_{xy} = 2\pi^2 \times 0.019047619 = 0.375984928 \quad (3.308)$$

$$K_y = 1.5 \times 0.8 = 1.2 \quad (3.309)$$

$$K_{Nx} = 2\pi^2 \times 0.001587301587 = 0.031332077 \quad (3.310)$$

Substituting Equations (3.307) to (3.310) into Equation (3.196) gives the non – dimensional critical buckling load for CCCC rectangular plate under vibration using trigonometric–polynomial shape functions expressed in Equation (3.311)

$$\bar{N}_x = \frac{\left[1.236940838 + \frac{2}{P^2}(0.375984928) + \frac{1.2}{P^4}\right]}{0.031332077} (1 - n^2) \quad (3.311)$$

3.2.3.3: Determination of the non – dimensional critical buckling load (\bar{N}_x)

Of CCSS Rectangular Plate.

(i) The complete trigonometric shape functions for CCSS rectangular plates given in Equations (3.149) and (3.150) are expressed in Equations (3.312) and (3.313) respectively.

$$h_x = (k - kR - k \cos kR + \sin kR) \quad (3.312)$$

$$h_y = (k - kQ - k \cos kQ + \sin kQ) \quad (3.313)$$

Squaring the shape functions, h_x and h_y in equations (3.312) and (3.313) gives Equations (3.314) and (3.315) respectively.

$$h_x^2 = (k^2 - 2k^2R - 2k^2 \cos kR + 2k \sin kR + k^2R^2 + 2k^2R \cos kR - 2kR \sin kR + k^2 \cos^2 kR - 2k \cos kR \sin kR + \sin^2 kR) \quad (3.314)$$

$$h_y^2 = (k^2 - 2k^2Q - 2k^2 \cos kQ + 2k \sin kQ + k^2Q^2 + 2k^2Q \cos kQ - 2kQ \sin kQ + k^2 \cos^2 kQ - 2k \cos kQ \sin kQ + \sin^2 kQ) \quad (3.315)$$

The shape function derivatives are given in Equations (3.316) to (3.319)

$$\left(\frac{dh_x}{dR}\right)^2 = (k^2 - 2k^3 \sin kR - 2k^2 \cos kR + k^4 \sin^2 kR + 2k^3 \sin kR \cos kR + k^2 \cos^2 kR) \quad (3.316)$$

$$\left(\frac{dh_y}{dQ}\right)^2 = (k^2 - 2k^3 \sin kQ - 2k^2 \cos kQ + k^4 \sin^2 kQ + 2k^3 \sin kQ \cos kQ + k^2 \cos^2 kQ) \quad (3.317)$$

$$\left(\frac{d^2h_x}{dR^2}\right)^2 = (k^6 \cos^2 kR - 2k^5 \sin kR \cos kR + k^4 \sin^2 kR) \quad (3.318)$$

$$\left(\frac{d^2h_y}{dQ^2}\right)^2 = (k^6 \cos^2 kQ - 2k^5 \sin kQ \cos kQ + k^4 \sin^2 kQ) \quad (3.319)$$

Integrating Equations (3.314) to (3.319) give Equations (3.320) to (3.325) respectively.

$$\int_0^1 h_x^2 dR = 19.42379403 \quad (3.320)$$

$$\int_0^1 h_y^2 dQ = 19.42379403 \quad (3.321)$$

$$\int_0^1 \left(\frac{dh_x}{dR}\right)^2 dR = 230.4800172 \quad (3.322)$$

$$\int_0^1 \left(\frac{dh_y}{dQ}\right)^2 dQ = 230.4800172 \quad (3.323)$$

$$\int_0^1 \left(\frac{d^2h_x}{dR^2}\right)^2 dR = 4376.113774 \quad (3.324)$$

$$\int_0^1 \left(\frac{d^2h_y}{dQ^2}\right)^2 dQ = 4376.113774 \quad (3.325)$$

Substituting the integrands obtained in Equations (3.320) to (3.325) into Equations (3.37) to (3.40) accordingly give Equations (3.326) to (3.329)

$$K_x = 4376.113774 \times 19.42379403 = 85000.7326 \quad (3.326)$$

$$K_{xy} = 230.4800172 \times 230.4800172 = 53121.03833 \quad (3.327)$$

$$K_y = 19.42379403 \times 4376.113774 = 85000.7326 \quad (3.328)$$

$$K_{Nx} = 230.4800172 \times 19.42379403 = 4476.796382 \quad (3.329)$$

Substituting Equations (3.326) to (3.329) into Equation (3.196) gives the non – dimensional critical buckling load for CCSS rectangular plate under vibration using complete trigonometric shape functions expressed in Equation (3.330)

$$\bar{N}_x = \frac{\left[85000.7326 + \frac{2}{p^2}(53121.03833) + \frac{85000.7326}{p^4}\right]}{4476.796382} (1 - n^2) \quad (3.330)$$

(ii) The complete polynomial shape functions for CCSS rectangular plates given in Equations (3.193) and (3.194) are expressed in Equations (3.331) and (3.332) respectively.

$$h_x = (1.5R^2 - 2.5R^3 + R^4) \quad (3.331)$$

$$h_y = (1.5Q^2 - 2.5Q^3 + Q^4) \quad (3.332)$$

Squaring the shape functions, h_x and h_y in Equations (3.331) and (3.332) gives Equations (3.333) and (3.334) respectively.

$$h_x^2 = (2.25R^4 - 7.5R^5 + 9.25R^6 - 5R^7 + R^8) \quad (3.333)$$

$$h_y^2 = (2.25Q^4 - 7.5Q^5 + 9.25Q^6 - 5Q^7 + Q^8) \quad (3.334)$$

The shape function derivatives are given in Equations (3.335) to (3.338)

$$\left(\frac{dh_x}{dR}\right)^2 = (9R^2 - 45R^3 + 80.25R^4 - 60R^5 + 16R^6) \quad (3.335)$$

$$\left(\frac{dh_y}{dQ}\right)^2 = (9Q^2 - 45Q^3 + 80.25Q^4 - 60Q^5 + 16Q^6) \quad (3.336)$$

$$\left(\frac{d^2h_x}{dR^2}\right)^2 = (9 - 90R + 297R^2 - 360R^3 + 144R^4) \quad (3.337)$$

$$\left(\frac{d^2h_y}{dQ^2}\right)^2 = (9 - 90Q + 297Q^2 - 360Q^3 + 144Q^4) \quad (3.338)$$

Integrating Equations (3.333) to (3.338) give Equations (3.339) to (3.344) respectively.

$$\int_0^1 h_x^2 dR = 0.00753968254 \quad (3.339)$$

$$\int_0^1 h_y^2 dQ = 0.00753968254 \quad (3.340)$$

$$\int_0^1 \left(\frac{dh_x}{dR}\right)^2 dR = 0.085714285 \quad (3.341)$$

$$\int_0^1 \left(\frac{dh_y}{dQ}\right)^2 dQ = 0.085714285 \quad (3.342)$$

$$\int_0^1 \left(\frac{d^2h_x}{dR^2}\right)^2 dR = 1.8 \quad (3.343)$$

$$\int_0^1 \left(\frac{d^2h_y}{dQ^2}\right)^2 dQ = 1.8 \quad (3.344)$$

Substituting the integrands obtained in Equations (3.339) to (3.344) into Equations (3.37) to (3.40) accordingly give Equations (3.345) to (3.348)

$$K_x = 1.8 \times 0.00753968254 = 0.013571428 \quad (3.345)$$

$$K_{xy} = 0.085714285 \times 0.085714285 = 0.007346938653 \quad (3.346)$$

$$K_y = 0.00753968254 \times 1.8 = 0.013571428 \quad (3.347)$$

$$K_{Nx} = 0.085714285 \times 0.00753968254 = 0.000646258498 \quad (3.348)$$

Substituting Equations (3.345) to (3.348) into Equation (3.196) gives the non-dimensional critical buckling load for CCSS rectangular plate under vibration using complete polynomial shape functions expressed in Equation (3.349)

$$\bar{N}_x = \frac{\left[0.013571428 + \frac{2}{P^2}(0.007346938653) + \frac{0.013571428}{P^4}\right]}{0.000646258498} (1 - n^2) \quad (3.349)$$

(iii) The trigonometric- polynomial shape functions for CCSS rectangular plates given in Equations (3.149) and (3.194) are expressed in Equations (3.350) and (3.351) respectively.

$$h_x = (k - kR - k \cos kR + \sin kR) \quad (3.350)$$

$$h_y = (1.5Q^2 - 2.5Q^3 + Q^4) \quad (3.351)$$

Squaring the shape functions, h_x and h_y in Equations (3.350) and (3.351) gives Equations (3.352) and (3.353) respectively.

$$h_x^2 = (k^2 - 2k^2R - 2k^2 \cos kR + 2k \sin kR + k^2R^2 + 2k^2R \cos kR - 2kR \sin kR + k^2 \cos^2 kR - 2k \cos kR \sin kR + \sin^2 kR) \quad (3.352)$$

$$h_y^2 = (2.25Q^4 - 7.5Q^5 + 9.25Q^6 - 5Q^7 + Q^8) \quad (3.353)$$

The shape function derivatives are given in Equations (3.354) to (3.357)

$$\left(\frac{dh_x}{dR}\right)^2 = (k^2 - 2k^3 \sin kR - 2k^2 \cos kR + k^4 \sin^2 kR + 2k^3 \sin kR \cos kR + k^2 \cos^2 kR) \quad (3.354)$$

$$\left(\frac{dh_y}{dQ}\right)^2 = (9Q^2 - 45Q^3 + 80.25Q^4 - 60Q^5 + 16Q^6) \quad (3.355)$$

$$\left(\frac{d^2h_x}{dR^2}\right)^2 = (k^6 \cos^2 kR - 2k^5 \sin kR \cos kR + k^4 \sin^2 kR) \quad (3.356)$$

$$\left(\frac{d^2h_y}{dQ^2}\right)^2 = (9 - 90Q + 297Q^2 - 360Q^3 + 144Q^4) \quad (3.357)$$

Integrating Equations (3.352) to (3.357) give Equations (3.358) to (3.363) respectively.

$$\int_0^1 h_x^2 dR = 19.42379403 \quad (3.358)$$

$$\int_0^1 h_y^2 dQ = 0.00753968254 \quad (3.359)$$

$$\int_0^1 \left(\frac{dh_x}{dR}\right)^2 dR = 230.4800172 \quad (3.360)$$

$$\int_0^1 \left(\frac{dh_y}{dQ}\right)^2 dQ = 0.085714285 \quad (3.361)$$

$$\int_0^1 \left(\frac{d^2h_x}{dR^2}\right)^2 dR = 4376.113774 \quad (3.362)$$

$$\int_0^1 \left(\frac{d^2h_y}{dQ^2}\right)^2 dQ = 1.8 \quad (3.363)$$

Substituting the integrands obtained in Equations (3.358) to (3.363) into Equations (3.37) to (3.40) accordingly give Equations (3.364) to (3.367)

$$K_x = 4376.113774 \times 0.00753968254 = 32.99450861 \quad (3.364)$$

$$K_{xy} = 230.4800172 \times 0.085714285 = 19.75542988 \quad (3.365)$$

$$K_y = 19.42379403 \times 1.8 = 34.96282925 \quad (3.366)$$

$$K_{Nx} = 230.4800172 \times 0.00753968254 = 1.737746162 \quad (3.367)$$

Substituting Equations (3.364) to (3.367) into Equation (3.196) gives the non – dimensional critical buckling load for CCSS rectangular plate under vibration using trigonometric - polynomial shape functions expressed in Equation (3.368)

$$\bar{N}_x = \frac{\left[32.9950861 + \frac{2}{P^2}(19.75542988) + \frac{34.96282925}{P^4} \right]}{1.737746162} (1 - n^2) \quad (3.368)$$

3.2.3.4: Determination of the non – dimensional critical buckling load (\bar{N}_x) Of CSCS Rectangular Plate.

(i) The complete trigonometric shape functions for CSCS rectangular plates given in Equations (3.103) and (3.122) are expressed in Equations (3.369) and (3.370) respectively.

$$h_x = \sin \pi R \quad (3.369)$$

$$h_y = (1 - \cos 2\pi Q) \quad (3.370)$$

Squaring the shape functions, h_x and h_y in Equations (3.369) and (3.370) gives Equations (3.371) and (3.372) respectively.

$$h_x^2 = \sin^2 \pi R \quad (3.371)$$

$$h_y^2 = (1 - 2\cos 2\pi Q + \cos^2 2\pi Q) \quad (3.372)$$

The shape function derivatives are given in Equations (3.373) to (3.376)

$$\left(\frac{dh_x}{dR}\right)^2 = \pi^2 \cos^2 \pi R \quad (3.373)$$

$$\left(\frac{dh_y}{dQ}\right)^2 = 4\pi^2 \sin^2 2\pi Q \quad (3.374)$$

$$\left(\frac{d^2h_x}{dR^2}\right)^2 = \pi^4 \sin^2 \pi R \quad (3.375)$$

$$\left(\frac{d^2h_y}{dQ^2}\right)^2 = 16\pi^4 \cos^2 2\pi Q \quad (3.376)$$

Integrating Equations (3.371) to (3.376) give Equations (3.377) to (3.382) respectively.

$$\int_0^1 h_x^2 dR = 0.5 \quad (3.377)$$

$$\int_0^1 h_y^2 dQ = 1.5 \quad (3.378)$$

$$\int_0^1 \left(\frac{dh_x}{dR}\right)^2 dR = 0.5\pi^2 \quad (3.379)$$

$$\int_0^1 \left(\frac{dh_y}{dQ}\right)^2 dQ = 2\pi^2 \quad (3.380)$$

$$\int_0^1 \left(\frac{d^2h_x}{dR^2}\right)^2 dR = 0.5\pi^4 \quad (3.381)$$

$$\int_0^1 \left(\frac{d^2 h_y}{dQ^2}\right)^2 dQ = 0.5\pi^4 \quad (3.382)$$

Substituting the integrands obtained in Equations (3.377) to (3.382) into Equations (3.37) to (3.40) accordingly give Equations (3.383) to (3.386)

$$K_x = 0.5\pi^4 \times 1.5 = 73.05681828 \quad (3.383)$$

$$K_{xy} = 0.5\pi^2 \times 2\pi^2 = 94.40909103 \quad (3.384)$$

$$K_y = 0.5 \times 8\pi^4 = 389.6363641 \quad (3.385)$$

$$K_{Nx} = 0.5\pi^2 \times 1.5 = 7.402203301 \quad (3.386)$$

Substituting Equations (3.383) to (3.386) into Equation (3.196) gives the non – dimensional critical buckling load for CSCS rectangular plate under vibration using complete trigonometric shape functions expressed in Equation (3.387)

$$N_x = \frac{\left[73.05681828 + \frac{2}{P^2}(94.40909103) + \frac{389.6363641}{P^4}\right]}{7.402203301} (1 - n^2) \quad (3.387)$$

