SOME DEDUCTIONS FROM THE FACTORIZATION OF FINITE SIMPLE GROUPS

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CERTIFICATION

This is to certify that this work entitled "Some deductions from the factorization of finite simple groups" was carried out by Agigor-Mike, Precious Ugonwanyi (20134868768) in partial fulfillment of the requirements for the award of the degree of Master of Science (M.Sc.-) in Pure Mathematics in the Department of Mathematics, Federal University of Technology, Owerri, Imo State, Nigeria.

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DEDICATION

This work is dedicated to Almighty JEHOVAH, my help in ages past and hope for years to come; CHRIST my solid rock, the song that I sing and my eternal destination; and the HOLY GHOST my comforter and my teacher.

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NOTATIONS

NOTATIONS

$\mathbb{N},\ \mathbb{Z}$	natural numbers and integers
$\mathbb{Q},\ \mathbb{R},\ \mathbb{C}$	rational, real and complex numbers
$\mathbb{F}_q$	field with q elements
K^d	vector space of dimension d over K
$Sym\ (\Omega), Alt\ (\Omega)$	symmetric and alternating groups on Ω
S_n, A_n	symmetric and alternating groups of degree n
C_n	cyclic group of order n
$GL_d(K),\ SL_d(K) or\ SL(d,K)$	Linear groups over K
$PGL_d(K) or\ PGL(d,K),\ PSL_d(K) or PSL(d,K)$	projective groups over K
$Sp_{2m}(K), Sp_{2m}(2)$	sympletic groups over K
$PGU_3(q), PSU_3(q)$	unitary groups over K
$Sz(2^s),\ \ R(3^s)$	Suzuki and Ree groups
$M_{10}, \ldots\ldots, M_{24}$	Mathieu groups
$Inn(\Phi), Ker(\Phi)$	image and kernel of Φ
$Aut\ (X)$	automorphism group of X
$Inn(G)$	inner automorphism group of G
$Out(G)$	outer automorphism group of G
$Soc(G)$	socle of G
$N_G(H)$	normalizer of H in G
$C_G(H)$	centralizer of H in G
$H \leq G,\ N \lhd G$	subgroup , normal subgroup
$G \times H,\ G^m$	direct product, direct power
$G \rtimes H$	semidirect product
σ_i	ith projection map
$\lvert A\rvert,\ \lvert x\rvert$	the order of the set A, the order of the element x
$Z(G)$	the centre of the group G

ABSTRACT

This work looked into the factorization of minimal normal subgroups of innately transitive groups. It was shown that ifπ_j, $j \in \mathbb{N}$, is a finite set of primes and if M is a non-abelian characteristically simple group with simple normal subgroups T_1, $T_2, \cdots T_k$ and $(M, \{K_1, K_2\})$ is a full factorization of M then (i) the pair $(T_i, \{\sigma_i(K_1), \sigma_i(K_2)\})$ for $i = 1, \dots, k$ forms π_j-groups (ii) the subgroups $\sigma_i(K_1)$ and $\sigma_i(K_2)$ cannot have normal π-complements in T_i and that $\sigma_i(K_1)$ and $\sigma_i(K_2)$ are not π- complements of each other. Again from the results of Baddeley and Praeger it showed that K_1 and K_2 are full diagonal subgroups of M. If K_1 and K_2 are self normalizing subgroups in M and not isomorphic to A_5, $PSL_2(11), M_{11}$, $Sp_2(q^2).2$, $Sp_2(q^2)$, then K_1 and K_2 are maximal nilpotent subgroups of M. Further, K_1 and K_2 are conjugates and therefore solvable groups. It further showed that if G is a finite simple group with a transitive minimal normal subgroup M where M is characteristically simple and can be expressed as $M = T_1 \times T_2 \times \cdots \times T_k \cong T^k$ and if K_1 and K_2 are proper subgroups of M, then $(M, \{K_1, K_2\})$ is a full factorization such that for all i, the pair $(T_i, \{\sigma_i(K_1), \sigma_i(K_2)\})$ is a full factorization and $\sigma_i(K_1)$ and $\sigma_i(K_2)$ are conjugates $K_1 \cap K_2 = \{1\}$, $C_M(K_1 \cap K_2) = \{1\}$ and $Z(K_1 \cap K_2) = C_M(K_1 \cap K_2) \cap (K_1 \cap K_2) = \{1\}$.

Keywords: Minimal normal subgroups, finite simple groups, centralizers, normalizers.

CHAPTER ONE

INTRODUCTION

1.1 Background Of The Study

The study of permutation groups is an old subject with a rich history, stretching back to the origins of group theory in the early 19th century. Galois introduced the notion of a group in his study of the permutations of roots of polynomial equations (the familiar Galois group of the polynomial), and groups of substitutions (what we now call Permutation groups) were a focus of interest for much of the 19th century. Of course, the modern notion of a permutation group is extremely flexible, and they arise naturally throughout mathematics, with important applications across the sciences (for example, combinatorics in mathematics) (Burness, 2014).

For instance, given any mathematical object or structure Σ (for example vector space, group, graph, topological space, etc) based on a set of points Ω (for example, vectors, group elements, vertices, points, etc.) then the set $Aut(\Sigma)$ of automorphisms (or symmetries) of Σ (that is, the bijective maps $f\colon \Omega \rightarrow \Omega$ such that f and f^{-1} preserve the structure of Σ) is a permutation group on Ω. That is, $Aut(\Sigma)$ is a group of bijections from Ω to itself.

A more abstract (modern) concept of a group is a simple result of Cayley's Theorem which states that, every group can be viewed as (imbedded into) a permutation group on some set (Burness, 2014).

The largest achievement in finite (abstract) group theory in the last half century (and possibly ever) is the classification of all finite simple groups. Its proof spans thousands of pages and uses the research of hundreds of mathematicians. The classification has been used to solve many open problems in group theory. One example is the factorization of finite simple and characteristically simple groups (Fawcett, 2009).

Finite simple groups are important to mathematicians because in a certain sense they are the basic building blocks of all finite groups, somewhat similar to the way the prime numbers are the basic building blocks of the natural numbers. However, a significant difference with respect to the case of integer factorization is that such "building blocks" do not necessarily determine uniquely a group, since there might be many non-isomorphic groups with the same composition series or, put in another way, the extension problem does not have a unique solution (Dixon and Mortimer, 1996).

In this work, we study the factorizations of simple and characteristically simple groups and prove some results about normalizers of subgroups of characteristically simple groups already studied by Baddeley *et al.,* (2004). We go ahead to extend their results and further prove some results about centralizers of subgroups of characteristically simple groups.

1.2 Statement Of The Problem

Group theory, a branch of Abstract Algebra has been a subject of debate and confusion from inception. With the introduction of the concept of a normal subgroup by Evaristes Galois in 1832 (Warwick, 2014), and the discovery of simple groups, came new higher challenges namely:

(i) What axioms can a group possess and it will be classified as simple?

(ii) Are these simple groups factorable and if possible how?

(iii) Does the factorization of these simple groups give us better or more insights into the structure of these groups?

(iv) What is the nature of the normalizers and centralizers of proper subgroups of some finite simple groups?

(v) What is the role of the factorization of finite simple groups in explaining or understanding these concepts of normalizers and centralizers?

(vi) How does the knowledge of the nature of the normalizers and centralizers of these factored groups help us in understanding the nature of the groups?

These and more are the issues we wish to investigate in this work.

1.3 Objectives Of The Study

Factorization of simple and characteristically simple group has been a favourite topic of research among mathematicians for years now. Our major objectives in this study include among others to:

(i) investigate the implications of factorization of characteristically simple groups

(ii) extend the results of the normalizers of subgroups of characteristically simple groups to those of centralizers.

1.4 Significance Of The Study

Simple groups are important because they play a role in group theory somewhat analogous to that which the primes play in number theory or the elements do in chemistry; that is, they serve as the "building blocks" for all groups. These "building blocks" are called the composition factors of the group. These composition factors are simple groups. In a certain sense, a group can be reconstructed from its composition factors and many of the properties of a group are determined by the nature of its composition factors. Many questions about finite groups can be reduced (by induction) to questions about simple groups. This fact makes the classification theorem to have applications in many branches of mathematics. Factorization of groups, on the other hand also play a constitutive role in group theory and geometry. Ever since the classification of the finite simple groups was first announced in February, 1981, it has answered many questions and has been consequential to so many theorems. Simple group classification has stunning applications in Algebraic Graph Theory and many new applications are accompanied by deep new results on the structure and properties of the simple groups. There are recent exciting developments related to expander graphs. Expander graphs are graphs or networks which are simultaneously sparse and highly connected. They have important applications for design and analysis of robust communication networks, for the theory of error-correcting codes, the theory of pseudo-randomness and many other uses, beautifully surveyed in Hoory *et al.*, (2006).

1.5 Scope Of The Study

We based our work on the results stemming from full factorization and strong multiple factorizations of finite simple groups.

1.6 Definition Of Some Basic Group Theoretic Terms

The main references for this section are: Conway *et al.,* (1985), Dixon and Mortimer (1996), Gorenstein and Solomon (1994), Ash (200), Dummit *et al.,* (2004), Bamberg (2008), Fawcett (2009), and Obi (2014).

1.6.1 Simple Groups

A group G is a simple group if and only if there are no non-trivial normal subgroups of G. That is, a group G is simple if and only if $N \trianglelefteq G$ implies that $N = \{e\}\ or\ N = G$.

Let G be a group. A composition series of a group G is a sequence of subgroups $\{1\} = G_n \trianglelefteq G_{n-1} \trianglelefteq \cdots \trianglelefteq G_0 = G$

where G_{i+1} is a maximal normal subgroup of G_i for each i, G_i/G_{i+1} is simple for each i. The factors G_i/G_{i+1} are called composition factors, and n is the length of the series. Moreover, it is easy to see that every finite group has a composition series: G must contain a maximal normal subgroup G_1; if G_1 is trivial, we are done, and if not, then G_1 contains a maximal normal subgroup, and so on. This process must terminate since G is finite. A finite group is solvable if all of its composition factors have prime order, thus, solvable groups are not simple.

A finite simple group is either cyclic of prime order, the alternating group A_n, when $n \geq 5$, a group of Lie type, or one of 26 sporadic groups.

As mentioned earlier in the introduction, the original proof was based on extensive research by numerous mathematicians, the completion of this immense result was first announced by Gorenstein in 1983 .

The proof is now being rewritten in a more concise and self-contained form.

We will very briefly outline the various types of finite simple groups.

If G is simple and abelian, then G is a cyclic group of prime order.

The alternating group A_n is a nonabelian simple group for $n \geq 5$.

The simple groups of Lie type can be characterized as groups of fixed points of endomorphisms of linear algebraic groups over an algebraically closed field of characteristic p, and they consist of several infinite families of groups.

Some of the groups of Lie type involve families of well-known classical groups: Linear groups, Unitary groups, Symplectic groups and Orthogonal groups.

We give few details on the classical groups of Lie type.

Consider the General linear group of $n \times n$ invertible matrices over the finite field $\mathbb{F}_q$, denoted by $GL_n(q)$. The Special linear group, denoted by $SL_n(q)$, is the set of all matrices of determinant one, and is actually a normal subgroup of $GL_n(q)$. The Projective Special Linear Group, denoted by $PSL_n(q)$, simply $SL_n(q)/Z\big(SL_n(q)\big)$. In fact, $Z\big(SL_n(q)\big)$ consists of the scalar matrices of $SL_n(q)$.

$PSL_n(q)$ is simple if $n \geq 2$ except $n = 2$ and $q = 2$ or 3; it is called a linear group within the world of finite simple groups.

The General Unitary Group $GU_n(q)$ is the group of matrices $M \in GL_n(q^2)$ such that $M^{-1} = (\bar{M})^t$, where $\bar{M}$ is simply M with every entry raised to the q-th power. The Special Unitary Group $SU_n(q)$ is then the subgroup of $GU_n(q)$ consisting of those matrices with determinant one, and the Projective Special Unitary group $PSU_n(q)$ is $SU_n(q)$ factored out by its scalar matrices.

$PSU_n(q)$ is simple if $n \geq 2$ except when $q = 2$ and $n = 2$ or 3 or when $q = 3$ and $n = 2$; it is called a Unitary Group within the world of finite simple groups.