(ii) The complete polynomial shape functions for CSCS rectangular plates given in Equations (3.168) and (3.181) are expressed in Equations (3.388) and (3.389) respectively.

$$h_x = (R - 2R^3 + R^4) \quad (3.388)$$

$$h_y = (Q^2 - 2Q^3 + Q^4) \quad (3.389)$$

Squaring the shape functions, h_x and h_y in Equations (3.388) and (3.389) gives Equations (3.390) and (3.391) respectively.

$$h_x^2 = (R^2 - 4R^4 + 2R^5 + 4R^6 - 4R^7 + R^8) \quad (3.390)$$

$$h_y^2 = (Q^2 - 4Q^4 + 2Q^5 + 4Q^6 - 4Q^7 + Q^8) \quad (3.391)$$

The shape function derivatives are given in Equations (3.392) to (3.395)

$$\left(\frac{dh_x}{dR}\right)^2 = (1 - 12R^2 + 8R^3 + 36R^4 - 48R^5 + 16R^6) \quad (3.392)$$

$$\left(\frac{dh_y}{dQ}\right)^2 = (4Q^2 - 24Q^3 + 52Q^4 - 48Q^5 + 16Q^6) \quad (3.393)$$

$$\left(\frac{d^2h_x}{dR^2}\right)^2 = 144 (R^4 - 2R^3 + R^2) \quad (3.394)$$

$$\left(\frac{d^2h_y}{dQ^2}\right)^2 = (4 - 48Q + 192Q^2 - 288Q^3 + 144Q^4) \quad (3.395)$$

Integrating Equations (3.390) to (3.395) give Equations (3.396) to (3.401) respectively.

$$\int_0^1 h_x^2 dR = 0.049206349 \quad (3.396)$$

$$\int_0^1 h_y^2 dQ = 0.001587301587 \quad (3.397)$$

$$\int_0^1 \left(\frac{dh_x}{dR}\right)^2 dR = 0.485714285 \quad (3.398)$$

$$\int_0^1 \left(\frac{dh_y}{dQ}\right)^2 dQ = 0.019047619 \quad (3.399)$$

$$\int_0^1 \left(\frac{d^2h_x}{dR^2}\right)^2 dR = 4.8 \quad (3.400)$$

$$\int_0^1 \left(\frac{d^2h_y}{dQ^2}\right)^2 dQ = 0.8 \quad (3.401)$$

Substituting the integrands obtained in Equations (3.396) to (3.401) into Equations (3.37) to (3.40) accordingly gives Equations (3.402) to (3.405)

$$K_x = 4.8 \times 0.001587301587 = 0.007619047618 \quad (3.402)$$

$$K_{xy} = 0.485714285 \times 0.019047619 = 0.009251700644 \quad (3.403)$$

$$K_y = 0.049206349 \times 0.8 = 0.039365079 \quad (3.404)$$

$$K_{Nx} = 0.485714285 \times 0.001587301587 = 0.0007709750554 \quad (3.405)$$

Substituting Equations (3.402) to (3.405) into Equation (3.196) gives the non – dimensional critical buckling load for CSCS rectangular plate under vibration using complete polynomial shape functions expressed in Equation (3.406)

$$\bar{N}_x = \frac{\left[0.007619047618 + \frac{2}{P^2}(0.009251700644) + \frac{0.039365079}{P^4}\right]}{0.0007709750554} (1 - n^2) \quad (3.406)$$

(iii) The trigonometric - polynomial shape functions for CSCS rectangular plates given in Equations (3.103) and (3.181) are expressed in Equations (3.407) and (3.408) respectively.

$$h_x = \sin \pi R \quad (3.407)$$

$$h_y = (Q^2 - 2Q^3 + Q^4) \quad (3.408)$$

Squaring the shape functions, h_x and h_y in Equations (3.407) and (3.408) gives Equations (3.409) and (3.410) respectively.

$$h_x^2 = \sin^2 \pi R \quad (3.409)$$

$$h_y^2 = (Q^4 - 4Q^5 + 6Q^6 + 4Q^7 + Q^8) \quad (3.410)$$

the shape function derivatives are given in Equations (3.411) to (3.414)

$$\left(\frac{dh_x}{dR}\right)^2 = \pi^2 \cos^2 \pi R \quad (3.411)$$

$$\left(\frac{dh_y}{dQ}\right)^2 = (4Q^2 - 24Q^3 + 52Q^4 - 48Q^5 + 16Q^6) \quad (3.412)$$

$$\left(\frac{d^2h_x}{dR^2}\right)^2 = \pi^4 \sin^2 \pi R \quad (3.413)$$

$$\left(\frac{d^2h_y}{dQ^2}\right)^2 = (4 - 48Q + 192Q^2 - 288Q^3 + 144Q^4) \quad (3.414)$$

Integrating Equations (3.409) to (3.414) give Equations (3.415) to (3.420) respectively.

$$\int_0^1 h_x^2 dR = 0.5 \quad (3.415)$$

$$\int_0^1 h_y^2 dQ = 0.001587301587 \quad (3.416)$$

$$\int_0^1 \left(\frac{dh_x}{dR}\right)^2 dR = 0.5\pi^2 \quad (3.417)$$

$$\int_0^1 \left(\frac{dh_y}{dQ}\right)^2 dQ = 0.019047619 \quad (3.418)$$

$$\int_0^1 \left(\frac{d^2h_x}{dR^2}\right)^2 dR = 0.5\pi^4 \quad (3.419)$$

$$\int_0^1 \left(\frac{d^2h_y}{dQ^2}\right)^2 dQ = 0.8 \quad (3.420)$$

Substituting the integrands obtained in Equations (3.415) to (3.420) into Equations (3.37) to (3.40) accordingly give Equations (3.421) to (3.424)

$$K_x = 0.5\pi^4 \times 0.001587301587 = 0.077308802 \quad (3.421)$$

$$K_{xy} = 0.5\pi^2 \times 0.019047619 = 0.093996232 \quad (3.422)$$

$$K_y = 0.5 \times 0.8 = 0.4 \quad (3.423)$$

$$K_{Nx} = 0.5\pi^2 \times 0.001587301587 = 0.007833019364 \quad (3.424)$$

Substituting equations (3.421) to (3.424) into Equation (3.196) gives the non – dimensional critical buckling load for CSCS rectangular plate under vibration using trigonometric - polynomial shape functions expressed in Equation (3.425)

$$\bar{N}_x = \frac{\left[0.077308802 + \frac{2}{P^2}(0.093996232) + \frac{0.4}{P^4}\right]}{0.007833019364} (1 - n^2) \quad (3.425)$$

(iv) The polynomial - trigonometric shape functions for CSCS rectangular plates given in Equations (3.168) and (3.122) are expressed in Equations (3.426) and (3.427) respectively.

$$h_x = (R - 2R^3 + R^4) \quad (3.426)$$

$$h_y = (1 - \cos 2\pi Q) \quad (3.427)$$

Squaring the shape functions, h_x and h_y in Equations (3.426) and (3.427) gives Equations (3.428) and (3.429) respectively.

$$h_x^2 = (R^2 - 4R^4 + 2R^5 + 4R^6 - 4R^7 + R^8) \quad (3.428)$$

$$h_y^2 = (1 - 2\cos 2\pi Q + \cos^2 2\pi Q) \quad (3.429)$$

the shape function derivatives are given in Equations (3.430) to (3.433)

$$\left(\frac{dh_x}{dR}\right)^2 = (1 - 12R^2 + 8R^3 + 36R^4 - 48R^5 + 16R^6) \quad (3.430)$$

$$\left(\frac{dh_y}{dQ}\right)^2 = 4\pi^2 \sin^2 2\pi Q \quad (3.431)$$

$$\left(\frac{d^2h_x}{dR^2}\right)^2 = 144(R^4 - 2R^3 + R^2) \quad (3.432)$$

$$\left(\frac{d^2h_y}{dQ^2}\right)^2 = 16\pi^4 \cos^2 2\pi Q \quad (3.433)$$

Integrating Equations (3.428) to (3.433) give Equations (3.434) to (3.439) respectively.

$$\int_0^1 h_x^2 dR = 0.049206349 \quad (3.434)$$

$$\int_0^1 h_y^2 dQ = 1.5 \quad (3.435)$$

$$\int_0^1 \left(\frac{dh_x}{dR}\right)^2 dR = 0.485714285 \quad (3.436)$$

$$\int_0^1 \left(\frac{dh_y}{dQ}\right)^2 dQ = 2\pi^2 \quad (3.437)$$

$$\int_0^1 \left(\frac{d^2h_x}{dR^2}\right)^2 dR = 4.8 \quad (3.438)$$

$$\int_0^1 \left(\frac{d^2h_y}{dQ^2}\right)^2 dQ = 8\pi^4 \quad (3.439)$$

Substituting the integrands obtained in Equations (3.434) to (3.439) into Equations (3.37) to (3.40) accordingly give Equations (3.440) to (3.443)

$$K_x = 4.8 \times 1.5 = 7.2 \quad (3.440)$$

$$K_{xy} = 0.485714285 \times 2\pi^2 = 9.58761569 \quad (3.441)$$

$$K_y = 0.049206349 \times 8\pi^4 = 38.34516583 \quad (3.442)$$

$$K_{Nx} = 0.485714285 \times 1.5 = 0.728571427 \quad (3.443)$$

Substituting Equations (3.440) to (3.443) into Equation (3.196) gives the non – dimensional critical buckling load for CSCS rectangular plate under vibration using complete polynomial shape functions expressed in Equation (3.444)

$$\bar{N}_x = \frac{\left[7.2 + \frac{2}{P^2}(9.58761569) + \frac{38.34516583}{P^4}\right]}{0.728571427} (1 - n^2) \quad (3.444)$$

3.2.3.5: Determination of the non – dimensional critical buckling load (\bar{N}_x) Of CSSS Rectangular Plate.

(i) The complete trigonometric shape functions for CSSS rectangular plates given in Equations (3.103) and (3.150) are expressed in Equations (3.445) and (3.446) respectively.

$$h_x = \sin \pi R \quad (3.445)$$

$$h_y = (k - kQ - k \cos kQ + \sin kQ) \quad (3.446)$$

Squaring the shape functions, h_x and h_y in Equations (3.445) and (3.446) gives Equations (3.447) and (3.448) respectively.

$$h_x^2 = \sin^2 \pi R \quad (3.447)$$

$$h_y^2 = (k^2 - 2k^2Q - 2k^2 \cos kQ + 2k \sin kQ + k^2Q^2 + 2k^2Q \cos kQ - 2kQ \sin kQ + k^2 \cos^2 kQ - 2k \cos kQ \sin kQ + \sin^2 kQ) \quad (3.448)$$

The shape function derivatives are given in Equations (3.449) to (3.452)

$$\left(\frac{dh_x}{dR}\right)^2 = \pi^2 \cos^2 \pi R \quad (3.449)$$

$$\left(\frac{dh_y}{dQ}\right)^2 = (k^2 - 2k^3 \sin kQ - 2k^2 \cos kQ + k^4 \sin^2 kQ + 2k^3 \sin kQ \cos kQ + k^2 \cos^2 kQ) \quad (3.450)$$

$$\left(\frac{d^2h_x}{dR^2}\right)^2 = \pi^4 \sin^2 \pi R \quad (3.451)$$

$$\left(\frac{d^2h_y}{dQ^2}\right)^2 = (k^6 \cos^2 kQ - 2k^5 \cos kQ \sin kQ + k^4 \sin^2 kQ) \quad (3.452)$$

Integrating Equations (3.447) to (3.452) give Equations (3.453) to (3.458) respectively.

$$\int_0^1 h_x^2 dR = 0.5 \quad (3.453)$$

$$\int_0^1 h_y^2 dQ = 19.42379403 \quad (3.454)$$

$$\int_0^1 \left(\frac{dh_x}{dR}\right)^2 dR = 0.5\pi^2 \quad (3.455)$$

$$\int_0^1 \left(\frac{dh_y}{dQ}\right)^2 dQ = 230.4800172 \quad (3.456)$$

$$\int_0^1 \left(\frac{d^2h_x}{dR^2}\right)^2 dR = 0.5\pi^4 \quad (3.457)$$

$$\int_0^1 \left(\frac{d^2h_y}{dQ^2}\right)^2 dQ = 4376.113774 \quad (3.458)$$

Substituting the integrands obtained in Equations (3.453) to (3.458) into Equations (3.37) to (3.40) accordingly give Equations (3.459) to (3.462)

$$K_x = 0.5\pi^4 \times 19.42379403 = 946.0270604 \quad (3.459)$$

$$K_{xy} = 0.5\pi^2 \times 230.4800172 = 1137.373296 \quad (3.460)$$

$$K_y = 0.5 \times 4376.113774 = 2188.056887 \quad (3.461)$$

$$K_{Nx} = 0.5\pi^2 \times 19.42379403 = 95.85258152 \quad (3.462)$$

Substituting Equations (3.459) to (3.462) into Equation (3.196) gives the non – dimensional critical buckling load for CSSS rectangular plate under vibration using complete trigonometric shape functions expressed in Equation (3.463)

$$\bar{N}_x = \frac{\left[946.0270604 + \frac{2}{P^2}(1137.373296) + \frac{2188.056887}{P^4}\right]}{95.85258152} (1 - n^2) \quad (3.463)$$

(ii) The complete polynomial shape functions for CSSS rectangular plates given in Equations (3.168) and (3.194) are expressed in Equations (3.464) and (3.465) respectively.

$$h_x = (R - 2R^3 + R^4) \quad (3.464)$$

$$h_y = (1.5Q^2 - 2.5Q^3 + Q^4) \quad (3.465)$$

Squaring the shape functions, h_x and h_y in Equations (3.464) and (3.465) gives Equations (3.466) and (3.467) respectively.

$$h_x^2 = (R^2 - 4R^4 + 2R^5 + 4R^6 - 4R^7 + R^8) \quad (3.466)$$

$$h_y^2 = (2.25Q^4 - 7.5Q^5 + 9.25Q^6 - 5Q^7 + Q^8) \quad (3.467)$$

The shape function derivatives are given in Equations (3.468) to (3.471)

$$\left(\frac{dh_x}{dR}\right)^2 = (1 - 12R^2 + 8R^3 + 36R^4 - 48R^5 + 16R^6) \quad (3.468)$$

$$\left(\frac{dh_y}{dQ}\right)^2 = (9Q^2 - 45Q^3 + 80.25Q^4 - 60Q^5 + 16Q^6) \quad (3.469)$$

$$\left(\frac{d^2h_x}{dR^2}\right)^2 = 144(R^4 - 2R^3 + R^2) \quad (3.470)$$

$$\left(\frac{d^2h_y}{dQ^2}\right)^2 = (9 - 90Q + 297Q^2 - 360Q^3 + 144Q^4) \quad (3.471)$$

Integrating Equations (3.466) to (3.471) give Equations (3.472) to (3.477) respectively.

$$\int_0^1 h_x^2 dR = 0.049206349 \quad (3.472)$$

$$\int_0^1 h_y^2 dQ = 0.00753968254 \quad (3.473)$$

$$\int_0^1 \left(\frac{dh_x}{dR}\right)^2 dR = 0.485714285 \quad (3.474)$$

$$\int_0^1 \left(\frac{dh_y}{dQ}\right)^2 dQ = 0.085714285 \quad (3.475)$$

$$\int_0^1 \left(\frac{d^2h_x}{dR^2}\right)^2 dR = 4.8 \quad (3.476)$$

$$\int_0^1 \left(\frac{d^2h_y}{dQ^2}\right)^2 dQ = 1.8 \quad (3.477)$$

Substituting the integrands obtained in Equations (3.472) to (3.477) into Equations (3.37) to (3.40) accordingly give Equations (3.478) to (3.481)

$$K_x = 4.8 \times 0.00753968254 = 0.036190476 \quad (3.478)$$

$$K_{xy} = 0.485714285 \times 0.085714285 = 0.041632652 \quad (3.479)$$

$$K_y = 0.049206349 \times 1.8 = 0.088571428 \quad (3.480)$$

$$K_{Nx} = 0.485714285 \times 0.00753968254 = 0.003662131514 \quad (3.481)$$

Substituting Equations (3.478) to (3.481) into Equation (3.196) gives the non – dimensional critical buckling load for CSSS rectangular plate under vibration using complete polynomial shape functions expressed in Equation (3.482)

$$\bar{N}_x = \frac{\left[0.036190476 + \frac{2}{P^2}(0.041632652) + \frac{0.088571428}{P^4}\right]}{0.003662131514} (1 - n^2) \quad (3.482)$$

(iii) The trigonometric - polynomial shape functions for CSSS rectangular plates given in Equations (3.103) and (3.194) are expressed in Equations (3.483) and (3.484) respectively.

$$h_x = \sin \pi R \quad (3.483)$$

$$h_y = (1.5Q^2 - 2.5Q^3 + Q^4) \quad (3.484)$$

Squaring the shape functions, h_x and h_y in Equations (3.483) and (3.484) gives Equations (3.485) and (3.486) respectively.

$$h_x^2 = \sin^2 \pi R \quad (3.485)$$

$$h_y^2 = (2.25Q^4 - 7.5Q^5 + 9.25Q^6 - 5Q^7 + Q^8) \quad (3.486)$$

The shape function derivatives are given in Equations (3.487) to (3.490)

$$\left(\frac{dh_x}{dR}\right)^2 = \pi^2 \cos^2 \pi R \quad (3.487)$$

$$\left(\frac{dh_y}{dQ}\right)^2 = (9Q^2 - 45Q^3 + 80.25Q^4 - 60Q^5 + 16Q^6) \quad (3.488)$$

$$\left(\frac{d^2h_x}{dR^2}\right)^2 = \pi^4 \sin^2 \pi R \quad (3.489)$$

$$\left(\frac{d^2h_y}{dQ^2}\right)^2 = (9 - 90Q + 297Q^2 - 360Q^3 + 144Q^4) \quad (3.490)$$

Integrating Equations (3.485) to (3.490) give Equations (3.491) to (3.496) respectively.

$$\int_0^1 h_x^2 dR = 0.5 \quad (3.491)$$

$$\int_0^1 h_y^2 dQ = 0.00753968254 \quad (3.492)$$

$$\int_0^1 \left(\frac{dh_x}{dR}\right)^2 dR = 0.5\pi^2 \quad (3.493)$$

$$\int_0^1 \left(\frac{dh_y}{dQ}\right)^2 dQ = 0.085714285 \quad (3.494)$$

$$\int_0^1 \left(\frac{d^2h_x}{dR^2}\right)^2 dR = 0.5\pi^4 \quad (3.495)$$

$$\int_0^1 \left(\frac{d^2h_y}{dQ^2}\right)^2 dQ = 1.8 \quad (3.496)$$

Substituting the integrands obtained in Equations (3.491) to (3.496) into Equations (3.37) to (3.40) accordingly give Equations (3.497) to (3.500)

$$K_x = 0.5\pi^4 \times 0.00753968254 = 0.367216811 \quad (3.497)$$

$$K_{xy} = 0.5\pi^2 \times 0.085714285 = 0.422983042 \quad (3.498)$$

$$K_y = 0.5 \times 1.8 = 0.9 \quad (3.499)$$

$$K_{Nx} = 0.5\pi^2 \times 0.00753968254 = 0.037206841 \quad (3.500)$$

Substituting Equations (3.497) to (3.500) into Equation (3.196) gives the critical buckling load for CSSS rectangular plate under vibration using trigonometric - polynomial shape functions expressed in Equation (3.501)

$$\bar{N}_x = \frac{\left[0.367216811 + \frac{2}{P^2}(0.422983042) + \frac{0.9}{P^4}\right]}{0.037206841} (1 - n^2) \quad (3.501)$$

(iv) The polynomial - trigonometric shape functions for CSSS rectangular plates given in Equations (3.168) and (3.150) are expressed in Equations (3.502) and (3.503) respectively.

$$h_x = (R - 2R^3 + R^4) \quad (3.502)$$

$$h_y = (k - kQ - k \cos kQ + \sin kQ) \quad (3.503)$$

Squaring the shape functions, h_x and h_y in Equations (3.502) and (3.503) gives Equations (3.504) and (3.505) respectively.

$$h_x^2 = (R^2 - 4R^4 + 2R^5 + 4R^6 - 4R^7 + R^8) \quad (3.504)$$

$$h_y^2 = (k^2 - 2k^2Q - 2k^2 \cos kQ + 2k \sin kQ + k^2Q^2 + 2k^2Q \cos kQ - 2kQ \sin kQ + k^2 \cos^2 kQ - 2k \cos kQ \sin kQ + \sin^2 kQ) \quad (3.505)$$

The shape function derivatives are given in Equations (3.506) to (3.509)

$$\left(\frac{dh_x}{dR}\right)^2 = (1 - 12R^2 + 8R^3 + 36R^4 - 48R^5 + 16R^6) \quad (3.506)$$

$$\left(\frac{dh_y}{dQ}\right)^2 = (k^2 - 2k^3 \sin kQ - 2k^2 \cos kQ + k^4 \sin^2 kQ + 2k^3 \sin kQ \cos kQ + k^2 \cos^2 kQ) \quad (3.507)$$

$$\left(\frac{d^2h_x}{dR^2}\right)^2 = 144(R^4 - 2R^3 + R^2) \quad (3.508)$$

$$\left(\frac{d^2h_y}{dQ^2}\right)^2 = (k^6 \cos^2 kQ - 2k^5 \cos kQ \sin kQ + k^4 \sin^2 kQ) \quad (3.509)$$

Integrating Equations (3.504) to (3.509) give Equations (3.510) to (3.515) respectively.

$$\int_0^1 h_x^2 dR = 0.049206349 \quad (3.510)$$

$$\int_0^1 h_y^2 dQ = 19.42379403 \quad (3.511)$$

$$\int_0^1 \left(\frac{dh_x}{dR}\right)^2 dR = 0.485714285 \quad (3.512)$$

$$\int_0^1 \left(\frac{dh_y}{dQ}\right)^2 dQ = 230.4800172 \quad (3.513)$$

$$\int_0^1 \left(\frac{d^2h_x}{dR^2}\right)^2 dR = 4.8 \quad (3.514)$$

$$\int_0^1 \left(\frac{d^2h_y}{dQ^2}\right)^2 dQ = 4376.113774 \quad (3.515)$$

Substituting the integrands obtained in Equations (3.510) to (3.515) into Equations (3.37) to (3.40) accordingly give Equations (3.516) to (3.519)

$$K_x = 4.8 \times 19.42379403 = 93.23421134 \quad (3.516)$$

$$K_{xy} = 0.485714285 \times 230.4800172 = 111.9474368 \quad (3.517)$$

$$K_y = 0.049206349 \times 4376.113774 = 215.3325816 \quad (3.518)$$

$$K_{Nx} = 0.485714285 \times 19.42379403 = 9.434414229 \quad (3.519)$$

Substituting Equations (3.516) to (3.519) into Equation (3.196) gives the non – dimensional critical buckling load for CSSS rectangular plate under vibration using complete trigonometric shape functions expressed in Equation (3.520)

$$\bar{N}_x = \frac{\left[93.23421134 + \frac{2}{P^2}(111.9474368) + \frac{215.3325816}{P^4} \right]}{9.434414229} (1 - n^2) \quad (3.520)$$

3.2.3.6: Determination of the non – dimensional critical buckling load (\bar{N}_x) Of CCCS Rectangular Plate.

(i) The complete trigonometric shape functions for CCCS rectangular plates given in Equations (3.149) and (3.122) are expressed in Equations (3.521) and (3.522) respectively.

$$h_x = (k - kR - k \cos kR + \sin kR) \quad (3.521)$$

$$h_y = (1 - \cos 2\pi Q) \quad (3.522)$$

Squaring the shape functions, h_x and h_y in Equations (3.521) and (3.522) gives Equations (3.523) and (3.524) respectively.

$$h_x^2 = (k^2 - 2k^2R - 2k^2 \cos kR + 2k \sin kR + k^2R^2 + 2k^2R \cos kR - 2kR \sin kR + k^2 \cos^2 kR - 2k \cos kR \sin kR + \sin^2 kR) \quad (3.523)$$

$$h_y^2 = (1 - 2\cos 2\pi Q + \cos^2 2\pi Q) \quad (3.524)$$

The shape function derivatives are given in Equations (3.525) to (3.528)

$$\left(\frac{dh_x}{dR}\right)^2 = (k^2 - 2k^3 \sin kR - 2k^2 \cos kR + k^4 \sin^2 kR + 2k^3 \sin kR \cos kR + k^2 \cos^2 kR) \quad (3.525)$$

$$\left(\frac{dh_y}{dQ}\right)^2 = 4\pi^2 \sin^2 2\pi Q \quad (3.526)$$

$$\left(\frac{d^2h_x}{dR^2}\right)^2 = (k^6 \cos^2 kR - 2k^5 \sin kR \cos kR + k^4 \sin^2 kR) \quad (3.527)$$

$$\int_0^1 \left(\frac{d^2h_y}{dQ^2}\right)^2 dQ = 16\pi^4 \cos^2 2\pi Q \quad (3.528)$$

Integrating Equations (3.523) to (3.528) give Equations (3.529) to (3.534) respectively.

$$\int_0^1 h_x^2 dR = 19.42379403 \quad (3.529)$$

$$\int_0^1 h_y^2 dQ = 1.5 \quad (3.530)$$

$$\int_0^1 \left(\frac{dh_x}{dR}\right)^2 dR = 230.4800172 \quad (3.531)$$

$$\int_0^1 \left(\frac{dh_y}{dQ}\right)^2 dQ = 2\pi^2$$

(3.532)

$$\int_0^1 \left(\frac{d^2 h_x}{dR^2}\right)^2 dR = 4376.113774 \quad (3.533)$$

$$\int_0^1 \left(\frac{d^2 h_y}{dQ^2}\right)^2 dQ = 8\pi^4 \quad (3.534)$$

Substituting the integrands obtained in Equations (3.529) to (3.534) into Equations (3.37) to (3.40) accordingly give Equations (3.535) to (3.538)

$$K_x = 4376.113774 \times 1.5 = 6564.170661 \quad (3.535)$$

$$K_{xy} = 230.4800172 \times 2\pi^2 = 4549.493184 \quad (3.536)$$

$$K_y = 19.42379403 \times 8\pi^4 = 15136.43297 \quad (3.537)$$

$$K_{Nx} = 230.4800172 \times 1.5 = 345.7200258 \quad (3.538)$$

Substituting Equations (3.535) to (3.538) into Equation (3.196) gives the non – dimensional critical buckling load for CCCS rectangular plate under vibration using complete trigonometric shape functions expressed in Equation (3.539)

$$\bar{N}_x = \frac{\left[6564.170661 + \frac{2}{P^2}(4549.493184) + \frac{15136.43297}{P^4}\right]}{345.7200258} (1 - n^2) \quad (3.539)$$

(ii) The complete polynomial shape functions for CCCS rectangular plates given in Equations (3.193) and (3.181) are expressed in Equations (3.540) and (3.541) respectively.

$$h_x = (1.5R^2 - 2.5R^3 + R^4) \quad (3.540)$$

$$h_y = (Q^2 - 2Q^3 + Q^4) \quad (3.541)$$

Squaring the shape functions, h_x and h_y in Equations (3.540) and (3.541) gives Equations (3.542) and (3.543) respectively.

$$h_x^2 = (2.25R^4 - 7.5R^5 + 9.25R^6 - 5R^7 + R^8) \quad (3.542)$$

$$h_y^2 = (Q^4 - 4Q^5 + 6Q^6 + 4Q^7 + Q^8) \quad (3.543)$$

The shape function derivatives are given in Equations (3.544) to (3.547)

$$\left(\frac{dh_x}{dR}\right)^2 = (9R^2 - 45R^3 + 80.25R^4 - 60R^5 + 16R^6) \quad (3.544)$$

$$\left(\frac{dh_y}{dQ}\right)^2 = (4Q^2 - 24Q^3 + 52Q^4 - 48Q^5 + 16Q^6) \quad (3.545)$$

$$\left(\frac{d^2h_x}{dR^2}\right)^2 = (9 - 90R + 297R^2 - 360R^3 + 144R^4) \quad (3.546)$$

$$\left(\frac{d^2h_y}{dQ^2}\right)^2 = (4 - 48Q + 192Q^2 - 288Q^3 + 144Q^4) \quad (3.547)$$