Symplectic group $Sp_{2n}(q)$ is a subgroup of $SL_{2n}(q)$. Projective Symplectic group $PSp_{2n}(q) \coloneqq Sp_{2n}(q)/Z\big(Sp_{2n}(q)\big)$, where $Z\big(Sp_{2n}(q)\big) = \{\pm E_n\}$.

General Orthogonal groups are $GO_{2n+1}(q)$, $GO^+_{2n}(q), GO^-_{2n}(q)$.

As in the linear case: $SO_n(q), PGO_n(q), PSO_n(q)$, where

$\mathrm{Z}\big(GO_n(q)\big) = \{\pm E_n\}$, and where $g.J.g^{tr} = J$ for J being the Gram matrix, implies $\det(g)^2 = 1$ for all $g \in GO_n(q)$. But $PSO_n(q)$ is in general not perfect.

The finite groups of Lie type also involve families of the Exceptional groups: for q a prime power, $f \geq 1$;

$E_6(q), E_7(q), E_8(q)$, $F_4(q)$, $G_2(q)(q \neq 2)$; Steinberg groups ${}^2E_6(q^2)$; Steinberg triality groups ${}^3D_4(q^3)$; Suzuki groups ${}^2B_2(2^{2f+1})$; Small Ree Groups ${}^2G_2(3^{2f+1})$; Large Ree Groups ${}^2F_4(2^{2f+1})$; Tits group ${}^2F_4(2)'$.

There are also 26 sporadic groups which do not fit into any infinite family of non-abelian simple groups. The first five of these groups were discovered by Mathieu in 1860's but most of the remaining sporadic groups were discovered through attempts to prove the classification of the finite simple groups.

The list of the sporadic groups include:

- Mathieu groups M_{11}, M_{12}, M_{22}, M_{23}, M_{24}
- Leech lattice groups:

  Conway groups Co_1, Co_2, Co_3.

  Mc Laughlin group McL. Higman-Sims group HS. Suzuki group Suz. Hall-Janko group J_2.
- Fischer groups Fi_{22}, Fi_{23}, Fi_{24}.
- Monstrous groups:

  Fischer –Griess Monster M

  Baby Monster B. Thompson group Th.

  Harda – Norton group HN. Held group He.
- Pariahs:

  Janko groups J_1, J_3, J_4. $O'Nan$ group ON

  Lyons group Ly. Rudvalis group Ru.

1.6.2 Characteristically Simple Group

A subgroup H of a group G is called characteristic if H is invariant under all automorphisms of G. That is to say, a subgroup H of G is characteristic in G, denoted by H char G, if $H\gamma = H$ for all $\gamma \in Aut(G)$. To show that H char G, it suffices to show that $H\gamma \leq H$ for all $\gamma \in Aut(G)$ (since then $H\gamma^{-1} \leq H$, which implies that $H = (H\gamma^{-1})\gamma \leq H\gamma$). A group G is characteristically simple if it has no proper non-trivial characteristic subgroup. The structures of the finite characteristically simple groups are well known. They are a direct product of finitely many isomorphic copies of a simple group. Characteristically simple groups are sometimes also referred to as elementary groups. In particular, a finite solvable group is characteristically simple if and only if it is an elementary abelian group. Characteristically simple is a weaker condition than being a simple group, as simple groups must not have any proper non-trivial normal subgroups, which include characteristic subgroup. That is to say that simple groups are also characteristically simple group.

Note that since conjugation by an element of G is an automorphism of G, $H\ char\ G$ implies that H is normal in G (the converse is not necessarily true).

A nontrivial group G is characteristically simple if G has no proper nontrivial characteristic subgroups.

1.6.3 Group Factorization

A group factorization is a pair $(G, \{A, B\})$ where G is a group and A, B are subgroups of G such that $B = G$; and every element $g \in G$ can be represented uniquely as $g = ab$ where $a \in A$, $b \in B$. In other words a group is said to be factorizable if $G = AB$ for some proper subgroups A and B of G. The expression $G = AB$ is called a factorization of G. A factorization is called non-trivial if both A and B are proper subgroups. In this thesis we only consider non-trivial factorizations.

1.6.4 Cyclic Group

A cyclic group is a group that is generated by a single element. A group G is called *cyclic* if there exists an element $g \in G$ such that $G = \langle g \rangle$. It is finite if for some least positive integer n, $g^n = e$. Otherwise it is infinite. If G is a finite group whose order is a prime number, then G is a cyclic group.

1.6.5 Matrix Group

The Matrix group or Linear group consists of invertible matrices of a given order n over a field K, which is closed under product and inverse. Such a group acts on the n-dimensional vector space K^n by linear transformation. This action makes matrix groups conceptually similar to permutation groups, and the geometry of the action may be usefully exploited to establish properties of a group G.

1.6.6 Abstract Group

Most groups considered in the first stage of the development of group theory were concrete, having been realized through numbers, permutations, or matrices. A typical way of specifying an abstract group is through a presentation by generators and relations, $G = \langle S/R \rangle$. A significant source of abstract group is given by the construction of a factor group, or quotient group G/H of a group G by a normal subgroup H. The change of perspective from concrete to abstract groups makes it natural to consider properties of groups that are independent of a particular realization, as well as the classes of groups with such given property: finite groups, periodic groups, simple groups, solvable groups and so on.

1.6.7 Lie Group

A Lie group is a group that is also a differentiable manifold, with the property that the group operations are compatible with the smooth structure. Lie groups are named after Sophus Lie, who laid the foundations of the theory of continuous transformation groups. The term *groups de Lie* first appeared in French in 1893 in the thesis of Lie's student Arthur Tresse. Lie groups represent the best-developed theory of continuous symmetry of mathematical objects and structures, which makes them indispensable tools for many parts of contemporary mathematics, as well as in modern theoretical physics. They provide a natural framework for analyzing the discrete symmetries of algebraic equations. An extension of Galois's theory to the case of continuous symmetry groups was one of Lie's principal motivations.

1.6.8 Commutator

Let G be a group, let $x, y \in G$ and let A, B be non-empty subsets of .

1. Define $[x, y] = x^{-1}y^{-1}xy$, called the commutator of x and y.

2. Define $[A, B] = \langle [a, b] | \, a \in A, b \in B \rangle$, the group generated by commutators of elements from A and from B.

3. Define $G' = \langle [x, \; y] | \, x, \; y \in G \rangle$, the subgroup of G generated by commutators of elements from G, called the commutator subgroup *of G.*

We say that the commutator of x and y is 1 if and only if x and y commutes which explains the terminology.

1.6.9 **Perfect Group**

A group A is called perfect if $A = A'$ (that is, A equals it own commutator subgroup).

1.6.10 **Almost Simple Group**

A group is said to be almost simple if it has a non-abelian simple unique minimal normal subgroup. If a group G has a normal subgroup K such that $C_G(K) = 1$ (for example, if G is almost simple with minimal normal subgroup K), then G can be embedded in $Aut(K)$ such that its image contains $Inn(K)$. Put in another way, Let K be a non-abelian finite simple group. A group G such that $K \leq G \leq Aut(K)$ is called almost Simple.

1.6.11 **Maximal Normal Subgroup**

Let N be a normal subgroup of a group G, then N is said to be maximal if N is not properly contained in any other proper normal subgroup of G.

1.6.12 **Minimal Normal Subgroups**

Let G be a group. A nontrivial normal subgroup N of G is said to be a minimal normal subgroup of G if N is the only nontrivial normal subgroup of G contained in G. If G is finite and nontrivial then G is guaranteed to have minimal normal subgroups. Every minimal normal subgroup of a finite group G is a product of isomorphic simple groups.

1.6.13 **Elementary Abelian p – Group**

Let p be a prime. An elementary abelian p-group is an abelian group G in which every nontrivial element has order p. Then G is a finite elementary abelian p-group if and only if $G \cong \mathbb{Z}_p \times \ldots \times \mathbb{Z}_p$.

1.6.14 **Sylow Subgroups**

Let G be a group and let p be a prime

(i) A group of order p^α for some $\alpha \geq 0$ is called a $p-group$**.** Subgroups of G which are $p-$ groups are called p $-$ subgroups.

(ii) If G is a group of order $p^\alpha m$, where $p \nmid m$, then a subgroup of order p^α is called a Sylow p – subgroup of G

(iii) The set of Sylow p – subgroups of G will be denoted by $Syl_p(G)$ and the number of Sylow p – subgroups of G will be denoted by $n_p(G)$ (or just n_p when G is clear from the context).

Suppose that G is a finite group such that $|G| = p^k m$ where p is a prime and $p|m$. A Sylow p $-$subgroup of G is a maximal p $-$subgroup of G. Since G must contain an element of order p by Cauchy's Theorem, every finite group has a Sylow p $-$subgroup. Clearly if P$\leq G$ and $|P| = p^k$, then P is a Sylow p $-$subgroup of G.

1.6.15 **Centralizers And Normalizers**

Let G be a group, let x and y be elements of G. The commutator of x and y is $[x, y] := x^{-1}y^{-1}xy$. If $[x, y] = 1$ then x and y commutes.

The centralizer of x in G is $C_G(x) := \{g \in G: gx = xg\}$, that is, the set of all elements of G that commute with x.

If $H \leq G$, then the centralizer of H in G is

$C_G(H) := \{g \in G: gh = hg \ \ \forall\ h \in H\}$.

Both $C_G(x)$ and $C_G(H)$ are subgroups of G.

Moreover, if $H \trianglelefteq G$, then $C_G(H) \trianglelefteq G$ since if $x \in G$ and $a \in C_G(H)$, then for all $h \in G$, $ghg^{-1} \in H$, and thus

$$(g^{-1}ag)^{-1}h(g^{-1}ag) = g^{-1}a^{-1}(ghg^{-1})ag = g^{-1}(ghg^{-1})g = h$$

Let H and K be subgroups of G. If $K \leq C_G(H)$, we say that K centralizes H. Define $[H,K] := \langle\{[h,k] : h \in H, k \in K\}\rangle$.

Then H and K centralize each other if and only if $[H,K] = \{1\}$.

Let H and G be groups. The normalizer of H in G is

$N_G(H) := \{g \in G: g^{-1}Hg = H\}$.

Then $N_G(H)$ is a subgroup of G, and if $H \leq G$, then H is clearly a normal subgroup of $N_G(H)$.

In fact $N_G(H)$ is the largest subgroup of G in which H is normal. Also H is said to normalize $K \leq G$ if $H \leq N_G(K)$, and H is said to be self-normalizing in G if $N_G(H) = H$.

Lastly, note that $C_G(H) \trianglelefteq N_G(H)\ for\ all\ H \leq G$.

The centre of a group G, denoted by $Z(G)$, is the set of all elements of G that commute with every element of G. Thus, $Z(G)$ is clealy a normal subgroup of G (that is, $Z(G) \trianglelefteq G$). Now G is abelian if and only if $G = Z(G)$, and so the centre of a simple nonabelian group must be trivial.

Also if $H \leq G$, then $Z(H) = C_G(H) \cap H$

The group of all isomorphisms of a group G onto itself is called the automorphism group of G, and is denoted by $Aut\ (G)$.

Let $\varphi_h: G \to G$ be defined by $g\varphi_h = h^{-1}gh$. Then $\varphi_h \in Aut\ (G)$ and is called an inner automorphism of G. The inner automorphism group of G, denoted by $Inn(G)$, is the normal subgroup of $Aut(G)$ consisting of all inner automorphisms of *G.*

Moreso, if $\varphi: G \to Inn(G)$ is defined by $g \longmapsto \varphi_g$, it is an onto homomorphism with kernel $Z(G)$. Hence, $G/Z(G) \cong Inn(G)$, in particular, if T is simple and non-abelian, then $T \cong Inn(T)$.

The outer automorphism group of *G,* denoted by $Out(G)$, is simply the quotient group $Aut(G)/Inn(G)$. Consider briefly the outer automorphism group of a finite simple group: If G is cyclic of prime order p; it is not hard to see that $Out(G) \cong Aut(G) \cong \mathbb{Z}_p^*$, the multiplicative group of units of the ring $\mathbb{Z}_p$, which is abelian.