Integrating Equations (3.542) to (3.547) give Equations (3.548) to (3.553) respectively.

$$\int_0^1 h_x^2 dR = 0.00753968254 \quad (3.548)$$

$$\int_0^1 h_y^2 dQ = 0.001587301587 \quad (3.549)$$

$$\int_0^1 \left(\frac{dh_x}{dR}\right)^2 dR = 0.085714285 \quad (3.550)$$

$$\int_0^1 \left(\frac{dh_y}{dQ}\right)^2 dQ = 0.019047619 \quad (3.551)$$

$$\int_0^1 \left(\frac{d^2h_x}{dR^2}\right)^2 dR = 1.8 \quad (3.552)$$

$$\int_0^1 \left(\frac{d^2h_y}{dQ^2}\right)^2 dQ = 0.8 \quad (3.553)$$

Substituting the integrands obtained in Equations (3.548) to (3.553) into Equations (3.37) to (3.40) accordingly give Equations (3.554) to (3.557)

$$K_x = 1.8 \times 0.001587301587 = 0.002857142857 \quad (3.554)$$

$$K_{xy} = 0.085714285 \times 0.019047619 = 0.001632653044 \quad (3.555)$$

$$K_y = 0.00753968254 \times 0.8 = 0.006031746032 \quad (3.556)$$

$$K_{Nx} = 0.085714285 \times 0.001587301587 = 0.0001360544206 \quad (3.557)$$

Substituting Equations (3.554) to (3.557) into Equation (3.196) gives the non – dimensional critical buckling load for CCCS rectangular plate under vibration using complete polynomial shape functions expressed in Equation (3.558)

$$\bar{N}_x = \frac{\left[0.002857142857 + \frac{2}{p^2}(0.001632653044) + \frac{0.006031746032}{p^4}\right]}{0.0001360544206} (1 - n^2) \quad (3.558)$$

(iii) The trigonometric - polynomial shape functions for CCCS rectangular plates given in Equations (3.149) and (3.181) are expressed in Equations (3.559) and (3.560) respectively.

$$h_x = (k - kR - k \cos kR + \sin kR) \quad (3.559)$$

$$h_y = (Q^2 - 2Q^3 + Q^4) \quad (3.560)$$

Squaring the shape functions, h_x and h_y in Equations (3.559) and (3.560) gives Equations (3.561) and (3.562) respectively.

$$h_x^2 = (k^2 - 2k^2R - 2k^2 \cos kR + 2k \sin kR + k^2R^2 + 2k^2R \cos kR - 2kR \sin kR + k^2 \cos^2 kR - 2k \cos kR \sin kR + \sin^2 kR) \quad (3.561)$$

$$h_y^2 = (Q^4 - 4Q^5 + 6Q^6 - 4Q^7 + Q^8) \quad (3.562)$$

The shape function derivatives are given in Equations (3.563) to (3.566)

$$\left(\frac{dh_x}{dR}\right)^2 = (k^2 - 2k^3 \sin kR - 2k^2 \cos kR + k^4 \sin^2 kR + 2k^3 \sin kR \cos kR + k^2 \cos^2 kR) \quad (3.563)$$

$$\left(\frac{dh_y}{dQ}\right)^2 = (4Q^2 - 24Q^3 + 52Q^4 - 48Q^5 + 16Q^6) \quad (3.564)$$

$$\left(\frac{d^2h_x}{dR^2}\right)^2 = (k^6 \cos^2 kR - 2k^5 \sin kR \cos kR + k^4 \sin^2 kR) \quad (3.565)$$

$$\left(\frac{d^2h_y}{dQ^2}\right)^2 = (4 - 48Q + 192Q^2 - 288Q^3 + 144Q^4) \quad (3.566)$$

Integrating Equations (3.561) to (3.566) give equations (3.567) to (3.572) respectively.

$$\int_0^1 h_x^2 dR = 19.42379403 \quad (3.567)$$

$$\int_0^1 h_y^2 dQ = 0.001587301587 \quad (3.568)$$

$$\int_0^1 \left(\frac{dh_x}{dR}\right)^2 dR = 230.4800172 \quad (3.569)$$

$$\int_0^1 \left(\frac{dh_y}{dQ}\right)^2 dQ = 0.019047619 \quad (3.570)$$

$$\int_0^1 \left(\frac{d^2h_x}{dR^2}\right)^2 dR = 4376.113774 \quad (3.571)$$

$$\int_0^1 \left(\frac{d^2h_y}{dQ^2}\right)^2 dQ = 0.8 \quad (3.572)$$

Substituting the integrands obtained in Equations (3.567) to (3.572) into Equations (3.37) to (3.40) accordingly give Equations (3.573) to (3.576)

$$K_x = 4376.113774 \times 0.001587301587 = 6.946212338 \quad (3.573)$$

$$K_{xy} = 230.4800172 \times 0.019047619 = 4.390095555 \quad (3.574)$$

$$K_y = 19.42379403 \times 0.8 = 15.53903522 \quad (3.575)$$

$$K_{Nx} = 230.4800172 \times 0.001587301587 = 0.365841297 \quad (3.576)$$

Substituting Equations (3.573) to (3.576) into Equation (3.196) gives the non – dimensional critical buckling load for CCCS rectangular plate under vibration using trigonometric - polynomial shape functions expressed in Equation (3.577)

$$\bar{N}_x = \frac{\left[6.946212338 + \frac{2}{P^2}(4.390095555) + \frac{15.53903522}{P^4}\right]}{0.365841297} (1 - n^2) \quad (3.577)$$

TABLE 3.1 STIFFNESS FACTORS OF THE SIX RECTANGULAR PLATES

Plate boundary condition	Complete trigonometric shape functions and its derivatives				Complete polynomial shape functions and its derivatives				Trigonometric - polynomial shape functions and its derivatives			
	K _x	K _{xy}	K _y	KN _x	K _x	K _{xy}	K _y	KN _x	K _x	K _{xy}	K _y	KN _x
SSSS	24.3523	24.3523	24.3223	2.4674	0.2362	0.2359	0.2362	0.0239	2.3966	2.3970	2.400	0.2428
CCCC	1168.9091	389.6364	1168.9091	29.6088	0.0013	0.0004	0.0013	0.000030	1.2369	0.3760	1.200	0.0313
CCSS	85000.7326	53121.0383	85000.7326	4476.7964	0.0136	0.0073	0.0136	0.00065	32.9945	19.7554	34.9628	1.7377
CSCS	73.0568	94.40909	389.6364	7.4022	0.0076	0.0093	0.039,4	0.00077	0.0773	0.0940	0.4000	0.0078
									7.2000	9.5876	38.3451	0.7286
CSSS	946.0270	1137.3733	2188.0569	95.8526	0.0362	0.0416	0.08857	0.0037	0.3672	0.4230	0.9000	0.0372
									93.2342	111.9474	215.3326	9.4344
CCCS	6564.1707	4549.4932	15136.4330	345.7200	0.002857	0.0016	0.0060	0.00014	6.9462	4.3901	15.5390	0.3658

3.2.4 Comparison of Results of this study with those from previous studies

Percentage difference tool was used in comparing the results from this study with those from Ibearugbulem et al. (2014). Percentage difference is defined as given in equation (3.578):

$$\text{Percentage (\%) difference} = \frac{|N_P - N_E|}{N_E} \times \frac{100}{1} \quad (3.578)$$

Where

N_P = Result of Present study

N_E = Result of Ibearugbulem et al. (2014)

CHAPTER FOUR

RESULTS AND DISCUSSION

4.1 RESULT PRESENTATION

The results obtained from the methods treated in chapter three were presented here.

4.1.1 Result of Critical Buckling Load Equation

The governing equation for critical buckling load of a rectangular plate subjected to vibration as obtained in this study is presented in equation (4.1) as:

$$N_x = \frac{\frac{D}{a^2} \left[K_x + \frac{2}{P^2} K_{xy} + \frac{K_y}{P^4} \right]}{K_{Nx}} (1 - n^2) = \frac{D}{a^2} \cdot \bar{N}_x \quad (4.1)$$

Where;

N_x is the critical buckling load

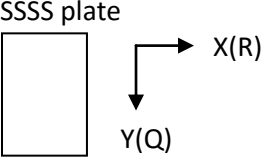
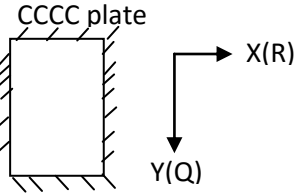
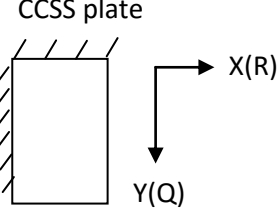
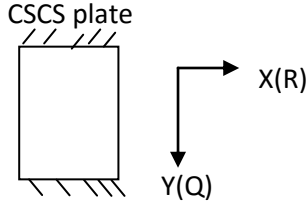
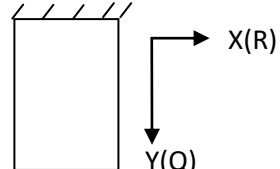
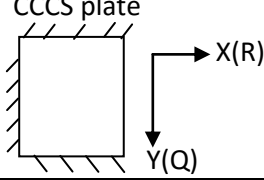
$\frac{D}{a^2}$ is a constant

\bar{N}_x is the non – dimensional critical buckling load

4.1.2 Results of the Derived Shape Functions for the Rectangular Plates

The various shape functions for the rectangular plates derived in this work were presented on table 4.1.

TABLE 4.1 VARIOUS SHAPE FUNCTIONS FOR THE SIX (6) RECTANGULAR PLATES

PLATE BOUNDARY CONDITIONS	SHAPE FUNCTIONS (h_x and h_y)
 <p>SSSS plate</p>	$h_x = \sin \pi R$ and $h_y = \sin \pi Q$ $h_x = (R - 2R^3 + R^4)$ and $h_y = (Q - 2Q^3 + Q^4)$ $h_x = \sin \pi R$ and $h_y = (Q - 2Q^3 + Q^4)$
 <p>CCCC plate</p>	$h_x = (1 - \cos 2\pi R)$ and $h_y = (1 - \cos 2\pi Q)$ $h_x = (R^2 - 2R^3 + R^4)$ and $h_y = (Q^2 - 2Q^3 + Q^4)$ $h_x = (1 - \cos 2\pi R)$ and $h_y = (Q^2 - 2Q^3 + Q^4)$
 <p>CCSS plate</p>	$h_x = (k - kR - k \cos kR + \sin kR)$ and $h_y = (k - kQ - k \cos kQ + \sin kQ)$ $h_x = (1.5R^2 - 2.5R^3 + R^4)$ and $h_y = (1.5Q^2 - 2.5Q^3 + Q^4)$ $h_x = (k - kR - k \cos kR + \sin kR)$ and $h_y = (1.5Q^2 - 2.5Q^3 + Q^4)$
 <p>CSCS plate</p>	$h_x = \sin \pi R$ and $h_y = (1 - \cos 2\pi Q)$ $h_x = (R - 2R^3 + R^4)$ and $h_y = (Q^2 - 2Q^3 + Q^4)$ $h_x =$ $\sin \pi R$ and $h_y = (Q^2 - 2Q^3 + Q^4)$ $h_x = (R - 2R^3 + R^4)$ and $h_y = (1 - \cos 2\pi Q)$
 <p>CSSS plate</p>	$h_x = \sin \pi R$ and $h_y = (k - kQ - k \cos kQ + \sin kQ)$ $h_x = (R - 2R^3 + R^4)$ and $h_y = (1.5Q^2 - 2.5Q^3 + Q^4)$ $h_x = \sin \pi R$ and $h_y = (1.5Q^2 - 2.5Q^3 + Q^4)$ $h_x = (R - 2R^3 + R^4)$ and $h_y = (k - kQ - k \cos kQ + \sin kQ)$
 <p>CCCS plate</p>	$h_x = (k - kR - k \cos kR + \sin kR)$ and $h_y = (1 - \cos 2\pi Q)$ $h_x = (1.5R^2 - 2.5R^3 + R^4)$ and $h_y = (Q^2 - 2Q^3 + Q^4)$ $h_x = (k - kR - k \cos kR + \sin kR)$, $h_y = (Q^2 - 2Q^3 + Q^4)$

NOTE: $k = 4.49340946$

4.1.3 Results of Non – dimensional Critical Buckling Loads of various Rectangular Plates

The non – dimensional critical buckling loads of various rectangular plates at aspect ratios ($1 \leq P \leq 2$) and resonating / vibration frequencies ($0 \leq n \leq 1$) were presented on Table 4.2, Table 4.3, Table 4.4, Table 4.5, Table 4.6 and Table 4.7 and also in figure A, Figure B, Figure C, Figure D, Figure E and Figure F.

Table 4.2a: Non – dimensional Critical Buckling Load for Rectangular SSSS Plate under Vibration using complete trigonometric shape function.