Suppose that $n \geq 5$. Let $\pi \in S_n$, and as usual, let $\varphi_\pi: S_n \to S_n$ be conjugation by π. We can easily map S_n into $Aut(A_n)$ by $\pi \longmapsto \varphi_\pi|_{A_n}$; this is clearly an embedding since $C_{S_n}(A_n)$ is trivial. Moreover, if $n \neq 6$, then this map is onto. But $A_n$is simple and nonabelian, so $A_n \cong Inn(A_n)$, which gives us that $|Out(A_n)| = [S_n: A_n] = 2$.

1.6.16 **Group Actions**

Let G be a group and Ω a nonempty set. Let S_Ω denote the symmetric group on Ω.

An action of G on Ω is a homomorphism $\phi: G \to S_\Omega$ from G into the symmetric group (the image of the action ϕ which is written as G^Ω is a permutation group and can be identified with G if the action is faithful). Now Ω is said to be a *G-space* (G-set) if there exists a function mapping from $\Omega \times$ G to Ω that satisfies $(\alpha^g)^h = \alpha^{gh}$ and $\alpha' = \alpha$ for all $\alpha \in \Omega$ and $g, h \in G$, where the image of (α, g) is denoted by α^g.

If ϕ is an action of G on Ω, then $\alpha^g \coloneqq \alpha(g\phi)$ satisfies the two conditions of a G-space, so that Ω is a G-space.

On the other hand, if Ω is a G-space and $g \in G$, let $\pi_g := \Omega \to \Omega$ be defined by $\alpha \longmapsto \alpha^g$. Then $\pi_g \in S_\Omega$ for all $g \in G$, and thus $\phi: g \longmapsto \pi_g$ is then an action of G on Ω.

Thus the two concepts of an action of a group on a set are equivalent.

Now G is a permutation group on Ω if it is a subgroup of S_Ω. The image of an action ϕ is called the permutation group induced on Ω by G, denoted by G^Ω. An action is faithful if $Ker(\phi) = \{1\}$, or equivalently Ω is a faithful G-space of whenever $\alpha^g = \alpha^h$ for all $\alpha \in \Omega$, we have that $g = h$.

In this case, G acts as a permutation group on Ω as $G \cong G\phi \leq S_\Omega$. In the light of the fact that we have two equivalent definitions of an action, the action of $g \in S_\Omega$ will either be written on the right as αg or in the form α^g, depending on the context.

Note that, the full symmetric group S_Ω of set Ω acts faithfully. In fact, any subgroup of a group acting faithfully is also faithful. A permutation group is a subgroup of some S_Ω. Hence every permutation group is faithful. The converse is also true.

Let us consider some basic definitions about group actions.

Let Ω be G-space, and let $\alpha, \beta \in \Omega$. Define a relation $\sim$ on Ω by $\alpha \sim \beta$ if there exists a $g \in G$ with $\alpha^g = \beta$. Then $\sim$ is an equivalence relation whose equivalence classes we call orbits of G.

Let α be in an orbit of G, then the orbit can be written as

$\{\alpha^g : g \in G\} =: \theta_{\mathrm{G}}(\alpha)$, which we call the orbit of α. G acts transitively on each orbit.

Now G is said to be transitive, or Ω is said to be a transitive G-space, if there is only one orbit, namely Ω. We call the group G^ϕ of permutations induced on an orbit ϕ by G a transitive constituent of G. Thus, the study of arbitrary permutation group "reduces" to that of transitive groups.

(However, it should be noted that G is not determined by its transitive constituents. It is contained in their Cartesian product, and is in fact a subcartesian product).

The stabilizer of α in G is $G_\alpha \coloneqq \{g \in G: \alpha^g = \alpha\}$, which is a subgroup of G.

The setwise stabilizer of $\Gamma \subseteq \Omega$ in G is $G_\Gamma \coloneqq \{g \in G: \Gamma^g = \Gamma\}$, which is also a subgroup of G, and of course when $\Gamma = \{\alpha\}$, $G_\Gamma = G_\alpha$.

Now G is said to be semiregular if $G_\alpha = \{1\}$ for all $\alpha \in \Omega$, and $G$is said to be regular if it is both transitive and semiregular.

Note: if $G \leq S_\Omega$ is transitive, then clearly every subgroup of S_Ω containing G is also transitive. Similarly, if $G \leq S_\Omega$ is semiregular, then every subgroup of G is semiregular.

If we define an action of G on G by right multiplication; that is
$x^g = xg \; for \; g, \; x \in G$. This is called the right regular representation *of G.* The left regular representation *of* G is given by the action $x^g = g^{-1}x \; of \; G$ on itself. Both actions are regular.

G also acts on itself by conjugation; that is, $x^g := g^{-1}x \; for \; all \; g, x \in G$.
This action is very important and is used often.
It is routine to verify that these three definitions (examples) do give rise to legitimate actions.

Let $H \leq G$. The right coset space of H in G, denoted by $G\backslash H$, is simply the set of right cosets of H in G. the backslash is used to avoid confusion with the quotient G/H. It is routine to verify that $G\backslash H$ is transitive $G - space$ with action $(Ha)^g \coloneqq Hag$ for all $H\,a \in G\backslash H$ and $g \in G$. Moreover, $G_{H_g} = g^{-1}Hg$ for all $g \in G$ since $h \in G_{H_g} \Leftrightarrow Hgh = Hg \Leftrightarrow ghg^{-1} \in H \Leftrightarrow h \in g^{-1}Hg$. In particular, $G_H = H$.

It can equally be seen that the Kernel of this action is $\bigcap_{g \in G} g^{-1}Hg$, which is called the core of H in G. Note that the core of H in G is a normal subgroup of G contained in H. If the core of H is trivial, then H is said to be core – free.

Hence, the action of G on the coset space $G\backslash H$ is faithful if and only if H is a core – free subgroup of G contained in H must be trivial. The left coset space of H in G is defined analogously.

1.6.17 Socle Of a Group

The socle of a group G, denoted by $soc(G)$, is defined to be the subgroup generated by the set of all minimal normal subgroups of G. Where $soc(G) \coloneqq \{1\}$ if G has no minimal normal subgroups (which can only occur if G is infinite or trivial). Note that $soc(G)$ is a normal subgroup of G.

1.6.18 Solvable Series

Let G be a group. A solvable series of a group G is a sequence of subgroups

$$\{1\} = G_n \trianglelefteq G_{n-1} \trianglelefteq \cdots \trianglelefteq G_0 = G$$

where G_i/G_{i+1} is abelian for all $i \in \{0, \dots, n-1\}$.

A group G is said to be solvable if G has a solvable series.

1.6.19 Nilpotent Groups

Let G be a group, a central series of a group G is a sequence of subgroups

$$\{1\} = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_n = G$$

where $G_{i+1}/G_i \leq Z(G/G_i)$ for all $i \in \{0, \dots, n-1\}$.

A group is said to be nilpotent if G has a central series. Clearly a nilpotent group is solvable. A group is nilpotent if all its Sylow subgroups are normal.

1.6.20 Innately Transitive Groups

A finite group G is Quasiprimitive if all of its minimal normal subgroups are transitive. We have the following weaker notion:

Let G be a finite permutation group. If there exists a transitive minimal normal subgroup M of G, then we say that G is innately transitive and that M is a plinth for G.

1.6.21 π – Group

Let G be a finite group and π a finite set of primes. A subgroup H of G is called a π-group if the order $|H|$ of H is divisible only by the primes in π, H is called a π'-group if $|H|$ is not divisible by any prime in π. A normal π-complement of a π-subgroup H of a finite group G is a π'-group K such that $G = HK,\ K \lhd G$ and $H \cap K = \{e\}$.

1.6.22 Direct Product

Let A and B be sets. The Cartesian cross produt of A with B, denoted by $A \times B$ is the set of all ordered pairs (x, y) where $x \in A$ and $y \in B$.

Now considering the groups G_1 and G_2, the external direct product (that is, the Cartesian product of groups) of G_1 with G_2 is also denoted in the same manner $G_1 \times G_2$. The set of elements that constitute $G_1 \times G_2$ is precisely the same as the Cartesian cross product of G_1 with G_2.

However, the external direct product of G_1 with G_2 is endowed with an operator (which makes it a group). If x_1 and x_2 are in G_1 and y_1 and y_2 are in G_2, $(x_1, y_1) * (x_2, y_2)$ is defined to be $(x_1.x_2,\ y_1.y_2)$ where the "." in the first coordinate is the G_1 operator and the "." in the second coordinate is the G_2 operator.

1.6.23 Centralizers in Direct Product

We review a few facts about centralizers of subgroups in direct products that we will use in our analysis.

Let $G_1, G_2, \dots, G_n$ be groups and the direct product $G_1 \times G_2 \times \dots \times G_n$ be their Cartesian product. If $(g_1, g_2, \dots, g_n) \in G_1 \times G_2 \times \dots \times G_n$, then the centralizer of $(g_1, g_2, \dots, g_n)$ is simply the product of the centralizers of $g_1, g_2, \dots, and\ g_n$, that is,

$$C_{G_1 \times G_2 \times \dots \times G_n}(g_1, g_2, \dots, g_n) = C_{G_1}(g_1) \times C_{G_2}(g_2) \times \dots \times C_{G_n}(g_n)$$

Next we look at related literatures to our work.

CHAPTER TWO

LITERATURE REVIEW

Here, related topics are discussed and a brief history of the classification of the finite simple groups presented and the literatures associated with the factorization of finite simple groups.

2.1 Classification of Finite Simple Groups

Gallian (1976), Solomon (2001) and Lee (2016) served as major references for this section.

Group theory which arose from the study of polynomial equations (by Galois and Abel) has become an active field of research today with the classification of all finite simple groups being its largest achievement in the last century. In this section we present a brief history of the classification of the finite simple groups.

In 1832, Galois first introduced normal subgroups, formulated the definition of a simple group and observed that the alternating group A_n , $n \geq 5$ (of even permutations) on 5 symbols and the Projective Special Linear Groups $PSL_2(F_p), (p \geq 5)$ are simple. However, the first major results in the theory were due to Camille Jordan (1838 – 1922). In 1870, Jordan published "Traite des Substitutions", the first book ever written on group theory. In this book he established the existence of five infinite families of finite simple groups. One of these families, which we denote by A_n, consists of the alternating permutation groups on $n \geq 5$ symbols. Jordan formed the other four families by using matrices with entries from finite fields. One of these may be described as follows: For $m \geq 2$, the special linear group $SL_m(q)$ is the multiplicative group of $m \times m$ matrices of determinant 1 with entries from the field with p^n elements (where p is a prime number and n is a positive integer) and the projective special linear group $PSL_m(q)$ is the factor group $SL_m(q)/Z(SL_m(q))$ where

$Z(SL_m(q))$, the centre of $SL_m(q)$, is the subgroup of $SL_m(q)$ consisting of all scalar matrices with determinant 1. Jordan proved that $PSL_m(q)$ is simple when (m, q) is not $(2, 2)$ or $(2, 3)$. The other three families have been given the names orthogonal, unitary and symplectic groups and following Herman Weyl, mathematicians refer to these four families collectively as the classical simple groups (Gallian, 1976).

In 1892, Holder in a quest to classify finite simple groups initiated what was called the Range Problem; namely the complete determination of all simple groups whose orders are in a given range. Here both the existence and uniqueness questions must be considered; that is, it must be determined which integers in the range are the orders of simple groups, and, for each such integer, all possible simple groups of that order must also be determined (up to isomorphism). He proved that the only two simple groups whose orders lie between 1 and 200 are A_5 of order 60 and $PSL_2(7)$ of order 168. Since then, two very different approaches have been applied by mathematicians to tackle the project of classifying finite simple groups. One was to find all the possible simple groups manually, like treasure hunting (range problem). It was rewarding but also very risky. The other approach was to rigorously restrict the conditions for a group G to be simple. It was more systematic and reliable, but not as exciting as the former (Lee, 2016).

F.N. Cole (1861 – 1927), the first American-born mathematician to publish in group theory, followed Holder's lead in 1892 when he examined the integers between 201 and 500 for simple groups. He was not totally successful for he was unable to prove that A_6 was the unique simple group of order 360; nor was he able to show that 432 was not the order of a simple group. He overcame these difficulties a year later, however, when he completed the determination of all the simple groups with orders in the range 1 to 660. In addition to the ones in this range already found by

Jordan, Cole discovered one more, $PSL_2(8)$ having order 504. This provided the first example of a simple group not known to Jordan and the first proof of the simplicity of one of the groups $PSL(m,\ q)$ with q not prime (Gallian, 1976).

Cole's discovery of the simplicity of $PSL_2(8)$ had far-reaching consequences because that same year E.H. Moore (1862 – 1932), the first mathematics Department Chairman of the University of Chicago, used it for the starting point of his investigations which resulted in a proof that the family of groups $PSL_2(q)$ are all simple except when $= 2\ or\ 3$. William Burnside (1852 – 1927) also obtained this result shortly after Moore. Moore's paper, in turn, led his first Ph.D. student, Leonard E. Dickson (1874 – 1954), to the complete generalization of Jordan's original result when in 1897 he proved that the family of groups $PSL_m\,(q)$, $m \geq 2$, consisted of simple groups except when $q = 2\ or\ 3$ (Gallian, 1976).

In 1895, Burnside obtained several powerful arithmetic tests for simple groups. By far the most important of these is the fact that a simple group of even order must be divisible by one of 12, 16 or 56. In proving this result, he showed that an even order simple group cannot have a cyclic Sylow 2-subgroup. This theorem appears to be the first non-simplicity criterion which is based on the structure of the Sylow 2-subgroups. In the past two or more decades much of the research in simple group theory has dealt with the problem of classifying all simple groups whose Sylow 2-subgroups have a specified structure. All the Sylow 2-subgroups of a group are isomorphic. In obtaining their results Burnside, Cole, and Holder utilized permutation representations of groups. Certain permutation groups – transitive, doubly transitive, and primitive – play an especially important role in simple group theory (Gallian, 1976).

A homomorphism from a group into a group of matrices with entries from some field is called a representation of the group. If T is a representation of

G , the character of this representation is the function X from G to the field defined by $X(g) = \text{trace}\left(T(g)\right)$ for g *in* $G.$ There exist numerous arithmetical relations on the characters of a group G which are intimately related to the structure of $G.$ Thus knowledge of the characters of a group reveals much information about the group itself. The theory of group characters has profoundly influenced the search for simple groups. In the period from 1897 to 1905 Dickson made many fundamental contributions to the theory of simple groups. In a series of papers appearing from 1897 to 1899 he extended Jordan's results on the simplicity of the orthogonal, Unitary and Symplectic groups over fields of prime order to arbitrary finite fields. In his classical book Linear Groups*,* Dickson listed all the isomorphisms between the simple groups he knew. For example, A_5, $PSL_2\,(4)$ and $PSL_2(5)$ are defined differently but isomorphic. These classical linear groups over the field of complex numbers are Lie groups (because, roughly, they possess a smooth geometric structure) and Wihelm Killing (1847 – 1923) and Elie Cartan (1869 – 1951) proved (1888 - 1894) that besides the simple lie groups corresponding to the classical groups there are only five additional simple ones called exceptional lie groups (Gallian, 1976).

In 1861, Mathieu discovered a family of five transitive permutation groups. This remarkable family has become very important in both the theory of simple groups and coding theory as well as in permutation group theory. In 1895, while determining all transitive permutation groups on 10 or 11 symbols, Cole observed that the smallest Mathieu group (order 7920*)* is simple and by 1900 G. A. Miller (1863 – 1951) had shown the other four are also simple groups (Gallian, 1976).

In 1902, Frobenius determined all transitive permutation groups of degree $p + 1$ and order $p(p^2 - 1)/2$ where p is a prime. Since any simple group of order $p(p^2 - 1)/2$ can be represented as a transitive permutation group on the $p + 1$ conjugate Sylow p-subgroups. It follows from Frobenius theorem that $PSL_2(p)$ is the only simple group of order $p(p^2 - 1)/2$. In 1904, Burnside used Character theory to prove that every group of order $p^a q^b$ where p and q are primes and a and b are non-negative integers is solvable. This theorem represented the final generalization of a large number of special cases which had been established by Sylow, Frobenius, Burnside, Jordan and Cole; it has become the classic example of the power of character theory. A character-free proof of Burnside's theorem has also emerged (Matsuyama, 1973).

Simple group theory was revitalized in 1955 when Chevalley in his celebrated paper "Sur certains groups simples" introduced a new approach to classifying simple groups which provided a uniform method for investigating three kinds of the classical linear groups and Dickson's simple groups of Lie type as well. This led to the discovery of new simple groups (the smallest of these has order $2^{24}.3^6.5^2.7^2.13.17$). The formulas for the orders of the Chevalley groups over finite fields were derived by using topological properties of the lie group of the same type. During the period 1958 – 1959 Chevalley's methods were extended and modified by Robert Steinberg, Jacques Tits and D. Hertzig to obtain additional new infinite families of simple groups and the classical groups not handled by Chevalley. Shortly thereafter, Suzuki, while in the process of classifying a certain type of doubly transitive permutation groups also discovered another new infinite family. Analyzing the Suzuki groups, Rimhal Ree noticed that when interpreted from a lie-theoretical point of view, they were closely related to a certain family of Chevalley group. He then showed that the method of Steinberg could be used to construct the Suzuki groups. This in turn led him to investigate two other similar situations and eventually discovered his two families of simple groups. The Suzuki and Ree groups together with those of Chevalley and Steinberg are collectively referred to as the Simple groups of Lie type. These, together with the alternating groups A_n ($n \geq 5$) account for all but 26 or so of the finite simple groups known to date (Gallian, 1976).

A simple group which no one has yet been able to fit in to an infinite class of simple groups in a natural way is called a Sporadic simple group. For example, A_n and $PSL_n(q)$ are infinite families of simple groups while the five Mathieu groups are sporadic. In the classification schemes of Dickson, Dieudonne, and Artin only the five Mathieu groups are sporadic until Zvonimir Janko found one of order 175,560 (which was latter called J_1)in 1966 (Janko, 1966). Janko and Hall in 1967 also proved the existence of a sporadic simple group of order 604, 800. Some of the sporadic simple groups have been discovered in the course of solving certain problems in permutation group theory while others have turned up as the automorphism group of a distance – transitive graph. An important technique which has led to the discovery of a number of sporadic simple groups involves the notion of the centralizer of an involution (i.e. of an element of order 2) (Gallian, 1976).

In 1973, Fischer, Thompson and Griess predicted an enormous simple group called the "Monster" M, which is not defined in terms of generators and relations. Thompson also proved that if this group exists, then this group contains other sporadic groups as subquotients of this group. It was Griess in 1982 that finally constructed the monster and hence proved its existence. They have been able to calculate its order to be

808,017,424,794,512,875,886,459,904,961,710,757,005,754,368,000,000, 000 = $2^{46}.3^{20}.5^{9}.7^{6}.11^{2}.13^{3}.17.19.23.29.31.41.59.71$ (Lee, 2016).

A certain section of M (that is, a group of the form H/K where H and K are subgroups of M and K is normal in H), the "Baby monster" B, is also a new simple group. Fischer has completed the order of B to be

4, 154, 781, 481, 226, 426, 191, 177, 580, 544, 000, 000,
= $2^{41}.3^{13}.5^{6}.7^{2}.11.13.17.19.23.41.47$ (Gallian, 1976).

Most simple groups contain other simple groups as subgroups. For example $A_5 \subset A_6 \subset A_7 \subset$ .... On the other hand, a minimal simple group is one, all of whose proper subgroups are solvable. It follows then that every simple group has a minimal simple group as a section. Minimal simple groups are therefore basic. The normalizer of a nonidentity solvable subgroup of a group G is called a local subgroup of G and an N-group is one in which all local subgroups are solvable. Evidently, every minimal simple group is also an N-group (Gallian, 1976). In 1963, Thompson concluded his classification of N-groups, with only many exceptions. The simple N-groups were $PSL_2(q)$ $(q > 3)$ and the Suzuki group.

Gorenstein in 1972 announced a program for completing the classification of finite simple groups which consisted of 16 steps and involved probably more individuals than in any other single area of simple group theory. In 1983, Gorenstein announced that the finite simple groups had all been classified, but this was premature as he had been misinformed about the proof of the classification of quasithin groups. The completed proof of the classification was announced by Aschbacher in 2004 after Aschbacher and Smith published a 1221 page proof for the missing quasithin case (Lee, 2016).

Among the numerous results of this Gorenstein's program we list a few thus;

In 1974, Tits shows that groups with BN pairs of rank at least 3 are groups of Lie type. Aschbacher classifies the groups with a proper 2 – generated core. In 1976, O'Nan introduces the O'Nan group. Janko introduced the Janko group J_4, the last sporadic group to be discovered. In 1977, Aschbacher characterizes the groups of Lie type of odd characteristic in his classical involution theorem. In1978, Timmesfeld proved the O_2 extraspecial theorem. In 1982, Mc Bride proved the signalizer functor theorem for all finite groups. Griess constructs the monster group by hand. In 2004, Aschbacher and Smith published their work on quasithin groups (which are mostly groups of Lie type of rank at most 2 over fields of even characteristic), filling the last gap in the classification known at that time (Solomon, 2001).

Harada and Solomon in 2008 filled a minor gap in the classification by describing groups with a standard component that is a cover of the Mathieu group M_{22}, a case that was accidentally omitted from the proof of the classification due to an error in the calculation of the Schur multiplier of M_{22}. Gonthier and collaborators in 2012 announced a computer-checked version of the Feit-Thompson theorem using the Coq proof assistant (Solomon, 2012).

Because of the extreme length of the proof of the classification theorem, much effort has been devoted to finding a simpler proof. As of 2005, six volumes of the second proof have been published (Gorenstein, Lyons and Solomon, 2005). In 2012 Solomon estimated that the project would need another 5 volumes, but said that progress on them was slow. It is estimated that the new proof will eventually fill approximately 5,000 pages (Solomon, 2012).

In the 1970s, John Conway, Robert Curtis, Simon Norton, Richard Parker and Robert Wilson decided to record all the known simple groups in a book called the ATLAS of Finite Groups. This book was first published in December 1985 and reprinted with corrections in 2003. It lists basic information about 93 finite simple groups from A_5 to the Monster (Lee, 2016).

2.2 Factorization of Finite Simple Groups

The factorization of finite simple groups is one out of the numerous consequences of the classification of the finite simple groups. A group is said to be factorizable if $G = AB$ for some proper subgroups A and B of G. There is surprisingly large number of mathematical questions related to factorization of groups. These relatively new area of research has caught the interest of some researchers.