Non – dimensional critical buckling load (\bar{N}_x)											
Resonating frequency ratio (n)											
Aspect ratio (P=b/a)	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
1.0	39.478	39.084	37.899	35.925	33.162	29.609	25.266	20.134	14.212	7.501	0
1.1	32.924	32.595	31.607	29.961	27.656	24.693	21.071	16.791	11.853	6.256	0
1.2	28.337	28.054	27.204	25.787	23.803	21.253	18.136	14.452	10.201	5.384	0
1.3	25.005	24.755	24.005	22.755	21.004	18.754	16.003	12.753	9.002	4.751	0
1.4	22.510	22.285	21.609	20.484	18.908	16.882	14.406	11.480	8.104	4.277	0
1.5	20.592	20.386	19.768	18.739	17.297	15.444	13.179	10.502	7.413	3.913	0
1.6	19.086	18.895	18.323	17.368	16.032	14.315	12.215	9.734	6.871	3.626	0
1.7	17.881	17.703	17.166	16.272	15.020	13.411	11.444	9.120	6.437	3.397	0
1.8	16.902	16.733	16.226	15.381	14.198	12.677	10.817	8.620	6.084	3.211	0
1.9	16.095	15.934	15.451	14.646	13.520	12.071	10.301	8.208	5.794	3.058	0
2.0	15.421	15.267	14.804	14.033	12.954	11.566	9.870	7.865	5.552	2.930	0

**Table 4.2b: Non-dimensional Critical Buckling Load for Rectangular SSSS
Plate under Vibration using complete polynomial shape function**

Non – dimensional critical buckling load (\bar{N}_x)											
Resonating frequency ratio (n)											
Aspect ratio (P=b/a)	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
1.0	39.507	39.112	37.926	35.951	33.186	29.630	25.284	20.148	14.222	7.506	0
1.1	32.948	32.618	31.630	29.982	27.676	24.711	21.087	16.803	11.861	6.260	0
1.2	28.358	28.074	27.224	25.806	23.821	21.268	18.149	14.462	10.209	5.388	0
1.3	25.024	24.774	24.023	22.772	21.020	18.768	16.015	12.762	9.009	4.755	0
1.4	22.527	22.302	21.626	20.500	18.923	16.895	14.417	11.489	8.110	4.280	0
1.5	20.609	20.403	19.784	18.754	17.311	15.456	13.190	10.510	7.419	3.916	0
1.6	19.102	18.911	18.338	17.383	16.046	14.326	12.225	9.724	6.877	3.629	0
1.7	17.897	17.718	17.181	16.286	15.033	13.423	11.454	9.127	6.443	3.400	0
1.8	16.917	16.748	16.240	15.394	14.210	12.688	10.827	8.628	6.090	3.214	0
1.9	16.109	15.948	15.465	14.660	13.532	12.070	10.310	8.216	5.799	3.061	0
2.0	15.435	15.281	14.818	14.046	12.966	11.577	9.879	7.872	5.557	2.933	0

**Table 4.2c: Non-dimensional Critical Buckling Load for Rectangular SSSS
Plate under Vibration using trigonometric-polynomial shape function.**

Non – dimensional critical buckling load (\bar{N}_x)											
Resonating frequency ratio (n)											
Aspect ratio (P=b/a)	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
1.0	39.495	39.100	37.915	35.941	33.176	29.621	25.277	20.143	14.218	7.504	0
1.1	32.936	32.607	31.619	29.972	27.666	24.702	21.079	16.797	11.857	6.258	0
1.2	28.346	28.062	27.212	25.795	23.810	21.259	18.141	14.456	10.204	5.386	0
1.3	25.012	24.762	24.011	22.761	21.010	18.759	16.008	12.756	9.004	4.752	0
1.4	22.515	22.290	21.614	20.488	18.912	16.886	14.409	11.483	8.105	4.278	0
1.5	20.596	20.390	19.772	18.742	17.301	15.447	13.182	10.504	7.415	3.913	0
1.6	19.089	18.899	18.326	17.371	16.035	14.317	12.217	9.736	6.872	3.627	0
1.7	17.884	17.705	17.169	16.275	15.023	13.413	11.446	9.121	6.438	3.398	0
1.8	16.904	16.735	16.228	15.383	14.200	12.678	10.819	8.621	6.086	3.212	0
1.9	16.097	15.936	15.453	14.648	13.521	12.073	10.302	8.209	5.795	3.058	0
2.0	15.423	15.269	14.806	14.035	12.955	11.567	9.871	7.866	5.552	2.930	0

**Table 4.3a:Non-dimensional Critical Buckling Load for Rectangular CCCC
Plate under Vibration using complete trigonometric shape function.**

Non – dimensional critical buckling load (\bar{N}_x)											
Resonating frequency ratio (n)											
Aspect ratio (P=b/a)	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
1.0	105.278	104.225	101.067	95.803	88.433	78.958	67.378	53.692	37.900	20.003	0
1.1	88.194	87.312	84.666	80.256	74.083	66.145	56.444	44.979	31.750	16.757	0
1.2	76.794	76.026	73.722	69.883	64.507	57.596	49.148	39.165	27.646	14.591	0
1.3	68.874	68.186	66.119	62.676	57.854	51.656	44.080	35.126	24.795	13.086	0
1.4	63.183	62.551	60.656	57.497	53.074	47.387	40.437	32.223	22.746	12.005	0
1.5	58.974	58.384	56.615	53.666	49.538	44.230	37.743	30.077	21.231	11.205	0
1.6	55.783	55.225	53.552	50.763	46.858	41.837	35.701	28.449	20.082	10.599	0
1.7	53.312	52.779	51.180	48.514	44.782	39.984	34.120	27.189	19.192	10.129	0
1.8	51.362	50.849	49.308	46.740	43.144	38.522	32.872	26.195	18.490	9.759	0
1.9	49.798	49.300	47.806	45.316	41.831	37.345	31.871	25.397	17.927	9.462	0
2.0	48.526	48.040	46.585	44.158	40.761	36.394	31.056	24.748	17.469	9.220	0

**Table 4.3b:Non-dimensional Critical Buckling Load for Rectangular CCCC
Plate under Vibration using complete polynomial shape function.**

Non – dimensional critical buckling load (\bar{N}_x)											
Resonating frequency ratio (n)											
Aspect ratio (P=b/a)	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
1.0	108.000	106.920	103.680	98.280	90.720	81.000	69.120	55.080	38.880	20.520	0
1.1	90.521	89.616	86.900	82.374	76.038	67.891	57.934	46.166	32.588	17.199	0
1.2	78.921	78.132	75.764	71.818	66.294	59.191	50.510	40.250	28.412	14.995	0
1.3	70.907	70.197	68.070	64.525	59.562	53.180	45.380	36.162	25.526	13.472	0
1.4	65.178	64.526	62.571	59.312	54.749	48.883	41.714	33.241	23.464	12.384	0
1.5	60.963	60.353	58.524	55.476	51.209	45.722	39.016	31.091	21.947	11.583	0
1.6	57.784	57.206	55.472	52.583	48.538	43.338	36.982	29.470	20.802	10.979	0
1.7	55.333	54.780	53.120	50.353	46.480	41.500	35.413	28.220	19.920	10.513	0
1.8	53.408	52.874	51.272	48.602	44.863	40.056	34.181	27.238	19.227	10.148	0
1.9	51.871	51.352	49.796	47.203	43.572	38.903	33.197	26.454	18.674	9.855	0
2.0	50.625	50.119	48.600	46.069	42.525	37.969	32.400	25.819	18.225	9.619	0

Table 4.3c: Non-dimensional Critical Buckling Load for Rectangular CCCC Plate under Vibration using trigonometric-polynomial shape function.

Non – dimensional critical buckling load (\bar{N}_x)											
Resonating frequency ratio (n)											
Aspect ratio (P=b/a)	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
1.0	101.778	100.760	97.707	92.618	85.493	76.333	65.138	51.907	36.640	19.338	0
1.1	85.472	84.617	82.053	77.780	71.797	64.104	54.702	43.591	30.770	16.240	0
1.2	74.615	73.869	71.630	67.900	62.677	55.961	47.754	38.054	26.861	14.177	0
1.3	67.089	66.418	64.406	61.051	56.355	50.317	42.937	34.216	24.152	12.747	0
1.4	61.693	61.076	59.225	56.141	51.822	46.270	39.483	31.463	22.209	11.722	0
1.5	57.710	57.133	55.402	52.516	48.477	43.283	36.935	29.432	20.776	10.965	0
1.6	54.697	54.150	52.510	49.775	45.946	41.023	35.006	27.896	19.691	10.393	0
1.7	52.369	51.845	50.274	47.655	43.990	39.276	33.516	26.708	18.853	9.950	0
1.8	50.534	50.029	48.513	45.986	42.449	37.901	32.342	25.772	18.192	9.602	0
1.9	49.065	48.575	47.103	44.650	41.215	36.799	31.402	25.023	17.664	9.322	0
2.0	47.872	47.393	45.957	43.564	40.213	35.904	30.638	24.415	17.234	9.096	0

Table 4.4a: Non-dimensional Critical Buckling Load for Rectangular CCSS Plate under Vibration using complete trigonometric shape function.

Non – dimensional critical buckling load (\bar{N}_x)											
Resonating frequency ratio (n)											
Aspect ratio (P=b/a)	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
1.0	61.706	61.089	59.237	56.152	51.833	46.279	39.492	31.470	22.214	11.724	0
1.1	51.568	51.053	49.506	46.927	43.317	38.676	33.004	26.300	18.565	9.798	0
1.2	44.624	44.178	42.839	40.608	37.484	33.468	28.559	22.758	16.065	8.479	0
1.3	39.677	39.280	38.090	36.106	33.329	29.758	25.393	20.235	14.284	7.539	0
1.4	36.037	35.677	34.596	32.794	30.271	27.028	23.064	18.379	12.973	6.911	0
1.5	33.285	32.952	31.954	30.289	27.959	24.964	21.302	16.975	11.983	6.324	0
1.6	31.154	30.843	29.908	28.350	26.170	23.366	19.939	15.889	11.216	5.919	0
1.7	29.472	29.177	28.293	26.819	24.756	22.104	18.862	15.031	10.610	5.600	0
1.8	28.120	27.839	26.995	25.589	23.621	21.090	17.997	14.341	10.123	5.343	0
1.9	27.018	26.748	25.937	24.586	22.695	20.263	17.291	13.779	9.726	5.133	0
2.0	26.107	25.846	25.062	23.757	21.930	19.580	16.708	13.314	9.398	4.960	0

**Table 4.4b: Non-dimensional Critical Buckling Load for Rectangular CCSS
Plate under Vibration using complete polynomial shape function.**

Non – dimensional critical buckling load (\bar{N}_x)											
Resonating frequency ratio (n)											
Aspect ratio (P=b/a)	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
1.0	64.737	64.089	62.147	58.910	54.379	48.553	41.432	33.016	23.305	12.300	0
1.1	54.134	53.593	51.969	49.262	45.473	40.601	34.646	27.608	19.488	10.285	0
1.2	46.917	46.448	45.040	42.694	39.410	35.188	30.027	23.928	16.890	8.914	0
1.3	41.806	41.388	40.134	38.044	35.117	31.355	26.756	21.321	15.050	7.943	0
1.4	38.067	37.686	36.544	34.641	31.976	28.550	24.363	19.414	13.704	7.233	0
1.5	35.253	34.901	33.843	32.081	29.613	26.440	22.562	17.979	12.691	6.698	0
1.6	33.086	32.755	31.762	30.108	27.792	24.814	21.175	16.874	11.911	6.286	0
1.7	31.382	31.068	30.126	28.557	26.361	23.536	20.084	16.005	11.297	5.963	0
1.8	30.018	29.718	28.817	27.316	25.215	22.514	19.212	15.309	10.806	5.703	0
1.9	28.910	28.621	27.753	26.308	24.284	21.682	18.502	14.744	10.407	5.493	0
2.0	27.997	27.717	26.877	25.477	23.517	20.998	17.918	14.278	10.079	5.319	0

**Table 4.4c: Non-dimensional Critical Buckling Load for Rectangular CCSS
Plate under Vibration using trigonometric – polynomial shape function.**

Non – dimensional critical buckling load (\bar{N}_x)											
Resonating frequency ratio (n)											
Aspect ratio (P=b/a)	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
1.0	61.843	61.225	59.370	56.278	51.948	46.383	39.580	31.540	22.264	11.750	0
1.1	51.520	51.005	49.459	46.883	43.277	38.640	32.973	26.275	18.547	9.789	0
1.2	44.479	44.034	42.700	40.476	37.363	33.359	28.467	22.684	16.013	8.451	0
1.3	39.485	39.090	37.906	35.931	33.168	29.614	25.270	20.137	14.215	7.502	0
1.4	35.825	35.466	34.392	32.600	30.093	26.869	22.928	18.271	12.897	6.807	0
1.5	33.066	32.736	31.744	30.090	27.776	24.800	21.163	16.864	11.904	6.283	0
1.6	30.939	30.629	29.701	28.154	25.988	23.204	19.801	15.779	11.138	5.878	0
1.7	29.263	28.971	28.093	26.630	24.581	21.947	18.729	14.924	10.535	5.560	0
1.8	27.921	27.642	26.804	25.408	23.454	20.941	17.869	14.240	10.052	5.305	0
1.9	26.829	26.561	25.756	24.414	22.536	20.122	17.171	13.683	9.658	5.098	0
2.0	25.929	25.669	24.891	23.595	21.780	19.446	16.594	13.224	9.334	4.926	0

Table 4.5a: Non – dimensional Critical Buckling Load for Rectangular CSCS Plate under Vibration using complete trigonometric shape function.

Non – dimensional critical buckling load (\bar{N}_x)											
Resonating frequency ratio (n)											
Aspect ratio (P=b/a)	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
1.0	88.016	87.136	84.495	80.094	73.933	66.012	56.330	44.888	31.686	16.723	0
1.1	66.903	66.234	64.227	60.882	56.199	50.177	42.818	34.121	24.085	12.712	0
1.2	52.969	52.439	50.850	48.201	44.494	39.726	33.900	27.014	19.069	10.064	0
1.3	43.393	42.959	41.658	39.488	36.450	32.545	27.772	22.131	15.622	8.245	0
1.4	36.586	36.220	35.123	33.293	30.732	27.440	23.415	18.659	13.171	6.951	0
1.5	31.604	31.288	30.340	28.760	26.548	23.703	20.227	16.118	11.378	6.005	0
1.6	27.866	27.587	26.751	25.358	23.407	20.899	17.834	14.212	10.032	5.294	0
1.7	24.998	24.748	23.998	22.749	20.999	18.749	15.999	12.749	8.999	4.750	0
1.8	22.757	22.529	21.847	20.709	19.116	17.068	14.564	11.606	8.192	4.324	0
1.9	20.975	20.765	20.136	19.087	17.619	15.731	13.424	10.697	7.551	3.985	0
2.0	19.537	19.341	18.755	17.778	16.411	14.652	12.503	9.964	7.033	3.718	0

Table 4.5b: Non -dimensional Critical Buckling Load for Rectangular CSCS Plate under Vibration using complete polynomial shape function.