In 1992, Amberg *et al.*, put up the question of finding all the factorizable groups (Amberg *et al.,* 1992). Of course not all groups are factorizable, for example in 1990, Liebeck *et al*. announced that the Conway's simple group Co_2 of order $2^{18}.3^{6}.5^{3}.7.11.23$ is not factorizable. Similarly an infinite group whose proper subgroups are finite does not have a proper factorization. Therefore we always search for a special kind of factorization. A factorization G = AB is called maximal if both factors A and B are maximal subgroups of G. Liebeck *et al*., found all the maximal factorizations of all finite simple groups and their automorphism groups. This special kind of factorization is useful because every factorization of a finite group is contained in a maximal factorization (Liebeck *et al*., 1990). In 1984, Arad and Fisman gave simple groups G with factorization $G = AB$ and with the additional condition $(|A|,|B|) = 1$ are determined. In this case we also have $A \cap B = 1$ the trivial group. A factorization G = AB with the condition $A \cap B = 1$ is called an exact factorization (Arad and Fisman, 1984). In 1980, Wiegold and Williamson found all the exact factorizations of the alternating and the symmetric groups (Wiegold and Williamson, 1980). But later in 1989, all the factorizations of the alternating and the symmetric groups were found where both factors are simple groups (Wall, 1989). In 2000 Etingof *et al.* gave an interesting application of exact factorization. They showed that an exact factorization of a finite group leads to the construction of a biperfect Hopf algebra, and then they found such a factorization for the Mathieu group M_{24}. This factorization is of the form $M_{24} = AB$, where $A \cong M_{23}$ and $B \cong 2^4; A_7$, both perfect groups (Etingof *et al.,* 2000). In 1963, Kegel and Luneburg classified all finite groups G = AB, $A \cong B \cong A_5$, and in 1975, Scott got factorizable groups where one factor is a non-abelian simple group and the other factor is isomorphic to the alternating group on 5 letters. In 1992, Walls classified factorizations of finite groups in the case which one factor of a factorizable group is simple and the other factor is almost simple (Walls, 1992). In 2001, Darafsheh and Rezaeezadeh determined all finite groups G = AB, where $A \cong A_6$ and B is isomorphic to the symmetric group on $n \geq 5$ letters are determined (Darafsheh and Rezaeezadeh, 2001). Also in 2003, Darafsheh and Rezaeezadeh determined the structure of a finite factorizable group with one factor a simple group and the other factor isomorphic to the symmetric group on 6 letters (Darafsheh and Rezaeezadeh, 2003). In 2002, Darafsheh determined the structure of a finite factorizable groups G = AB where $A \cong A_7$ and $B \cong S_n$ (Darafsheh, 2002) Motivated by this paper Darafsheh in 2005, in his work title "Factorization of simple groups involving the alternating group" found the structure of the finite simple factorizable groups G = AB such that A is a non-abelian simple group and B$\cong A_7$, the symmetric group on seven letters (Darafsheh, 2005). In 2008, Darafsheh and Rezaeezadeh in their work titled "Factorization of Groups Involving Symmetric and Alternating Groups" obtained the structure of finite groups of the form $G = AB$, where B is a group isomorphic to the symmetric group of n letters S_n, $n \geq 5$ and A is a group isomorphic to the alternating group on 6 letters (Darafsheh and Rezaeezadeh, 2008).

Other key researchers in this area include C. E. Praeger, C. Schneider, R. W. Baddeley and others. They have published a lot of literatures in this area. In 1998, Baddeley and Praeger, published a work titled, "On classifying All Full Factorizations and Multiple Factorizations of the Finite Almost Simple Groups." They were able to classify all full factorizations and multiple factorizations of finite almost simple groups in that paper. These results were consequences of the classification of all maximal factorizations of almost simple groups and are needed for applications to the theory of quasiprimitive permutation groups (Baddeley and Praeger, 1998).

In 2001, Praeger and Schneider published a paper titled “Factorization of characteristically simple groups”. The work stemmed from a study of finite primitive and quasiprimitive permutation groups in which the problem of classifying all factorizations $(T, \{A, B\})$ of finite simple groups T such that $|T|$, $|A|$, and $|B|$ are divisible by the same primes plays a vital role. Such a factorization is called a full factorization of the simple group T. In this paper they characterized three different kinds of factorization of finite characteristically simple group; and solved some factorization problems for finite characteristically simple group using factorization of elementary abelian 2-groups (Praeger and Schneider, 2001).

Also in 2005, Baddeley, Praeger and Schneider published a work on “Innately Transitive Subgroups of Wreath Products in Product Action”. In this work, they studied the class of finite, innately transitive permutation groups that can be embedded into wreath products in product action. They carried out this investigation by observing that such a wreath product preserves a natural Cartesian decomposition of the underlying set. They identified several different types of Cartesian decompositions that can be preserved by an innately transitive group with a non-abelian plinth. These different types of decompositions led to different types of embeddings of the acting group into wreath products in product action (Baddeley*et al.*, 2005).

In 2007, Baddeley, Praeger and Schneider extended their previous work on product actions and published a new paper titled “Intransitive Cartesian Decompositions Preserved by Innately Transitive Permutation Groups.” In the work they proved that the transitivity assumption made in their previous paper was not too restrictive. In this work they found that the innately transitive group can have at most three orbits on an invariant

Cartesian decomposition. They used a lot of results about characteristically simple groups, and the factorizations of characteristically simple groups. They also went ahead to prove several results about normalizers of subgroups of characteristically simple groups (Baddeley *et al.,* 2007).

In 2010, Ngulde in his paper titled "Factorization of Simple and Characteristically Simple Groups", reviewed the results of Robert *et al*., (2007) on the factorization of characteristically simple groups. In this work he used the results of full factorization of characteristically simple groups to derive some facts concerning the normalizers of the subgroups that occur (Ngulde, 2010).

Motivated by these papers the need arose to outline some implications of these results about normalizers of these subgroups and also extend a little to find out the nature of the centralizers of the subgroups of characteristically simple groups and some applications of these results.

Next, we look at the methods to achieve the identified need.

CHAPTER THREE

METHODOLOGY

3.1 Background Theorems

In this chapter we present some theorems from group theory that will be used throughout this thesis. Robinson (1996), Rotman (1995), Cameron(1999) and Fawcett (2009) served as general references for this chapter.

3.1.1 **Group Actions**

Let G be a group and Ω a nonempty set. Let S_Ω denote the symmetric group on Ω.

We now discuss centralizers in permutation groups which illustrates some of the above concepts. That is that a permutation group is transitive if and only if its centralizer in the symmetric group is semiregular and vice versa.

Proposition 3.1 (TSUZUKI, 1982)

Let G be a permutation group on Ω

(i) *If $C_{S_\Omega}(G)$ is transitive on Ω, then G is semiregular*

(ii) *If G is transitive on Ω, then $C_{S_\Omega}(G)$ is semiregular*

3.1.2 **Sylow Subgroups** (Dummit et al, 2004)

Theorem 3.2 (Cauchy) (Dummit et al, 2004)

If G is a finite group and p is a prime where p divides the order of G, then G contains an element of order p.

Lemma 3.3 (Dummit et al, 2004)

Let P be a Sylow $p-$subgroup of a finite group G. Then $N_G(P)/P$ contains no element of order p.

Proposition 3.4(Dummit et al, 2004)

Let P be a Sylow $p-$subgroup of a group G. Then $N_G(P)$ is self – normalizing in G.

Lemma 3.5(Dummit et al, 2004)

If P is a Sylow p −subgroup of a group G, and Q is a p − subgroup of G which normalizes P, then Q is a subgroup of P.

Theorem 3.6 (Sylow's Theorem) (Dummit et al, 2004)

Let G be a group of order $p^\alpha m$, where p is a prime not dividing m

(i) Sylow p −subgroups of G exist i.e $Syl_p(G) \neq \phi$

(ii) If P is a Sylow p −subgroup of G and Q is any p −subgroup of G, then there exists $g \in G$ such that $Q \leq gPg^{-1}$, that is, Q is contained in some conjugate of P. In particular, any two Sylow p −subgroups of G are conjugate in G.

(iii) The number n_p of Sylow p −subgroups of G is of the form $1 + kp$, that is $n_p \equiv 1 (\text{mod } p)$

Further, n_p is the index in G of the normalize $N_G(P)$, for any Sylow p −subgroup P, hence n_p divides m.

3.1.3 **Subdirect Products**

Bamberg (2008) served as a general reference for this section.

Let $G := G_1 \times G_2 \times \ldots\ G_k$ be a direct product of groups G_i. Let $\rho_i: G \longrightarrow G_i$ be the projection map for each i. A group H is a subdirect product of G if there exists an embedding $\phi: H \longrightarrow G$ such that $\phi\rho_i: H \longrightarrow G_i$is as onto homomorphism for all i.

If H is actually a subgroup of G, then of course we may take ϕ to be the inclusion map, and we call the subdirect product H a subdirect subgroup of G.

If H is a subgroup of G and the restriction of ρ_i to H (that is $\rho_i|_H$) is injective $(1 - 1)$ for all i, then H is called a diagonal subgroup of G(where H is not necessarily subdirect).

Lastly, if H is a subgroup of G, then H is a full diagonal subgroup of G if it is both a subdirect subgroup and a diagonal subgroup.

If $h := (h_1, h_2, \ldots, h_k)$ is any element of a subgroup H of G, then

$$h = (h_1, h_2, \ldots, h_k) = (h\rho_1, h\rho_2, \ldots, h\rho_k)$$

Thus $H = \{(h\rho_1, h\rho_2, \ldots, h\rho_k)\}: h \in H$

If H is a full diagonal subgroup of G, note that $\rho_i|_H$ is then an isomorphism of H onto G_i for each i. Consequently, all of the G_i must themselves be isomorphic to one another.

Lemma 3.7 (Bamberg, 2008)

Let $G = T_1 \times T_2 \times \ldots \times T_k$ be a direct product of simple nonabelian groups $(k \geq 1)$. Let H be a subgroup of G and $I := \{1, \ldots, k\}$.

(i) If H is a full diagonal subgroup of G, then H is self – normalizing in G

(ii) If H is a subdirect subgroup of G, then H is a direct product $\prod H_j$, where H_j is a full diagonal subgroup of some subproduct $\prod_{i \in I_j} T_i$ such that I is partitioned by the I_j

(iii) If H is a nontrivial normal subgroup of G, then $H = \prod_{j \in J} T_j$ where J is some nonempty subset of I.

Lemma 3.8 (Baddeley and Praeger, 2002)

Suppose that G_1, G_2 are groups; let N_1 and N_2 be normal subgroups of G_1 and G_2, respectively, such that $G_1/N_1 \cong G_2/N_2$ and let $\varphi: G_1/N_1 \to G_2/N_2$ be an isomorphism. Then the set

$H(\varphi) = \{(g_1, g_2) | g_1 \in G_1,\ g_2 \in G_2, \varphi(g_1 N_1) = g_2 N_2\}$ is a subdirect subgroup of $G_1 \times G_2$

Proposition 3.9 (Fawcett, 2009)

Let G be a group that normalizes $N := T_1 \times \ldots \times T_k$ where the T_i are all simple and nonabelian. Then G acts by conjugation on the set $\{T_1, \ldots, T_k\}$.

3.1.4 Minimal Normal Subgroups

The next few propositions and theorems illuminate the structure of a minimal normal subgroup.

Proposition 3.10 (Bamberg, 2008)

Any two distinct minimal normal subgroups of a group G must intersect trivially. It follows that any two distinct minimal normal subgroups centralize each other.

Recall from proposition 3.9 that G acts by conjugation on $\{T_1, \dots, T_k\}$ if the T_i are all simple and nonabelian and if $T_1 \times \dots \times T_k$ is normalized in G

Proposition 3.11 (Bamberg, 2008)

Let G be a group. Suppose that $N := T_1 \times \dots \times T_k$ is a normal subgroup of G where T_i are all simple and nonabelian. Then G acts transitively by conjugation on $\{T_1, \dots, T_k\}$ if and only if N is a minimal normal subgroup of G.

We have just seen that a minimal normal subgroup can be a direct product of isomorphic simple groups (they are isomorphic because they are conjugate). It turns out that, at least in a finite group, every minimal normal subgroup is a direct product of isomorphic simple groups.

Proposition 3.12(Fawcett, 2009)

Let G be a group

(i) *If H char K and $K \trianglelefteq G$, then $H \trianglelefteq G$*

(ii) *If N is a minimal normal subgroup of G, then N is characteristically simple.*

Theorem 3.13 (Rotman 1982)

A finite characteristically simple group G is a direct product of isomorphic simple groups.

Corollary 3.14 (Fawcett, 2009)

A minimal normal subgroup of a finite group is a direct product of isomorphic simple groups.

Every minimal normal subgroup of a finite group G is a product of isomorphic simple groups. More often than not, we are concerned with the case when all of these simple groups are nonabelian. The next proposition gives a handy condition for proving when a product of simple non-abelian groups is actually the socle of a finite group G.

Proposition 3.15 (Bamberg, 2008)

Let G be a finite group with subgroup $M := T_1 \times \ldots \times T_k$ where $k \geq 1$ and T_i is simple and nonabelian for all i. Then M is the socle of G if and only if $C_G(M) = \{1\}$ and $M \trianglelefteq G$.

3.1.5 Innately Transitive Groups (Bamberg, 2008)

The following proposition gives us necessary and sufficient conditions for an innately transitive group to be quasiprimitive.

Theorem 3.16 (Bamberg, 2008)

Let G be a finite innately transitive permutation group on a set Ω with plinth M. Then G is quasiprimitive if and only if $C_G(M)$ is transitive or $C_G(M) = 1$.

Proposition 3.17 (Bamberg, 2008)

Let G be a finite innately transitive permutation group on Ω with nonabelian and nonsimple plinth M, and let $\alpha \in \Omega$. If M_α is a subdirect subgroup of M, then G is quasiprimitive.

We now proceed to outline briefly the theorems (results) and ideas of full factorization and strong multiple factorization of finite simple and characteristically simple groups as obtained by Baddeley and Praeger (1998) and Praeger and Schnieder (2002)

3.1.6 Full Factorization

Let $M = T_1 \times T_2 \times \cdots \times T_k$ be a finite, non-abelian, characteristically simple group where $T_1, \cdots, T_k$ are pairwise isomorphic, simple normal subgroups, and let K_1 and K_2 be proper subgroups of M. Then a factorization $M = K_1K_2$ is said to be a full factorization if, for each $i \in \{1, \ldots, k\}$

(a) the subgroups of $\sigma_i(K_1)$, $\sigma_i(K_2)$ are proper subgroups of T_i;

(b) the orders $|\sigma_i(K_1)|$, $|\sigma_i(K_2)|$, and $|T_i|$ are divisible by the same set of primes.

Theorem 3.18 Praeger and Schneider (2002)

Suppose that $k \geq 1$ and $T_1, \cdots, T_k$ are pairwise isomorphic, finite, non-abelian simple groups, and set $M = T_1 \times T_2 \times \cdots \times T_k$. If $M = K_1K_2$ is a full factorization, then

$$\sigma_1(K_j)' \times \ldots \times \sigma_k(K_j)' \leq K_j \quad for\ j \in \{1,2\}.$$

Further, for each $i \in \{1, \ldots, k\}$ the pair $(T_i, \{\sigma_i(K_1), \sigma_i(K_2)\})$ is a full factorization, and thus occurs as $(T, \{A, B\})$ in one of the lines of Table 1.

Table 1: Full Factorizations $\{A, B\}$ of Finite Simple Groups

	T	A	B
1	A_6	A_5	A_5
2	M_{12}	M_{11}	$M_{11}, PSL_2(11)$
3	$P\Omega_8^+(q), q \geq 3$	$\Omega_7(q)$	$\Omega_7(q)$
4	$P\Omega_8^+(2)$	$Sp_6(2)$	$A_7, A_8, S_7, S_8, Sp_6(2), \mathbb{Z}_2^6 \rtimes A_7, \mathbb{Z}_2^6 \rtimes A_8$
		A_9	$A_8, S_8, Sp_6(2), \mathbb{Z}_2^6 \rtimes A_7, \mathbb{Z}_2^6 \rtimes A_8$
5	$Sp_4(q), q \geq 4\ even$	$Sp_2(q^2).2$	$Sp_2(q^2).2,\ Sp_2(q^2)$

Source: Praeger and Schneider (2002)

An important subfamily of full factorizations consists of the factorizations of non-abelian, finite simple groups with two isomorphic subgroups. Consider the following lemma.

Lemma 3.19 (Baddeley *et al.*, 2006)

Let T be a finite simple group and let A, B, be proper subgroups of T such that $|A| = |B|$ and let $T = AB$. Then the following hold:

(i) The isomorphism types of T, A and B are as in Table 2 and A, B are isomorphic, maximal subgroups of T.

(ii) There is an automorphism $\vartheta \in Aut\ T$ that interchanges A and B.

(iii) *We have*

$N_T(A' \cap B') = N_T(A \cap B) = A \cap B$ *and*

$C_T(A' \cap B') = C_T(A \cap B) = 1$

Table 2: Factorizations of Finite Simple Groups in Lemma 3.19

	T	**A, B**
1	A_6	A_5
2	M_{12}	M_{11}
3	$P\Omega_8^+(q), q \geq 3$	$\Omega_7(q)$
4	$Sp_4(q), q \geq 4\ even$	$Sp_2(q^2).2$

Source: Baddeley, *et al.* (2006)

3.1.7 Strong Multiple Factorization

Let $M = T_1 \times \ldots \times T_k$ be a finite, non-abelian, characteristically simple group. For example $K_1, \ldots, K_l$ of M, the pair $(M, \{K_1, \ldots, K_l\})$ is said to be a strong multiple factorization if, for all $i \in \{1, \ldots, k\}$ and all pairwise distinct $j_1, j_2, j_3 \in \{1, \ldots, k\}$,

(a) $\sigma_i(K_1), \ldots, \sigma_i(K_l)$ are proper subgroups of T_i, and

(b) $K_{j_1}(K_{j_2} \cap K_{j_3}) = K_{j_2}(K_{j_1} \cap K_{j_3}) = K_{j_3}(K_{j_1} \cap K_{j_2}) = M$

The following theorem, combining (Baddeley and Praeger (1988), Table V) and (Praeger and Schneider (2002), Theorem 1.7 and Corollary 1.8), gives a characterization of strong multiple factorizations of characteristically simple groups.

Theorem 3.20 (Praeger and Schneider, 2002)

A strong multiple factorization of a finite characteristically simple group contains exactly three subgroups. If M is a non-abelian, characteristically simple group with simple normal subgroups $T_1, \ldots, T_k$ and $(M, \{K_1, K_2, K_3\})$ is a strong multiple factorization, then

$\sigma_1(K_i)' \times \ldots \times \sigma_k(K_i)' \leq K_i$ *for $i = 1, 2, 3$ and for $i = 1, \ldots, k$, the pair $(T_i, \{\sigma_i(K_1), \sigma_i(K_2), \sigma_i(K_3)\})$ is a strong multiple factorization and thus occurs as $(T, \{A, B, C\})$ in one of the lines of Table 3. Further, if one of the lines 1 – 2 of Table 3 is valid then $\sigma_1(K_i) \times \ldots \times \sigma_k(K_i) = K_i$ for $1, 2, 3$.*

Table 3: Strong Multiple Factorization $\{A, B, C\}$ of Finite Simple Groups T

	T	A	B	C
1	$Sp_{4a}(q), a \geq 2$	$Sp_{2a}(4).2$	$O^-_{4a}(2)$	$O^+_{4a}(2)$
2	$P\Omega^+_8(3)$	$\Omega_7(3)$	$\mathbb{Z}^6_3 \rtimes PSL_4(3)$	$P\Omega^+_8(2)$
3	$Sp_6(2)$	$G_2(2)$	$O^-_6(2)$	$O^+_6(2)$
		$G_2(2)'$	$O^-_6(2)$	$O^+_6(2)$
		$G_2(2)$	$O^-_6(2)'$	$O^+_6(2)$
		$G_2(2)$	$O^-_6(2)$	$O^+_6(2)'$

Source: Praeger and Schneider (2002)

3.1.8 Normalizers in Direct Products

Here, we collect together some facts about normalizers of subgroups in direct products that were used by Baddeley and Praeger (2002) in their analysis.

It is easy to see that the normalizer in a direct product $G_1 \times \ldots \times G_k$ of a subgroup H is contained in $N_{G_1}(\sigma_1(H)) \times \ldots \times N_{G_k}(\sigma_k(H))$, that is to say that $N_{G_1 \times \ldots \times G_k}(H) \leq N_{G_1}(\sigma_1(H)) \times \ldots \times N_{G_k}(\sigma_k(H))$.

Moreover, if H is the direct product $\sigma_1(H) \times \ldots \times \sigma_k(H)$ that is, $H = \sigma_1(H) \times \ldots \times \sigma_k(H)$, then

$$N_{G_1 \times \ldots \times G_k}(H) = N_{G_1}(\sigma_1(H)) \times \ldots \times N_{G_k}(\sigma_k(H)).$$

The following simple lemma extends this observation to a more general situation.

Lemma 3.21 (Baddeley *et al.,* 2006)

Let $G_1, \ldots, G_k$ be groups, set $G = G_1 \times \ldots \times G_k$ and for $i = 1, \ldots, k$, let H_i be a subgroup of G_i. Let H be a subgroup of G such that $H_1 \times \ldots \times H_k \lhd H$, the factor $N_G(H_1 \times \ldots \times H_k)/(H_1 \times \ldots \times H_k)$ is abelian and $N_{G_i}(\sigma_i(H)) = N_{G_i}(H_i)$

Then $N_G(H) = N_G(H_1 \times \ldots \times H_k) = N_{G_1}(H_1) \times \ldots \times N_{G_k}(H_k)$.

The results above are used to derive some facts concerning normalizers of the subgroups that occur in Table 1.

TABLE 4: Factorizations Of Finite Simple Groups In Proposition 3.22

	T	A	B
1	A_6	A_5	A_5
2	M_{12}	M_{11}	$M_{11}, PSL_2(11)$
3	$P\Omega_8^+(q)$	$\Omega_7(q)$	$\Omega_7(q)$
4	$Sp_4(q), q \geq 4\ even$	$Sp_2(q^2) \cdot 2$	$Sp_2(q^2) \cdot 2$

Source: Praeger and Schneider (2002)

Proposition 3.22

Suppose that $M = T_1 \times \ldots \times T_k \cong T^k$ *is a characteristically simple group and* $(M, \{K_1, K_2\})$ *is a full factorization such that, for all i, the pair* $(T_i, \{\sigma_i(K_1),\ \sigma_i(K_2)\})$ *is a full factorization and thus is as* $(T, \{A, B\})$ *in one of the rows of Table 4.*

(a) *If* $T \cong A_6$, M_{12} *or* $P\Omega_8^+(q)$, *then* K_1, K_2 *and* $K_1 \cap K_2$ *are self-normalizing in* M.

(b) *If* $(Sp_4(q), \{Sp_4(q^2) \cdot 2,\ Sp_4(q^2) \cdot 2\})$ *is a full factorization, then for* $j = 1, 2$, *we have* $N_M(K_j) = \prod_i \sigma_i(K_j)$ *and* $N_M(K_1 \cap K_2) = N_M(K_1) \cap N_M(K_2)$.

Proof

(a) In this case the $\sigma_i(K_j)$ are perfect and by Theorem 3.20, $K_j = \prod_i \sigma_i(K_j)$ for $j = 1, 2$ and Table 4 shows that $\sigma_i(K_j)$ is self normalizing in T_i for all $i \in \{1, \ldots, k\}$ and $j \in \{1, 2\}$. Therefore K_1 and K_2 are self-normalizing. That is to say,

$N_M(K_j) = N_M\left(\prod_{i=1}^k \sigma_i(K_j)\right)$ for $j = 1, 2$

Recall that $M = T_1 \times \ldots \times T_k$. Therefore we have

$$N_{T_i}\left(\sigma_i(K_j)\right) = N_{T_1 \times \ldots \times T_k}\left(\sigma_i(K_j)\right) \quad \text{for } j = 1, 2$$

$$= N_{T_1 \times \ldots \times T_k}\left(\sigma_1(K_j) \times \ldots \times \sigma_k(K_j)\right)$$

$$= N_{T_1}\left(\sigma_1(K_j)\right) \times \ldots \times N_{T_k}\left(\sigma_k(K_j)\right)$$

Since $\sigma_i(K_j)$ is self-normalizing in T_i that is $N_{T_i}\left(\sigma_i(K_j)\right) = \sigma_i(K_j)$ we have that

$$N_{T_1}\left(\sigma_1(K_j)\right) \times \ldots \times N_{T_k}\left(\sigma_k(K_j)\right) = \sigma_1(K_j) \times \ldots \times \sigma_k(K_j) = K_j$$

Hence,

$N_M(K_j) = K_j$ for $j = 1, 2$.

Therefore K_1 and K_2 are self-normalizing. Further,

$$K_1 \cap K_2 = \prod_{i=1}^{k} \sigma_i(K_1 \cap K_2) = \prod_{i=1}^{k} \sigma_i(K_1) \cap \sigma_i(K_2).$$

Using Lemma 3.19, we obtain that

$N_{T_i}(\sigma_i(K_1) \cap \sigma_i(K_2)) = \sigma_i(K_1) \cap \sigma_i(K_2) = N_{T_1 \times \ldots \times T_k}\left(\sigma_i(K_j)\right)$ for all i.