Non – dimensional critical buckling load (\bar{N}_x)											
Resonating frequency ratio (n)											
Aspect ratio (P=b/a)	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
1.0	84.941	84.092	81.544	77.296	71.351	63.706	54.362	43.320	30.579	16.139	0
1.1	64.591	63.945	62.007	58.778	54.256	48.443	41.338	32.941	23.253	12.272	0
1.2	51.172	50.661	49.125	46.567	42.985	38.379	32.750	26.098	18.422	9.723	0
1.3	41.961	41.541	40.282	38.184	35.247	31.470	26.855	21.400	15.106	7.973	0
1.4	35.418	35.064	34.002	32.231	29.751	26.564	22.668	18.063	12.751	6.729	0
1.5	30.635	30.328	29.409	27.878	25.733	22.976	19.606	15.624	11.028	5.821	0
1.6	27.048	26.778	25.966	24.614	22.721	20.286	17.311	13.795	9.737	5.821	0
1.7	24.300	24.057	23.328	22.113	20.412	18.225	15.552	12.393	8.748	4.617	0
1.8	22.154	21.932	21.267	20.160	18.609	16.615	14.178	11.298	7.975	4.209	0
1.9	20.448	20.244	19.631	18.608	17.177	15.336	13.087	10.429	7.361	3.885	0
2.0	19.074	18.883	18.311	17.357	16.022	14.305	12.207	9.727	6.866	3.624	0

Table 4.5c: Non-dimensional Critical Buckling Load for Rectangular CSCS Plate under Vibration using trigonometric-polynomial shape function.

Non – dimensional critical buckling load (\bar{N}_x)											
Resonating frequency ratio (n)											
Aspect ratio (P=)	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
1.0	84.935	84.086	81.538	77.291	71.346	63.702	54.359	43.317	30.577	16.138	0
1.1	64.583	63.937	62.000	58.771	54.250	48.437	41.333	32.937	23.250	12.271	0
1.2	51.163	50.651	49.116	46.558	42.977	38.372	32.744	26.093	18.419	9.721	0
1.3	41.950	41.531	40.272	38.175	35.238	31.463	26.848	21.395	15.102	7.971	0
1.4	35.407	35.053	33.991	32.221	29.742	26.556	22.661	18.058	12.747	6.727	0
1.5	30.623	30.317	29.398	27.867	25.724	22.968	19.599	15.618	11.024	5.818	0
1.6	27.037	26.766	25.955	24.603	22.711	20.277	17.303	13.789	9.733	5.137	0
1.7	24.288	24.045	23.317	22.102	20.402	18.216	15.544	12.387	8.744	4.615	0
1.8	22.142	21.920	21.256	20.149	18.599	16.606	14.171	11.292	7.971	4.207	0
1.9	20.436	20.232	19.619	18.597	17.166	15.327	13.079	10.422	7.357	3.883	0
2.0	19.061	18.871	18.299	17.346	16.011	14.296	12.199	9.721	6.862	3.622	0

Table 4.5d: Non – dimensional Critical Buckling Load for Rectangular CSCS Plate under Vibration using polynomial-trigonometric shape function.

Non – dimensional critical buckling load (\bar{N}_x)											
Resonating frequency ratio (n)											
Aspect ratio (P=b/a)	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
1.0	88.832	87.944	85.279	80.837	74.619	66.624	56.852	45.304	31.979	16.878	0
1.1	67.581	66.905	64.878	61.499	56.768	50.686	43.252	34.466	24.329	12.840	0
1.2	53.541	53.005	51.399	48.722	44.474	40.156	34.266	27.306	19.275	10.173	0
1.3	43.883	43.444	42.128	39.934	36.862	32.912	28.085	22.380	15.798	8.338	0
1.4	37.011	36.640	35.530	33.680	31.089	27.758	23.687	18.875	13.324	7.032	0
1.5	31.976	31.166	30.697	29.098	26.860	23.982	20.465	16.308	11.511	6.075	0
1.6	28.194	27.912	27.066	25.657	23.683	21.145	18.044	14.379	10.150	6.075	0
1.7	25.291	25.058	24.279	23.015	21.244	18.968	16.186	12.898	9.105	4.805	0
1.8	23.019	22.789	22.098	20.947	19.336	17.264	14.379	11.740	8.287	4.374	0
1.9	21.211	20.999	20.363	19.302	17.818	15.909	13.575	10.818	7.636	4.030	0
2.0	19.752	19.554	18.961	17.961	16.591	14.814	12.641	10.073	7.111	3.753	0

**Table 4.6a: Non-dimensional Critical Buckling Load for Rectangular CSSS
Plate under Vibration using complete trigonometric shape function.**

Non – dimensional critical buckling load (\bar{N}_x)											
Resonating frequency ratio (n)											
Aspect ratio (P=b/a)	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
1.0	56.429	55.864	54.171	51.350	47.400	42.321	36.114	28.779	20.314	10.721	0
1.1	45.078	44.627	43.275	41.021	37.865	33.808	28.850	22.990	16.228	8.565	0
1.2	37.359	36.985	35.864	33.996	31.381	28.019	23.909	19.053	13.449	7.098	0
1.3	31.905	31.585	30.628	29.033	26.800	23.928	20.419	16.271	11.486	6.062	0
1.4	27.920	27.641	26.803	25.407	23.453	20.940	17.869	14.239	10.051	5.305	0
1.5	24.926	24.677	23.929	22.683	20.938	18.695	15.953	12.712	8.973	4.736	0
1.6	22.623	22.397	21.718	20.587	19.003	16.967	14.479	11.538	8.144	4.298	0
1.7	20.814	20.606	19.982	18.941	17.484	15.611	13.321	10.615	7.493	3.955	0
1.8	19.369	19.175	18.594	17.626	16.270	14.527	12.396	9.878	6.973	3.680	0
1.9	18.195	18.013	17.467	16.558	15.284	13.646	11.645	9.280	6.550	3.457	0
2.0	17.229	17.057	16.540	15.679	14.473	12.922	11.027	8.787	6.203	3.274	0

**Table 4.6b: Non – dimensional Critical Buckling Load for Rectangular CSSS
Plate under Vibration using complete polynomial shape function.**

Non – dimensional critical buckling load (\bar{N}_x)											
Resonating frequency ratio (n)											
Aspect ratio (P=b/a)	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
1.0	56.804	56.237	54.533	51.693	47.716	42.604	36.355	28.971	20.450	10.793	0
1.1	45.192	44.740	43.385	41.125	37.962	33.894	28.923	23.048	16.269	8.587	0
1.2	37.335	36.962	35.842	33.975	31.362	28.002	23.895	19.041	13.441	7.094	0
1.3	31.804	31.486	30.532	28.942	26.716	23.853	20.355	16.220	11.450	6.043	0
1.4	27.779	27.501	26.667	25.278	23.334	20.834	17.778	14.167	10.000	5.278	0
1.5	24.765	24.517	23.774	22.536	20.803	18.574	15.850	12.630	8.915	4.705	0
1.6	22.454	22.230	21.556	20.433	18.862	16.841	14.371	11.452	8.084	4.266	0
1.7	20.646	20.439	19.820	18.787	17.342	15.484	13.213	10.529	7.432	3.923	0
1.8	19.204	19.012	18.436	17.475	16.131	14.403	12.290	9.794	6.913	3.649	0
1.9	18.037	17.856	17.315	16.413	15.151	13.526	11.543	9.199	6.493	3.427	0
2.0	17.078	16.907	16.395	15.541	14.346	12.809	10.930	8.710	6.148	3.245	0

Table 4.6c: Non – dimensional Critical Buckling Load for Rectangular CSSS Plate under Vibration using trigonometric-polynomial shape function.

Non – dimensional critical buckling load (\bar{N}_x)											
Resonating frequency ratio (n)											
Aspect ratio (P=b/a)	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
1.0	56.796	56.228	54.524	51.684	47.708	42.597	36.349	28.966	20.446	10.791	0
1.1	45.182	44.730	43.375	41.115	37.953	33.886	28.916	23.043	16.265	8.585	0
1.2	37.324	36.951	35.831	33.965	31.352	27.993	23.888	19.035	13.437	7.092	0
1.3	31.793	31.475	30.521	28.931	26.706	23.844	20.347	16.214	11.445	6.041	0
1.4	27.767	27.489	26.656	25.268	23.324	20.825	17.771	14.161	9.996	5.276	0
1.5	24.753	24.505	23.763	22.525	20.792	18.565	15.842	12.624	8.911	4.703	0
1.6	22.442	22.218	21.544	20.422	18.851	16.832	14.363	11.445	8.079	4.264	0
1.7	20.633	20.427	19.808	18.776	17.332	15.475	13.205	10.523	7.428	3.920	0
1.8	19.191	18.990	18.424	17.464	16.121	14.394	12.282	9.788	6.909	3.646	0
1.9	18.024	17.849	17.303	16.402	15.140	13.518	11.535	9.192	6.489	3.425	0
2.0	17.066	16.895	16.383	15.530	14.335	12.799	10.922	8.703	6.144	3.243	0

Table 4.6d: Non – dimensional Critical Buckling Load for Rectangular CSSS Plate under Vibration using polynomial-trigonometric shape function.

Non – dimensional critical buckling load (\bar{N}_x)											
Resonating frequency ratio (n)											
Aspect ratio (P=b/a)	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
1.0	56.438	55.874	54.181	51.359	47.408	42.329	36.120	28.783	20.318	10.723	0
1.1	45.085	44.634	43.281	41.027	37.871	33.813	28.854	22.993	16.230	8.566	0
1.2	37.370	36.996	35.875	34.006	31.391	28.027	23.917	19.059	13.453	7.100	0
1.3	31.916	31.597	30.640	29.044	26.810	23.937	20.426	16.277	11.490	6.064	0
1.4	27.932	27.652	26.814	25.418	23.463	20.949	17.876	14.245	10.055	5.307	0
1.5	24.938	24.928	23.941	22.694	20.948	18.704	15.960	12.719	8.978	4.738	0
1.6	22.635	22.409	21.730	20.598	19.014	16.976	14.487	11.544	8.149	4.301	0
1.7	20.827	20.618	19.994	18.952	17.494	15.620	13.329	10.622	7.498	3.957	0
1.8	19.381	19.187	18.606	17.640	16.280	14.536	12.404	9.884	6.977	3.682	0
1.9	18.208	18.026	17.479	16.569	15.294	13.656	11.653	9.286	6.555	3.459	0
2.0	17.242	17.069	16.552	15.690	14.483	12.931	11.035	8.799	6.207	3.276	0

Table 4.7a: Non-dimensional Critical Buckling Load for Rectangular CCCS Plate under Vibration using complete trigonometric shape function.

Non – dimensional critical buckling load (\bar{N}_x)											
Resonating frequency ratio (n)											
Aspect ratio (P= b/a)	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
1.0	89.088	88.197	85.525	81.070	74.834	66.816	57.016	45.435	32.072	16.927	0
1.1	70.642	69.936	67.816	64.284	59.339	52.982	45.211	36.027	25.431	13.422	0
1.2	58.378	57.794	56.043	53.124	49.038	43.784	37.362	29.773	21.016	11.092	0
1.3	49.890	49.391	47.874	45.400	41.907	37.417	31.929	25.444	17.960	9.479	0
1.4	43.812	43.374	42.059	39.869	36.802	32.859	28.040	22.344	15.772	8.324	0
1.5	39.333	38.939	37.759	35.793	33.039	29.499	25.173	20.060	14.160	7.473	0
1.6	35.948	35.589	34.511	32.713	30.197	26.961	23.007	18.334	12.941	6.830	0
1.7	33.336	33.003	32.002	30.336	28.002	25.002	21.335	17.001	12.001	6.334	0
1.8	31.281	30.968	30.030	28.466	26.276	23.461	20.020	15.953	11.261	5.943	0
1.9	29.637	29.341	28.452	26.970	24.895	22.228	18.968	15.115	10.669	5.631	0
2.0	28.303	28.020	27.171	25.756	23.775	21.227	18.114	14.435	10.189	5.378	0

Table 4.7b: Non-dimensional Critical Buckling Load for Rectangular CCCS Plate under Vibration using complete polynomial shape function.

Non – dimensional critical buckling load (\bar{N}_x)											
Resonating frequency ratio (n)											
Aspect ratio (P= b/a)	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
1.0	89.333	88.440	85.760	81.293	75.040	67.000	57.173	45.560	32.160	16.973	0
1.1	71.115	70.404	68.270	64.715	59.737	53.336	45.514	36.269	25.601	13.512	0
1.2	59.047	58.456	56.685	53.732	49.599	44.285	37.790	30.114	21.257	11.219	0
1.3	50.724	50.216	48.695	46.158	42.608	38.043	32.463	25.869	18.260	9.637	0
1.4	44.785	44.337	42.994	40.755	37.620	33.589	28.663	22.840	16.123	8.509	0
1.5	40.424	40.020	38.807	36.786	33.956	30.318	25.871	20.616	14.553	7.681	0
1.6	37.140	36.768	35.654	33.797	31.197	27.855	23.769	18.941	13.370	7.057	0
1.7	34.613	34.266	33.228	31.497	29.075	25.959	22.152	17.652	12.461	6.576	0
1.8	32.631	32.304	31.325	29.694	27.410	24.473	20.884	16.642	11.747	6.200	0
1.9	31.050	30.740	29.808	28.256	26.082	23.286	19.872	15.836	11.178	5.900	0
2.0	29.771	29.473	28.580	27.091	25.008	22.328	19.053	15.183	10.718	5.656	0

Table 4.7c: Non-dimensional Critical Buckling Load for Rectangular CCCS Plate under Vibration using trigonometric-polynomial shape function.

Non – dimensional critical buckling load (\bar{N}_x)											
Resonating frequency ratio (n)											
Aspect ratio (P=b/a)	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
1.0	85.462	84.607	82.043	77.770	71.788	64.096	54.696	43.585	30.766	16.238	0
1.1	65.833	67.154	65.119	61.728	56.979	50.874	43.413	34.595	24.420	12.888	0
1.2	56.137	55.576	53.892	51.085	47.155	42.103	35.928	28.630	20.209	10.666	0
1.3	48.060	47.579	46.137	43.734	40.370	36.045	30.758	24.510	17.302	9.131	0
1.4	42.288	41.866	40.597	38.482	35.522	31.716	27.065	21.567	15.224	8.035	0
1.5	38.044	37.663	36.522	34.620	31.957	28.533	24.348	19.402	13.696	7.228	0
1.6	34.843	34.495	33.449	31.707	29.268	26.132	21.814	17.770	12.544	6.620	0
1.7	32.377	32.053	31.082	29.463	27.197	24.283	20.721	16.512	11.656	6.152	0
1.8	30.441	30.136	29.223	27.701	25.570	22.830	19.482	15.525	10.959	5.784	0
1.9	28.894	28.605	27.739	26.294	24.271	21.671	18.492	14.736	10.402	5.490	0
2.0	27.642	27.365	26.536	25.154	23.219	20.731	17.691	14.097	9.951	5.252	0

4.1.4 Results of Comparison of the Non – dimensional Critical Buckling

Loads from this present study and those of previous study.

The comparison of the non – dimensional buckling loads at zero vibration between this present study and previous study were presented on Table 4.8, Table 4.9, Table 4.10, Table 4.11, Table 4.12 and Table 4.13.

Table 4.8: Non – dimensional critical buckling loads (\bar{N}_x) for SSSS Rectangular plates under uniform unilateral stress (n = 0)

Aspect ratio (P =b/a)	Ibearugbulem et al. (2014) N_p	Present result N_E	Percentage difference $\frac{ N_p - N_E }{N_E} \times \frac{100}{1}$
1.0	39.507	39.507	0.000
1.1	32.948	32.948	0.000
1.2	28.358	28.358	0.000
1.3	25.025	25.024	0.004
1.4	22.528	22.527	0.004
1.5	20.609	20.609	0.000
1.6	19.103	19.102	0.005
1.7	17.898	17.897	0.006
1.8	16.917	16.917	0.000
1.9	16.110	16.109	0.006
2.0	15.436	15.435	0.006

Table 4.9: Non - dimensional critical buckling loads (\bar{N}_x) for CCCC Rectangular plates under uniform unilateral stress (n = 0)

Aspect ratio (P = b/a)	Ibearugbulem et al. (2014) N_P	Present study N_E	Percentage difference $\frac{ N_P - N_E }{N_E} \times \frac{100}{1}$
1.0	108.000	108.667	0.614
1.1	90.521	91.082	0.616
1.2	78.921	79.415	0.622
1.3	70.907	71.357	0.631
1.4	65.178	65.598	0.640
1.5	60.963	61.362	0.650
1.6	57.784	58.167	0.658
1.7	55.333	55.706	0.670
1.8	53.408	53.773	0.679
1.9	51.871	52.229	0.685
2.0	50.625	50.979	0.694

Table 4.10: Non - dimensional critical buckling loads for CCSS rectangular Plates under uniform unilateral stress (n = 0)

Aspect ratio (P = b/a)	Ibearugbulem et al. (2014) N_P	Present study N_E	Percentage difference $\frac{ N_P - N_E }{N_E} \times \frac{100}{1}$
1.0	64.736	64.737	0.002
1.1	54.133	54.134	0.002
1.2	46.916	46.917	0.002
1.3	41.806	41.806	0.000
1.4	38.066	38.067	0.003
1.5	35.253	35.253	0.000
1.6	33.074	33.086	0.036
1.7	31.382	31.382	0.000
1.8	30.018	30.018	0.000
1.9	28.910	28.910	0.000
2.0	27.997	27.997	0.000

Table 4.11: Non - dimensional critical buckling loads for CSCS rectangular**Plates under uniform unilateral stress (n = 0)**

Aspect ratio (P = b/a)	Ibearugbulem et al. (2014) N_P	Present study N_E	Percentage difference $\frac{ N_P - N_E }{N_E} \times \frac{100}{1}$
1.0	85.064	84.941	0.145
1.1	64.688	64.591	0.150
1.2	51.251	51.172	0.154
1.3	42.028	41.961	0.159
1.4	35.477	35.418	0.166
1.5	30.687	30.635	0.169
1.6	27.096	27.048	0.177
1.7	24.344	24.300	0.181
1.8	22.195	22.154	0.185
1.9	20.488	20.448	0.195
2.0	19.111	19.074	0.194

Table 4.12: Non – dimensional critical buckling loads for CSSS rectangular**Plates under uniform unilateral stress (n = 0)**

Aspect ratio (P = b/a)	Ibearugbulem et al. (2014) N_P	Present study N_E	Percentage difference $\frac{ N_P - N_E }{N_E} \times \frac{100}{1}$
1.0	56.802	56.804	0.004
1.1	45.190	45.192	0.004
1.2	37.334	37.335	0.003
1.3	31.803	31.804	0.003
1.4	27.777	27.779	0.007
1.5	24.764	24.765	0.004
1.6	22.454	22.454	0.000
1.7	20.645	20.646	0.005
1.8	19.203	19.204	0.005
1.9	18.036	18.037	0.006
2.0	17.078	17.078	0.000

**Table 4.13: Non - dimensional critical buckling loads for CCCS rectangular
Plates under uniform unilateral stress (n = 0)**

Aspect ratio (P = b/a)	Ibearugbulem et al. (2014) N_P	Present study N_E	Percentage difference $\frac{ N_P - N_E }{N_E} \times \frac{100}{1}$
1.0	89.354	89.333	0.024
1.1	71.131	71.115	0.022
1.2	59.060	59.047	0.022
1.3	50.735	50.724	0.022
1.4	44.795	44.785	0.022
1.5	40.433	40.424	0.022
1.6	37.148	37.140	0.022
1.7	34.620	34.613	0.020
1.8	32.637	32.631	0.018
1.9	31.056	31.050	0.019
2.0	29.777	29.771	0.020

4.2 Discussion of Results

4.2.1 Discussion of result for Critical Buckling Equation of a Rectangular Plate under Vibration.

Comparing the critical buckling load equation of previous study given in Equation (2.1) as:

$$N_x = \frac{\frac{D}{a^2} \left[K_x + \frac{2}{P^2} K_{xy} + \frac{K_y}{P^4} \right]}{K_{Nx}} \quad (4.2)$$

With that of this present study presented in Equation (4.1) as:

$$N_x = \frac{\frac{D}{a^2} \left[K_x + \frac{2}{P^2} K_{xy} + \frac{K_y}{P^4} \right]}{K_{Nx}} (1 - n^2) \quad (4.3)$$

It is noticed that at zero vibration ($n = 0$), the critical buckling equations are the same but that of this present study differ from the one given by previous study at vibration frequencies ($n \neq 0$). Hence, this research work considered the effect of vibration on buckling of rectangular plates.

4.2.2 Discussion of results for derived Shape Functions for Rectangular Plates.

The various shape functions of the different rectangular plates such as SSSS, CCCC, CCSS, CSCS, CSSS and CCCS plates given in Table 4.1 were derived using complete trigonometric, complete polynomial, trigonometric – polynomial or polynomial - trigonometric functions. Ibearugbulem et al. (2016) used polynomial - trigonometric shape functions and complete polynomial shape

functions to determine the non – dimensional critical buckling loads for SSSS and CCCC rectangular plates respectively. A comparison of their results with this present study showed that other shape functions were derived for other rectangular plates limited to that of their study.

4.2.3 Discussion of results for Non – dimensional Critical Buckling Loads

Of various Rectangular Plates

It was seen in Table 4.2, Table 4.3, Table 4.4, Table 4.5, Table 4.6 and Table 4.7 that as aspect ratio ($P = b/a$) increases from 1.0 to 2.0 at each resonating frequency ratio (n) ranging from 0 to 0.9, non – dimensional critical buckling load, \bar{N}_x decreases, thereby causing the plate to get more slender. Furthermore, as resonating frequency ratio (n) increases from 0 to 0.9 at each aspect ratio ($P = b/a$) ranging from 1.0 to 2.0, non – dimensional critical buckling load also decreases, thereby weakening the strength of the plate and hence requires relatively less effort to cause it to buckle. At resonating frequency ratio (n = 1) at all aspect ratios, non – dimensional critical buckling load, $\bar{N}_x = 0$. Hence, the plate buckles without any buckling load.

It was also noticed from the tables mentioned above that plate of particular boundary condition shows very slight differences in their critical buckling loads at a specific aspect ratio and resonating frequency ratio as a result of various shape functions but in the graphical representation given in Figures A, Figure B,

Figure C, Figure D, Figure E and Figure F, the non – dimensional critical buckling loads showed no differences. This shows that the various shape functions of plate of particular boundary condition seems to agree with one another in rectangular plates.

4.2.4 Discussion of results for comparison of the non – dimensional

Buckling loads from this present study and those of previous study.

We can see in Table 4.8, Table 4.9, Table 4.10, Table 4.11, Table 4.12 and Table 4.13 that the non – dimensional critical buckling load results of this present work agree very well with the established (previous) results of uniaxially loaded rectangular plates subjected to uniform stress along the X – axis ($n = 0$) for various aspect ratios ($1 \leq P \leq 2$). Given that non – dimensional critical buckling load results at $n = 0$ agreed, it also follows that the results of non – dimensional critical buckling load for other n – values and their corresponding aspect ratios presented in Table 4.2b, Table 4.3b, Table 4.4b, Table 4.5b, Table 4.6b and Table 4.7b for which there are no existing results in literature are also corrected.

CHAPTER FIVE

CONCLUSIONS AND RECOMMENDATIONS

5.1 CONCLUSIONS

From the results obtained and the discussions made, it can be concluded that the deficiencies and difficulties facing using other methods of plate analysis have been overcome by split – deflection method.

The critical buckling load equation for thin rectangular plates under vibration has been derived.

The various shape functions for rectangular plates such as complete trigonometric, complete polynomial, trigonometric – polynomial and polynomial – trigonometric functions has been derived.

The non – dimensional critical buckling loads for various rectangular plates have been obtained.

Increase in aspect ratio decreases buckling load of rectangular plate.

Increase in vibration decreases buckling load of rectangular plate.

Finally, the results of this study excellently agree with solution from previous studies.

5.2 RECOMMENDATIONS

Based on the conclusions made, the following recommendations are drawn;

- (i) This research work could be used for reference purposes in future researches.
 - (ii) Similar research works could consider analyzing thin rectangular plates to cover aspect ratios outside ($1 \leq P \leq 2$) and resonating frequency ratios ($0 \leq n \leq 1$)
 - (iii) The boundary conditions considered in this research work are either clamped or simply supported or a combination of clamped and simply supported edges, other researchers could consider analyzing similar works on plates having free edges.
 - (iv) This research work considered thin plates, other researchers could consider analyzing similar works on thick plates.
 - (v) This research work considered thin plates with small deflection, other researcher could consider analyzing similar works thin plates with large deflection.
 - (vi) This research work considered rectangular plates, other researchers could consider analyzing similar works on circular or elliptical shaped plates.
 - (vii) This present method could be extended to refined plate theory analysis
- (RPT)

5.3 CONTRIBUTIONS TO KNOWLEDGE

This work has contributed the following to knowledge;

(i) This work introduced new deflection functions as product of two beam deflection equations with one being in the polynomial family and the other in trigonometric family. General variational approach was applied to derive two governing equations through which the trigonometric deflection functions that give exact solutions in thin rectangular plate analysis were obtained. The deflection equations are for ccss, cscs, csss and cccs plates. These deflection functions improve the efficiency of the plate analysis in terms of accuracy and speed.

S/N	Plate	Deflection function
1	ccss	$w = A(k - kR - K\cos kR + \sin kR)(1.5Q^2 - 2.5Q^3 + Q^4)$ (Where: $k = 4.49340946$) Or $w = A(1.5R^2 - 2.5R^3 + R^4)(k - kQ - K\cos kQ + \sin kQ)$
2	cscs	$w = A\sin\pi R(Q^2 - 2Q^3 + Q^4)$ Or $w = A(R - 2R^3 + R^4)(1 - \cos 2\pi Q)$
3	csss	$w = A(R - 2R^3 + R^4)(k - kR - K\cos kR + \sin kR)$ (Where: $k = 4.49340946$) Or $w = A\sin\pi R(1.5Q^2 - 2.5Q^3 + Q^4)$
4	cccs	$w = A(k - kR - K\cos kR + \sin kR)(1 - \cos 2\pi Q)$ (Where: $k = 4.49340946$) Or $w = A(1.5R^2 - 2.5R^3 + R^4)(1 - \cos 2\pi Q)$

(ii) This work also came up with a formula for calculating the buckling load of a rectangular thin plate under vibration. The formula is:

$$N_x = \frac{\frac{D}{a^2} \left[K_x + \frac{2}{P^2} K_{xy} + \frac{K_y}{P^4} \right]}{K_{Nx}} (1 - n^2)$$

Where n is the ratio of forcing frequency to natural frequency. Forcing frequency is the frequency of the plate in the presence of the forcing force whereas the natural frequency is the frequency of the plate in the absence of excitation (perturbation) force. Unlike design codes that provides factor of safety for practical applications in considering the effect of vibration on buckling of plates which is based on assumption, this work provides exact approach in determining buckling of thin rectangular plate under known vibrational frequency.

(iii) It has provided the non – dimensional critical buckling loads for thin rectangular plates under vibration with various boundary conditions.

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APPENDICES

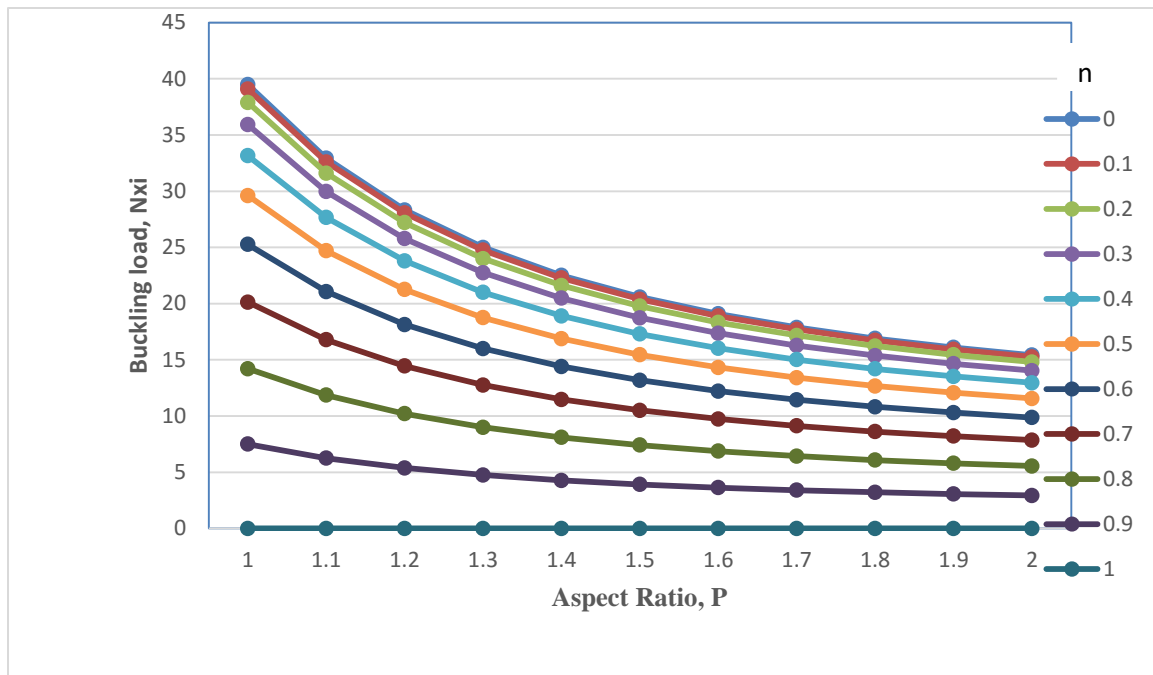


Figure A: Non – dimensional Critical buckling load for SSSS rectangular Plate under Vibration.