Thus,

$$N_M(K_1 \cap K_2) = N_M\left(\prod_{i=1}^{k} \sigma_i(K_1) \cap \sigma_i(K_2)\right)$$

$$= \prod_{i=1}^{k} N_{T_i}(\sigma_i(K_1) \cap \sigma_i(K_2))$$

$$= \prod_{i=1}^{k} \sigma_i(K_1) \cap \sigma_i(K_2)$$

$$= \prod_{i=1}^{k} \sigma_i(K_1 \cap K_2) = K_1 \cap K_2$$

(b) Now assume that $T \cong Sp_4(q)$ for some, $q \geq 4, even$ and let $j \in \{1, 2\}$. By theorem 3.20, $K_j' = \prod_i \sigma_i(K_j)'$ and we can read from Table 4 that

$N_{T_i}(\sigma_i(K_j)') = N_{T_i}\left(\sigma_i(K_j)\right) = \sigma_i(K_j)$ for all $i \in \{1, \ldots, k\}$

As $N_M(K_j')/K_j'$ is elementary abelian and $N_M(K_j') \geq K_j \geq K_j'$. Lemma 3.19 gives $N_M(K_j') = N_M(K_j)$.

On the other hand,

$$N_M(K_j') = \prod_{i=1}^{k} N_{T_i}(\sigma_i(K_j)') = \prod_{i=1}^{k} N_{T_i}\left(\sigma_i(K_j)\right) = \prod_{i=1}^{k} \sigma_i(K_j)$$

Now theorem 3.18 shows that $K_1' \cap K_2' = \prod_i \sigma_i(K_1{}' \cap K_2{}')$.

We also obtain from theorem 3.18 and lemma 3.19 that

$$N_{T_i}\big(\sigma_i(K_1 \cap K_2)\big) = N_{T_i}\big(\sigma_i(K_1{}' \cap K_2{}')\big)$$

$$= N_{T_i}\big(\sigma_i(K_1)\big) \cap N_{T_i}\big(\sigma_i(K_2)\big)$$

Thus,

$$N_M(K_1' \cap K_2') = \prod_{i=1}^{k} N_{T_i}\big(\sigma_i(K_1{}' \cap K_2{}')\big)$$

$$= \prod_{i=1}^{k}\Big(N_{T_i}\big(\sigma_i(K_1)\big) \cap N_{T_i}\big(\sigma_i(K_2)\big)\Big)$$

$$= \prod_{i=1}^{k} N_{T_i}\big(\sigma_i(K_1)\big) \cap \prod_{i=1}^{k} N_{T_i}\big(\sigma_i(K_2)\big)$$

$$= N_M(K_1) \cap N_M(K_2)$$

As $N_M(K_1' \cap K_2')/(K_1' \cap K_2')$ is abelian, and

$K_1' \cap K_2' \leq K_1 \cap K_2 \leq N_M(K_1' \cap K_2')$,

Lemma 3.19 implies that

$N_M(K_1 \cap K_2) = N_M(K_1' \cap K_2') = N_M(K_1) \cap N_M(K_2)$

3.1.9 Centralizers in Direct Product

We review a few facts about centralizers of subgroups in direct products that we will use in our analysis.

Proposition 3.23

The centre of a direct product is the direct product of the centres that is $Z(G_1 \times G_2 \times \ldots \times G_n) = Z(G_1) \times Z(G_2) \times \ldots \times Z(G_n)$

Note that $C_G(A) \leq N_G(A)$.

In the next section we extend the work of Baddeley *et al.,* (2002) to investigating the centralizers of the subgroups that occur in full factorizations.

3.2 Main Propositions

Now, suppose G is an innately transitive permutation group acting on a set or G-space Ω with plinth M. M is a minimal normal subgroup of G, and so M is a nonabelian characteristically simple group and hence can be written in the form $M = T_1 \times T_2 \times \cdots \times T_k$ where the T_i's (i.e. $T_1, \cdots T_k$) are finite simple normal groups each isomorphic to the same simple group T. It is important to note that the group G acts transitively by conjugation on the set $\{T_1, T_2, \cdots T_k\}$. Since T is a group isomorphic to T_i for $i = 1, \ldots, k$; we also identify M with T^k that is to say $M \cong T^k$. For $i = 1, \ldots, k$; let $\sigma_i : M \rightarrow T_1$ denote the ith projection map.

Proposition 3.24

Let π_j, $j \in \mathbb{N}$, be finite sets of primes. If M is a non-abelian, characteristically simple group with simple normal subgroups $T_1 \ldots, T_k$ and $(M, \{K_1, K_2\})$ is a full factorization then

(a) the pairs $(T_i, \{\sigma_i(K_1), \sigma_i(K_2)\})$ $(for\ i = 1, \ldots, k)$ forms π_j-groups.

(b) the subgroups $\sigma_i(K_1)$ and $\sigma_i(K_2)$cannot have normal π-complements in T_i. In particular $\sigma_i(K_1)$ and $\sigma_i(K_2)$ are not π- complements of each other.

Proof

(a) From the idea of full factorization, $\sigma_i(K_1)$ and $\sigma_i(K_2)$ are subgroups of T_i. We see also from Table 5 that the order $|T_i|$ of T_i, the order $|\sigma_i(K_1)|$ of $\sigma_i(K_1)$, and the order $|\sigma_i(K_2)|$of $\sigma_i(K_2)$ are divisible by the same set of unique primes. Hence, it follows readily that the pair $(T_i, \{\sigma_i(K_1), \sigma_i(K_2)\})$ forms π_j*-groups for* $i = 1, \ldots, k$

(b) We proceed to prove that of (b) by contradiction. Suppose that $\sigma_i(K_1)$ is the π-complement group of $\sigma_i(K_2)$ then, $\sigma_i(K_1)$ is not divisible by any primes in π (which is a contradiction). Hence $\sigma_i(K_1)$ is not a π-complement of $\sigma_i(K_2)$. Similarly, $\sigma_i(K_2)$ is not a π - complement of $\sigma_i(K_1)$.

For different sets of unique primes say π_j, $j \in \mathbb{N}$ the pair $(T_i, \{\sigma_i(K_1), \sigma_i(K_2)\})$ are said to be π_j – groups if the order $|T_i|$ of T_i, the order $|\sigma_i(K_1)|$ of $\sigma_i(K_1)$, and the order $|\sigma_i(K_2)|$ of $\sigma_i(K_2)$ are divisible by the same set of unique primes.

Proposition 3.25

From the results of Baddeley and Praeger in Proposition 3.22, we make the following deductions

(i) K_1 and K_2 are full diagonal subgroups of M

(ii) The group G is quasiprimitive

(iii) If $K_1, K_2 \ncong A_5$, $PSL_2(11)$, M_{11}, $\Omega_7(q)$, $Sp_2(q^2).2$, $Sp_2(q^2)$; then, since K_1 and K_2 are self-normalizing in M, it follows that

  (a) K_1 and K_2 are maximal nilpotent subgroups of M and K_1 and K_2 are conjugates and subsequently

  (b) K_1 and K_2 are solvable groups.

Proof

Let $M = T_1 \times \ldots \times T_k$ be a finite, non-abelian, characteristically simple group, where $T_1, \ldots, T_k$ are the simple normal subgroups of M each isomorphic to the same simple group T. Let $\sigma_i: M \to \prod_{i=1}^{k} T_i$ be a natural projection map from M to $\prod_{i=1}^{k} T_i$. Also let $K_1 = \sigma_1(K_1) \times \ldots \times \sigma_k(K_1)$ and $K_2 = \sigma_1(K_2) \times \ldots \times \sigma_k(K_2)$. Let $\phi: K_1 \to M$ be an embedding such that $\phi \circ \sigma_i: K_1 \to T_i$ is an epimorphism for $i \in \{1, .., k\}$. The projection map σ_i is an isomorphism.

(i) Since K_1 and K_2 are subgroups of M and ϕ is an inclusion map, then by Lemma 3.7, we conclude that $K_1$and K_2 are subdirect subgroups of M
Since σ_i is an isomorphism, then $\sigma_i|_{K_1}$ and $\sigma_i|_{K_2}$ will also be isomorphisms. Hence K_1 and K_2 are diagonal subgroups of M.
Since K_1 and K_2 are both subdirect and diagonal subgroups of M it follows that they are full diagonal subgroups. K_1 and K_2 are direct products of their projections; that is, $K_1 = \sigma_1(K_1) \times \ldots \times \sigma_k(K_1)$ and $K_2 = \sigma_1(K_2) \times \ldots \times \sigma_k(K_2)$.

(ii) This follows from the fact that K_1 and K_2 are subdirect subgroups of the minimal simple non-abelian group M.

(iii) (a) From the fact that M is a non-solvable group and $K_1$and K_2 are self-normalizing in M, it then follows readily that $K_1$and K_2 are maximal, nilpotent subgroups of M. This is because it is only maximal, nilpotent subgroups of a non-solvable group that can be self-normalizing in the non-solvable group.

Furthermore, it follows readily using a deep result of Thompson in 1960, that any two such maximal nilpotent subgroups of M are conjugates. Therefore K_1 and K_2 are conjugates.

(b) K_1 and K_2 are nilpotent subgroups and therefore solvable, this follows from the fact that every nilpotent group is solvable.

Also, every minimal simple group has proper subgroups that are all solvable. Now M is a minimal non-abelian simple group with K_1 and K_2 as its proper subgroups; therefore, K_1 and K_2 are solvable groups.

Proposition 3.26

Suppose that G is a finite simple group with a transitive minimal normal subgroup M . As said *M is a characteristically simple group and can be expressed as $M = T_1 \times \ldots \times T_k \cong T^k$. Suppose K_1 and K_2 are proper subgroups of M and $(M, \{K_1, K_2\})$ is a full factorization such that for all i, the pair $(T_i, \{\sigma_i(K_1), \sigma_i(K_2)\})$ is a full factorization and thus is as $(T, \{A, B\})$ in one of the rows of Table 4.*

If T is as in one of the rows 1 – 3 of Table 4 (that is if $T \cong A_6$, M_{12}, or $P\Omega_8^+(q)$) then

(a) $\sigma_i(K_1)$ *and* $\sigma_i(K_2)$ *are conjugates*

(b) $K_1 \cap K_2 = 1$

(c) $C_M(K_1 \cap K_2) = 1$

(d) $Z(K_1 \cap K_2) = C_M(K_1 \cap K_2) \cap (K_1 \cap K_2) = 1$

Proof

(a) $\sigma_i(K_1)$ and $\sigma_i(K_2)$ are conjugates since A_6 has two conjugacy classes of subgroups isomorphic to A_5 (Line 1 of Table 4). K_1 and K_2 are conjugates hence their projections $\sigma_i(K_1)$ and $\sigma_i(K_2)$ are also conjugates.

(b) K_1 and K_2 are maximal distinct subgroups of M. Hence the only element in their intersection is the identity element, 1.

(c) Recall that $\sigma_i(K_j)$ are perfect and $K_j = \prod_i \sigma_i(K_j)$ for $j = 1, 2$ (by theorem 3.20). this implies that

$K_1 = \prod_{i=1}^{k} \sigma_i(K_1) = \sigma_1(K_1) \times \ldots \times \sigma_k(K_1)$

$K_2 = \prod_{i=1}^{k} \sigma_i(K_2) = \sigma_1(K_2) \times \ldots \times \sigma_k(K_2)$

Therefore,

$$K_1 \cap K_2 = \prod_{i=1}^{k} \sigma_i(K_1 \cap K_2) = \prod_{i=1}^{k} \sigma_i(K_1) \cap \sigma_i(K_2).$$

Now,

$$C_M(K_1 \cap K_2) = C_M\left(\prod_{i=1}^{k} \sigma_i(K_1) \cap \sigma_i(K_2)\right)$$

Recall that $M = T_1 \times \ldots \times T_k$, Hence we have

$$C_M(K_1 \cap K_2) = \prod_{i=1}^{k} C_{T_i}\big(\sigma_i(K_1) \cap \sigma_i(K_2)\big)$$

$$= C_{T_1}\big(\sigma_1(K_1) \cap \sigma_1(K_2)\big) \times \ldots \times C_{T_k}\big(\sigma_k(K_1) \cap \sigma_k(K_2)\big)$$

Hence, from Lemma 3.19, we have that

$$C_M(K_1 \cap K_2) = C_{T_1}(\sigma_1(K_1)' \cap \sigma_1(K_2)') \times \ldots \times C_{T_k}(\sigma_k(K_1)' \cap \sigma_k(K_2)')$$

$= 1 \times .. \times 1 \ = 1$

Hence, $C_M(K_1 \cap K_2) = 1$ as required.