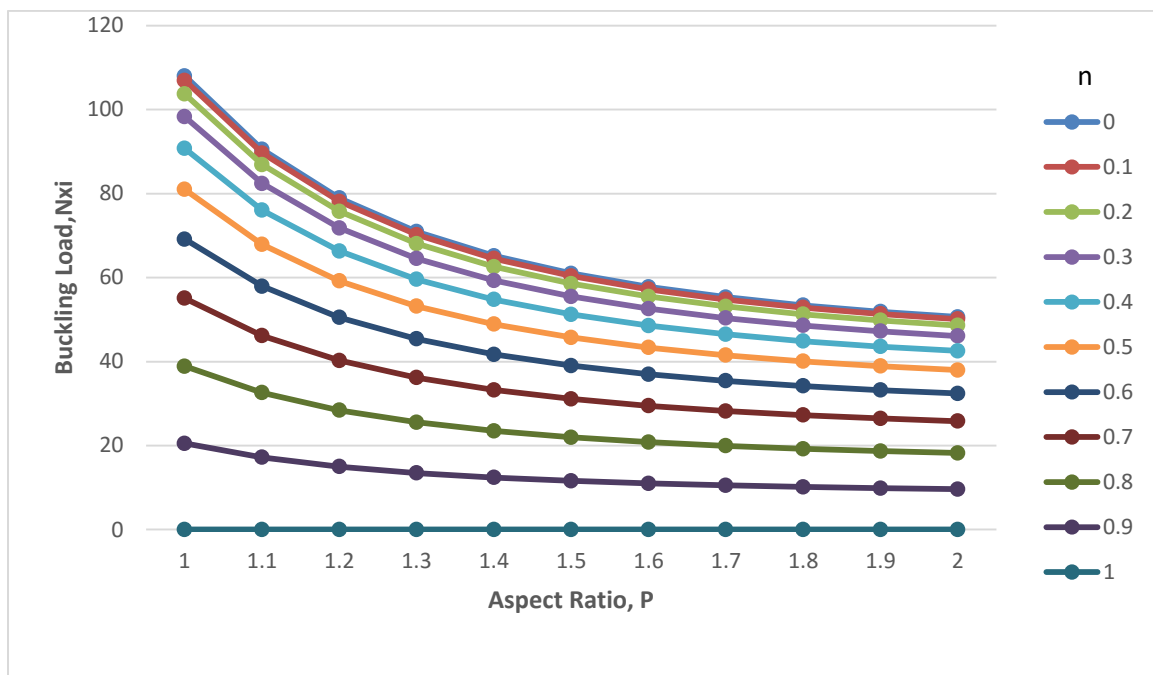


Figure B: Non - dimensional Critical buckling load for CCCC rectangular Plate under Vibration.

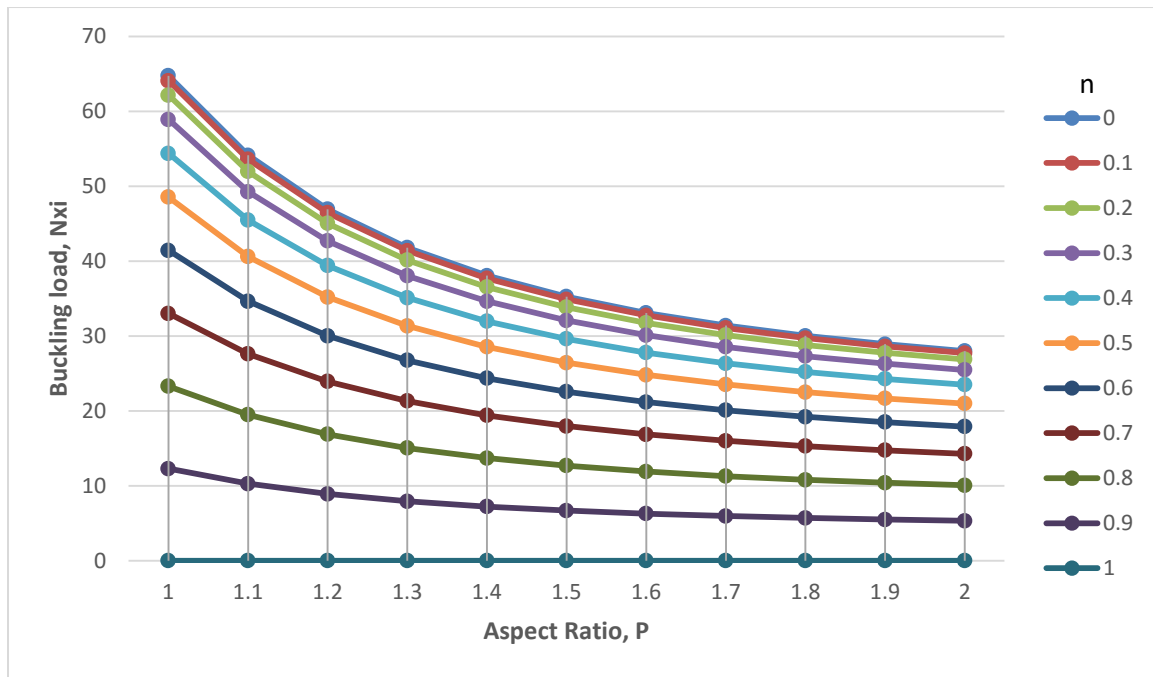


Figure C: Non – dimensional Critical buckling load for CCSS rectangular Plate under Vibration.

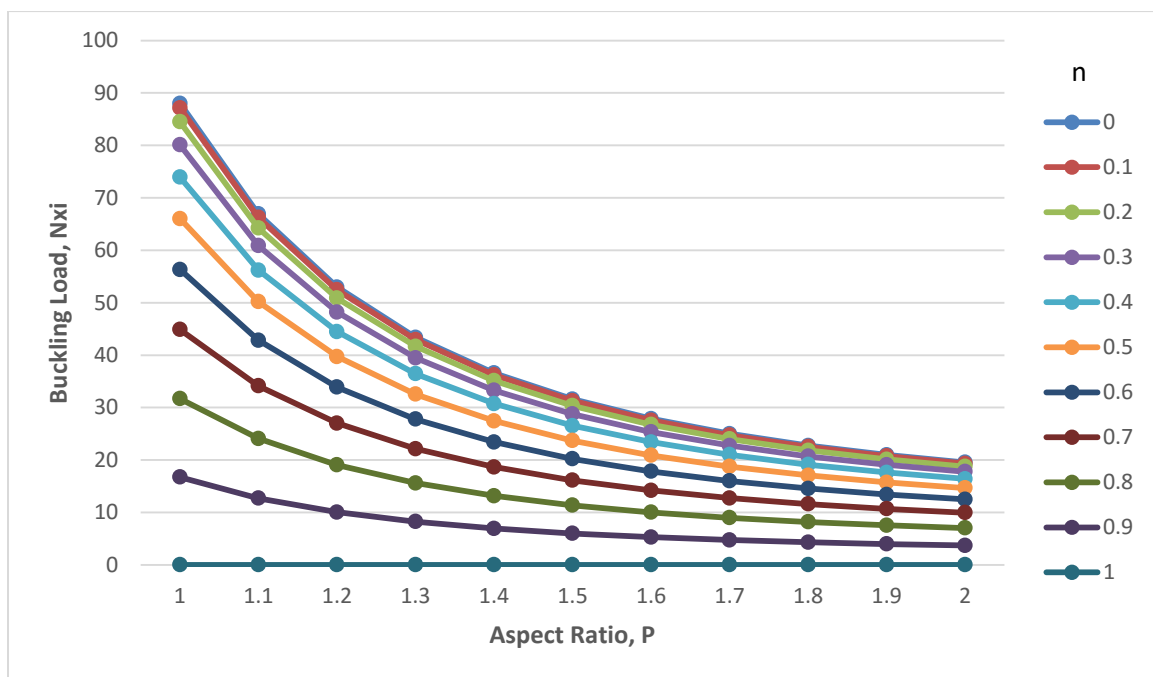


Figure D: Non – dimensional Critical buckling load for CSCS rectangular Plate under Vibration.

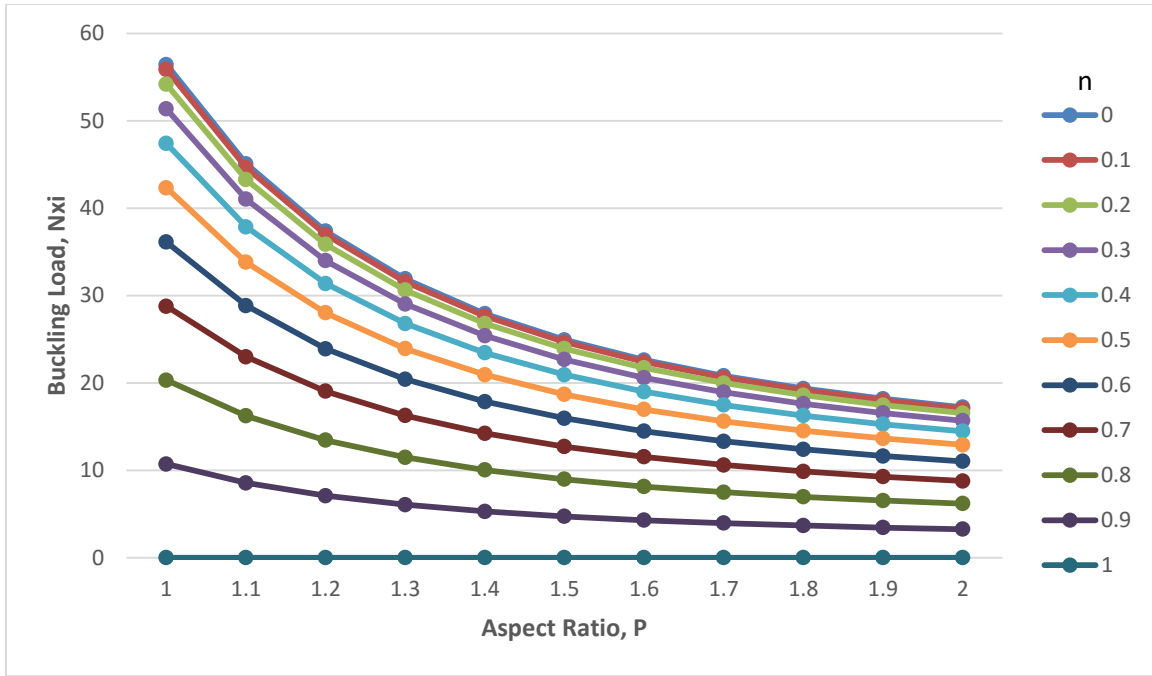


Figure E: Non – dimensional Critical buckling load for CSSS rectangular plate Under Vibration.

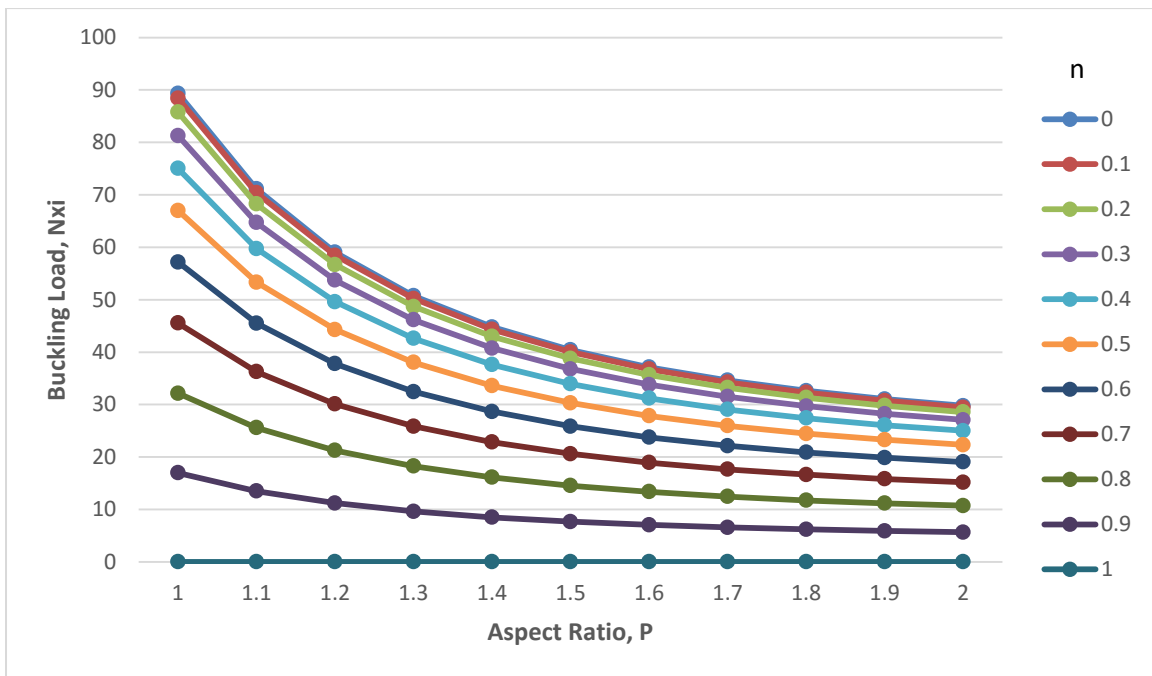


Figure F: Non – dimensional Critical buckling load for CCCS rectangular Plate under Vibration.