(d) This flows from the fact that the centre of a simple non-abelian simple group must be trivial.

In the next chapter we outline the results we have obtained so far from this study and also state some of their applications.

CHAPTER FOUR

RESULTS AND APPLICATIONS

Here, we present our results and some of their applications.

4.1 Results

Result I

From table 1 about the full factorization of finite simple and characteristically simple groups the following deductions were obtained:

If $(T, \{A, B\})$ is a full factorization of a finite simple group T with subgroups A and B then we have that

(i) $T = AB$, such that given any $t \in T$,there exists $a \in A$ and $b \in B$ such that $t = ab$.

(ii) The order $|T|$ of T , the order $|A|$ of A and the order $|B|$ of B are all divisible by the same set of unique prime numbers.

Using the above point and table 1 we have the following interpretations in table 5 of full factorizations of specific finite simple groups with the set of unique primes playing a vital role.

Table 5: Interpretations

	T	***A***	***B***	***UNIQUE PRIMES***
1	A_6 $\|A_6\| = 360$ $= 2^3 \times 3^2 \times 5$	A_5 $\|A_5\| = 60$ $= 2^2 \times 3 \times 5$	A_5 $\|A_5\| = 60$ $= 2^2 \times 3 \times 5$	2,3 *and* 5
2	M_{12} $\|M_{12}\| = 95040$ $= 2^6 \times 3^2 \times 5 \times 11$	M_{11} $\|M_{11}\| = 7920$ $= 2^4 \times 3^2 \times 5 \times 11$	M_{11} $\|M_{11}\| = 7920$ $= 2^4 \times 3^2 \times 5 \times 11$ $PSL_2(11)$ $\|PSL_2(11)\| = 660$ $= 2^2 \times 3 \times 5 \times 11$	2,3,5 *and* 11
3	$Sp_4(q), q \geq 4$ *even* $\|Sp_4(q)\| = 979200$ $= 2^8 \times 3^2 \times 5^2 \times 17$	$Sp_2(q^2).2$ $\|Sp_2(q^2).2\| = 16320$ $= 2^6 \times 3 \times 5^2 \times 17$	$Sp_2(q^2).2$ $\|Sp_2(q^2).2\| = 16320$ $= 2^6 \times 3 \times 5^2 \times 17$ $Sp_2(q^2)$ $\|Sp_2(q^2)\| = 8160$ $= 2^5 \times 3 \times 5^2 \times 17$	2,3,5 *and* 17

Example 4.1: From Table 5;

Let $\pi_1 = \{2,3,5\}$, $\pi_2 = \{2,3,5,11\}$, $\pi_3 = \{2,3,5,17\}$

Then the pair $(A_6, \{A_5, A_5\})$ is a π_1- group, since $|A_6|$, $|A_5|$, $|A_5|$ are divisible by the same set of primes π_1. In the same way, the pair $(M_{12}, \{M_{11}, PSL_2(11)\})$ is a π_2- group, and so on.

Result II

Deductions from the work of Baddeley *et al.,* (2006) about normalizers of subgroups that occurs in factorization.

Let G be an innately transitive group with a minimal normal subgroup M. where, $M \cong M_{11} \times M_{11}$. Let $K_{1,} = M_{11}$ and $K_2 = M_{11}$ be proper subgroups of M. Then

$$N_M(K_1) = N_M(M_{11}) = N_{M_{11}\times M_{11}}\left(\left(\sigma_1(M_{11}) \times \sigma_2(M_{11})\right)\right)$$

$$= N_{M_{11}}\left(\sigma_1(M_{11})\right) \times N_{M_{11}}\left(\sigma_2(M_{11})\right)$$

$$= \left(\sigma_1(M_{11})\right) \times \left(\sigma_2(M_{11})\right)$$

$$= M_{11} = K_1$$

This implies that M_{11} is self-normalizing.

Result III

The result we obtained about the centralizers of the subgroups that occur in the factorization.

Let G be an innately transitive group with a minimal normal subgroup M. where, $M \cong A_5 \times A_5$. Let $K_{1,} = A_5$ and $K_2 = A_5$ be proper subgroups of M. Then

$$C_M(K_1 \cap K_2) = C_M(A_5 \cap A_5)$$

$$= C_{A_5\times A_5}\left(\left(\sigma_1(A_5) \cap \sigma_1(A_5)\right) \times \left(\sigma_2(A_5) \cap \sigma_2(A_5)\right)\right)$$

$$= C_{A_5}\left(\sigma_1(A_5) \cap \sigma_1(A_5)\right) \times C_{A_5}\left(\sigma_2(A_5) \cap \sigma_2(A_5)\right)$$

$$= C_{A_5}\left(\sigma_1(A_5)\right) \times C_{A_5}\left(\sigma_2(A_5)\right)$$

$$= C_{A_5}\big(\sigma_1(A_5) \times \sigma_2(A_5)\big)$$
$$= C_{A_5}(A_5)$$
$$= Z(A_5) = 1$$

4.2 Application

Factorization of finite simple groups is very useful in the Cartesian decomposition of permutation groups. A Cartesian decomposition of a finite set Ω is a way of identifying Ω with a Cartesian product $\Gamma_1 \times \ldots \times \Gamma_l$ of smaller sets Γ_i. It is an equivalence class of identifications of Ω with a Cartesian product under a certain notion of equivalence. If Ω is a finite set, then a set $\mathcal{E} = \{\Gamma_1, \ldots, \Gamma_l\}$ of partitions of Ω is said to be a Cartesian decomposition of Ω if $|\gamma_1 \cap \ldots \cap \gamma_l| = 1$ for all $\gamma_1 \in \Gamma_1, \ldots, \gamma_l \in \Gamma_l$. Let G be an innately transitive group. The Cartesian decompositions preserved by G correspond to certain families of subgroups, called Cartesian systems, of a normal subgroup M of G. Many Cartesian decompositions correspond to direct decompositions of M.

Example 4.2

Let $\Omega = \{a, b\} \times \{a, b, c\}$. Then the identity map $(i, j) \longmapsto (i, j)$ of Ω is a bijection whose corresponding Cartesian decomposition contains the two partitions given by the column and rows of the following grid.

(a, a)	(a, b)	(a, c)
(b, a)	(b, b)	(b, c)

Hence the Cartesian decomposition corresponding to the identity map on Ω consists of the following two partitions, namely the rows of the grid,

$$\Gamma_1 = \big\{\{(a, a), (a, b), (a, c)\}, \{(b, a), (b, b), (b, c)\}\big\}$$

and the columns of the grid

$$\Gamma_2 = \big\{\{(a, a), (b, a)\}, \{(a, b), (b, b)\}, \{(a, c), (b, c)\}\big\}$$

The group $G \cong S_2 \times S_3$ in its natural action on Ω is the full stabilizer of $\{\Gamma_1, \Gamma_2\}$ in $Sym\, \Omega$ and is transitive on Ω.

Example 4.3

Let $G \cong PGL_2(9)$ and consider the unique transitive action of G on a set Ω of 36 points. The group G is innately transitive on Ω, because G has a unique minimal normal subgroup $M \cong A_6$ and M is transitive on Ω. Moreover if $w \in \Omega$, then $M_w \cong D_{10}$. Now M has subgroups K_1, K_2 both isomorphic to A_5, such that $\{K_1,\ K_2\}$ is a Cartesian system of subgroups in M with respect to w. Hence G preserves a Cartesian decomposition $\{\Gamma_1,\ \Gamma_2\}$ of Ω, where each Γ_i has six parts of six 6.

Finally we summarize and conclude our work. We also give recommendations for further studies and outline our contributions to knowledge.

CHAPTER FIVE

SUMMARY, CONCLUSION AND RECOMMENDATIONS

5.1 Summary

The finite simple groups are important to mathematicians because in a certain sense, they are the basic building blocks of all finite groups, somewhat similar to the way prime numbers are the building blocks of integers. These "building blocks" are called the composition factors of the group. These composition factors are simple groups. In a certain sense, a group can be reconstructed from its composition factors and many of the properties of a group are determined by the nature of its composition factors. Many questions about finite groups can be reduced (by induction) to questions about simple groups. This fact makes the classification theorem to have applications in many branches of mathematics. Factorization of groups, on the other hand also play a constitutive role in group theory and geometry.

Our main objective in this thesis was to study the factorization of simple and characteristically simple groups. After presenting a lot of group theoretic concepts related to this research, we proved some facts about normalizers of subgroups; extended the result to that of the centralizers of the subgroups that occurred and also outlined some important consequences of these results. Most of our results depended on the correctness of the finite simple group classification. For instance, a lot of information on the factorization of simple and characteristically simple groups that depend on this classification is used in this work.

From the results we obtained that the subgroups K_1 and K_2 are self-normalizing in M that is $N_M(K_1) = K_1$, $N_M(K_2) = K_2$ and $N_M(K_1 \cap K_2) = K_1 \cap K_2$. Also, the centralizer of $K_1 \cap K_2$ is $\{1\}$ that is $C_M(K_1 \cap K_2) = \{1\}$. The normalizers of the subgroups, K_1, K_2 and $K_1 \cap K_2$ form a group. Likewise their centralizers also form a group (though a trivial group). The centralizer subgroup $C_M(K_1 \cap K_2)$ which corresponds to the centre of $K_1 \cap K_2$ is a normal subgroup of the normalize subgroup $N_M(K_1 \cap K_2)$ which buttresses the fact that the centre of every non-abelian group is trivial, that is $\{1\}$, since it is the only normal subgroup of K_1, K_2 apart from the groups themselves.

5.2 Conclusion

The factorization of finite simple and characteristically simple groups are very important in the study of finite groups. We used the result to explain some facts about normalizers and centralizers of the subgroups that occur.

The results obtained in this thesis helps to bring to limelight some important facts concerning minimal simple groups.

The fact that M is a minimal non-abelian simple group with K_1 and K_2 as proper subgroups; K_1 and K_2 being self normalizing in M; if $K_1, K_2 \ncong A_5$, $PSL_2(11), M_{11}, Sp_2(q^2).2, Sp_2(q^2)$, then it implies that they are maximal nilpotent subgroups hence solvable, if they are not isomorphic to any of the simple groups highlighted in table 1.

This further implies that every minimal simple group has solvable proper subgroups. We also obtained that the centralizer of $K_1 \cap K_2$ in M is the identity element implying K_1 and K_2 are distinct maximal subgroups of M. It is noteworthy that determining all relevant factorizations of characteristically simple group is still a difficult task.

5.3 Recommendations

The simple groups we investigated in the thesis are those whose factorizations were readily available. The subgroups of these factorizations were almost simple or perfect, which made possible our investigations of their normalizers and centralizers. We, therefore recommend that researchers should extend this work by taking into the consideration the simple groups whose factorizations are not readily available.

5.4 Contribution to Knowledge

This study has contributed to existing knowledge in the following ways:

i. It has shown that given an innately transitive group G which contains a minimal normal subgroup M. If K_1 and K_2 are proper self-normalizing subgroups of M not isomorphic to A_5, $PSL_2(11), M_{11}$, $Sp_2(q^2).2$, $Sp_2(q^2)$, then K_1 and K_2 are nilpotent subgroups and hence solvable.

ii. It has also shown that the centralizer subgroup of K_1 and K_2 (described in (i) above) in M, that is, $C_M(K_1 \cap K_2)$ is equal to the trivial subgroup $\{1\}$.

iii. Thus, the centralizer subgroup $C_M(K_1 \cap K_2)$ which corresponds to the centre of $K_1 \cap K_2$ is a normal subgroup of the normalizer subgroup $N_M(K_1 \cap K_2)$ which buttresses the fact that the centre of every non-abelian group is trivial, that is, $\{1\}$, since it is the only normal subgroup of K_1, K_2 apart from the groups themselves.

iv. The study has been able to present the study of simple groups and its factorization especially full factorization involving unique primes in a clear format so that anyone with some basic knowledge of Algebra will comprehend.

v. It has also been able to group the full factorizations of some simple groups into π_j- groups.

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